Notre Dame Journal of Formal Logic
Volume XII, Number 2, April 1971

SOME EMBEDDING THEOREMS FOR MODAL LOGIC

DAVID MAKINSON

We* shall prove some embedding theorems for modal logic, that is, theorems to the effect that every consistent modal logic satisfying certain general conditions is a sublogic of certain particular logics. Our results are related to those of McKinsey [1] and Tarski [2].

We begin with terminology. *Formulae* are understood to be built from a denumerable list of elementary letters by means of the operators $\neg$, $\wedge$, $\Box$, with other operators introduced as usual. By a *modal logic* we mean any set S of formulae that contains all the tautologies in $\neg$ and $\wedge$ and is closed under the operations of substitution (of arbitrary formulae for elementary letters) and detachment $(\alpha, \alpha \supset \beta / \beta)$. We say that a set S of formulae is closed under *congruence* if whenever $(\alpha \equiv \beta) \in S$ then $(\Box\alpha \equiv \Box\beta) \in S$; closed under *monotony* if whenever $(\alpha \supset \beta) \in S$ then $(\Box\alpha \supset \Box\beta) \in S$; closed under *antitony* if whenever $(\alpha \supset \beta) \in S$ then $(\Box\beta \supset \Box\alpha) \in S$.

By a *modal algebra* we mean a structure $\mathfrak{A} = \langle A, -, \cap, * \rangle$ where $\langle A, -, \cap \rangle$ is a Boolean algebra and $*$ is a unary operation over A. A modal algebra is said to be *monotonic* if for all $x, y \in A$, $x \leq y$ implies $*x \leq *y$, and is said to be *antitonic* if for all $x, y \in A$, $x \leq y$ implies $*y \leq *x$. Among the modal algebras there are clearly just four that can be obtained by adding a unary operation to the two-element Boolean algebra: we shall call these the *unit* algebra ($*1 = 1$, $*0 = 1$), the *identity* algebra ($*1 = 1$, $*0 = 0$), the *complement* algebra ($*1 = 0$, $*0 = 1$), and the *zero* algebra ($*1 = 0$, $*0 = 0$). Each of these four algebras determines a corresponding set of formulae, consisting of just those formulae that are valid in the algebra, that is, just those formulae α such that for every homomorphism h from formulae into that algebra, $h(\alpha) = 1$. It is easy to verify that each of these four sets of formulae is a modal logic in the sense defined, is closed either under monotony or under antitony, and can be axiomatized in a trivial way: we refer to these four sets of formulae as the *unit*, *identity*, *complement*, and *zero* modal logics respectively.

---

*Work for this paper was carried out while the author was on contract with the Organization of American States, Department of Scientific Affairs.

Received March 1, 1970

Theorem 1. *Let S be any consistent modal logic that is closed under congruence and contains the theses $\Box(p \vee \neg p)$ and $\neg\Box(p \wedge \neg p)$. Then S is a sublogic of the identity logic.*

Proof: We use an algebraic argument. Define a relation over formulae by putting $\alpha \simeq \beta \pmod{S}$ if $(\alpha \equiv \beta) \in S$. It can be verified that since S is a modal logic in the sense defined, the relation is an equivalence and is congruential with respect to the operators $\neg$ and $\wedge$. Further, since S is closed under congruence, the relation is clearly congruential with respect to the modal operator $\Box$. Hence we can form a Lindenbaum algebra $|S|$ of S as the quotient structure determined by the relation. It can be verified that $|S|$ is a modal algebra $\langle A, -, \cap, *\rangle$ which, since S is consistent, has at least two elements. Also, for every formula α, $\alpha \in S$ if and only if α is valid in $|S|$—that is, if and only if $h(\alpha) = 1$ for every homomorphism form formulae into $|S|$.

Moreover, since $\Box(p \vee \neg p) \in S$ and $\neg\Box(p \wedge \neg p) \in S$ we have $*1 = 1$ and $*0 = 0$. Thus the set $\{1, 0\}$ consisting of the unit and zero elements of $|S|$ is closed under all three operations $\neg$, $\wedge$, $\Box$, and so forms a subalgebra of $|S|$ which clearly coincides with the identity algebra. Now on universal algebraic grounds every formula that is valid in $|S|$ is valid in all of its subalgebras. So since $|S|$ is characteristic for S, we have that S is a sublogic of the identity logic.

Theorem 2. *Let S be any consistent modal logic that is closed under monotony. Then S is a sublogic of the identity logic, or the zero logic, or the unit logic.*

Proof: Clearly any modal logic that is closed under monotony is closed under congruence, and we can form a characteristic Lindenbaum algebra in the same way as in the proof of theorem 1. At this point the argument divides into three cases.

Case 1. Suppose that $\Box(p \vee \neg p) \in S$ and $\neg\Box(p \wedge \neg p) \in S$. Then the conditions of theorem 1 are satisfied and so S is a sublogic of the identity logic.
Case 2. Suppose that $\Box(p \vee \neg p) \notin S$. Then $*1 \neq 1$ and so by standard results on Boolean algebras there is an ultrafilter X of $|S|$ with $-*1 \in X$. Now it can be verified that since S is closed under monotony, $|S|$ is monotonic. Thus for all $x \in |S|$ we have $x \leq 1$ and so $*x \leq *1$ and so $-*1 \leq -*x$ and so $-*x \in X$ and so $*x \notin X$. We define a function h from $|S|$ into the zero algebra as follows: if $x \in X$ put $h(x) = 1$, and if $x \notin X$ put $h(x) = 0$. Since X is an ultrafilter, h is homomorphic with respect to the Boolean operations. Further, for each $x \in |S|$ we have $h(*x) = 0 = *h(x)$ and so h is a homomorphism from $|S|$ into, and indeed clearly onto, the zero algebra. Now on universal algebraic grounds every formula that is valid in $|S|$ is valid in all of its homomorphic images, and so since $|S|$ is characteristic for S, we have that S is a sublogic of the zero logic.
Case 3. Suppose that $\neg\Box(p \wedge \neg p) \notin S$. Then $*0 \neq 0$, and so by standard results on Boolean algebras there is an ultrafilter X of $|S|$ with $*0 \in X$. We can use an argument similar to that of the second case to show that S is a sublogic of the unit logic.

Theorem 3. *Let S be any consistent modal logic that is closed under antitony. Then S is a sublogic of the complement logic, or the unit logic, or the zero logic.*

Proof: We can use the same kind of argument as for theorem 2.

REFERENCES

[1] McKinsey, J. C. C., "On the number of complete extensions of the Lewis systems of sentential calculus," *The Journal of Symbolic Logic*, vol. 9 (1944), pp. 42-45.

[2] Tarski, A., "On extensions of incomplete systems of the sentential calculus," in *Logic, Semantics, Metamathematics*, transl. by J. Woodger, Oxford: Clarendon Press (1956).

American University of Beirut
Beirut, Lebanon