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SOME ETA-IDENTITIES ARISING FROM THETA SERIES

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1. Introduction and results.

For Im(z) > 0 the Dedekind η-function is defined by the product formula

$$\eta(z) = e\left(\frac{z}{24}\right)\prod_{n=1}^{\infty}(1 - e(nz))$$

where we use the notation $e(w) = e^{2\pi i w}$ for any w in C. For rational integers m and n, $\left(\frac{m}{n}\right)$ denotes the Legendre-Jacobi-Kronecker symbol. In the following theorem I list seven identities for η. They will be derived from theta series identities in my previous paper [4].

THEOREM. *The η-function satisfies the identities*

(1) $$\frac{\eta^5(2z)}{\eta^2(z)} = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n}{3}\right)ne\left(\frac{n^2 z}{3}\right),$$

(2) $$\frac{\eta^5(z)}{\eta^2(2z)} = \sum_{n>0\,\mathrm{odd}}\left(\frac{n}{3}\right)ne\left(\frac{n^2 z}{24}\right),$$

(3) $$\frac{\eta^2(z)\eta^2(4z)}{\eta(2z)} = \sum_{n=1}^{\infty}\left(\frac{n}{3}\right)ne\left(\frac{n^2 z}{3}\right),$$

(4) $$\frac{\eta^9(2z)}{\eta^3(z)\eta^3(4z)} = \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)ne\left(\frac{n^2 z}{8}\right),$$

(5) $$\frac{\eta^{13}(2z)}{\eta^5(z)\eta^5(4z)} = \sum_{n=1}^{\infty}\left(\frac{-6}{n}\right)ne\left(\frac{n^2 z}{24}\right),$$

(6) $$\frac{\eta^6(3z)}{\eta^2(6z)} = \frac{\eta^3(2z)\eta^2(9z)}{\eta(18z)} + 3\frac{\eta^3(18z)\eta^2(z)}{\eta(2z)},$$

(7) $$\frac{\eta^7(6z)}{\eta(3z)} = \frac{\eta^3(4z)\eta^9(18z)}{\eta^3(9z)\eta^3(36z)} + \frac{\eta^3(36z)\eta^9(2z)}{\eta^3(z)\eta^3(4z)}.$$

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The identities (1), (2) and (3) appear as special cases of the Macdonald identities in the theory of affine Lie algebras; see the introduction and the appendix in [6]. The identities (2) and (3) have already been deduced by Gordon (formulas (11) and (12) in [1]) from a "quintuple product identity", just as Euler's and Jacobi's identities for η and η^3 follow from the Jacobi triple product identity. Klyachko [3] exhibited a new proof of (2) as a corollary from his results on projective representations of symmetric groups over fields of characteristic p. Klyachko also rediscovered the identity

$$\frac{\eta(2z)\eta^2(3z)}{\eta(z)\eta(6z)} = \sum_{\substack{n\in Z\\ n\equiv 1 (\mathrm{mod}\, 6)}} e\left(\frac{n^2 z}{24}\right)$$

which is contained as an example in Kac [2] and which I cannot prove by my methods. There are more η-identities in the lists of examples in Kac [2] and in Lepowsky [5] which I cannot prove by my methods. On the other hand, these lists do not contain (4) or (5). I do not know whether the identities (4) to (7) are new.

The identities (1) to (5) resemble the Jacobi identity

$$\eta^3(z) = \sum_{n=1}^{\infty} \left(\frac{-1}{n}\right) n e\left(\frac{n^2 z}{8}\right)$$

while the above mentioned Kac-Klyachko identity and several others in [2], [5] resemble the Euler identity

$$\eta(z) = \sum_{n>0 \text{ odd}} \left(\frac{3}{n}\right) e\left(\frac{n^2 z}{24}\right).$$

The proof of the Theorem is summarized as follows. In a recent paper [4] I listed many modular forms on the theta group Γ_ϑ which are represented by theta series with a grössencharacter attached to an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ with $d \in \{1, 2, 3, 6\}$. These results can as well be stated for the conjugate group $\Gamma_0(2)$. Some of the partial series of these theta series, representing modular forms of weight 2 or 3 on $\Gamma_0(2)$, turn out to split into a product of two simple series. One of the factors can be identified, by means of Jacobi's identity or some previously proved identity, with a well known function. In this way all of the results in the Theorem will follow.

The classical theta function

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e\left(\frac{n^2 z}{2}\right)$$

will occur in the proof. It is a modular form of weight $\frac{1}{2}$ on the theta group Γ_ϑ

which is generated by the transformations

$$z \to z + 2 \text{ and } z \to -\frac{1}{z}$$

of the upper half plane. The congruence group $\Gamma_0(2)$ consists of all transformations of the upper half plane with matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbf{Z})$ satisfying $c \equiv 0 (\bmod 2)$. We have

$$\Gamma_\vartheta = M^{-1}\Gamma_0(2)M \text{ with } M(z) = \frac{1}{2}(z + 1).$$

It follows that $f(z)$ is a modular form of weight k on Γ_ϑ if and only if $g(z) = f(2z - 1)$ is a modular form of weight k on $\Gamma_0(2)$. The identity

$$\vartheta(2z - 1) = \eta^2(z)/\eta(2z)$$

is well known and will be needed; it is due to Gauss.

2. Proofs.

2.1. From [4, Theorem 11] we know that

$$\vartheta^{\frac{9}{2}}\eta^{\frac{3}{2}}(z) + 8\vartheta^{-\frac{3}{2}}\eta^{\frac{15}{2}}(z) = \Theta_3(\chi_8, z) = \frac{1}{4}\cdot\sum_{\mu}\chi_8(\mu)\mu^2 e\left(\mu\bar{\mu}\frac{z}{16}\right)$$

where the summation is on all μ in the ring $\mathbf{Z}[i]$ of Gaussian integers and χ_8 is a Dirichlet character modulo 8 on $\mathbf{Z}[i]$ which can be defined by its values $\chi_8(i) = -1$, $\chi_8(3) = 1$, $\chi_8(2 + i) = -i$ at the integers i, 3, $2 + i$ whose residue classes modulo 8 generate the group of coprime residues modulo 8 in $\mathbf{Z}[i]$. We obtain $\vartheta^{-\frac{3}{2}}\eta^{\frac{15}{2}}$ by restricting the summation to $\mu\bar{\mu} \equiv 5 (\bmod 8)$. In that case we have $\chi_8(\bar{\mu}) = -\chi_8(\mu)$, and we can choose $\mu = a + bi$ uniquely among its associates such that a is positive and odd, and $b \equiv 2 (\bmod 4)$. Since $\mu^2 - \bar{\mu}^2 = 4abi$, we get

$$\vartheta^{-\frac{3}{2}}\eta^{\frac{15}{2}}(z) = \sum_{\substack{m > 0 \text{ odd} \\ n > 0 \text{ odd}}} i\chi_8(m + 2ni)mne\left(\left(m^2 + 4n^2\right)\frac{z}{16}\right)$$

We replace z by $2z - 1$, use the Gauss identity, and obtain

$$\frac{\eta^9(2z)}{\eta^3(z)} = \sum_{\substack{m > 0 \text{ odd} \\ n > 0 \text{ odd}}} e\left(\frac{5 - m^2 - 4n^2}{16}\right)i\chi_8(m + 2ni)mne\left(\left(m^2 + 4n^2\right)\frac{z}{8}\right).$$

The definition of χ_8 yields

$$e\left(\frac{5 - m^2 - 4n^2}{16}\right)i\chi_8(m + 2ni) = \left(\frac{-2}{m}\right)\cdot\left(\frac{-1}{n}\right)$$

for m, n odd. Thus the double series splits into a product of simple series,

$$\frac{\eta^9(2z)}{\eta^3(z)} = \sum_{m=1}^{\infty}\left(\frac{-2}{m}\right)me\left(\frac{m^2z}{8}\right)\cdot\sum_{n=1}^{\infty}\left(\frac{-1}{n}\right)ne\left(\frac{n^2z}{2}\right).$$

The series on n is $\eta^3(4z)$. This proves (4).

2.2. From [4, Theorem 20] we know that

$$4\sqrt{-6}\vartheta^{\frac{5}{2}}\eta^{\frac{7}{2}} \text{ and } 8\sqrt{-6}\vartheta^{\frac{1}{2}}\eta^{\frac{11}{2}}$$

are components in a theta series

$$\Theta_3(\varphi_{24}, z) = \frac{1}{2}\sum_{\mu}\varphi_{24}(\mu)\mu^2 e\left(\mu\bar{\mu}\frac{z}{48}\right)$$

attached to the field $\mathbb{Q}(\sqrt{-6})$. Here, μ runs through a set I of ideal numbers for this field, and φ_{24} is a certain character of the group of coprime residue classes modulo 24 in I. We obtain $\vartheta^{\frac{5}{2}}\eta^{\frac{7}{2}}$ by restricting the summation to all $\mu \equiv m + n\sqrt{-6}$ which satisfy $\mu\bar{\mu} \equiv 7 \pmod{24}$, i.e., $m \equiv \pm 1 \pmod 6$ and $n \equiv 1 \pmod 2$. For these μ we have $\varphi_{24}(\bar{\mu}) = -\varphi_{24}(\mu)$, and from $\mu^2 - \bar{\mu}^2 = 4mn\sqrt{-6}$ we get

$$\begin{aligned}\vartheta^{\frac{5}{2}}\eta^{\frac{7}{2}}(z) &= \frac{1}{8\sqrt{-6}}\sum_{\mu\bar{\mu}\equiv 7(24)}\varphi_{24}(\mu)\mu^2 e\left(\mu\bar{\mu}\frac{z}{48}\right)\\ &= \sum_{\substack{m,n>0\\ m\equiv\pm1(6), n\equiv 1(2)}}\varphi_{24}(m+n\sqrt{-6})mne\left(\left(m^2+6n^2\right)\frac{z}{48}\right).\end{aligned}$$

We replace z by $2z - 1$ and obtain

$$\eta^5(z)\eta(2z) = \sum_{\substack{m,n>0\\ m\equiv\pm1(6), n\equiv 1(2)}} e\left(\frac{7-m^2-6n^2}{48}\right)\varphi_{24}(m+n\sqrt{-6})mne\left(\left(m^2+6n^2\right)\frac{z}{24}\right).$$

The definition of φ_{24} yields

$$e\left(\frac{7-m^2-6n^2}{48}\right)\varphi_{24}(m+n\sqrt{-6}) = \left(\frac{m}{3}\right)\left(\frac{-1}{n}\right)$$

for m, n as in the summation. Thus the double series splits into a product of simple series,

$$\eta^5(z)\eta(2z) = \sum_{m>0 \text{ odd}}\left(\frac{m}{3}\right)me\left(\frac{m^2z}{24}\right)\cdot\sum_{n=1}^{\infty}\left(\frac{-1}{n}\right)ne\left(\frac{n^2z}{4}\right).$$

The series on n is $\eta^3(2z)$. This proves (2).

Identity (2) can also be deduced, essentially in the same way, from a formula for $\vartheta^{\frac{7}{2}}\eta^{\frac{1}{2}}$ in [4, Theorem 19]. Here, a theta series of weight 2 appears which is attached to another character modulo 24 on I.

Dealing now with $\vartheta^{\frac{1}{2}}\eta^{\frac{11}{2}}$ in the same way, we obtain

$$\vartheta^{\frac{1}{2}}\eta^{\frac{11}{2}}(z) = \sum_{\substack{m,n>0 \\ m\equiv 1(2), n\equiv \pm 1(3)}} \varphi_{24}(m\sqrt{3} + 2n\sqrt{-2})\, mne\left(\left(3m^2 + 8n^2\right)\frac{z}{48}\right)$$

and

$$\eta(z)\eta^5(2z) =$$

$$= \sum_{\substack{m,n>0 \\ m\equiv 1(2), n\equiv \pm 1(3)}} e\left(\frac{11 - 3m^2 - 8n^2}{48}\right)\varphi_{24}\left(m\sqrt{3} + 2n\sqrt{-2}\right)\mathrm{mne}\left(\left(3m^2 + 8n^2\right)\frac{z}{24}\right)$$

$$= \sum_{m=1}^{\infty}\left(\frac{-1}{m}\right)me\left(\frac{m^2 z}{8}\right)\cdot\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n}{3}\right)ne\left(\frac{n^2 z}{3}\right).$$

This proves (1).

2.3. The identity (1) can also be deduced from a formula for $\vartheta^{-1}\eta^5$ in [4, Theorem 12]. According to this theorem we have

$$\vartheta^3\eta(z) + 4\vartheta^{-1}\eta^5(z) = \Theta_2(\chi_{12}, z) = \frac{1}{4}\sum_{\mu}\chi_{12}(\mu)\,\mu e\left(\mu\bar{\mu}\frac{z}{24}\right)$$

where μ runs through $\mathsf{Z}[i]$ and χ_{12} is a certain Dirichlet character modulo 12 on $\mathsf{Z}[i]$. We obtain $\vartheta^3\eta$ by restricting the summation to $\mu\bar{\mu} \equiv 1 \pmod{12}$. In that case we can choose $\mu = a + bi$ among its associates such that a is positive and either $a \equiv \pm 1 \pmod 6$, $b \equiv 0 \pmod 6$, or $a \equiv 3 \pmod 6$, $b \equiv \pm 2 \pmod 6$. Combining the contributions of μ and $\bar{\mu}$, looking at the definition of χ_{12}, and replacing z by $2z - 1$, we get

$$\frac{\eta^6(z)}{\eta^2(2z)} = \sum_{\substack{m>0 \\ m\equiv \pm 1(6)}}\left(\frac{-1}{m}\right)me\left(\frac{m^2 z}{12}\right)\sum_{n\in\mathsf{Z}}(-1)^n e(3n^2 z)$$

$$+ 3\sum_{m=1}^{\infty}\left(\frac{-1}{m}\right)me\left(\frac{6m^2 z}{8}\right)\sum_{\substack{n\in\mathsf{Z} \\ n\equiv \pm 1(3)}}(-1)^n e\left(\frac{n^2 z}{3}\right).$$

From Jacobi's identity we infer that

$$\sum_{\substack{m>0 \\ m\equiv \pm 1(6)}}\left(\frac{-1}{m}\right)me\left(\frac{m^2 z}{12}\right) = \eta^3\left(\frac{2z}{3}\right) + + 3\eta^3(6z).$$

Similarly, the series on n can be expressed by values of ϑ. Thus a final replacement of z by $3z$ yields the identity (6).

2.4. From [4, Theorem 13] we know that

$$\vartheta^5\eta(z) + 8\vartheta\eta^5(z) = \Theta_3(\tilde{\chi}_{12}, z) = \frac{1}{4}\sum_{\mu} \tilde{\chi}_{12}(\mu)\mu^2 e\left(\mu\bar{\mu}\frac{z}{24}\right)$$

where μ runs through $\mathsf{Z}[i]$ and $\tilde{\chi}$ is another Dirichlet character modulo 12 on $\mathsf{Z}[i]$. We obtain $\vartheta\eta^5$ by restricting the summation to $\mu\bar{\mu} \equiv 5 \pmod{12}$. In that case we have $\tilde{\chi}_{12}(\bar{\mu}) = -\tilde{\chi}_{12}(\mu)$, and we can choose $\mu = a + bi$ uniquely among its associates such that

$$a > 0, a \equiv \pm 1 \pmod 6, b \equiv \pm 2 \pmod 6).$$

Combining the contributions of μ and $\bar{\mu}$, replacing z by $2z - 1$, and looking at the definition of $\tilde{\chi}_{12}$, we get

$$\eta^2(z)\eta^4(2z) = \sum_{m>0\,\mathrm{odd}} \left(\frac{m}{3}\right) me\left(\frac{m^2 z}{12}\right) \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) ne\left(\frac{n^2 z}{3}\right).$$

Because of (2), the series on m is $\eta^5(2z)/\eta^2(4z)$. This proves (3).

2.5. From [4, Theorem 15] we know that

$$24\vartheta^{-\frac{1}{2}}\eta^{\frac{13}{2}} \text{ and } 16\vartheta^{-\frac{5}{2}}\eta^{\frac{17}{2}}$$

are components in a theta series

$$\Theta_3(\chi_{24}, z) = \frac{1}{4}\sum_{\mu} \chi_{24}(\mu)\mu^2 e\left(\mu\bar{\mu}\frac{z}{48}\right)$$

where μ runs through $\mathsf{Z}[i]$ and χ_{24} is a certain Dirichlet character modulo 24 on $\mathsf{Z}[i]$. We obtain $\vartheta^{-\frac{1}{2}}\eta^{\frac{13}{2}}$ by restricting the summation to $\mu\bar{\mu} \equiv 13 \pmod{24}$. In that case we have $\chi_{24}(\bar{\mu}) = -\chi_{24}(\mu)$, and we can choose $\mu = a + bi$ uniquely among its associates such that a is positive and either $a \equiv \pm 1 \pmod 6$, $b \equiv 6 \pmod{12}$, or $a \equiv 3 \pmod 6$, $b \equiv \pm 2 \pmod{12}$. Combining the contributions of μ and $\bar{\mu}$, replacing z by $2z - 1$, and looking at the definition of χ_{24}, we get

$$\frac{\eta^7(2z)}{\eta(z)} = \sum_{\substack{m>0 \\ m \equiv \pm 1(3)}} \left(\frac{-2}{m}\right) me\left(\frac{m^2 z}{24}\right) \sum_{n=1}^{\infty} \left(\frac{-1}{n}\right) ne\left(\frac{3n^2 z}{2}\right)$$
$$+ \sum_{m=1}^{\infty} \left(\frac{-2}{m}\right) me\left(\frac{3m^2 z}{8}\right) \sum_{\substack{n>0 \\ n \equiv \pm 1(3)}} \left(\frac{-1}{n}\right) ne\left(\frac{n^2 z}{6}\right).$$

As in subsection 2.3, the series on n can be expressed by values of η^3. Similarly, we use identity (4) to express the series on m by combinations of η-values. Thus a final replacement of z by $3z$ yields the identity (7).

Now the summation in $\Theta_3(\chi_{24}, z)$ is restricted to $\mu\bar{\mu} \equiv 17 \pmod{24}$. Then the same procedure as above yields

$$\frac{\eta^{11}(2z)}{\eta^5(z)} = \sum_{m=1}^{\infty}\left(\frac{-6}{m}\right)me\left(\frac{m^2z}{24}\right)\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n}{3}\right)ne\left(\frac{2n^2z}{3}\right).$$

Because of (1), the series on n is $\eta^5(4z)/\eta^2(2z)$. This proves (5).

3. Remarks.

3.1. There are some more identities which can be deduced from the results in [4] and which express combinations of η-values explicitely as Fourier series. However, these identities concern non-cusp forms and look less spectacular. Two examples of this kind are

$$\frac{\eta^8(2z)}{\eta^4(z)} = \sum_{n>0\,\mathrm{odd}}\left(\sum_{d\mid n} d\right)e\left(\frac{nz}{2}\right),$$

$$\frac{\eta^{12}(2z)}{\eta^6(z)} \sum_{\substack{n>0\\ n\equiv 3(4)}}\left(-\frac{1}{8}\sum_{d\mid n}\left(\frac{-1}{d}\right)d^2\right)e\left(\frac{nz}{4}\right).$$

In the theta series of section 2 we can restrict the summation to subseries in which the character-values at μ and $\bar{\mu}$ agree. Then we obtain identities which express combinations of η-values as double series which do not split into a product of two simple series, but which nevertheless may be of some interest. For example, if we restrict the summation in $\Theta_3(\chi_{24}, z)$ to $\mu\bar{\mu} \equiv 5 \pmod{24}$ then we get the identity

$$\frac{\eta^7(z)}{\eta(2z)} = \sum_{m,n>0\,\mathrm{odd}}\left(\frac{-3}{m}\right)\left(\frac{6}{n}\right)\frac{4m^2-n^2}{3}e\left(\left(4m^2+n^2\right)\frac{z}{24}\right).$$

3.2. Several easy consequences can be deduced from the Theorem. For example, it follows from (1), (2), (3) that

$$\frac{\eta^5(2z)}{\eta^2(z)} + \frac{\eta^2(z)\eta^2(4z)}{\eta(2z)} = 2\frac{\eta^5(8z)}{\eta^2(16z)},$$

$$\frac{\eta^5(2z)}{\eta^2(z)} - \frac{\eta^2(z)\eta^2(4z)}{\eta(2z)} = 4\frac{\eta^2(4z)\eta^2(16z)}{\eta(8z)}.$$

3.3. We can also deduce some identities for the representation of integers by ternary quadratic forms. Let us write

$$(\eta^3(2z))^3 = \eta^3(z)\cdot\eta^3(4z)\cdot\frac{\eta^9(2z)}{\eta^3(z)\eta^3(4z)} = \frac{\eta^5(z)}{\eta^2(2z)}\cdot\frac{\eta^5(4z)}{\eta^2(2z)}\cdot\frac{\eta^{13}(2z)}{\eta^5(z)\eta^5(4z)}.$$

Here we insert (1), (2), (4), (5), and Jacobi's identity, and we compare coefficients. Then we obtain

$$\sum_{x^2+y^2+t^2=n}\left(\frac{-1}{xyt}\right)xyt=\sum_{x^2+4y^2+t^2=2n}\left(\frac{-1}{xy}\right)\left(\frac{-2}{t}\right)xyt=$$
$$\sum_{x^2+16y^2+t^2=6n}(-1)^{y-1}\left(\frac{xy}{3}\right)\left(\frac{-6}{t}\right)xyt$$

where x, y, t run through all positive integers which satisfy the stated equalities. These identities can be used to check the results against errors. Similarly, we write

$$(\eta(2z))^2\cdot\eta^3(2z)=(\eta(4z))^2\cdot\frac{\eta^5(2z)}{\eta^2(4z)}=(\eta(z))^2\cdot\frac{\eta^5(2z)}{\eta^2(z)}$$

and use (1), (2) and Euler's and Jacobi's identity. This yields

$$\sum_{x^2+y^2+3t^2=n}\left(\frac{3}{xy}\right)\left(\frac{-1}{t}\right)t=\sum_{2x^2+2y^2+t^2=n}\left(\frac{3}{xy}\right)\left(\frac{t}{3}\right)t=$$
$$\sum_{x^2+y^2+8t^2=2n}(-1)^{t-1}\left(\frac{3}{xy}\right)\left(\frac{t}{3}\right)t$$

where x, y, t run through positive integers and x, y are odd.

3.4. Jacobi's identity for η^3 can also be recovered from the results in [4]: use the formula for $\vartheta^{\frac{5}{2}}\eta^{\frac{3}{2}}$ in [4, Theorem 16].

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