Generic pro-p Hecke algebras, the Hecke algebra of $\mathrm{PGL}_2(\mathbb{Z})$, and the cohomology of root data

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Generic pro-p Hecke algebras

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Abstract

This is a contribution to the theory of Hecke algebras. A class of algebras called *generic pro-p Hecke algebras* is introduced, enlarging the class of generic Hecke algebras by considering certain extensions of (extended) Coxeter groups. Examples of generic pro-p Hecke algebras are given by pro-p-Iwahori Hecke algebras and Yokonuma-Hecke algebras. The notion of an *orientation* of a Coxeter group is introduced and used to define 'Bernstein maps' intimately related to Bernstein's presentation and to Cherednik's cocycle. It is shown that certain relations in the Hecke algebra hold true, equivalent to Bernstein's relations in the case of Iwahori-Hecke algebras.

For a certain subclass called *affine pro-p Hecke algebras*, containing Iwahori-Hecke and pro-p-Iwahori Hecke algebras, an explicit canonical and integral basis of the center is constructed and finiteness results are proved about the center and the module-structure of the algebra over its center, recovering results of Bernstein-Zelevinsky-Lusztig and Vignéras.

Zusammenfassung

Es wird ein Beitrag zur Theorie der Hecke-Algebren geleistet. Speziell wird eine Klasse von Algebren eingeführt, die *generischen pro-p Hecke-Algebren*, welche die Klasse der generischen Hecke-Algebren erweitert durch Übergang von Coxetergruppen zu Erweiterungen solcher durch abelsche Gruppen. Beispiele sind gegeben durch pro-p-Iwahori Hecke-Algebren und Yokonuma-Hecke Algebren. Es wird der Begriff der *Orientierung* einer Coxetergruppe eingeführt und benutzt um sogenannte *Bernsteinabbildungen* definieren, welche eng verwandt sind mit der Bernsteinpräsentierung und dem Cherednik-Kozykel. Sodann wird gezeigt, dass zwischen den Bildern der Bernsteinabbildungen gewisse Relationen herrschen, welche sich im Spezialfall der Iwahori-Hecke Algebra auf die bekannten Bernsteinrelationen reduzieren.

Ferner wird für die Unterklasse der *affinen pro-p Hecke-Algebren*, welche sowohl die Iwahori-Hecke als auch die pro-p-Iwahori Hecke-Algebren umfassen, eine kanonische und ganzzahlige Basis des Zentrums konstruiert und es werden Endlichkeitssätze über das Zentrum, aufgefasst als Algebra, und über die Hecke-Algebra selbst, aufgefasst als Modul über dem Zentrum, bewiesen. Dabei werden bereits bekannte Ergebnisse von Bernstein-Zelevinsky-Lusztig und Vignéras verallgemeinert.

Contents

0 Introduction

The present article is a contribution to the theory of Hecke algebras, continuing previous work [Sch09] of the author. It is concerned with a recent addition to the diverse family of algebras that go under the name 'Hecke algebra', the 'pro-p Hecke algebras'. The story of these algebras begins with the two articles [Vig06], [Vig05] of Vignéras, both of which generalize the theory of the center of affine Hecke algebras ('Bernstein's presentation') of Bernstein-Zelevinsky and Lusztig [Lus89], but in different directions.

The first article [Vig06] develops an integral version of this theory, removing the restrictions on the ring of coefficients. Recall that affine Hecke algebras $H_q(W,S)$ are defined with respect to a base ring R by generators $\{T_w\}_{w\in W}$ and relations

$$T_w T_{w'} = T_{ww'} \quad \text{if} \quad \ell(w) + \ell(w') = \ell(ww') \tag{0.0.1}$$

$$T_s^2 = q_s + (q_s - 1)T_s \quad (s \in S) \tag{0.0.2}$$

depending on the choice of an extended affine Weyl group W (associated to some root datum $(X, \Phi, X^\vee, \Phi^\vee)$), a set $S \subseteq W$ of simple reflections (defining a length function $\ell : W \to \mathbb{N}$), and a family $\{q_s\}_{s\in S} \subseteq R$ of parameters subject to the constraint

$$q_s = q_t \quad \text{if } s, t \in S \text{ are conjugate in } W$$

This general definition imposes no restrictions on the nature of the ring R whatsoever. However in [Lus89], it was assumed that the ring of coefficients be $R = \mathbb{C}$ and that the parameters q_s are invertible[1]. Traditionally, this wasn't seen as a restriction because the results of [Lus89] were usually applied in the context of *complex* representations of reductive groups and the *classical* Langlands program.

Let us briefly recall how affine Hecke algebras are related to reductive groups. Given a split[2] connected reductive group G over a nonarchimedean local field F and an Iwahori subgroup $I \le G(F)$, a standard construction from representation theory yields the associated *Iwahori-Hecke algebra* $H(G(F), I)$ over R. This algebra has an R-basis indexed by the set $I\backslash G(F)/I$ of double cosets with product structure given by convolution. More conceptually, the algebra $H(G(F), I)$ identifies with the endomorphism ring of the R-linear $G(F)$-representation $\mathrm{ind}_I^{G(F)} \mathbb{1}$ induced from the trivial representation of I. By Frobenius reciprocity, this induced representation also represents the functor of I-invariants of $G(F)$-representations, and the latter therefore lifts to a functor

$$\{R\text{-linear } G(F)\text{-representations}\} \longrightarrow \{\text{Right-}H(G(F), I)\text{-modules}\} \tag{0.0.3}$$

relating representations of reductive groups to modules of Iwahori-Hecke algebras. Finally, affine Hecke algebras and Iwahori-Hecke algebras are related via the *Iwahori-Matsumoto presentation* of $H(G(F), I)$, which defines an isomorphism

$$H_q(W,S) \xrightarrow{\sim} H(G(F), I)$$

where W is the extended Weyl group of the root datum of G and the parameters q_s all equal the cardinality $q = p^r$ of the residue field of F.

[1] The results of [Lus89] are actually applicable to any ring R as long as the q_s are invertible and admit square roots.

[2] The splitness assumption is necessary in order to dispose of the Iwahori-Matsumoto presentation; although it was a folklore result that Iwahori-Hecke algebras of non-split groups admit an Iwahori-Matsumoto presentation with unequal parameters, there was no proof or even a precise result available until the appearance of [Vig16].

Note that when $R = \mathbb{C}$, the parameters q_s are invertible and the results of [Lus89] are applicable. On the other hand if R is a field of characteristic p, the q_s are all equal to zero. In this case there is no hope of applying the Bernstein-Zelevinsky theory as presented in [Lus89], since the relevant constructions depend *explicitly* on the invertibility of the parameters q_s.

In particular, Bernstein-Zelevinsky's description of the center of affine Hecke algebras had not been available to the *mod p Langlands program*—which aims to study representations of reductive groups in precisely this equal-characteristic situation—when it emerged in the early 2000s. The purpose of [Vig06] was to remedy this fact by developing an integral version of the theory in [Lus89]. The surprising result of [Vig06] was that one could completely avoid any invertibility assumptions and make the results carry over to arbitrary coefficient rings by simply replacing the Bernstein-Zelevinsky basis $\{\widetilde{\theta}_x T_w\}$ with an integral variant $\{E_w\}$ differing from it only by explicit scalar factors.

Still, it was not clear how useful Hecke algebras would be in the study of mod p representations because, in contrast to the case of characteristic zero, the functor of I-invariants is not exact in characteristic p. However, it was soon observed that a certain variant of the Iwahori subgroup enjoys a remarkable property in characteristic p that almost makes up for the lack of exactness. This property goes back to the following elementary fact: a p-group that acts on a nonzero $\mathbb{F}_p$-vector space must have a nonzero fix point. It follows at once that the same holds true more generally for *pro-p* groups acting *smoothly*, i.e. with open stabilizers, and for arbitrary coefficient rings of characteristic p. Thus, if one replaces an Iwahori subgroup I by its maximal open normal pro-p subgroup $I(1) \leq I$, the analogue

$$\{R\text{-linear } G(F)\text{-representations}\} \longrightarrow \{\text{Right-}H(G(F), I(1))\text{-modules}\}$$

of the functor (0.0.3) above sends *nonzero smooth* representations to *nonzero* modules (while still being not exact of course). This remarkable property has some important consequences. For example, it implies the following practical irreducibility criterion: a $G(F)$-representation V generated by its $I(1)$-invariants $V^{I(1)}$ is irreducible if the $H(G, I(1))$-module $V^{I(1)}$ is simple.

The subgroup $I(1) \leq I$ and the algebra $H(G(F), I(1))$ were introduced by Vignéras in the second article [Vig05], where they were named 'pro-p-Iwahori group' and 'pro-p-Iwahori Hecke algebra' respectively. Since the appearance of [Vig05], these 'higher congruence analogues' of the Iwahori-Hecke algebras have proven to be of ever-growing importance in the mod p Langlands program (see also [Fli11] for an application in the classical context).

Having removed the restrictions on the ring of coefficients in [Vig06], in [Vig05] Vignéras re-developed this new integral Bernstein-Zelevinsky theory in the context of pro-p-Iwahori Hecke algebras (of split groups). Surprisingly, the results carried over almost verbatim. However, the methods of proof were different as [Vig05] dealt with the concrete convolution Hecke algebras $H(G(F), I(1))$ and not with abstract Hecke algebras $H_q(W, S)$ defined by generators and relations. As a result, the proofs in [Vig05] were less elementary as they assumed some familiarity with reductive groups. A more serious consequence was that one could not take advantage of a reduction argument (the 'specialization argument', see remark 1.10.3) available in the abstract setting that often allows one to reduce statements to the case of invertible parameters.

For these reasons, it seemed desirable if there was a 'pro-p analogue' of the affine Hecke algebras. Our first contribution to this subject was to verify that such an analogue exists: the *generic pro-p Hecke algebras* that formed the subject of [Sch09].

More precisely, generic pro-p Hecke algebras are the pro-p analogues of *generic* Hecke algebras. The latter generalize affine Hecke algebras by allowing (W, S) to be abstract 'extended Coxeter groups' instead of just extended affine Weyl groups. The generic pro-p Hecke algebras which are analogous to affine Hecke algebras and to which the Bernstein-Zelevinsky method applies are the *affine pro-p Hecke algebras* (see definition 2.1.4).

Like generic Hecke algebras, generic pro-p Hecke algebras are associated to a 'Coxeter-like' group $W^{(1)}$ equipped with a length function $\ell : W^{(1)} \to \mathbb{N}$ and a set of parameters, and are equipped with a linear basis $\{T_w\}$ indexed by $W^{(1)}$ such that relations similar to (0.0.1), (0.0.2) above (see definition 1.3.4 for details) hold true.

However, there are two essential differences. First of all, the $W^{(1)}$ aren't extended Coxeter groups but *extensions*

$$1 \longrightarrow T \longrightarrow W^{(1)} \longrightarrow W \longrightarrow 1$$

of extended Coxeter groups by abelian groups (where the group T is not to be confused with the basis $\{T_w\}$). In particular even for affine pro-p Hecke algebras, the representation as a group of isometries of a real affine space the groups $W^{(1)}$ come equipped with is in general[3] not faithful. In other words, the groups $W^{(1)}$ are

[3]The general case being $T \neq 1$; in the degenerate case $T = 1$, the notion of generic pro-p Hecke algebras reduces to that of generic Hecke algebras, making the latter a special case of the former.

only ‘geometric up to T’, which adds an extra layer of difficulty to many statements whose analogues for affine Hecke algebras have purely geometric proofs. But, this difficulty also forces one to recognize structures that remain hidden in the classical case. Namely, of great importance is the existence of a family $(n_s)_{s\in S}$ of lifts of the simple reflections $s \in S$ to the group $W^{(1)}$ which satisfy the braid relations

$$\underbrace{n_s n_t n_s \dots}_{m \text{ factors}} = \underbrace{n_t n_s n_t \dots}_{m \text{ factors}} \quad \text{if } st \text{ is of order } m < \infty \tag{0.0.4}$$

For generic Hecke algebras, the canonical choice $n_s = s$ renders this point trivial. But, even in this case it is ultimately the existence of this family (cf. the proof of theorem 1.6.1) that allows the construction of the ‘Bernstein functions’ θ on which the Bernstein-Zelevinsky theory rests. Showing the existence of such lifts $n_s \in W^{(1)}$ in examples associated to reductive groups is nontrivial (cf. lemma 2.2.3) and related to describing normalizers of maximal tori in split reductive groups, a problem that has been studied in depth by Tits [Tit66]. In fact, his ‘groupes de Coxeter étendu’ are almost the same as our ‘pro-p Coxeter groups’ (see section 1.8).

The second difference is that the analogues of the quadratic relations (0.0.2) are more delicate and that generic pro-p Hecke algebras must therefore be viewed as objects $\mathcal{H}^{(1)} = \mathcal{H}^{(1)}(a,b)$ that depend[4] on *two* families $\{a_s\}_s$, $\{b_s\}_s$ of parameters, instead of one family $\{q_s\}_s$, and are thus actually pro-p analogues of the two-parameter generic Hecke algebras $H_{a,b}(W,S)$ which are defined like $H_q(W,S)$ but with the quadratic relation $T_s^2 = q_s + (q_s - 1)T_s$ replaced by $T_s^2 = a_s + b_s T_s$. But whereas working with two parameters is a convenience in the classical case, in the pro-p case it becomes a necessity because the parameters b_s appearing in the quadratic relations

$$T_{n_s}^2 = a_s T_{n_s^2} + b_s T_{n_s}$$

are no longer elements of the ground ring R but elements of the group ring $R[T]$ of T (viewed as a subalgebra of $\mathcal{H}^{(1)}$ by identifying an element $t \in T$ with the basis element $T_t \in \mathcal{H}^{(1)}$). Thus, there is no sensible one-parameter version of the generic pro-p Hecke algebras as the parameters a_s, b_s never satisfy a simple relation like $b_s = a_s - 1$ in interesting examples. Yet, even for generic Hecke algebras it is fruitful to let a_s and b_s vary independently because then (and only then!) it is possible to reduce statements to the case $a_s = 1$ using the ‘specialization argument’ mentioned before, where formulas take on a particularly simple form.

With these abstract versions of the pro-p-Iwahori Hecke algebras at hand, the next goal then becomes to redevelop the integral Bernstein-Zelevinsky theory of [Vig05] using generic algebra methods as in [Lus89] and [Vig06]. Recall that the method of Bernstein-Zelevinsky[5] rests on the decomposition

$$W = X \rtimes W_0$$

of W into a semi-direct product of a finite group W_0 (‘Weyl group’) and a ‘large’ abelian subgroup X (‘lattice of translations’) provided by the root datum $(X, \Phi, X^\vee, \Phi^\vee)$ giving rise to W, with the group W_0 acting on X; because the group law on the abelian subgroup X is traditionally written additively, one uses the exponential notation τ^x when viewing an $x \in X$ as an element of W, in order to avoid confusion. With this convention, the action of W_0 can be written as

$$w(x) = w\tau^x w^{-1}$$

To the commutative subgroup X now corresponds a commutative subalgebra $\mathcal{A} \subseteq H_q(W,S)$ via a group homomorphism[6]

$$\widetilde{\theta} : X \to H_q(W,S)^\times$$

such that the $\widetilde{\theta}(x)$, $x \in X$ form a basis of $\mathcal{A}$ and W_0 acts on it via algebra automorphisms permuting the basis elements. The main result of the Bernstein-Zelevinsky method is the equality

$$Z(H_q(W,S)) = \mathcal{A}^{W_0}$$

between the center of the Hecke algebra and the invariants of this commutative subalgebra under the action of the Weyl group. Proving this equality usually proceeds in two steps (cf. [Lus89]). First, one establishes that $\mathcal{A}^{W_0}$ lies in the center using the *Bernstein relations*, and then one shows—using this inclusion—that equality must hold. In [Lus83] the last step is justified by referring to a ‘Nakayama argument’ (without providing details). Here and in [Sch09], we follow the mentioned outline but replace the ‘Nakayama argument’ with a combination of an induction (theorem 2.6.3) and an explicit computation (proposition 2.5.4) that shows that

[4]The dependence on the group $W^{(1)}$ is suppressed in the notation.

[5]The description of the center of affine Hecke algebras for ‘constant parameters’ was obtained by Bernstein and Zelevinsky in an unpublished work. Their results were generalized by Lusztig in [Lus89], which is the canonical reference for the theory.

[6]$\widetilde{\theta}$ corresponds to the map denoted by θ in [Lus89]. We use the notation $\widetilde{\theta}$ in order to be consistent with the notation for the *normalized Bernstein maps* $\widetilde{\theta}_{\mathbf{o}}$ to be introduced later that generalize $\widetilde{\theta}$ and have ‘unnormalized’ counterparts denoted by $\theta_{\mathbf{o}}$.

the subalgebra $\mathcal{A}$ equals its own centralizer (in the '*split* case'; in the '*non-split* case' it is a proper subalgebra of $\mathcal{A}$ in general). This step isn't difficult although somewhat convoluted (especially in the 'non-split case').

The essential difficulty of the Bernstein-Zelevinsky method lies in establishing the Bernstein relations. In [Lus89], they are stated in the following form (restated here for two parameters). Given a reflection $s = s_\alpha \in W_0 \cap S$ attached to a simple root $\alpha : X \to \mathbb{Z}$ and an element $x \in X$, we have

$$\widetilde{\theta}(x)T_s - T_s\widetilde{\theta}(s(x)) = \begin{cases} b_s \dfrac{\widetilde{\theta}(x) - \widetilde{\theta}(s(x))}{1 - \widetilde{\theta}(-\alpha^\vee)} & \text{if } \alpha(X) = \mathbb{Z} \\ a_s^{1/2}\left(a_s^{-1/2}b_s + a_{s'}^{-1/2}b_{s'}\widetilde{\theta}(-\alpha^\vee)\right)\dfrac{\widetilde{\theta}(x) - \widetilde{\theta}(s(x))}{1 - \widetilde{\theta}(-2\alpha^\vee)} & \text{if } \alpha(X) = 2\mathbb{Z} \end{cases} \tag{0.0.5}$$

where $\alpha^\vee \in X$ denotes the dual coroot of α and $s' \in S$ is any simple reflection conjugate to the affine reflection $s_{\alpha,1} = \tau^{-\alpha^\vee} s_\alpha \tau^{\alpha^\vee}$ in W. The homomorphism $\widetilde{\theta}$ is defined as

$$\widetilde{\theta}(x) = \widetilde{T}_y \widetilde{T}_z^{-1}$$

where $y, z \in X$ are any two elements lying in the dominant cone that satisfy $x = y - z$, and the $\widetilde{T}_w$ are normalized versions of the T_w determined by $\widetilde{T}_s = a_s^{-1/2} T_s$ and the analogues of the braid relations (0.0.1).

In [Vig05], Vignéras established analogues of the Bernstein relations for pro-p-Iwahori Hecke algebras; her proof closely followed Lusztig's intricate computational proof [Lus89]. Shortly after her article appeared, Görtz published a simple geometric proof [Gör07] of the Bernstein relations for affine Hecke algebras. When we learnt of his article, we hoped that his geometric approach might work for pro-p-Iwahori Hecke algebras too. His proof was based on Ram's theory of alcove walk algebras [Ram06]. The main input of that theory to Görtz' proof is a geometric interpretation of the elements $\widetilde{\theta}(x)$ based on identifying formal expressions in the Hecke algebra like[7]

$$\widetilde{T}_{s_1}^{\varepsilon_1} \widetilde{T}_{s_2}^{\varepsilon_2} \ldots \widetilde{T}_{s_r}^{\varepsilon_r} \quad (\varepsilon_i \in \{\pm 1\})$$

with 'coloured' or 'signed' galleries (i.e. 'unfolded alcove walks' in the terminology of [Ram06]) in the Coxeter complex starting at the base alcove C, the above expression corresponding to the gallery

$$\Gamma = (C_0 = C,\ C_1 = s_1C,\ C_2 = s_1s_2C,\ \ldots,\ C_r = s_1 \ldots s_rC)$$

from C to wC, where $w = s_1 \ldots s_r$ and the colour of the arrow from C_{i-1} to C_i is determined by the sign ε_i, as in figure 1. Expanding $\widetilde{T}_y$ and $\widetilde{T}_z$ in the definition of $\widetilde{\theta}(x) = \widetilde{T}_y\widetilde{T}_z^{-1}$ into a product of generators $\widetilde{T}_s$, it is easy to see that $\widetilde{\theta}(x)$ is given by *some* coloured gallery in this way.

The key point however is this: fixing an 'orientation' (see definition 1.5.1), there is a canonical way to colour every ordinary (uncoloured) gallery starting at C in such a way that *any* two such coloured galleries having the *same endpoint* define the same element in the Hecke algebra. For the 'spherical orientation' attached to the dominant Weyl chamber (see definition 2.4.1), this is the content of the following theorem, quoted verbatim from [Gör07] (W corresponding to 'W_a', and C to '$\mathbf{a}$' in his notation):

0.0.1 Theorem. *Let $w \in W_a$. For an expression*

$$w = s_{i_1} \ldots s_{i_n} \tag{1.1.1}$$

of w as a product in the generators (which does not have to be reduced), consider the element

$$\Psi(w) := T_{s_{i_1}}^{\varepsilon_1} \ldots T_{s_{i_n}}^{\varepsilon_n}$$

in the affine Hecke algebra, where the $\varepsilon_\nu \in \{\pm 1\}$ are determined as follows. Let $\mathbf{b}$ be an alcove far out in the anti-dominant chamber ("far out" depends on w, and the result will then be independent of $\mathbf{b}$, see section [...] for a precise definition). For each ν, consider the alcove $\mathbf{c}_\nu := s_{i_1} \ldots s_{i_{\nu-1}}\mathbf{a}$, and denote by H_ν the affine root hyperplane containing its face of type i_ν. We set

$$\varepsilon_\nu := \begin{cases} 1 & \text{if } \mathbf{c}_\nu \text{ is on the same side of } H_\nu \text{ as } \mathbf{b} \\ -1 & \text{otherwise} \end{cases}$$

Then the element $\Psi(w)$ is independent of the choice of the expression (1.1.1).

([Gör07, Theorem 1.1.1])

[7] In order to avoid some minor subtleties arising from the group Ω of elements of length zero, we assume that $\Omega = 1$, i.e. that the extended affine Weyl group W coincides with the affine Weyl group.

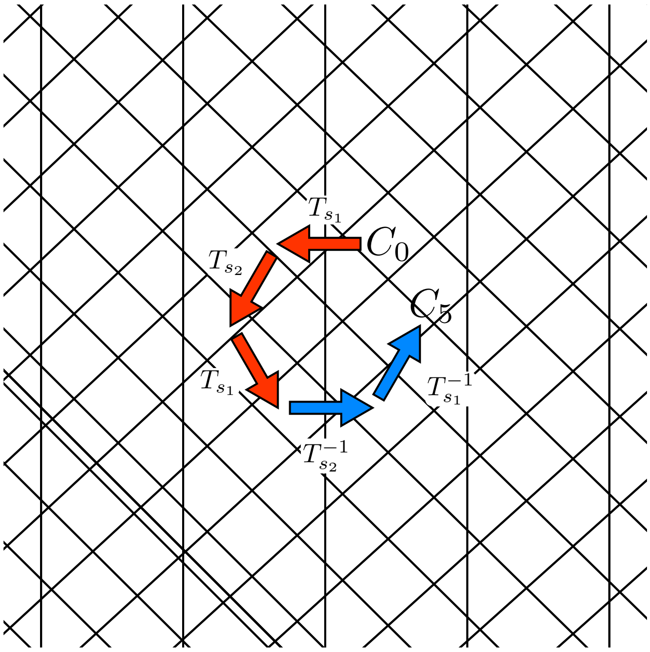

Figure 1: The coloured gallery $\Gamma = (C_0, \dots, C_5)$ corresponding to the expression $T_{s_1} T_{s_2} T_{s_1} T_{s_2}^{-1} T_{s_1}^{-1}$ in the affine Coxeter complex of type $\widetilde{A}_2$.

The theorem also holds true with the T_s replaced by the $\widetilde{T}_s$, and the galleries corresponding to the expressions

$$\widetilde{T}_{s_1} \dots \widetilde{T}_{s_n} \widetilde{T}_{t_m}^{-1} \dots \widetilde{T}_{t_1}^{-1}$$

arising from expanding $\widetilde{T}_y$ and $\widetilde{T}_z$ in $\widetilde{\theta}(x) = \widetilde{T}_y \widetilde{T}_z^{-1}$ into *reduced* products (i.e. $\ell(y) = n$, $\ell(z) = m$)

$$\widetilde{T}_y = \widetilde{T}_{s_1} \dots \widetilde{T}_{s_n} \quad \text{and} \quad \widetilde{T}_z = \widetilde{T}_{t_1} \dots \widetilde{T}_{t_m}$$

of the generators $\widetilde{T}_s$ are easily seen to be coloured according to the method given in the theorem. Therefore, $\widetilde{\theta}(x)$ is given by *any* canonically coloured gallery from C to $x + C$, giving the Bernstein homomorphism $\widetilde{\theta}$ a very natural and intuitive interpretation in terms of alcove walks. This geometric interpretation fueled Görtz geometric proof of the Bernstein relations, reducing it essentially to a telescopic sum expansion of the left hand side, with each summand possessing a geometric interpretation as an alcove walk. Before we explain this in more detail, let us note some further consequences of the above theorem. These consequences played no explicit role in Görtz' original proof, but they allow us to recast it in a way that makes it adaptable to the pro-p case. In order to simplify the exposition, we will discuss everything in the affine case first, making only some brief indications on how the pro-p case differs, and then later discuss the pro-p case more fully.

The first thing to note is that the above theorem suggests to extend the Bernstein homomorphism to a map defined on all of W. Further, we will see in a moment that it is useful to explicitly denote the dependence on the orientation. Let us therefore write $\widetilde{\theta}_{\mathfrak{o}}(w)$ for the element defined by a gallery from C to $w(C)$ that is coloured according to the orientation $\mathfrak{o}$, and let $\mathfrak{o}$ denote the spherical orientation attached to the dominant Weyl chamber (hence $\widetilde{\theta} = \widetilde{\theta}_{\mathfrak{o}}$) in the following. The group W naturally acts on orientations from the right such that the signs assigned to a gallery Γ by $\mathfrak{o} \bullet w$ are the ones assigned to $w(\Gamma)$ by $\mathfrak{o}$. Granting the theorem, the definitions then immediately imply the following *cocycle rule* (called *product formula* in [Sch09]):

$$\widetilde{\theta}_{\mathfrak{o}}(ww') = \widetilde{\theta}_{\mathfrak{o}}(w) \widetilde{\theta}_{\mathfrak{o} \bullet w}(w') \tag{0.0.6}$$

The cocycle rule recovers the homomorphism property of the 'Bernstein homomorphism' because the subgroup X acts trivial on $\mathfrak{o}$ (indeed on all spherical orientations). Moreover, the cocycle rule implies the formula

$$\widetilde{\theta}_{\mathfrak{o}}(w)^{-1} = \widetilde{\theta}_{\mathfrak{o} \bullet w}(w^{-1})$$

Because the spherical orientation $\mathfrak{o}$ of the dominant Weyl chamber has the property that $\widetilde{\theta}_{\mathfrak{o}}(s) = \widetilde{T}_s^{-1}$ for all simple reflections $s \in W_0$, it follows from the cocycle rule that one can rewrite the second summand on the left hand side of (0.0.5) as

$$T_s \widetilde{\theta}(s(x)) = a_s^{1/2} \widetilde{\theta}_{\mathfrak{o} \bullet s}(s) \widetilde{\theta}_{\mathfrak{o}}(s(x)) = a_s^{1/2} \widetilde{\theta}_{\mathfrak{o} \bullet s}(s \tau^{s(x)}) = a_s^{1/2} \widetilde{\theta}_{\mathfrak{o} \bullet s}(\tau^x s)$$

The first summand can't be rewritten in this way, but since

$$\widetilde{\theta}_{\mathfrak{o}\bullet s}(x)T_s = a_s^{1/2}\widetilde{\theta}_{\mathfrak{o}\bullet s}(x)\widetilde{\theta}_{\mathfrak{o}\bullet s}(s) = a_s^{1/2}\widetilde{\theta}_{\mathfrak{o}\bullet s}(\tau^x s)$$

it follows that

$$\widetilde{\theta}(x)T_s - T_s\widetilde{\theta}(s(x)) = a_s^{1/2}\left(\widetilde{\theta}_{\mathfrak{o}}(x) - \widetilde{\theta}_{\mathfrak{o}\bullet s}(x)\right)\widetilde{\theta}_{\mathfrak{o}\bullet s}(s)$$

The proof of the Bernstein relations therefore comes down to computing the difference $\widetilde{\theta}_{\mathfrak{o}}(x) - \widetilde{\theta}_{\mathfrak{o}\bullet s}(x)$. To carry out this computation, one *chooses* an explicit expression $\tau^x = s_1 \dots s_r$ (not necessarily reduced) and writes this difference as a telescopic sum

$$\widetilde{\theta}_{\mathfrak{o}}(x) - \widetilde{\theta}_{\mathfrak{o}\bullet s}(x) = \widetilde{T}_{s_1}^{\varepsilon_1}\dots\widetilde{T}_{s_r}^{\varepsilon_r} - \widetilde{T}_{s_1}^{\varepsilon_1'}\dots\widetilde{T}_{s_r}^{\varepsilon_r'} = \sum_i \widetilde{T}_{s_1}^{\varepsilon_1}\dots\widetilde{T}_{s_{i-1}}^{\varepsilon_i}\left(\widetilde{T}_{s_i}^{\varepsilon_i} - \widetilde{T}_{s_i}^{\varepsilon_i'}\right)\widetilde{T}_{s_{i+1}}^{\varepsilon_{i+1}'}\dots\widetilde{T}_{s_r}^{\varepsilon_r'}$$

Since the sum needs only to be taken over the indices i where $\varepsilon_i \neq \varepsilon_i'$, one can use the quadratic relations in the Hecke algebra written in the symmetric form $\widetilde{T}_s - \widetilde{T}_s^{-1} = a_s^{-1/2}b_s$ to simplify each summand:

$$\widetilde{T}_{s_1}^{\varepsilon_1}\dots\widetilde{T}_{s_{i-1}}^{\varepsilon_i}\left(\widetilde{T}_{s_i}^{\varepsilon_i} - \widetilde{T}_{s_i}^{\varepsilon_i'}\right)\widetilde{T}_{s_{i+1}}^{\varepsilon_{i+1}'}\dots\widetilde{T}_{s_r}^{\varepsilon_r'} = \varepsilon_i a_{s_i}^{-1/2}b_{s_i}\widetilde{T}_{s_1}^{\varepsilon_1}\dots\widetilde{T}_{s_{i-1}}^{\varepsilon_i}\widetilde{T}_{s_{i+1}}^{\varepsilon_{i+1}'}\dots\widetilde{T}_{s_r}^{\varepsilon_r'}$$

The crucial point of the proof now is to recognize each summand as something defined a priori, without reference to the particular chosen expression $\tau^x = s_1 \dots s_r$. This step is very delicate in the pro-p case, which makes transposing Görtz' to this context nontrivial. In fact, it quickly became apparent to us that a purely geometric proof of the Bernstein relations for pro-p Hecke algebras *couldn't exist*, unless a miracle happened. This miracle is:

0.0.2 Theorem. *For any hyperplane $H \in \mathfrak{H}$ and any orientation $\mathfrak{o} \in \mathcal{O}$, there exists a unique element $\Xi_{\mathfrak{o}}(H) \in \mathcal{H}^{(1)}$, such that if $s \in S$, $w \in W^{(1)}$ with*

$$\pi(wn_s w^{-1}) = H$$

then

$$\Xi_{\mathfrak{o}}(H) = \sqrt{a_s}^{-1}w(b_s)\cdot\widetilde{\theta}_{\mathfrak{o}}(wn_s^{-1}w^{-1}) = \sqrt{a_s}^{-1}\widetilde{\theta}_{\mathfrak{o}}(wn_s^{-1}w^{-1})\cdot w(b_s)$$

(Proposition/definition 1.11.1)

For affine Hecke algebras no miracle beyond Görtz' theorem is needed to see that

$$\widetilde{T}_{s_1}^{\varepsilon_1}\dots\widetilde{T}_{s_{i-1}}^{\varepsilon_{i-1}}\widetilde{T}_{s_{i+1}}^{\varepsilon_{i+1}'}\dots\widetilde{T}_{s_r}^{\varepsilon_r'} = \widetilde{\theta}_{\mathfrak{o}}(s_1\dots\widehat{s_i}\dots s_r) = \widetilde{\theta}_{\mathfrak{o}}(s_{H_i}\tau^x) = \widetilde{\theta}_{\mathfrak{o}}(s_{H_i})\widetilde{\theta}_{\mathfrak{o}\bullet s}(x) \tag{0.0.7}$$

writing $s_{H_i} = (s_1\dots s_{i-1})s_i(s_1\dots s_{i-1})^{-1}$ for the reflection at the i-th (affine) hyperplane H_i crossed by the gallery $\Gamma = (C, s_1C, s_1s_2C, \dots, x+C)$, and denoting with $\widehat{s_i}$ the omission of the element s_i from the sequence. One therefore arrives at the formula

$$\widetilde{\theta}_{\mathfrak{o}}(x) - \widetilde{\theta}_{\mathfrak{o}\bullet s}(x) = \left(\sum_i \varepsilon_i a_{s_i}^{-1/2}b_{s_i}\widetilde{\theta}_{\mathfrak{o}}(s_{H_i})\right)\widetilde{\theta}_{\mathfrak{o}\bullet s}(x) \tag{0.0.8}$$

If the expression $\tau^x = s_1 \dots s_r$ is taken to be reduced, the H_i appearing in the above sum are precisely the hyperplanes separating C and $x + C$ at which the orientations $\mathfrak{o}$ and $\mathfrak{o} \bullet s$ disagree (i.e. those parallel to the hyperplane H_α defined by the root α). Since a_{s_i}, b_{s_i}, and the sign ε_i only depend on the hyperplane H_i, the whole sum is therefore purely geometric and independent of the chosen expression. The classical Bernstein relations (0.0.5) are now easily derived from (0.0.8) using the identity

$$\widetilde{\theta}_{\mathfrak{o}}(s_H)\widetilde{\theta}_{\mathfrak{o}\bullet s}(x)\widetilde{\theta}_{\mathfrak{o}\bullet s}(s) = \widetilde{\theta}_{\mathfrak{o}}(s_H s)\widetilde{\theta}_{\mathfrak{o}}(s(x))$$

and by recognizing $\sum_i \widetilde{\theta}_{\mathfrak{o}}(s_{H_i}s)$ as a geometric sum.

The proof sketched above is a reformulation of the proof of Görtz: his proof was direct and didn't involve formula (0.0.8). Although a general notion of orientation was defined there, the discussion in [Gör07] was restricted to the spherical orientation $\mathfrak{o}$ attached to the dominant Weyl chamber and its associated Bernstein map $\widetilde{\theta}_{\mathfrak{o}}$; in particular, neither the cocycle rule nor (0.0.8) appeared.

As we've already seen, the cocycle rule is important because it simplifies many computations. It also brings the connection to the work of Cherednik (see below), and forms the proper basis for the definition of the Bernstein maps in the case of extended or pro-p Coxeter groups. Its discovery—a lucky byproduct of pedantic notation—was the origin of [Sch09]. Its impetus led us to consider $\{\widetilde{\theta}_{\mathfrak{o}}(w)\}_{w\in W}$ and its integral version $\{\widehat{\theta}_{\mathfrak{o}}(w)\}_{w\in W}$ instead of the traditional Bernstein-Zelevinsky basis $\{\widetilde{\theta}_{\mathfrak{o}}(x)T_w\}_{x\in X,\ w\in W_0}$ and its integral analogue $\{E_w\}_{w\in W}$ defined in [Vig06]. This resulted in the following integral analogue of the cocycle rule (see corollary 1.10.5), generalizing the formula for the product $E_{w_0x}E_{x'}$ given in [Vig06]:

$$\widehat{\theta}_{\mathfrak{o}}(w)\widehat{\theta}_{\mathfrak{o}\bullet w}(w') = \overline{\mathbb{X}}(w,w')\widehat{\theta}_{\mathfrak{o}}(ww') \tag{0.0.9}$$

The factor $\overline{\mathbb{X}}(w,w')$ that appears in this formula played an important role in establishing the integral theory, both in [Vig06] and in [Sch09]. In [Vig06], it appeared[8] in the crucial 'lemme fondamental' ([Vig06, 1.2]), which was not explicitly mentioned in [Sch09] but which we recover here in lemma 1.7.10. In [Sch09], the factor $\overline{\mathbb{X}}(w,w')$ entered through its relation to another map γ ([Sch09, Lemma 3.3.26]) that was used to relate the integral Bernstein map $\widehat{\theta}_{\mathfrak{o}}$ to its non-integral counterpart $\theta_{\mathfrak{o}}$ but was given no further interpretation. Here, we show that the 'lemme fondamentale' and [Sch09, Lemma 3.3.26] can be seen as exhibiting $\overline{\mathbb{X}}$ as a 2-coboundary in two different ways (see remark 1.7.7 for details), the latter exhibiting $\overline{\mathbb{X}}$ as the coboundary of γ.

Another interesting consequence of the cocycle rule—further emphasizing the importance to consider *all* spherical orientations—was the realization that the basis of the center of the Hecke algebra provided by the orbit sums

$$z^{\mathfrak{o}}_{\gamma} = \sum_{x\in\gamma}\widetilde{\theta}_{\mathfrak{o}}(x), \quad \gamma\in W_0\backslash X$$

under the equality

$$Z(H_q(W,S)) = \mathcal{A}_{\mathfrak{o}}^{W_0}$$

is in fact *canonical*, i.e. *independent* of the choice of $\mathfrak{o}$. In fact, the independence of the element $z^{\mathfrak{o}}_{\gamma}$ from $\mathfrak{o}$ turns out to be equivalent to the fact that it lies in the center (see the proof of proposition 2.6.1).

Unfortunately, in [Sch09] we couldn't realize this geometric approach to pro-p Hecke algebras to its full potential as we didn't dipose of the 'miracle' needed transpose Görtz' proof into the context of pro-p Hecke algebras. In addition, the proof of the Bernstein relations we gave was flawed[9]. Moreover, the axiomatics of 'affine pro-p Hecke algebras' were too restrictive, as they only included the pro-p-Iwahori Hecke algebras of *split* reductive groups. And so, although we achieved our goal of developing an abstract theory of pro-p Hecke algebras and of re-deriving the integral Bernstein-Zelevinsky theory of [Vig05] in this context, [Sch09] remained incomplete in a technical and a moral sense.

Forunately, these issues are all resolved in this article. We give a new and purely geometric proof of the Bernstein relations for pro-p Hecke algebras, based on formula (0.0.8) derived above. First of all, there is no need to restrict to elements $x\in X$ in this formula: it remains true for any element of W. Second, its proof never explicitly used that W is an affine Weyl group. This fact only entered indirectly through the properties of the orientations $\mathfrak{o}$ and $\mathfrak{o}\bullet s$ used in deducing (0.0.7). By abstracting these properties (and using the 'miracle proposition' to extend to the pro-p case), we can thus prove the following generalization of (0.0.8), holding for any generic pro-p Hecke algebra whose parameters a_s are invertible and squares, generalizing the Bernstein relations for *affine* pro-p Hecke algebras obtained earlier by Vignéras [Vig16, Theorem 5.46]:

0.0.3 Theorem (Theorem 1.11.3)**.**

$$\widetilde{\theta}_{\mathfrak{o}}(w) - \widetilde{\theta}_{\mathfrak{o}'}(w) = \left(\sum_H \mathfrak{o}(1,H)\Xi_{\mathfrak{o}'}(H)\right)\widetilde{\theta}_{\mathfrak{o}}(w) \tag{0.0.10}$$

Here, $\mathfrak{o}$, $\mathfrak{o}'$ denotes a pair of *adjacent* orientations (see definition 1.11.2). Apart from spherical orientations $\mathfrak{o}_D$, $\mathfrak{o}_{D'}$ of affine Coxeter groups associated to adjacent Weyl chambers D, D', the hyperbolic Coxeter group $\mathrm{PGL}_2(\mathbb{Z})$ provides examples of such pairs (see remark 1.5.13) and therefore new examples of Bernstein relations. But even in the affine case, (0.0.10) is more general than the Bernstein relations in [Vig16] as w is allowed to be an arbitrary element of $W^{(1)}$. Moreover, phrasing the Bernstein relations in this abstract generality makes the proof cleaner and more transparent, especially in the pro-p case.

We will now discuss the contents of this article in detail. After recalling the notion of Coxeter groups and some common (and maybe not so common) related geometric terminology, we introduce in section 1.1 two

[8] In a slightly disguised form; see remark 1.7.11(ii) for details.

[9] We thank M.F. Vignéras for pointing this out to us.

(successive) generalizations of the notion of Coxeter groups, *extended* and *pro-p* Coxeter groups, designed to capture the essential properties of the groups appearing in the Bruhat decomposition for Iwahori and pro-p-Iwahori groups.

In section 1.2 we give a classification of 1-cocycles of pro-p Coxeter groups that is later used to construct the *Bernstein maps*. Section 1.3 that follows is fundamental: we define generic pro-p Hecke algebras and show that they behave like generic Hecke algebras (existence of a canonical linear basis, Iwahori-Matsumoto relations); we also give a first proper example of generic pro-p Hecke algebras, the *Yokonuma-Hecke* algebras. The Iwahori-Matsumoto presentation of generic pro-p Hecke algebras is proved in the following section 1.4, phrased in the language of braid groups.

Section 1.5 introduces another fundamental concept, the notion of *orientations* of Coxeter groups, which abstracts and generalizes similar notions considered earlier by [Gör07] and [Ram06], and besides the classification of 1-cocycles is the second ingredient in the construction of the Bernstein maps. The set $\mathcal{O}$ of all orientations of a Coxeter group W is shown to be endowed with the structure of a compact Hausdorff topological space acted upon by W, and the group W is embedded as a subset in $\mathcal{O}$ in two different ways, associating to an element $w \in W$ the orientation $\mathfrak{o}_w$ *towards* w and the orientation $\mathfrak{o}_w^{\mathrm{op}}$ *away* from w, both embeddings being exchanged under a canonical involution $\mathfrak{o} \mapsto \mathfrak{o}^{\mathrm{op}}$ on $\mathcal{O}$. It is shown that the images of these embeddings give *all* orientations when W is finite, but that there must exist other orientations when $W = (W, S)$ is infinite and $\#S < \infty$, the *boundary orientations*. The notion of orientation is then transferred in a natural way onto extended and pro-p Coxeter groups, such that the set of orientations of an extended or pro-p Coxeter group is in canonical bijection with the set of orientations of its underlying Coxeter group.

The third principal protagonist, the Bernstein maps, is introduced in the following section 1.6. The existence theorem, theorem 1.6.1, proven there should be seen as the equivalent of Görtz' theorem in our context. Section 1.7 discusses a certain 2-cocycle that is canonically associated to every Coxeter group and plays a prominent in the theory of Hecke algebras. We show that it can be written as 2-coboundary in *two different* ways, which is used to define integral and normalized Bernstein maps in section 1.10. The intermediate sections section 1.8 and section 1.9 are logically independent from the rest of the text, and should be regarded as optional: in section 1.8 we show that the 2-cocycle $\mathbb{X}$ can be used to classify pro-p Coxeter groups, recovering a result of Tits [Tit66, 3.4], and in section 1.9 we discuss the relation of the Bernstein map θ to a cocycle considered by Cherednik.

Finally, in section 1.11, we prove one of the main results of this article, the generalized Bernstein relations (0.0.10).

Whereas the first part dealt with general generic pro-p Hecke algebras, the second part of this article specializes to those generic pro-p Hecke algebras for which a meaningful analogue of the Bernstein-Zelevinsky theory can be developed. These are the *affine pro-p Hecke algebras*, the generic pro-p Hecke algebras whose underlying extended Coxeter group is equipped with the structure of an *affine extended Coxeter group*, a notion that is introduced in section 2.1 and which generalizes the class of extended affine Weyl groups to allow *all* groups that appear in the Bruhat decomposition of Iwahori groups of (possibly non-split) connected reductive groups over local fields. This makes it necessary to prove some well-known facts from the theory of root data in our more general context.

In section 2.2 we show that our theory is non-empty by introducing three examples of affine pro-p Hecke algebras: the *affine Hecke algebras* considered in the classical Bernstein-Zelevinsky theory [Lus89], the *pro-p Iwahori Hecke algebras* considered in the p-adic and mod-p Langlands program, and the *affine Yokonuma-Hecke algebras* from the theory of knot invariants. These examples have already appeared in the literature before (see [CS15], [Vig16]), and for the heavy-duty computations needed for the verification of the axioms in the case of pro-p-Iwahori Hecke algebra we refer to [Vig16]; however, this section provides some details not found in either source, including an effective version of the existence of the lifts $(n_s)_{s\in S}$, which the reader may find helpful.

Section 2.3 is devoted to the proof of some finiteness properties of affine extended Coxeter groups, which are the key to prove corresponding finiteness results for affine pro-p Hecke algebras. These results were basically already proven in [Sch09, 4.2.5], but the proofs were a bit ad hoc. Here, we give a more unified treatment by relating these finiteness properties to the (known) fact that the weak Bruhat order is a *well partial order*.

In section 2.4 we introduce *spherical orientations* of affine extended Coxeter groups W and prove that they are limits of nets of chamber orientations $\mathfrak{o}_w$, which makes them concrete examples of boundary orientations and gives a precise sense to the notion in Görtz' theorem, of the orientation 'attached to an alcove infinitely deep in the anti-dominant chamber'. The most important property of the spherical orientations is that the subgroup $X \leq W$ of 'translations' acts trivially on them, as the cocycle rule implies that the Bernstein map $\widetilde{\theta}_{\mathfrak{o}}$ induced an embedding of the group algebra of the stabilizer of $\mathfrak{o}$ (in $W^{(1)}$) embeds into the Hecke algebra. Thus we introduce in the following section 2.5 subalgebras $\mathcal{A}_{\mathfrak{o}}^{(1)} \subseteq \mathcal{H}^{(1)}$ for every spherical orientation, which are not far from being commutative (and are commutative for affine Yokonuma-Hecke algebras or pro-p-Iwahori Hecke algebras of *split* groups remark 2.5.2). The main result of this section is the computation of the centralizer of these subalgebras in the Hecke algebra; in particular, we prove that the centralizer of $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is a subalgebra of $\mathcal{A}_{\mathfrak{o}}^{(1)}$, which is an important step towards the computation of the center of $\mathcal{H}^{(1)}$.

In section 2.6, we use the Bernstein relations to show that the invariants of $\mathcal{A}_{\mathfrak{o}}^{(1)}$ under the natural action of $W^{(1)}$ are contained in the center of $\mathcal{H}^{(1)}$. Afterwards, we verify using explicit computations that this exhausts the center. In the final section 2.7, the results of the previous sections culminate in the *structure theorem* theorem 2.7.1, which elucidates the structure of $\mathcal{H}^{(1)}$ in terms of its center under very mild assumptions on the coefficient ring and the group $W^{(1)}$, satisfied in all the examples we consider. This generalizes similar results obtained by Vignéras in [Vig14] for pro-p Iwahori-Hecke algebras. In particular, the structure theorem covers the affine Yokonuma-Hecke algebras $\mathcal{H}_{d,n}^{(1)}$ of Juyumaya and Lambropoulou (section 2.2.4) in *all* cases (when d is a prime-power, $\mathcal{H}_{d,n}^{(1)}$ is isomorphic to the pro-p Iwahori-Hecke algebra of GL_n and the results of [Vig14] apply).

In the third part, section 3, we investigate the hyperbolic Coxeter group $\mathrm{PGL}_2(\mathbb{Z})$ and its Hecke algebra $\mathcal{H}$, and we describe some boundary orientations, attached to points of the 'actual' boundary $\mathbb{P}^1(\mathbb{R})$. Moreover, for the point at infinity $\infty \in \mathbb{P}^1(\mathbb{R})$, we define a subalgebra $\mathcal{A}_\infty$ and show that $\mathcal{H}$ is free as a right module over $\mathcal{A}_\infty$, with an explicit basis. Finally, we compute the intertwiners of the induced module $M_\chi = \chi \otimes_{\mathcal{A}_\infty} \mathcal{H}$ for $\chi : \mathcal{A}_\infty \longrightarrow R$ a character, proving that these intertwiners reduce to R and the module M_χ is therefore Schur-simple.

In the fourth and final part, we use the characterization of pro-p Coxeter groups in terms of the 2-cocycle $\mathbb{X}$ proven earlier, to investigate the question when the canonical exact sequence

$$1 \longrightarrow T \longrightarrow N \longrightarrow W_0 \longrightarrow 1$$

, given by the rational points T and N of a maximal split torus and its normalizer inside a split reductive group, splits. This question (which had been already resolved for almost-simple semisimple split groups) is answered up to rank eight using computer calculations, which compute the cohomology groups $H^k(W_0, X^\vee)$ and $H^k(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$. Using the theory of **FI**-modules and a theorem of Nagpal and Snowden, these computations in a finite number of cases are extended to prove the following theorem:

0.0.4 Theorem. *The dimension*

$$d_k(\ell) := \dim_{\mathbb{F}_2} H^k(S_{\ell+1}, Q_\ell^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$$

of the first cohomology group of the mod 2 reduction of the coroot lattice $Q_\ell^\vee$ of the root system A_ℓ is given in degrees $k=1$ by

$$d_1(\ell) = \begin{cases} 1 & \text{if } \ell = 1 \\ 0 & \text{if } \ell \geq 2, \text{ and } \ell \text{ even} \\ 2 & \text{if } \ell \geq 2, \text{ and } \ell \text{ odd} \end{cases}$$

in degree $k=2$ by

$$d_2(\ell) = \begin{cases} 1 & \text{if } \ell = 1 \\ 0 & \text{if } \ell = 2 \\ 2 & \text{if } \ell = 3 \\ 0 & \text{if } \ell \geq 4, \text{ and } \ell \text{ even} \\ 3 & \text{if } \ell \geq 4, \text{ and } \ell \text{ odd} \end{cases}$$

and in degree $k=3$ by

$$d_3(\ell) = \begin{cases} 1 & \text{if } \ell = 1 \\ 0 & \text{if } \ell = 2 \\ 3 & \text{if } \ell = 3 \\ 0 & \text{if } \ell = 4 \\ 5 & \text{if } \ell = 5 \\ 0 & \text{if } \ell \geq 6, \text{ and } \ell \text{ even} \\ 6 & \text{if } \ell \geq 6, \text{ and } \ell \text{ odd} \end{cases}$$

(Theorem 4.8.5)

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1 Generic pro-p Hecke algebras and Bernstein maps

1.1 Basic definitions and some geometric terminology

We recall some standard facts and terminology from the theory of Coxeter groups (cf. [Bou07, Ch. IV] or [Bro89, II]).

1.1.1 Definition. A **Coxeter group** $W = (W, S)$ consists of a group W and a set $S \subseteq W$ of generators of order 2 satisfying the *action condition.* That is, there exists an action

$$\rho : W \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(\mathfrak{H} \times \{\pm\})$$

on the set $\mathfrak{H} \times \{\pm 1\}$, where

$$\mathfrak{H} := \{wsw^{-1} : w \in W,\ s \in S\} \subseteq W$$

such that a generator $s \in S$ acts as

$$\rho(s)(H, \varepsilon) = \begin{cases} (sHs^{-1}, -\varepsilon) & : H = s \\ (sHs^{-1}, \varepsilon) & : H \neq s \end{cases}$$

1.1.2 Remark. There are several other equivalent definitions of the notion of a Coxeter group (see [Bro89, II.4]). In particular, given a group W and a set S of generators of order 2, the action condition is equivalent to both the *exchange condition* (**E**) and the *deletion condition* (**D**). The former states that given a reduced expression $w = s_1 \dots s_r$ and an element $s \in S$, either $\ell(sw) = \ell(w) + 1$ or

$$w = ss_1 \dots \widehat{s_i} \dots s_r \tag{E}$$

for some $1 \leq i \leq r$ (where $\widehat{s_i}$ denotes omission of s_i), the latter that if the expression $w = s_1 \dots s_r$ is *not* reduced, then

$$w = s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r \tag{D}$$

for some $1 \leq i < j \leq r$.

1.1.3 Terminology. If an action as in definition 1.1.1 exists, it is uniquely determined and called the *canonical action.* The set $\mathfrak{H}$ is called the set of *walls* or *hyperplanes.* When we want to view a hyperplane $H \in \mathfrak{H}$ as the reflection in W it corresponds to, we sometimes write s_H instead of H. Elements of W are also called *chambers.* A distinguished chamber is given by the neutral element $1 \in W$ and is called the *fundamental chamber.* Two chambers w, w' are called *adjacent* if $w^{-1}w' \in S$. A *gallery* from w to w' is a finite sequence $\Gamma = (w = w_0, \dots, w_r = w')$ such that w_i, w_{i+1} are adjacent. Galleries from the fundamental chamber to a chamber $w \in W$ correspond to expressions

$$w = s_1 \dots s_r$$

of w as a product of generators $s_i \in S$, the associated gallery being

$$\Gamma = (1, s_1, s_1 s_2, \dots, s_1 \dots s_r)$$

A wall H is said to *separate* $w_1, w_2 \in W$ if

$$\rho(w_2^{-1}w_1)(w_1^{-1}Hw_1, 1) = (w_2^{-1}Hw_2, -1)$$

Otherwise w_1, w_2 are said to *lie on the same side* with respect to H. The number of walls separating 1 and w is finite and equal to

$$\ell(w) := \min\{r \in \mathbb{N} : \exists s_1, \dots, s_r \in S \quad w = s_1 \dots s_r\}$$

which is called the *length* of w. An arbitrary expression

$$w = s_1 \dots s_r$$

is called *reduced* if $r = \ell(w)$. Given such a reduced expression, the set

$$\{s_1, s_1 s_2 s_1^{-1}, \ldots, (s_1 \ldots s_{r-1}) s_r (s_1 \ldots s_{r-1})^{-1}\}$$

is the set of hyperplanes separating 1 and w. More generally, we can define for any two chambers w, w' the *distance* $d(w, w')$ between w and w' as the length of the shortest gallery from w to w'. A gallery Γ is called a *geodesic* if its length equals the distance between its start- and endpoint. One can show that a gallery is a geodesic if and only if it does not cross a hyperplane twice. In particular the distance $d(w, w')$ equals the number of walls separating w and w'. Moreover, the distance is W-invariant and so in particular $d(w, w') = d(1, w^{-1}w') = \ell(w^{-1}w')$. A wall $H = w_0 s w_0^{-1}$ divides W into two equivalence classes under the relation of lying on the same side with respect to H, namely the *positive half-space*

$$U_H^+ = \{w \in W : \ell(s w_0^{-1} w) - \ell(w_0^{-1} w) = \ell(s w_0^{-1}) - \ell(w_0^{-1})\}$$

and the *negative half-space*

$$U_H^- = \{w \in W : \ell(s w_0^{-1} w) - \ell(w_0^{-1} w) = -(\ell(s w_0^{-1}) - \ell(w_0^{-1}))\}$$

By definition the positive half-space is the one containing the fundamental chamber. The map $(H, \varepsilon) \mapsto U_H^\varepsilon$ gives a bijection between $\mathfrak{H} \times \{\pm 1\}$ and the set of all *half-spaces*. This bijection is W-equivariant with respect to the natural actions and allows to identify these two W-sets.

The *(strong) Bruhat order* $<$ on W is the strict partial order in which $w < w'$ if and only if for some (every) reduced expression

$$w = s_1 \ldots s_r$$

there exist $1 \leq i_1 < \ldots < i_m \leq r$, $m < r$ such that

$$w' = s_{i_1} \ldots s_{i_m}$$

The order of the product $st \in W$ of two generators $s, t \in S$ will be denoted by $m(s, t)$ and is an element of $\{1, 2, \ldots, \infty\}$.

1.1.4 Remark. The inclusion $S \subseteq \mathfrak{H}$ induces a bijection

$$S/_\sim \xrightarrow{\sim} W \backslash \mathfrak{H}$$

where $\sim$ is the equivalence relation given by

$$s \sim t \quad \Leftrightarrow \quad \exists w \in W \quad w s w^{-1} = t$$

In the context of root systems and Iwahori-Hecke algebras one is naturally led to consider groups slightly more general than Coxeter groups. We will therefore introduce a nonstandard definition which axiomatizes extended Weyl groups.

1.1.5 Definition. An **extended Coxeter group** W consists of a group W, subgroups $W_{\text{aff}}, \Omega \leq W$, a subset $S \subseteq W_{\text{aff}}$ and a retraction $W \to \Omega$ of inclusion $\Omega \subseteq W$ such that

(i) The sequence

$$1 \longrightarrow W_{\text{aff}} \longrightarrow W \longrightarrow \Omega \longrightarrow 1$$

is exact.

(ii) (W_{aff}, S) is a Coxeter group.

(iii) The action of Ω on W_{aff} by conjugation restricts to an action on S.

In other words an extended Coxeter group W is a semidirect product $W = W_{\text{aff}} \rtimes \Omega$ of a Coxeter group (W_{aff}, S) and a group Ω acting on W_{aff} by automorphisms of Coxeter groups.

1.1.6 Notation. The action of $u \in \Omega$ on $w \in W_{\text{aff}}$ will be denoted by $u(w)$, uwu^{-1} or even $u \bullet w$.

1.1.7 Remark. The conjugation action of Ω on W_{aff} induces a right action on $\mathrm{Hom}_{\mathrm{Set}}(W_{\text{aff}}, \mathbb{N})$ by acting on arguments. The invariance of $S \subseteq W_{\text{aff}}$ is then equivalent to the length function $\ell : W_{\text{aff}} \to \mathbb{N}$ being fixed under the action of Ω. We may therefore uniquely extend ℓ to a function $W \to \mathbb{N}$ denoted by the same letter and satisfying

$$\ell(wu) = \ell(uw) = \ell(w), \quad w \in W_{\text{aff}}, u \in \Omega$$

1.1.8 Remark. The group W acts on the set $\mathfrak{H}$ of walls of (W_{aff}, S) by conjugation and we again have a bijection

$$S/\sim \xrightarrow{\sim} W\backslash\mathfrak{H}$$

where $\sim$ now refers to the equivalence relation given by

$$s \sim t \quad \Leftrightarrow \quad \exists w \in W \quad wsw^{-1} = t$$

Two elements $s, t \in S$ can be conjugate in W without being conjugate in W_{aff}. In the context of extended Coxeter groups, $\sim$ will by convention always refer the relation induced by conjugation in W.

1.1.9 Remark. By assumption we have an action $\rho : \Omega \to \text{Aut}_{\text{Grp}}(W_{\text{aff}})$ of Ω on W_{aff} by group automorphisms. On the other hand W_{aff} acts on itself via left translation $\lambda : W_{\text{aff}} \to \text{Aut}_{\text{Set}}(W_{\text{aff}})$. One has

$$\rho_u(\lambda_w(w')) = \rho_u(ww') = \rho_u(w)\rho_u(w') = \lambda_{\rho_u(w)}(\rho_u(w')) = \lambda_{u\bullet w}(\rho_u(w'))$$

for every $w' \in W_{\text{aff}}$. By the universal property of the semidirect product ρ and λ therefore combine in a unique way to an action of W on the *set* W_{aff}, which we would like to view as the set of chambers. It follows immediately that the stabilizer of the fundamental chamber $1 \in W_{\text{aff}}$ is Ω. We will occasionally view elements $w \in W$ as chambers via the orbit map $W \to W_{\text{aff}}, w \mapsto w \bullet 1$, that is $w = w'u$, $w' \in W_{\text{aff}}$, $u \in \Omega$ will be replaced by w'. Accordingly we will talk about walls separating two elements $w, w' \in W$ or the distance between w and w'. This is consistent with the definitions given so far in the sense that the distance between w, w' viewed as chambers is equal to $\ell(w^{-1}w')$.

1.1.10 Remark. One can extend the Bruhat order on W_{aff} to a strict partial order $<$ on W by letting

$$wu < w'u' :\ \Leftrightarrow \quad u = u' \wedge w < w'$$

for all $w, w' \in W_{\text{aff}}$ and $u, u' \in \Omega$. This relation is invariant under conjugation by Ω, but beware that in general

$$uw < u'w' \quad \not\Leftrightarrow \quad w < w'$$

Note that in [Sch09] (and a previous version of this article) we gave a definition of the Bruhat order that was non-standard. We thank C. Heiermann and U. Görtz who (independently) pointed this out to us.

1.1.11 Remark. Some caution has to be applied when dealing with length function on extended Coxeter groups. It is not true that for any $w, w' \in W$ and $u \in \Omega$

$$\ell(wuw') = \ell(ww')$$

If for example u permutes two distinguished generators $s \neq t$ then

$$\ell(sut) = \ell(s(utu^{-1})u) = \ell(ssu) = 1$$

whereas $\ell(st) = 2$. However, it remains true that for $s \in S$ and $w \in W$ either $\ell(sw) = \ell(w)+1$ or $\ell(sw) = \ell(w)-1$ and similarly either $\ell(ws) = \ell(w) + 1$ or $\ell(ws) = \ell(w) - 1$.

1.1.12 Remark. The examples motivating definition 1.1.5 are the *extended affine Weyl groups* associated to *root data*. This will be discussed later (see example 2.1.3), when we will introduce the stronger notion of *affine extended Coxeter groups*.

We will define generic pro-p Hecke algebras via a presentation à la Iwahori-Matsumoto. In this presentation the "Weyl group" will not be an extended Coxeter group, but a group of a more general type which naturally occurs when considering algebras of the form $\text{End}_G(\text{ind}_{I(1)}^G \mathbb{1})$. The following axioms are modelled on this particular case (cf. [Vig05, 1.2]).

1.1.13 Definition. A **pro-p Coxeter group** $W^{(1)}$ consists of an abelian group T, an extended Coxeter group W and a group extension

$$1 \longrightarrow T \longrightarrow W^{(1)} \xrightarrow{\pi} W \longrightarrow 1$$

together with a family $(n_s)_{s\in S}$ of lifts $n_s \in \pi^{-1}(s)$ of the generators $s \in S$ subject to the following "braid" condition. If $s, t \in S$ with $m(s, t) < \infty$ then

$$n_s n_t n_s \ldots = n_t n_s n_t \ldots \tag{1.1.1}$$

where the number of factors on both sides is $m(s, t)$.

1.1.14 Convention. To ease the notation we will in the following always assume that the map $T \hookrightarrow W^{(1)}$ is an inclusion.

1.1.15 Notation. In the above situation we have a canonical action of $W^{(1)}$ on T by conjugation. This action $W^{(1)} \times T \to T$ is denoted by $(w, t) \mapsto w(t)$. Since T is commutative this action factors over the projection $\pi : W^{(1)} \to W$. The induced action of W on T will also be denoted by $(w, t) \mapsto w(t)$.

1.1.16 Notation. Given a pro-p Coxeter group $W^{(1)}$ as above with associated extended Coxeter group W and length function $\ell : W \to \mathbb{N}$, we will by abuse of notation denote the composite function $\ell \circ \pi : W^{(1)} \to \mathbb{N}$ again by ℓ and refer to it also as "the length function".

1.1.17 Definition. There is a natural strict partial order on $W^{(1)}$ such that $\pi : W^{(1)} \longrightarrow W$ is monotone with respect to the order on W defined in remark 1.1.10. Namely, for $w, w' \in W^{(1)}$ the relation $w' < w$ holds if and only if there exists[10] a *reduced* expression

$$w = n_{s_1} \dots n_{s_r} u, \quad s_i \in S, \ u \in \Omega^{(1)}$$

and integers $1 \leq i_1 < \dots i_m \leq r$ such that

$$w' = n_{s_{i_1}} \dots n_{s_{i_m}} u \quad \text{and} \quad m < r$$

This relation will also be called the *(strong) Bruhat order.* One checks easily using the definition of the Bruhat order on W_{aff} (1.1.3) and its extension to W (remark 1.1.10) that π is monotone, i.e. that

$$w' < w \quad \Longrightarrow \quad \pi(w) < \pi(w')$$

Note however that the reverse implication does *not* hold, i.e. the order $<$ on $W^{(1)}$ is *not* the pullback of the Bruhat order on W along π, and so is different from the definition of the Bruhat order on $W^{(1)}$ given in [Vig05] (also used in [Vig16]) which we adopted in our previous work [Sch09]. The reason for our choice is that we want $<$ to be as weak as possible such that proposition 1.6.3 still holds.

1.1.18 Notation. Given any subset $X \subseteq W$ we will denote by $X^{(1)}$ the preimage of X under π. In particular we have

$$\Omega^{(1)} = \{w \in W^{(1)} : \ell(w) = 0\}$$

1.1.19 Notation. Via the quotient map $\Omega^{(1)} \twoheadrightarrow \Omega$, the Ω-action on S can be inflated to an action of $\Omega^{(1)}$. Following notation 1.1.6, this action will be denoted by

$$u(s) = u \bullet s, \quad u \in \Omega^{(1)}, \ s \in S$$

1.2 1-Cocycles of pro-p Coxeter groups

We recall that a 1-cocycle of a group G with values in a (possibly non-commutative) G-module M (i.e. a *group* endowed with a G-action by group automorphisms) is a map $\phi : G \to M$ satisfying the cocycle rule

$$\forall g, g' \in G \ \ \phi(gg') = \phi(g)g(\phi(g'))$$

Generalizing a result of Cherednik, we will now obtain an explicit description of the set $Z^1(G, M)$ of 1-cocycles when $G = W^{(1)}$ is a pro-p Coxeter group[11].

1.2.1 Lemma. *Let M be a $W^{(1)}$-module. Restriction defines an injective map*

$$Z^1(W^{(1)}, M) \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(S, M) \times Z^1(\Omega^{(1)}, M)$$

$$\phi \mapsto ((s \mapsto \phi(n_s)), (u \mapsto \phi(u)))$$

whose image consists of all pairs (σ, ρ) satisfying the following properties.

(i) $\sigma(s) n_s(\sigma(s)) = \rho(n_s^2)$ *for all* $s \in S$

[10] Equivalently, *for every* (even though we won't need this fact). This follows the fact that any two reduced expressions are connected by a finite chain of 'braid transformations' $\mathrm{III}^{(1)}$ (see proof of lemma 1.2.1).

[11] In a previous version of this lemma, we had assumed that T acts trivial on M. We thank M.F. Vignéras for suggesting to remove this hypothesis.

(ii) *For all* $u \in \Omega^{(1)}$, $s \in S$

$$\rho(u) \cdot u(\sigma(s)) = \sigma(u(s)) \cdot n_{u(s)}(\rho(ut_{s,u}))$$

where $t_{s,u} \in T$ *denotes the element defined by the equation* $un_s = n_{u(s)}ut_{s,u}$

(iii) *For all* $s, t \in S$ *with* $m(s,t) < \infty$, *the following two products with* $m(s,t)$ *factors are equal*

$$\sigma(s) \cdot n_s(\sigma(t)) \cdot (n_s n_t)(\sigma(s)) \cdot (n_s n_t n_s)(\sigma(t)) \ldots = \sigma(t) \cdot n_t(\sigma(s)) \cdot (n_t n_s)(\sigma(t)) \cdot (n_t n_s n_t)(\sigma(s)) \ldots$$

Proof. The map is obviously well-defined and also injective. In fact, let $\phi \in Z^1(W^{(1)}, M)$ be mapped to (σ, ρ). For any $w \in W^{(1)}$, we can find an expression

$$w = n_{s_1} \ldots n_{s_r} u \tag{1.2.1}$$

with $s_i \in S$ and $u \in \Omega^{(1)}$. The cocycle rule for ϕ now implies

$$\phi(w) = \sigma(s_1) \cdot n_{s_1}(\sigma(s_2)) \cdot (n_{s_1} n_{s_2})(\sigma(s_3)) \cdot \ \ldots \cdot (n_{s_1} \ldots n_{s_{r-1}})(\sigma(s_r)) \cdot (n_{s_1} \ldots n_{s_r})(\rho(u)) \tag{1.2.2}$$

Moreover, straightforward computations show that the cocycle rule for ϕ implies the conditions (i)-(iii) for the pair (σ, ρ).

We will now show that, starting with any pair (σ, ρ) satisfying (i)-(iii), equation (1.2.2) gives rise to a well-defined cocycle $\phi : W^{(1)} \to M$. In fact, to show that (1.2.2) gives a well-defined map $\phi : W^{(1)} \to M$ of sets independent of the choice of the expression (1.2.1), it suffices to assume (i), (iii) and the following condition (iv). It is implied by (ii) by taking $u = t \in T$, observing that in this case $ut_{s,u} = s^{-1}(t)$

$$\rho(t) \cdot t(\sigma(s)) = \sigma(s) n_s(\rho(s^{-1}(t))) \quad \forall s \in S,\ t \in T \tag{iv}$$

Now let

$$w = n_{\overline{s}_1} \ldots n_{\overline{s}_m} \overline{u}$$

be another expression for w. We verify that

$$\sigma(s_1) \cdot n_{s_1}(\sigma(s_2)) \cdot \ \ldots \cdot (n_{s_1} \ldots n_{s_r})(\rho(u)) = \sigma(\overline{s}_1) \cdot n_{\overline{s}_1}(\sigma(\overline{s}_2)) \cdot \ \ldots \cdot (n_{\overline{s}_1} \ldots n_{\overline{s}_m})(\rho(\overline{u})) \tag{1.2.3}$$

It suffices to show this when $u, \overline{u} \in T$. Indeed, assume the statement is true in this case. Then, since $W = W_{\mathrm{aff}} \rtimes \Omega$, reducing the equation

$$n_{s_1} \ldots n_{s_r} u = n_{\overline{s}_1} \ldots n_{\overline{s}_m} \overline{u} \tag{1.2.4}$$

via $\pi : W^{(1)} \to W$ shows that $s_1 \ldots s_r = \overline{s}_1 \ldots \overline{s}_m$ and $\pi(u) = \pi(\overline{u})$, and therefore $u\overline{u}^{-1} \in T$. Multiplying (1.2.4) by $\overline{u}^{-1}$ and using (1.2.3) for the case $u, \overline{u} \in T$ gives

$$\sigma(s_1) \cdot \ \ldots \cdot (n_{s_1} \ldots n_{s_{r-1}})(\sigma(s_r)) \cdot (n_{s_1} \ldots n_{s_r})(\rho(u\overline{u}^{-1})) = \sigma(\overline{s}_1) \cdot \ \ldots \cdot (n_{\overline{s}_1} \ldots n_{\overline{s}_{m-1}})(\sigma(\overline{s}_m)) \tag{1.2.5}$$

The cocycle property for ρ implies

$$\rho(u\overline{u}^{-1}) = \rho(u) \cdot u(\rho(\overline{u}^{-1})) = \rho(u) \cdot u(\overline{u}^{-1}(\rho(\overline{u})^{-1}))$$

Therefore

$$(n_{s_1} \ldots n_{s_r})(\rho(u\overline{u}^{-1})) = (n_{s_1} \ldots n_{s_r})(\rho(u)) \cdot (n_{\overline{s}_1} \ldots n_{\overline{s}_m})(\rho(\overline{u})^{-1})$$

Multiplying (1.2.5) from the right by $(n_{\overline{s}_1} \ldots n_{\overline{s}_m})(\rho(\overline{u}))$ therefore gives the desired equation (1.2.3).

We proceed now with the proof of (1.2.3) in the case $u, \overline{u} \in T$. Since the two words $s_1 \ldots s_r$ and $\overline{s}_1 \ldots \overline{s}_m$ in the generators define the same element in W_{aff}, by Tits' solution [Tit69] of the word problem for Coxeter groups we can transform $s_1 \ldots s_r$ into $\overline{s}_1 \ldots \overline{s}_r$ by applying a finite number of transformations of words in the generators $s \in S$ of the following form.

$$t_1 \ldots t_i t_{i+1} \ldots t_m \longmapsto t_1 \ldots t_i s s t_{i+1} \ldots t_m \tag{I}$$

$$t_1 \ldots t_i s s t_{i+1} \ldots t_m \longmapsto t_1 \ldots t_i t_{i+1} \ldots t_m \tag{II}$$

$$t_1 \ldots t_i \underbrace{sts\ldots}_{m(s,t)<\infty} t_{i+1} \ldots t_m \longmapsto t_1 \ldots t_i \underbrace{tst\ldots}_{m(s,t)<\infty} t_{i+1} \ldots t_m \tag{III}$$

Consider the following ‘companion’ transformations for expressions of the form $n_{t_1}\dots n_{t_m}u$ (with $t_i\in S$, $u\in T$)

$$(\mathrm{I}^{(1)})\qquad n_{t_1}\dots n_{t_i}n_{t_{i+1}}\dots n_{t_m}u\longmapsto n_{t_1}\dots n_{t_i}n_sn_sn_{t_{i+1}}\dots n_{t_m}(t_{i+1}\dots t_m)^{-1}(n_s^{-2})u$$

$$(\mathrm{II}^{(1)})\qquad n_{t_1}\dots n_{t_i}n_sn_sn_{t_{i+1}}\dots n_{t_m}u\longmapsto n_{t_1}\dots n_{t_i}n_{t_{i+1}}\dots n_{t_m}(t_{i+1}\dots t_m)^{-1}(n_s^{2})u$$

$$(\mathrm{III}^{(1)})\qquad n_{t_1}\dots n_{t_i}\underbrace{n_sn_tn_s\dots}_{m(s,t)}n_{t_{i+1}}\dots n_{t_m}u\longmapsto n_{t_1}\dots n_{t_i}\underbrace{n_tn_sn_t\dots}_{m(s,t)}n_{t_{i+1}}\dots n_{t_m}u$$

Taking the sequence of transformations of type (I)-(III) which transforms $s_1\dots s_r$ into $\overline{s}_1\dots\overline{s}_m$ and applying the corresponding sequence of transformations of type (I$^{(1)}$)-(III$^{(1)}$) to $n_{s_1}\dots n_{s_r}u$ will give an expression of the form $n_{\overline{s}_1}\dots n_{\overline{s}_m}t$ with $t\in T$. A simple computation shows that the transformations (I$^{(1)}$)-(III$^{(1)}$) do not change the element in $W^{(1)}$ which the expression defines. Therefore

$$n_{\overline{s}_1}\dots n_{\overline{s}_m}t=n_{s_1}\dots n_{s_r}u=n_{\overline{s}_1}\dots n_{\overline{s}_m}\overline{u}$$

and therefore $t=\overline{u}$. To prove (1.2.3), it is therefore enough to show that the element in M defined by the right hand side of (1.2.3) corresponding to an expression $n_{s_1}\dots n_{s_r}u$ does not change if we apply any transformation of type (I$^{(1)}$)-(III$^{(1)}$). For (III$^{(1)}$) this follows from property (iii). We now prove the invariance for transformations of type (I$^{(1)}$), leaving the dual case (II$^{(1)}$) to the reader. It obviously suffices to consider the case $i=0$, i.e. the transformation

$$n_{s_1}\dots n_{s_r}u\mapsto n_sn_sn_{s_1}\dots n_{s_r}(s_1\dots s_r)^{-1}(n_s^{-2})u$$

and to prove that

$$\sigma(s_1)\cdot\ \dots\cdot(n_{s_1}\dots n_{s_{r-1}})(\sigma(s_r))\cdot(n_{s_1}\dots n_{s_r})(\rho(u))$$

is equal to

$$\sigma(s)\cdot n_s(\sigma(s))\cdot(n_sn_s)(\sigma(s_1))\cdot\ \dots\cdot(n_sn_sn_{s_1}\dots n_{s_{r-1}})(\sigma(s_r))\cdot(n_sn_sn_{s_1}\dots n_{s_r})(\rho((s_1\dots s_r)^{-1}(n_s^{-2})u)$$

But, using property (i) and the following identity (implied by the cocycle property of ρ)

$$(n_sn_sn_{s_1}\dots n_{s_r})(\rho((s_1\dots s_r)^{-1}(n_s^{-2})u))=(n_sn_sn_{s_1}\dots n_{s_r})(\rho((s_1\dots s_r)^{-1}(n_s^{-2})))\cdot(n_{s_1}\dots n_{s_r})(\rho(u))$$

it follows that the last expression is equivalent to

$$\rho(n_s^2)\cdot n_s^2\left(\sigma(s_1)\cdot\ \dots\cdot(n_{s_1}\dots n_{s_{r-1}})(\sigma(s_r))\cdot(n_{s_1}\dots n_{s_r})(\rho((s_1\dots s_r)^{-1}(n_s^{-2})))\right)\cdot(n_{s_1}\dots n_{s_r})(\rho(u))$$

Thus it suffices to show that

$$\sigma(s_1)\cdot\ \dots\cdot(n_{s_1}\dots n_{s_{r-1}})(\sigma(s_r))$$

is equal to

$$\rho(n_s^2)\cdot n_s^2\left(\sigma(s_1)\cdot\ \dots\cdot(n_{s_1}\dots n_{s_{r-1}})(\sigma(s_r))\cdot(n_{s_1}\dots n_{s_r})(\rho((s_1\dots s_r)^{-1}(n_s^{-2})))\right)$$

But, this follows by repeated application of (iv), using that $\rho(n_s^2)\cdot n_s^2(\rho(n_s^{-2}))=1$.

Thus, we have shown the existence of a map $\phi:W^{(1)}\to M$ satisfying (1.2.2). It remains to show that ϕ is a 1-cocycle if condition (ii) is satisfied, i.e. that

$$\phi(ww')=\phi(w)\cdot w(\phi(w'))\tag{1.2.6}$$

holds for all $w,w'\in W^{(1)}$. First, we consider the case when w is as a product $w=n_{s_1}\dots n_{s_r}$ in the distinguished generators. In this case, (1.2.6) follows from (1.2.2). Next, we treat the case $w=u\in\Omega^{(1)}$. From the identity

$$u(n_s)=n_{u(s)}u(t_{s,u})$$

it follows by induction that

$$u(n_{s_1})\dots u(n_{s_i})=n_{u(s_1)}\dots n_{u(s_i)}u(t_{s_i,u}s_i^{-1}(t_{s_{i-1},u})\dots(s_2\dots s_i)^{-1}(t_{s_1,u}))\tag{1.2.7}$$

Using (1.2.7), we can now repeatedly apply property (ii) to compute $\phi(w) \cdot w(\phi(w'))$ for $w = u \in \Omega^{(1)}$ and $w' = n_{s_1} \dots n_{s_r} u'$:

$$\begin{aligned}
\phi(u) \cdot u(\phi(w')) &= \rho(u) \cdot u(\sigma(s_1) \cdot n_{s_1}(\sigma(s_2)) \cdot \ \dots \cdot (n_{s_1} \dots n_{s_{r-1}})(\sigma(s_r)) \cdot (n_{s_1} \dots n_{s_r})(\rho(u'))) \\
&= \rho(u) \cdot u(\sigma(s_1)) \cdot u(n_{s_1})(u(\sigma(s_2))) \cdot \ \dots \cdot u(n_{s_1} \dots n_{s_r})(u(\rho(u'))) \\
&= \sigma(u(s_1)) \cdot n_{u(s_1)}(\rho(ut_{s_1,u})) \cdot u(n_{s_1})(u(\sigma(s_2))) \cdot \ \dots \cdot u(n_{s_1} \dots n_{s_r})(u(\rho(u'))) \\
&= \sigma(u(s_1)) \cdot n_{u(s_1)}(\sigma(u(s_2))) \cdot (n_{u(s_1)} n_{u(s_2)})(\rho(ut_{s_2,u} s_2^{-1}(t_{s_1,u}))) \cdot \\
&\quad \cdot u(n_{s_1} n_{s_2})(u(\sigma(s_3))) \cdot \ \dots \cdot u(n_{s_1} \dots n_{s_r})(u(\rho(u'))) \\
&\ \ \vdots \\
&= \sigma(u(s_1)) \cdot n_{u(s_1)}(\sigma(u(s_2))) \cdot \ \dots \cdot (n_{u(s_1)} \dots n_{u(s_{r-1})})(\sigma(u(s_r))) \cdot \\
&\quad \cdot (n_{u(s_1)} \dots n_{u(s_r)})(\rho(ut_{s_r,u} s_r^{-1}(t_{s_{r-1},u}) \dots (s_2 \dots s_r)^{-1}(t_{s_1,u}))) \\
&\quad \cdot (u(n_{s_1} \dots n_{s_r}))(u(\rho(u')))
\end{aligned}$$

Using (1.2.7) again, we see that

$$u(n_{s_1} \dots n_{s_r})(u(\rho(u'))) = (n_{u(s_1)} \dots n_{u(s_r)} ut_{s_r,u} s_r^{-1}(t_{s_{r-1},u}) \dots (s_2 \dots s_r)^{-1}(t_{s_1,u}))(\rho(u'))$$

We can therefore apply the cocycle property of ρ to finally obtain that

$$\begin{aligned}
\phi(u) \cdot u(\phi(w')) = \sigma(u(s_1)) \cdot n_{u(s_1)}(\sigma(u(s_2))) \cdot \ \dots \cdot (n_{u(s_1)} \dots n_{u(s_{r-1})})(\sigma(u(s_r))) \cdot \\
\cdot (n_{u(s_1)} \dots n_{u(s_r)})(\rho(ut_{s_r,u} s_r^{-1}(t_{s_{r-1},u}) \dots (s_2 \dots s_r)^{-1}(t_{s_1,u}) u'))
\end{aligned} \tag{1.2.8}$$

Now

$$\begin{aligned}
uw' &= un_{s_1} \dots n_{s_r} u' \\
&= n_{u(s_1)} ut_{s_1,u} n_{s_2} \dots n_{s_r} u' = n_{u(s_1)} un_{s_2} \dots n_{s_r} (s_2 \dots s_r)^{-1}(t_{s_1,u}) u' \\
&\ \ \vdots \\
&= n_{u(s_1)} n_{u(s_2)} \dots n_{u(s_r)} ut_{s_r,u} s_r^{-1}(t_{s_{r-1},u}) \dots (s_2 \dots s_r)^{-1}(t_{s_1,u}) u'
\end{aligned}$$

and hence

$$\begin{aligned}
\phi(uw') = \sigma(u(s_1)) \cdot n_{u(s_1)}(\sigma(u(s_2)) \cdot \ \dots \cdot (n_{u(s_1)} \dots n_{u(s_{r-1})})(\sigma(u(s_r))) \cdot \\
\cdot (n_{u(s_1)} \dots n_{u(s_r)})(\rho(ut_{s_r,u} s_r^{-1}(t_{s_{r-1},u}) \dots (s_2 \dots s_r)^{-1}(t_{s_1,u}) u'))
\end{aligned} \tag{1.2.9}$$

Comparing (1.2.8) with (1.2.9) gives (1.2.6) for $w = u \in \Omega^{(1)}$ and $w' \in W^{(1)}$ arbitrary. The general case now follows by induction on $\ell(w)$. We have just proved the start of the induction $\ell(w) = 0$. Now let $\ell(w) = r > 0$ and write $w = n_{s_1} \dots n_{s_r} u$. Then

$$\begin{aligned}
\phi(ww') &= \phi(n_{s_1} n_{s_2} \dots n_{s_r} uw') \\
&= \phi(n_{s_1}) n_{s_1}(\phi(n_{s_2} \dots n_{s_r} uw')) \\
&= \phi(n_{s_1}) n_{s_1}(\phi(n_{s_2} \dots n_{s_r} u) \cdot (n_{s_2} \dots n_{s_r} u)(\phi(w'))) \\
&= \phi(n_{s_1} n_{s_2} \dots n_{s_r} u)(n_{s_1} n_{s_2} \dots n_{s_r} u)(\phi(w')) \\
&= \phi(w) w(\phi(w'))
\end{aligned}$$

where we used that $\ell(n_{s_2} \dots n_{s_r} u) = r - 1 < r$ in line 2 in order to apply the induction hypothesis. □

1.3 Construction of generic pro-p Hecke algebras

In this section we will construct the main object of this article. Throughout, $W^{(1)}$ will denote a fixed pro-p Coxeter group. The notation $W, W_{\text{aff}}, S, \Omega, \ell$ etc. will be conserved. We will also fix a commutative associative unital ring R. The monoid algebra of T over R will be denoted by $R[T]$. The action of W on T extends naturally to an action on $R[T]$ by R-algebra automorphisms.

1.3.1 Theorem. *Let $(a_s)_{s \in S}$ and $(b_s)_{s \in S}$ be families of elements $a_s \in R$ and $b_s \in R[T]$ subject to the following condition. Given $s, t \in S$ and $w \in W^{(1)}$ such that $s\pi(w) = \pi(w)t$, the following two equalities in R resp. $R[T]$ hold*[12]

$$a_s = a_t \qquad\qquad (n_s w n_t^{-1} w^{-1}) w(b_t) = b_s \tag{1.3.1}$$

[12]Note that $n_s w n_t^{-1} w^{-1} \in T$

Under this assumption, there exists a unique structure of an R-algebra on the free R-Module M with basis $\{T_w\}_{w\in W^{(1)}}$ which is compatible with the given R-module structure and such that the following two conditions hold

$$(1.3.2) \qquad \forall w, w' \in W^{(1)} \quad \ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \quad T_{ww'} = T_w T'_w$$

$$(1.3.3) \qquad \forall s \in S \quad T^2_{n_s} = a_s T_{n_s^2} + T_{n_s} b_s$$

Before we begin with the proof of theorem 1.3.1, we make a couple of remarks.

1.3.2 Remark. (i) As a consequence of the first condition, the natural embedding $R[T] \hookrightarrow M$ of R-modules will be a morphism of R-algebras because the length function vanishes on T. The R-algebra M will therefore carry a canonical structure of an $(R[T], R[T])$-bimodule so that the second condition makes sense.

(ii) The first condition implies the following basic commutation rule for $t \in T$ and $w \in W^{(1)}$

$$(1.3.4) \qquad T_w T_t = T_{wt} = T_{wtw^{-1}w} = T_{w(t)w} = T_{w(t)} T_w$$

This implies more generally that for any $b \in R[T]$ we have

$$(1.3.5) \qquad T_w b = w(b) T_w$$

(iii) Applying relation (1.3.1) for $w = n_s^{-1}$ and $s = t$ shows that

$$(1.3.6) \qquad n_s^{-1}(b_s) = b_s$$

(iv) In view of (1.3.5) and (1.3.6), the second relation could also have been written as

$$T^2_{n_s} = a_s T_{n_s^2} + b_s T_{n_s}$$

Proof of theorem 1.3.1. We will closely follow the proof in the classical case (cf. [Bou07, Ch. IV, Exercices §2, Ex. 23]). First, we show uniqueness. It suffices to prove that for all $w, w' \in W^{(1)}$ the expansion of the product $T_w T_{w'}$ in terms of the given basis can be effectively computed in terms of the coefficient families $(a_s)_s$ and $(b_s)_s$. If $\ell(w) > 0$, we can write $w = n_s \widetilde{w}$ with $\ell(w) = 1 + \ell(\widetilde{w})$. By (1.3.2)

$$T_w T_{w'} = T_{n_s} T_{\widetilde{w}} T_{w'}$$

By induction it therefore suffices to compute products of the form $T_u T_w$ for $u \in \Omega^{(1)}$ and $T_{n_s} T_w$. From (1.3.2) it follows immediately that $T_u T_w = T_{uw}$. We now show how to compute products of the form $T_{n_s} T_w$ by induction on $\ell(w)$. If $\ell(n_s w) = \ell(w) + 1$, again by (1.3.2) we find that $T_{n_s} T_w = T_{n_s w}$. If $\ell(n_s w) = \ell(w) - 1$, we can write $w = n_s \widetilde{w}$ with $\ell(w) = \ell(\widetilde{w}) + 1$, and so

$$T_{n_s} T_w = T_{n_s} T_{n_s} T_{\widetilde{w}} = (a_s T_{n_s^2} + T_{n_s} b_s) T_{\widetilde{w}} = a_s T_{n_s^2 \widetilde{w}} + T_{n_s} T_{\widetilde{w}} \widetilde{w}^{-1}(b_s)$$

We now show the existence of the algebra structure in question. The construction proceeds by defining an R-subalgebra $\Lambda \subseteq \mathrm{End}_R(M)$ and then showing that $\mathrm{ev}_{T_1} : \mathrm{End}_R(M) \to M$ induces an isomorphism $\Lambda \xrightarrow{\sim} M$ of R-modules. By transport of structure, we obtain an R-algebra structure on M which is then easily verified to have the required properties.

First, we will construct the structure of an $(R[\Omega^{(1)}], R[\Omega^{(1)}])$-bimodule on M. Such a structure is equivalent to giving morphisms $\lambda : R[\Omega^{(1)}] \to \mathrm{End}_R(M)$ and $\rho : R[\Omega^{(1)}]^{\mathrm{op}} \to \mathrm{End}_R(M)$ whose images commute. For $u \in \Omega^{(1)}$ we define $\lambda(u)$ and $\rho(u)$ on basis elements by

$$\lambda(u)(T_w) := T_{uw} \qquad\qquad \rho(u)(T_w) := T_{wu}$$

One verifies immediately that $\lambda(uu') = \lambda(u)\lambda(u')$ and $\rho(uu') = \rho(u')\rho(u)$ and hence we get well defined morphisms λ and ρ. From the definition it is immediate that the images of λ and ρ commute. With respect to this bimodule structure the following identity

$$T_w b = w(b) T_w$$

holds for all $b \in R[T] \subseteq R[\Omega^{(1)}]$ and $w \in W^{(1)}$.

We will now introduce for every $s \in S$ elements $\lambda_{n_s}, \rho_{n_s} \in \mathrm{End}_R(M)$, which will a posteriori turn out the be left respectively right multiplication by T_{n_s}. Put

$$\lambda_{n_s}(T_w) := \begin{cases} T_{n_s w} & : \ell(n_s w) = \ell(w) + 1 \\ a_s T_{n_s w} + b_s T_w & : \ell(n_s w) = \ell(w) - 1 \end{cases}$$

$$\rho_{n_s}(T_w) := \begin{cases} T_{w n_s} & : \ell(w n_s) = \ell(w) + 1 \\ T_{w n_s} a_s + T_w b_s & : \ell(w n_s) = \ell(w) - 1 \end{cases}$$

The products $b_s T_w$, $a_s T_{n_s w}$ etc. therefore refer to the $(R[\Omega^{(1)}], R[\Omega^{(1)}])$-bimodule structure already constructed. Also note that λ_{n_s} and ρ_{n_s} are linear with respect to the right respectively left $R[\Omega^{(1)}]$-module structure.

The main part of the proof consists of showing that the elements $\lambda_{n_s}, \rho_{n_t}$ commute for all $s, t \in S$. Fix $w \in W^{(1)}$ and $s, t \in S$. We make a case distinction according to the 6 possible constellations of $\ell(w), \ell(n_s w), \ell(w n_t)$ and $\ell(n_s w n_t)$

(i) $\ell(n_s w n_t) > \ell(n_s w) = \ell(w n_t) > \ell(w)$:

$$(\lambda_{n_s}\rho_{n_t})(T_w) = \lambda_{n_s}(T_{w n_t}) = T_{n_s w n_t} = \rho_{n_t}(T_{n_s w}) = (\rho_{n_t}\lambda_{n_s})(T_w)$$

(ii) $\ell(n_s w n_t) < \ell(n_s w) = \ell(w n_t) < \ell(w)$:

$$\begin{aligned}(\lambda_{n_s}\rho_{n_t})(T_w) &= \lambda_{n_s}(T_{w n_t} a_t + T_w b_t) = \lambda_{n_s}(T_{w n_t}) a_t + \lambda_{n_s}(T_w) b_t \\ &= a_s T_{n_s w n_t} a_t + b_s T_{w n_t} a_t + a_s T_{n_s w} b_t + b_s T_w b_t \\ &= a_s \rho_{n_t}(T_{n_s w}) + b_s \rho_{n_t}(T_w) = \rho_{n_t}(a_s T_{n_s w} + b_s T_w) \\ &= (\rho_{n_t}\lambda_{n_s})(T_w)\end{aligned}$$

(iii) $\ell(n_s w n_t) = \ell(w) < \ell(n_s w) = \ell(w n_t)$: By lemma 1.3.3, we have $s\pi(w) = \pi(w)t$ and hence that $n_s w n_t^{-1} w^{-1} \in T$. We can therefore invoke relation (1.3.1) to conclude that

$$\begin{aligned}(\lambda_{n_s}\rho_{n_t})(T_w) &= \lambda_{n_s}(T_{w n_t}) = a_s T_{n_s w n_t} + b_s T_{w n_t} \\ &= a_t T_{n_s w n_t} + (n_s w n_t^{-1} w^{-1}) w(b_t) \rho_{n_t}(T_w) \\ &= a_t T_{n_s w n_t} + \rho_{n_t}((n_s w n_t^{-1} w^{-1}) w(b_t) T_w) \\ &= a_t T_{n_s w n_t} + \rho_{n_t}((n_s w n_t^{-1} w^{-1}) T_w b_t) \\ &= a_t T_{n_s w n_t} + \rho_{n_t}(T_{n_s w n_t^{-1}} b_t) \\ &= a_t T_{n_s w n_t} + \rho_{n_t}((n_s w n_t^{-1})(b_t) T_{n_s w n_t^{-1}}) \\ &= a_t T_{n_s w n_t} + (n_s w n_t^{-1})(b_t) \rho_{n_t}(T_{n_s w n_t^{-1}}) \\ &= a_t T_{n_s w n_t} + (n_s w n_t^{-1})(b_t) T_{n_s w} \\ &= a_t T_{n_s w n_t} + T_{n_s w} n_t^{-1}(b_t) \\ &\overset{1.3.6}{=} T_{n_s w n_t} a_t + T_{n_s w} b_t \\ &= \rho_{n_t}(T_{n_s w}) = (\rho_{n_t}\lambda_{n_s})(T_w)\end{aligned}$$

(iv) $\ell(n_s w n_t) = \ell(w) > \ell(n_s w) = \ell(w n_t)$: Similar to (iii).

(v) $\ell(n_s w) < \ell(w) = \ell(n_s w n_t) < \ell(w n_t)$:

$$\begin{aligned}(\lambda_{n_s}\rho_{n_t})(T_w) &= \lambda_{n_s}(T_{w n_t}) = a_s T_{n_s w n_t} + b_s T_{w n_t} = \rho_{n_t}(a_s T_{n_s w} + b_s T_w) \\ &= (\rho_{n_t}\lambda_{n_s})(T_w)\end{aligned}$$

(vi) $\ell(n_s w) > \ell(w) = \ell(n_s w n_t) > \ell(w n_t)$: Similar to (v).

Let now $\Lambda \subseteq \mathrm{End}_R(M)$ be the R-subalgebra generated by $\{\lambda_{n_s}\}_{s \in S}$ and $\{\lambda_u\}_{u \in \Omega^{(1)}}$ and consider the evaluation homomorphism $\mathrm{ev}_{T_1} : \mathrm{End}_R(M) \to M$, $\mathrm{ev}_{T_1}(\phi) = \phi(T_1)$. We claim that restriction to Λ induces an isomorphism

$$\mathrm{ev}_{T_1}| : \Lambda \xrightarrow{\sim} M$$

of R-modules. If $s \in S$ and $w \in W^{(1)}$ are such that $\ell(n_s w) = 1 + \ell(w)$, then $\lambda_{n_s}(T_w) = T_{n_s w}$ by definition. From this it follows immediately that

$$\mathrm{ev}_{T_1}(\lambda_{n_{s_1}} \circ \ldots \circ \lambda_{n_{s_r}} \circ \lambda_u) = T_{n_{s_1} \ldots n_{s_r} u}$$

if $w = n_{s_1} \ldots n_{s_r} u$, $u \in \Omega^{(1)}$ is a reduced expression. This proves surjectivity. To show injectivity let $\phi \in \Lambda$ be such that $\phi(T_1) = 0$. It suffices to show by induction on $\ell(w)$ that $\phi(T_w) = 0$ for all $w \in W^{(1)}$. For $\ell(w) = 0$ we have $w = u \in \Omega^{(1)}$ and hence

$$\phi(T_u) = \phi(\rho_u(T_1)) = \rho_u(\phi(T_1)) = 0$$

Here we have used the fact that ρ_u commutes with all elements of Λ. If $\ell(w) > 0$, write $w = \widetilde{w} n_s$ with $\ell(w) = 1 + \ell(\widetilde{w})$. Then

$$\phi(T_w) = \phi(\rho_{n_s} T_{\widetilde{w}}) = \rho_{n_s}(\phi(T_{\widetilde{w}})) = 0$$

where we have made use of the fact that ρ_{n_s} commutes with the elements of Λ.

By transport of structure, we now get on M the structure of an R-algebra compatible with the given R-module structure. It remains to verify the conditions (1.3.2) and (1.3.3). Assume $\ell(ww') = \ell(w) + \ell(w')$ and let $w = un_{s_1} \ldots n_{s_r}$, $w' = n_{s_{r+1}} \ldots n_{s_{r+t}} u'$ be two reduced expressions. Then $\mathrm{ev}_{T_1}(\lambda_u \lambda_{n_{s_1}} \ldots \lambda_{n_{s_r}}) = T_w$ and $\mathrm{ev}_{T_1}(\lambda_{n_{s_{r+1}}} \ldots \lambda_{n_{s_{r+t}}} \lambda_u) = T_{w'}$ and hence

$$T_w T_{w'} = \mathrm{ev}_{T_1}(\lambda_u \lambda_{n_{s_1}} \ldots \lambda_{n_{s_r}} \lambda_{n_{s_{r+1}}} \ldots \lambda_{n_{s_{r+t}}} \lambda_{u'}) = T_{un_{s_1} \ldots n_{s_r} n_{s_{r+1}} \ldots n_{s_{r+t}} u'} = T_{ww'}$$

The validity of (1.3.3) is equivalent to

$$(\lambda_{n_s} \circ \lambda_{n_s})(T_1) = a_s \lambda_{n_s^2}(T_1) + (\lambda_{n_s} \circ \lambda_{b_s})(T_1)$$

But

$$\lambda_{n_s}^2(T_1) = \lambda_{n_s}(T_{n_s}) = a_s T_{n_s^2} + b_s T_{n_s}$$

by definition and

$$\begin{aligned} a_s \lambda_{n_s^2}(T_1) + (\lambda_{n_s} \circ \lambda_{b_s})(T_1) &= a_s T_{n_s^2} + \lambda_{n_s}(b_s) = a_s T_{n_s^2} + T_{n_s} b_s \\ &= a_s T_{n_s^2} + n_s(b_s) T_{n_s} = a_s T_{n_s^2} + b_s T_{n_s} \end{aligned}$$

□

1.3.3 Lemma. *Let W be an extended Coxeter group and $w \in W$, $s, t \in S$. If either*

$$\ell(sw) = \ell(wt) < \ell(w) = \ell(swt)$$

or

$$\ell(sw) = \ell(wt) > \ell(w) = \ell(swt)$$

then

$$swt = w$$

Proof. For the case of ordinary Coxeter groups we refer to [Hum00, Lemma 7.2]. We show why the statement carries over to the case of extended Coxeter groups. Assume for concreteness that we are in the first case. Write $w = w'u$ with $w' \in W_{\mathrm{aff}}$ and $u \in \Omega$. Then

$$\ell(sw') = \ell(sw) = \ell(wt) = \ell(w'u(t)u) = \ell(w'u(t))$$

and

$$\ell(w') = \ell(w) = \ell(swt) = \ell(sw'u(t)u) = \ell(sw'u(t))$$

According to the version of this lemma for Coxeter groups we conclude that

$$w' = sw'u(t)$$

and hence $w = swt$. □

1.3.4 Definition. The R-algebra constructed in theorem 1.3.1 is called **the generic pro-p Hecke algebra for the parameters** $a = (a_s)_s, b = (b_s)_s$ and is denoted by $\mathcal{H}_R^{(1)}(a, b)$.

1.3.5 Remark. Because of remark 1.1.8 and the condition (1.3.1), we can extend the family $(a_s)_{s\in S}$ to a family $(a_H)_{H\in\mathfrak{H}}$ by putting

$$a_H := a_s$$

if $s \in S$ is an element conjugate to $H \in \mathfrak{H}$ under W.

1.3.6 Remark. The condition (1.3.1) of theorem 1.3.1 is easily seen to be equivalent to the following two conditions.

(i) For any $s, t \in S$ which are conjugate under W we have

$$a_s = a_t$$

and for *some* $w \in W^{(1)}$ with $s\pi(w) = \pi(w)t$ we have

$$(n_s w n_t^{-1} w^{-1}) w(b_t) = b_s$$

(ii) For every $s \in S$ and every $t \in T$ we have

$$s(t)t^{-1}b_s = b_s$$

and for every $w \in W$ with $sw = ws$ we have

$$(n_s \widetilde{w} n_s^{-1} \widetilde{w}^{-1}) w(b_s) = b_s$$

for *some* lift $\widetilde{w} \in W^{(1)}$ of w under $\pi : W^{(1)} \to W$.

1.3.7 Remark. If the coefficient a_s is a unit in R, then the quadratic equation

$$T_{n_s}^2 = a_s T_{n_s^2} + T_{n_s} b_s$$

implies that T_{n_s} is a unit in $\mathcal{H}^{(1)}$, and in this case

$$T_{n_s}^{-1} = a_s^{-1}(T_{n_s^{-1}} - b_s T_{n_s^{-2}}) \tag{1.3.7}$$

Moreover, $T_{n_s^{-1}}$ is then also invertible and we can rewrite the quadratic equation in the following symmetric form

$$T_{n_s} - a_s^{-1} T_{n_s^{-1}}^{-1} = b_s \tag{1.3.8}$$

Proof. Since $n_s^2 \in T \subseteq \Omega^{(1)}$, the element $T_{n_s^2}$ is invertible with inverse $T_{n_s^{-2}}$. Moreover, it follows that

$$1 = a_s T_{n_s^2} T_{n_s^2}^{-1} a_s^{-1} = T_{n_s}(T_{n_s} - b_s) T_{n_s^{-2}} a_s^{-1} = T_{n_s}(T_{n_s^{-1}} - b_s T_{n_s^{-2}}) a_s^{-1}$$

where we have used that n_s^2 is of length zero in the last step. Thus $a_s^{-1}(T_{n_s^{-1}} - b_s T_{n_s^{-2}}) = (T_{n_s^{-1}} - b_s T_{n_s^{-2}}) a_s^{-1}$ is a right inverse to T_{n_s}. Since $T_{n_s} b_s = b_s T_{n_s}$, a similar computation shows that it is also a left inverse to T_{n_s}, and the formula eq. (1.3.7). The invertibility of $T_{n_s^{-1}}$ and eq. (1.3.8) both follow from the formula $T_{n_s^{-1}} = T_{n_s} T_{n_s^{-2}}$. $\square$

1.3.8 Example. The main examples of generic pro-p Hecke algebras that motivated their introduction and the terminology are the double coset convolution algebras $H(G, I^{(1)})$ associated to pro-p-Iwahori subgroups $I^{(1)} \leq G$ of reductive groups. These will be considered in detail in the next section (section 2.2.3). Let us therefore consider here other important examples.

(i) Every Coxeter group W can be viewed as a pro-p Coxeter group with $T = \Omega = 1$ and $n_s = s$. The generic pro-p Hecke algebra $\mathcal{H}_R^{(1)}((a_s)_s, (b_s)_s)$ then coincides with the classical generic Hecke algebra associated to the Coxeter group W and the families $(a_s)_s, (b_s)_s \in R$ of parameters. In the notation of [Bou07, Ch. IV, Exercices §2, Ex. 23] we have

$$\mathcal{H}_R^{(1)}((a_s)_s, (b_s)_s) = E_R((b_s), (a_s))$$

(ii) Given a Coxeter group W and an action

$$W \longrightarrow \mathrm{GL}_{\mathbb{Z}}(T)$$

on an abelian group T by group automorphisms, we get a pro-p Coxeter group $W^{(1)}$ with $\Omega = 1$ by forming the semi-direct product $W^{(1)} = T \rtimes W$ and letting $n_s = s$.

Generic pro-p Hecke algebras of this type include Yokonuma-Hecke algebras $Y_{d,n}$ $(d, n \in \mathbb{N})$. These are algebras over $R = \mathbb{C}[u^{\pm 1}, v]$ generated by elements (cf. [Jd15])

$$g_1, \dots, g_{n-1}, t_1, \dots, t_n$$

subject to the relations

$$\begin{aligned}
g_i g_j &= g_j g_i && \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 1 \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && \text{for all } i = 1, \dots, n-2 \\
t_i t_j &= t_j t_i && \text{for all } i, j = 1, \dots, n \\
g_i t_j &= t_{s_i(j)} g_i && \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n \\
t_j^d &= 1 && \text{for all } j = 1, \dots, n \\
g_i^2 &= u^2 + v e_i g_i && \text{for all } i = 1, \dots, n-1
\end{aligned}$$

where $s_i \in S_n$ denotes the transposition $(i\ i+1)$ and e_i is given by

$$e_i = \frac{1}{d} \sum_{0 \leq s < d} (t_i / t_{i+1})^s$$

In order to relate these to generic pro-p Hecke algebras, let W be the Coxeter group S_n with the standard generators $S = \{s_1, \dots, s_{n-1}\}$ and let T be the finite abelian group $T = (\mathbb{Z}/d\mathbb{Z})^n$. Then we get an isomorphism

$$Y_{d,n} \xrightarrow{\sim} \mathcal{H}_R^{(1)}((a_s)_s, (b_s)_s)$$

of R-algebras by sending g_i to $T_{n_{s_i}} = T_{s_i}$ and t_j to the element of T denoted by the same letter and given component-wise by $(t_j)_i = \overline{\delta_{ij}} \in \mathbb{Z}/d\mathbb{Z}$, if we let

$$a_{s_i} = u^2 \in R \qquad i = 1, \dots, n-1$$

and

$$b_{s_i} = \frac{v}{d} \sum_{s \in \mathbb{Z}/d\mathbb{Z}} (t_i / t_{i+1})^s \in R[T] \qquad i = 1, \dots, n-1$$

1.3.9 Remark. (i) Given a ring R and families $a = (a_s)_{s \in S} \in R$, $b = (b_s)_{s \in S} \in R[T]$ satisfying condition (1.3.1), it is clear that for any ring homomorphism $\varphi : R \to R'$ the image families $\varphi(a) = (\varphi(a_s))_{s \in S} \in R'$ and $\varphi(b) = (\varphi(b_s))_{s \in S} \in R'[T]$ again satisfy condition (1.3.1). Moreover, the natural homomorphism of R'-algebras

$$\begin{aligned}
\mathcal{H}_R^{(1)}(a, b) \otimes_R R' &\longrightarrow \mathcal{H}_{R'}^{(1)}(\varphi(a), \varphi(b)) \\
T_w \otimes x &\longmapsto \varphi(x) T_w
\end{aligned}$$

is an isomorphism, as it is a bijection on the canonical R'-bases on both sides.

(ii) Given a pro-p Coxeter group $W^{(1)}$, let $\mathcal{R}(W^{(1)})$ denote the following category. Objects of $\mathcal{R}(W^{(1)})$ consist of triples (R, a, b) where R is a ring and $a = (a_s)_{s \in S} \in R$ and $b = (b_s)_{s \in S} \in R[T]$ are parameters satisfying condition (1.3.1). A morphism $f : (R, a, b) \to (R', a', b')$ is a ring homomorphism $f : R \to R'$ preserving the parameters

$$f(a_s) = a'_s, \quad f[T](b_s) = b'_s \quad \forall s \in S$$

Here $f[T] : R[T] \to R'[T]$ denotes the induced ring homomorphism.

If the group T is finite, the category $\mathcal{R}(W^{(1)})$ has an initial object R^{univ} given as follows. Consider the polynomial ring

$$R = \mathbb{Z}[\{\mathbf{a}_s, \mathbf{b}_{s,t} : s \in S,\ t \in T\}]$$

in the formal variables $\mathbf{a}_s$ and $\mathbf{b}_{s,t}$. Let

$$\mathbf{b}_s := \sum_{t \in T} \mathbf{b}_{s,t} \cdot t \in R[T]$$

The families $(\mathbf{a}_s)_{s\in S} \in R$ and $(\mathbf{b}_s)_{s\in S} \in R[T]$ do not satisfy condition (1.3.1) in general. However, condition (1.3.1) is equivalent to a set of relations of the form

$$\mathbf{a}_s = \mathbf{a}_{s'} \quad \text{and} \quad \mathbf{b}_{s,t} = \mathbf{b}_{s',t'}$$

where (s, s') ranges over all pairs of W-conjugate elements of S and (t, t') ranges, for each pair (s, s'), over certain pairs of elements of T. Letting $p : R \twoheadrightarrow R^{\mathrm{univ}}$ denote the quotient of R by the ideal generated by these relations, we obtain a well-defined object

$$R^{\mathrm{univ}} = (R^{\mathrm{univ}}, a^{\mathrm{univ}}, b^{\mathrm{univ}}) := (R^{\mathrm{univ}}, (p(\mathbf{a}_s))_{s\in S}, (p[T](\mathbf{b}_s))_{s\in S})$$

of the category $\mathcal{R}(W^{(1)})$. It is clear from the construction that this object is initial. Moreover, by construction R^{univ} is the polynomial ring over $\mathbb{Z}$ on a set of formal variables, that is a quotient of the set $S \amalg (S \times T)$. In particular R^{univ} is noetherian if $\#S < \infty$.

(iii) By the above remarks, when T is finite, every generic pro-p Hecke algebra $\mathcal{H}^{(1)}_R(a, b)$ over a ring R is naturally obtained by base change

$$\mathcal{H}^{(1)}_{R^{\mathrm{univ}}}(a^{\mathrm{univ}}, b^{\mathrm{univ}}) \otimes_{R^{\mathrm{univ}}} R \xrightarrow{\sim} \mathcal{H}^{(1)}_R(a, b)$$

from the *universal* generic pro-p Hecke algebra $\mathcal{H}^{(1)}_{R^{\mathrm{univ}}}(a^{\mathrm{univ}}, b^{\mathrm{univ}})$ over R^{univ}. This allows to reduce many statements about generic pro-p Hecke algebras to the 'universal case'. In particular when we will study the structure of *affine* pro-p Hecke algebras (in which case S and T are finite) in section 2, this will allow us to reduce to the case of a noetherian coefficient ring R.

1.4 Presentations of generic pro-p Hecke algebras via braid groups

Generic Iwahori-Hecke algebras can be described as quotients of monoid algebras of braid monoids (see [GP00, 4.4.1]). The same holds true for generic pro-p Hecke algebras if one introduces the appropriate analogue of braid monoids in the context of pro-p Coxeter groups.

1.4.1 Definition. Let $W^{(1)}$ be a pro-p Coxeter group.

(i) The **(generalized) braid monoid $\mathfrak{B}(W^{(1)})$ associated to $W^{(1)}$** is the monoid with presentation

$$\mathfrak{B}(W^{(1)}) = \langle \{T_w\}_{w\in W^{(1)}} : T_{ww'} = T_w T_{w'} \text{ if } \ell(ww') = \ell(w) + \ell(w')\rangle$$

(ii) The **(generalized) braid group $\mathfrak{A}(W^{(1)})$ associated to $W^{(1)}$** is the group with presentation

$$\mathfrak{A}(W^{(1)}) = \langle \{T_w\}_{w\in W^{(1)}} : T_{ww'} = T_w T_{w'} \text{ if } \ell(ww') = \ell(w) + \ell(w')\rangle$$

By (1.3.2), the canonical map $\{T_w\}_{w\in W^{(1)}} \to \mathcal{H}^{(1)}_R(a, b)$ of sets extends to a morphism

$$\mathfrak{B}(W^{(1)}) \longrightarrow \mathcal{H}^{(1)}_R(a, b)$$

of monoids which in turn induces a morphism

$$R[\mathfrak{B}(W^{(1)})] \longrightarrow \mathcal{H}^{(1)}_R(a, b)$$

of R-algebras. Let $\mathfrak{b}$ denote the two-sided ideal in $R[\mathfrak{B}(W^{(1)})]$ generated by all elements of the form $T^2_{n_s} - a_s T_{n_s^2} - T_{n_s} b_s$, where s runs over all elements of S. By (1.3.3), we have an induced morphism

$$\phi : R[\mathfrak{B}(W^{(1)})]/\mathfrak{b} \longrightarrow \mathcal{H}^{(1)}_R(a, b)$$

1.4.2 Proposition. *The above map ϕ is an isomorphism of R-algebras.*

Proof. The proof is standard (cf. [GP00]). Obviously ϕ is surjective. It therefore suffices to show that ϕ has a left inverse ψ. Because $\mathcal{H}^{(1)}_R(a, b)$ is a free R-module over $\{T_w\}_{w\in W^{(1)}}$, we have a map ψ which associates to any element T_w of the basis the image of the generator T_w of the braid monoid in the quotient $R[\mathfrak{B}(W^{(1)})]/\mathfrak{b}$. Obviously the equation $(\psi \circ \phi)(x) = x$ is satisfied for said images of the generators of $\mathfrak{B}(W^{(1)})$. But because of the quadratic relations, these images already generate the quotient $R[\mathfrak{B}(W^{(1)})]/\mathfrak{b}$ as an R-module. Hence, $\psi \circ \phi = \mathrm{id}$. □

When the parameters a_s are units in R, the generic pro-p Hecke algebra has a second presentation in terms of $\mathfrak{A}(W^{(1)})$. In fact, in this case $T_{n_s} \in \mathcal{H}_R^{(1)}(a,b)^\times$ with inverse (see remark 1.3.7)

$$T_{n_s}^{-1} = a_s^{-1}(T_{n_s^{-1}} - b_s T_{n_s^{-2}})$$

This implies that for every $w \in W^{(1)}$ we have $T_w \in \mathcal{H}_R^{(1)}(a,b)^\times$, since

$$T_w = T_{n_{s_1}} \dots T_{n_{s_r}} T_u$$

if $w = n_{s_1} \dots n_{s_r} u$, $u \in \Omega^{(1)}$ is any expression with $r = \ell(w)$. Just as before, we get an induced morphism

$$\varphi : R[\mathfrak{A}(W^{(1)})]/\mathfrak{a} \longrightarrow \mathcal{H}_R^{(1)}(a,b)$$

of R-algebras, where $\mathfrak{a}$ denotes the two-sided ideal generated by $T_{n_s}^2 - a_s T_{n_s^2} - T_{n_s} b_s$, $s \in S$. The same arguments as in the previous proposition show that

1.4.3 Proposition. *The above morphism φ is an isomorphism of R-algebras.*

1.4.4 Example. (i) Continuing the examples given in example 1.3.8, if we take $W^{(1)} = W = S_n$ to be the symmetric group on n letters in the first example, the associated generalized braid group $\mathfrak{A}(W^{(1)})$ identifies canonically with the classical **Artin braid group** B_n on n strands. The above presentation then relates the representation theory of finite Hecke algebras associated to S_n to invariants of braids and hence of links, via the construction which associates to a braid its link closure.

(ii) In the second example of example 1.3.8 the generalized braid group $\mathfrak{A}(W^{(1)})$ of $W^{(1)} = (\mathbb{Z}/d\mathbb{Z})^n \rtimes S_n$ identifies canonically with the **d-modular framed braid group** $(\mathbb{Z}/d\mathbb{Z})^n \rtimes B_n$ on n strands, where B_n acts on $(\mathbb{Z}/d\mathbb{Z})^n$ by permutation. The above presentation then relates the representation theory of the Yokonuma Hecke algebra $Y_{d,n}$ to invariants of framed braids and links (see [Jd15]). The special interest in *framed* braids and links arises from the fact [Kir78] that 3-manifolds are classified up to homeomorphism (or equivalently, up to diffeomorphism) by framed links up to a certain equivalence.

1.5 Orientations of Coxeter groups

The following definition is motivated by theorem 0.0.1.

1.5.1 Definition. An **orientation** $\mathfrak{o}$ of a Coxeter group (W,S) is a map

$$\mathfrak{o} : W \times S \longrightarrow \{\pm 1\}$$

satisfying the following two properties:

(OR1) $\mathfrak{o}(ws,s) = -\mathfrak{o}(w,s)$ for all $w \in W$, $s \in S$.

(OR2) If $s,t \in S$ with $m(s,t) < \infty$ and $w \in W$ is arbitrary, then pair of sequences

$$(\mathfrak{o}(w,s), \mathfrak{o}(ws,t), \mathfrak{o}(wst,s), \mathfrak{o}(wsts,t), \dots), \quad (\mathfrak{o}(w,t), \mathfrak{o}(wt,s), \mathfrak{o}(wts,t), \mathfrak{o}(wtst,s), \dots)$$

is either of the form

$$(\underbrace{+,\dots,+}_{k}, \underbrace{-,\dots,-}_{m(s,t)-k}), \quad (\underbrace{-,\dots,-}_{m(s,t)-k}, \underbrace{+,\dots,+}_{k})$$

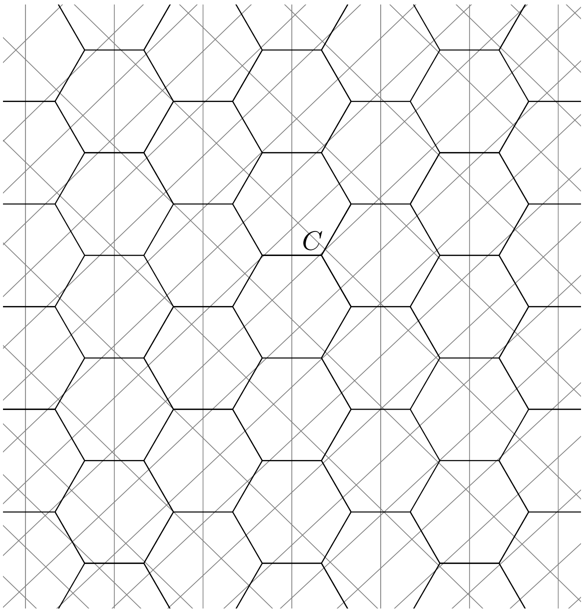
or

$$(\underbrace{-,\dots,-}_{k}, \underbrace{+,\dots,+}_{m(s,t)-k}), \quad (\underbrace{+,\dots,+}_{m(s,t)-k}, \underbrace{-,\dots,-}_{k})$$

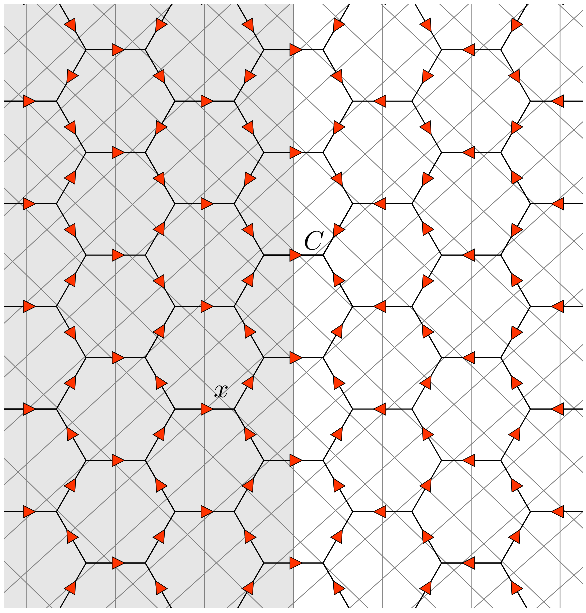
for some $0 \leq k \leq m(s,t)$.

The set of all orientations of a Coxeter group (W,S) **is denoted by** $\mathcal{O}(W,S)$, or simply by $\mathcal{O}$ if the underlying Coxeter group is understood.

1.5.2 Terminology. Viewing elements $w \in W$ of Coxeter groups as chambers according to the terminology introduced in 1.1.3, the sign $\mathfrak{o}(w,s)$ should be interpreted geometrically as follows. The sequence w, ws of adjacent chambers forms a gallery that crosses the hyperplane $H = wsw^{-1}$. We will say that $\mathfrak{o}(w,s)$ is the *sign given to this crossing by the orientation* $\mathfrak{o}$, or that it is the *sign attached to crossing* H at w by the orientation $\mathfrak{o}$. The axiom (**OR1**) therefore ensures that the sign attached to the opposite crossing ws, w is opposite.



(a) The undirected Cayley graph Γ.



(b) The directed Cayley graph $\Gamma_{\mathfrak{o}}$ given by the orientation $\mathfrak{o}$ towards C. On the 'cycle' γ (in grey and light yellow), the orientation coincides with the orientation towards $x \in \gamma$.

Figure 2: The Coxeter complex of the affine Coxeter group (W, S) of type $\widetilde{A}_2$ and its Cayley graph Γ. The orientation $\mathfrak{o}$ of (W, S) *towards* the chamber C (definition 1.5.7) determines an orientation of the Cayley graph, giving rise to the directed Cayley graph $\Gamma_{\mathfrak{o}}$. The condition (**OR2**) ensures that restricted to any 'cycle' $\gamma \subseteq \mathfrak{o}$, the orientation coincides with the orientation towards a chamber $x \in \gamma$.

1.5.3 Remark. Definition 1.5.1 is inherently symmetric: to any orientation $\mathfrak{o} : W \times S \to \{\pm\}$ one can associate its *opposite orientation* $\mathfrak{o}^{\mathrm{op}} : W \times S \to \{\pm\}$ given by $\mathfrak{o}^{\mathrm{op}}(w, s) = -\mathfrak{o}(w, s)$.

1.5.4 Remark. Definition 1.5.1 can be interpreted in terms of the (undirected) Cayley graph Γ of (W, S). Recall (cf. [AB08, Def. 1.73]) that Γ is the undirected graph with $\mathrm{Vert}(\Gamma) = W$ and $\{w_1, w_2\} \in \mathrm{Edge}(\Gamma)$ iff $w_1^{-1}w_2 \in S$. By (**OR1**), an orientation $\mathfrak{o}$ of W now determines an *orientation* of Γ *in the sense of graph theory*, i.e. it determines a directed graph $\Gamma_{\mathfrak{o}}$ whose underlying undirected graph equals Γ, if one lets

$$(w_1, w_2) \in \mathrm{Edge}(\Gamma_{\mathfrak{o}}) \quad \Leftrightarrow \quad w_1^{-1}w_2 \in S \wedge \mathfrak{o}(w_1, w_1^{-1}w_2) = +1$$

In terms of $\Gamma_{\mathfrak{o}}$, (**OR2**) means that every cycle $\gamma \subseteq \Gamma$ of the form

$$\gamma = \{w, ws, wst, wsts, \ldots, w(st)^{m(s,t)-1}\}, \quad s, t \in S,\ m(s,t) < \infty$$

is 'oriented towards' some vertex $w_0 \in \gamma$, as indicated in figure 2.

1.5.5 Remark. There exists a natural right action of W on the set of all orientations of (W, S). Given an orientation $\mathfrak{o}$ and $w \in W$, it follows easily that the function $\mathfrak{o} \bullet w$ defined by

$$(\mathfrak{o} \bullet w)(w', s) := \mathfrak{o}(ww', s)$$

is again an orientation of W. Moreover, it is clear that this action commutes with the involution $\mathfrak{o} \mapsto \mathfrak{o}^{\mathrm{op}}$.

1.5.6 Remark. The set $\mathcal{O}(W, S)$ of orientations of a Coxeter group (W, S) naturally carries the structure of a topological space, in fact that of a compact Hausdorff space. Namely, we can view it as a subspace of the mapping space $\{\pm\}^{W \times S}$ endowed with the compact-open topology[13], where $\{\pm\}$ and $W \times S$ are considered discrete. By definition, a basis of the topology on $\{\pm\}^{W \times S}$ is given by

$$U_{\{x_i\},\{y_i\}} = \{f \in \{\pm\}^{W \times S} : f(x_i) = y_i \quad \forall i = 1, \ldots, n\}$$

where $\{x_1, \ldots, x_n\} \subseteq W \times S$ and $\{y_1, \ldots, y_n\} \subseteq \{\pm\}$ are finite subsets.

[13]Which can also be viewed as the product space $\prod_{(w,s) \in W \times S} \{\pm\}$.

It is then easy to see that the set of orientations of (W,S) forms a *closed* (and hence compact) subspace of $\{\pm\}^{W\times S}$, as the conditions (**OR1**) and (**OR2**) involve only finitely many elements of $W\times S$ at a time. This is of course trivial when W is finite, but this fact will be useful later when we will construct *spherical orientations* of affine Coxeter groups, which are obtained as limits of orientations associated to chambers, in the sense of the definition below.

1.5.7 Definition. Given a chamber $w_0 \in W_{\text{aff}}$ let $\mathfrak{o}_{w_0} : W\times S \to \{\pm\}$ be the map defined by

$$\mathfrak{o}_{w_0}(w,s) := \begin{cases} +1 & : \ell(w_0^{-1}ws) < \ell(w_0^{-1}w) \\ -1 & : \ell(w_0^{-1}ws) > \ell(w_0^{-1}w) \end{cases}$$

Then $\mathfrak{o}_{w_0}$ is called the **orientation associated to the chamber** w_0 or the **orientation towards the chamber** w_0 (cf. figure 2).

1.5.8 Remark. The $\mathfrak{o}_{w_0}$ are indeed orientations in the sense of definition 1.5.1. In particular, the set of orientations of a Coxeter group is always non-empty. Indeed, by construction we have $\mathfrak{o}_{ww'} = \mathfrak{o}_{w'} \bullet w^{-1}$, so it suffices to verify that $\mathfrak{o}_1$ is an orientation. Obviously condition (**OR1**) holds true. An exercise in Coxeter groups [Bou07, Ch. IV, Exercices §1, Ex. 2] now shows that for any given $w\in W$ and $s,t\in S$ we can always find an element $w_0 \in w\langle s,t\rangle$ such that

$$\ell(w_0) = \ell(w') + \ell(w_0^{-1}w')$$

for every $w' \in w\langle s,t\rangle$. So approaching 1 is the same as moving further away from w_0. By remark 1.5.4, it follows that $\mathfrak{o}_1$ is an orientation.

1.5.9 Remark. The orientation $\mathfrak{o}_w$ defined in definition 1.5.7 is not the only orientation naturally attached to an element $w\in W$. One can just as well define an orientation 'away from the chamber w', which in fact is none other than the opposite orientation $\mathfrak{o}_w^{\text{op}}$, and so there is no need for a separate definition.

Moreover when W is a finite group, every orientation away from a chamber is in fact also an orientation towards another chamber, namely

$$\mathfrak{o}_w^{\text{op}} = \mathfrak{o}_{w_0 w}$$

if w_0 is the longest element of W. In contrast, for infinite groups orientations towards and away from chambers are disjoint in general (cf. proof of remark 1.5.10).

1.5.10 Remark. For a Coxeter group (W,S), the map

$$\begin{aligned} j : W &\longrightarrow \mathcal{O}(W,S) \\ w &\longmapsto \mathfrak{o}_w \end{aligned}$$

is *injective*. Moreover if S is finite, then W is *discrete* as a subset of $\mathcal{O}(W,S)$. In fact, in this case

$$\mathcal{O}_{\text{chamber}} := j(W) \cup j(W)^{\text{op}} = \{\mathfrak{o}_w, \mathfrak{o}_w^{\text{op}} : w\in W\} \subseteq \mathcal{O}$$

is discrete. In particular for an infinite Coxeter group (W,S) with $\#S<\infty$, the set

$$\mathcal{O}_{\text{boundary}} := \overline{\mathcal{O}_{\text{chamber}}} - \mathcal{O}_{\text{chamber}} \subseteq \mathcal{O}$$

of **boundary orientations** of W is non-empty.

Proof. The element $w\in W$ can be recovered from the orientation $\mathfrak{o}_w$ as the unique element w' satisfying

$$\mathfrak{o}_w(w',s) = -1$$

for all $s\in S$, which shows the injectivity of the map. Moreover if $S=\{s_1,\dots,s_n\}$ is finite, this also shows that

$$U_{\{x_i\},\{y_i\}} \cap W = \{\mathfrak{o}_w\}, \quad x_i = (w,s_i), y_i = -1$$

and therefore that W is discrete as a subset of $\mathcal{O}(W,S)$. Furthermore, if the above neighbourhood $U_{\{x_i\},\{y_i\}}$ contains $\mathfrak{o}_{w_0}^{\text{op}}$ for some $w_0\in W$, then

$$\mathfrak{o}_{w_0}(w,s) = \ell(w_0^{-1}w) - \ell(w_0^{-1}ws) = 1$$

for all $s\in S$. This implies (cf. [Bou07, Ch. IV, §1, exerc. 22b]) that $w_0^{-1}w$ is a longest element of W; in particular, the length on W is bounded. Since S is finite, it follows that W must be finite and so the space $\mathcal{O}$ is finite and discrete itself, and there is nothing to show.

Finally if W is infinite and $\#S<\infty$, the set $W\cup W^{\text{op}}$ is discrete and infinite, and therefore its (compact) closure must be a proper superset. ☐

1.5.11 Remark. (i) The above remark gives an abstract proof (relying on Tychonoff's theorem) that for an infinite Coxeter groups $W = (W, S)$ with $\#S < \infty$ the (compact, W-invariant, and ${}^{\mathrm{op}}$-invariant) set

$$\mathcal{O}_{\text{boundary}} = \overline{\mathcal{O}_{\text{chamber}}} - \mathcal{O}_{\text{boundary}} \subseteq \mathcal{O}$$

is non-empty. In section 2.4, we will construct some concrete examples of elements $\mathfrak{o} \in \mathcal{O}_{\text{boundary}}$ for affine Coxeter groups, the *spherical orientations* that lie at the heart of the Bernstein-Zelevinsky method. What makes these orientations useful for the study of Hecke algebras is the fact that they have a very large stabilizer (the commutative subgroup $X \leq W$ of translations) under the action of W, and via the unnormalized and normalized Bernstein maps θ, $\widetilde{\theta}$ (definition 1.6.2, definition 1.10.9) therefore give rise to embeddings (cf. proposition 1.6.3)

$$k[\mathrm{Stab}_W(\mathfrak{o})] \hookrightarrow \mathcal{H}^{(1)}$$

of the corresponding group algebra, given by $w \mapsto \theta_{\mathfrak{o}}(w)$ and $w \mapsto \widetilde{\theta}_{\mathfrak{o}}(w)$ respectively. Although these are the only boundary orientations that we will be concerned with, there exist many more (infinitely many) such orientations for affine Coxeter groups[14].

(ii) The set $\mathcal{O}_{\text{boundary}}$ seems to be particularly interesting in the case of hyperbolic[15] Coxeter groups. In particular, there seems to be a rich supply of orientations having non-trivial stabilizer, although it is not clear at the moment whether the corresponding subalgebras $k[\mathrm{Stab}_W(\mathfrak{o})] \subseteq \mathcal{H}^{(1)}$ yield any useful information about the structure of the Hecke algebras $\mathcal{H}^{(1)}$ attached to W. The set $\mathcal{O}_{\text{boundary}}$ also appears to be somewhat related to the *Gromov boundary* $\partial(W, S)$ of W (see [Dav08, 12.4]).

The richness of $\mathcal{O}_{\text{boundary}}$ in the hyperbolic case is illustrated by the example of the group $W = \mathrm{PGL}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z})/\{\pm\}$ of invertible 2x2 integer matrices modulo center, which is discussed in section 3.

1.5.12 Remark. We will show later in lemma 1.7.4 that orientations $\mathfrak{o} \in \mathcal{O}$ can also be viewed as choosing for every hyperplane $H \in \mathfrak{H}$ a 'positive' half-space $U^+_{\mathfrak{o},H} \in \{U^+_H, U^-_H\}$ such that $\mathfrak{o}(w, s) = 1$ iff $ws \in U^+_{\mathfrak{o},H}$ where $H = wsw^{-1}$. In view of this, it follows easily from unwinding definitions that the union

$$\mathcal{O}_G := \overline{j(W)} \cup \overline{j(W)^{\mathrm{op}}} = \overline{j(W) \cup j(W)^{\mathrm{op}}} = \overline{\mathcal{O}_{\text{chamber}}} \subseteq \mathcal{O}$$

identifies exactly with the *root hyperplane orientations* defined in [Gör07, Def. 2.3.1], i.e. those orientations $\mathfrak{o}$ having the property that all finite intersections

$$U^+_{\mathfrak{o},H_1} \cap \cdots \cap U^+_{\mathfrak{o},H_n} \neq \emptyset$$

of positive half-spaces with respect to $\mathfrak{o}$ are non-empty, or that all finite intersections

$$U^-_{\mathfrak{o},H_1} \cap \cdots \cap U^-_{\mathfrak{o},H_n} \neq \emptyset$$

of the corresponding negative half-space are non-empty. The example of the infinite dihedral group (the free group on generators s, t; $m(s, t) = \infty$) shows that the inclusion

$$\mathcal{O}_G \subseteq \mathcal{O}$$

is proper in general.

1.5.13 Remark. The two orientations attached to a rational point $x \in \mathbb{P}^1(\mathbb{Q})$ are *adjacent* in the sense of definition 1.11.2.

Proof. If $\mathfrak{o}$ is an orientation attached to a point $x \in \mathbb{P}^1(\mathbb{R})$ as above, then for all hyperplanes $H \in \mathfrak{H}$ that don't end in x (i.e. $x \notin \overline{H}$), the half-space bounded by H that is positive with respect to $\mathfrak{o}$ is the one containing x (in the 'obvious' sense). In particular, the orientation $\mathfrak{o}$ is determined on all those H by the point x alone. Therefore, if $\mathfrak{o}^+_x$, $\mathfrak{o}^-_x$ now denote the two distinct orientations attached to $x \in \mathbb{P}^1(\mathbb{Q})$, then $\mathfrak{o}^+_x, \mathfrak{o}^-_x$ disagree precisely at those H with $x \in \overline{H}$, and it's clear that for every such H, the reflection s_H permutes the two subsets in

$$\mathfrak{H} = \{H \in \mathfrak{H} : x \notin \overline{H}\} \amalg \{H \in \mathfrak{H} : x \in \overline{H}\}$$

amongst themselves, and moreover preserves the half-spaces bounded by the hyperplanes in the left set that are positive with respect to $\mathfrak{o}^+_x$ (equivalently $\mathfrak{o}^-_x$), and maps the half-spaces bounded by the hyperplanes in the right set that are positive with respect to $\mathfrak{o}^+_x$ to negative ones (i.e. positive with respect to $\mathfrak{o}^-_x$). $\square$

[14] The set of boundary orientations can easily be worked out for the group of type $\widetilde{A_2}$; apart from the six spherical orientations, it contains countably many orientations all of whose stabilizers are subgroups of X of rank 1.

[15] There are several inequivalent definitions of the term *hyperbolic Coxeter groups*, see [AB08, 10.4].

As a complement to remark 1.5.10, the following lemma shows that for *finite* Coxeter groups (W, S) chamber orientations *do* exhaust all possibilities. This lemma is not used in our results concerning affine pro-p Hecke algebras and may therefore be safely skipped.

1.5.14 Lemma. *Let $\mathfrak{o}$ be an orientation of (W, S), and suppose that $\#W < \infty$. Then*

$$\mathfrak{o} = \mathfrak{o}_w$$

for some $w \in W$.

Proof. Let us begin with a general observation. Given any (not necessarily finite) Coxeter group (W, S) and an orientation $\mathfrak{o}$, we can construct a function

$$\phi_{\mathfrak{o}} : W \longrightarrow \mathbb{Z}$$

as follows. Given a $w \in W$, let $w = s_1 \dots s_r$ be any expression as a product of generators, and put

$$\phi_{\mathfrak{o}}(w) = \sum_{i=1}^{r} \mathfrak{o}(s_1 \dots s_{i-1}, s_i)$$

In other words, $\phi_{\mathfrak{o}}(w)$ is the sum of the signs that $\mathfrak{o}$ associates to the gallery $\Gamma = (1, s_1, s_1 s_2, \dots, w)$ from 1 to w. We need to see that this sum is well-defined independent of the choice of Γ.

By Tits' solution of the word problem for Coxeter groups [Bro89, II.3C], any two expressions of w as a product of generators are related by a sequence of transformations of the following type.

(I) $s_1 \dots s_i s s s_{i+1} \dots s_r \mapsto s_1 \dots s_i s_{i+1} \dots s_r$

(II) $s_1 \dots s_i s_{i+1} \dots s_r \mapsto s_1 \dots s_i s s s_{i+1} \dots s_r$

(III) $s_1 \dots s_i \underbrace{sts\dots}_{m(s,t)} s_{i+1} \dots s_r \mapsto s_1 \dots s_i \underbrace{tst\dots}_{m(s,t)} s_{i+1} \dots s_r$ if $m(s,t) < \infty$.

Because (**OR1**) guarantees the invariance under the first two transformations and (**OR2**) guarantees the invariance under the third, it therefore follows that $\phi_{\mathfrak{o}}(w)$ is well-defined. Moreover, it is immediate from the definitions that

$$\phi_{\mathfrak{o}}(ww') = \phi_{\mathfrak{o}}(w) + \phi_{\mathfrak{o} \bullet w}(w')$$

which we can also write as

$$\phi_{\mathfrak{o} \bullet w} = \phi_{\mathfrak{o}} \bullet w - \phi_{\mathfrak{o}}(w) \tag{1.5.1}$$

For orientations $\mathfrak{o}$ of the form $\mathfrak{o} = \mathfrak{o}_w$, the function $\phi_{\mathfrak{o}}$ is easily seen to be given by

$$\phi_{\mathfrak{o}}(w') = \ell(w) - \ell(w^{-1}w')$$

Conversely, if $\phi_{\mathfrak{o}}$ is of the above form, it follows that $\mathfrak{o} = \mathfrak{o}_w$, and in this case w is determined as the unique element $w' \in W$ at which $\phi_{\mathfrak{o}}$ attains its global maximum.

Let us now assume that W is finite, and let w be such that $\phi_{\mathfrak{o}}(w)$ is maximal. Using (1.5.1) and replacing $\mathfrak{o}$ by $\mathfrak{o} \bullet w$, we may assume that $w = 1$. In order to show that $\mathfrak{o} = \mathfrak{o}_1$, it suffices by the above remark to prove that

$$\phi_{\mathfrak{o}}(w) = -\ell(w)$$

or equivalently, to prove that $\phi_{\mathfrak{o}}$ is monotonously decreasing along geodesics, i.e. to prove that for every *reduced* expression $s_1 \dots s_r$ the sequence

$$\phi_{\mathfrak{o}}(w_0), \phi_{\mathfrak{o}}(w_1), \dots, \phi_{\mathfrak{o}}(w_r) \quad \text{with} \quad w_i = s_1 \dots s_i \tag{1.5.2}$$

is (strictly) decreasing (note that two consecutive elements of the above sequence differ by ± 1).

We prove this using induction over r. For $r = 1$, this follows from the fact that ϕ has its (a priori not unique) global maximum at $w = 1$. Let now $r \geq 2$, and assume that the claim holds for sequences of length $< r$. In particular

$$\phi_{\mathfrak{o}}(w_i) = -i \quad \text{for } i < r$$

Suppose that we had $\phi_{\mathfrak{o}}(w_r) > \phi_{\mathfrak{o}}(w_{r-1})$, i.e. $\phi_{\mathfrak{o}}(w_r) = -(r-2)$, and put $s = s_{r-1}$, $t = s_r$. We then have the following situation

$$\phi_{\mathfrak{o}}(w_{r-2}) = -(r-2) \quad \overset{s}{|} \quad \phi_{\mathfrak{o}}(w_{r-1}) = -(r-1) \quad \overset{t}{|} \quad \phi_{\mathfrak{o}}(w_r) = -(r-2)$$

By remark 1.5.4, the restriction of $\mathfrak{o}$ to the 'loop' $w_{r-2} \cdot \langle s,t\rangle$ is given by the distance to a chamber, and therefore $\phi_{\mathfrak{o}}$ attains precisely one local minimum there. Thus, this minimum is attained at $w_{r-1} = w_{r-2}s$ and for all $k \leq m(s,t)-1$ we have that

$$\phi_{\mathfrak{o}}(w_{r-2}\underbrace{tst\ldots}_{k}) = -(r-2)+k \tag{1.5.3}$$

Note that $m(s,t) < \infty$ because W is finite, and $m(s,t) \geq 2$ because $s = t$ would contradict the reducedness of $s_1 \ldots s_r$. In particular, $\phi_{\mathfrak{o}}(w_{r-2}t) = -(r-3)$; because of our induction hypothesis, it follows that the expression $s_1 \ldots s_{r-2}t$ must be reducible, yielding an immediate contradiction if $r = 2$. If $r \geq 3$, we can apply the *deletion condition* (see remark 1.1.2) and the reducedness of the expression $s_1 \ldots s_{r-2}$ to conclude that

$$w_{r-2}t = s_1 \ldots s_{r-2}t = s_1 \ldots \widehat{s_j} \ldots s_{r-2}$$

for some $1 \leq j \leq r-2$. This subsequence $s_1 \ldots \widehat{s_j} \ldots s_{r-2}$ of length $r-3$ is again reduced, and its associated sequence of values of $\phi_{\mathfrak{o}}$ is again strictly decreasing (we don't need to use the induction hypothesis for this; this already follows from the fact that $\phi_{\mathfrak{o}}(1) = 0$ and $\phi_{\mathfrak{o}}(s_1 \ldots \widehat{s_j} \ldots s_{r-2}) = -(r-3)$).

We can therefore repeat the above argument with the expression $s_1 \ldots s_{r-2}$ replaced with $s_1 \ldots \widehat{s_j} \ldots s_{r-2}$, using equation (1.5.3) for $k = 2$ and the induction hypothesis to conclude that $s_1 \ldots \widehat{s_j} \ldots s_{r-2}s$ is reducible. We can keep iterating this argument as long as we are able to apply (1.5.3), that is, applying this argument k times we end up with an equation

$$w_{r-2}\underbrace{tst\ldots}_{k} = s_{j_1} \ldots s_{j_{r-2-k}}$$

for some sequence $1 \leq j_1 < \ldots < j_{r-2-k} \leq r-2$, such that either $k = r-2 < m(s,t)-1$ and the product on the right hand side is empty, or $k = m(s,t)-1$. In the first case, we would have

$$w_{r-2} = \underbrace{\ldots tst}_{r-2}$$

Again, using that the restriction of $\mathfrak{o}$ to the loop $\langle s,t\rangle$ of length $m(s,t)$ is given by the distance to a chamber, and that the restriction of $\phi_{\mathfrak{o}}$ to this loop therefore has a unique local minimum and a unique local maximum, both of which are lying opposite to each other, it follows that the maximum must be attained at $w = 1$ (!) and that the minimum must be attained at $w = w_{r-1}$. In particular, $r-1 = m(s,t)$ which is a contradiction.

In the second case, we would have a reduced (!) expression

$$w_{r-2} = s_{j_1} \ldots s_{j_{r-2-k}} \underbrace{\ldots tst}_{m(s,t)-1}$$

Since $w_r = w_{r-2}st$ and $\ell(w_r) = w_{r-2}+2$ by assumption, the expression

$$s_{j_1} \ldots s_{j_{r-2-k}} \underbrace{\ldots tst}_{m(s,t)-1} st$$

would be reduced. But already the subexpression

$$\underbrace{\ldots tst}_{m(s,t)-1} st$$

is reducible, yielding a contradiction. □

We will now extend the notion of an orientation to extended and pro-p Coxeter groups. The extension from extended to pro-p Coxeter groups is trivial, but the extension from Coxeter to extended Coxeter groups is a bit subtle because of the action of Ω.

1.5.15 Definition. Let W be an extended Coxeter group and $\mathfrak{o}$ be an orientation of W_{aff}. Then the map

$$\widetilde{\mathfrak{o}} : W \times S \longrightarrow \{\pm 1\}$$

given by $\widetilde{\mathfrak{o}}(wu,s) := \mathfrak{o}(w,u(s))$, $w \in W_{\text{aff}}$, $u \in \Omega$ is called the **orientation of W associated to $\mathfrak{o}$**.

A map $\mathfrak{o} : W \times S \longrightarrow \{\pm 1\}$ is called an **orientation** if it is associated to an orientation of W_{aff} in the above sense, and the set of all such orientations is denoted by $\mathcal{O}(W)$, or simply by $\mathcal{O}$ if W is understood.

1.5.16 Remark. There exists a natural right action of Ω on the set $\mathcal{O}(W_{\text{aff}}, S)$ of all orientations of W_{aff}. Given an orientation $\mathfrak{o} \in \mathcal{O}(W_{\text{aff}}, S)$ and $u \in \Omega$

$$(\mathfrak{o} \bullet u)(w, s) := \mathfrak{o}(uwu^{-1}, u(s))$$

again defines an orientation of W_{aff}. On the other hand by remark 1.5.5, we also have a natural right action of W_{aff} on $\mathcal{O}(W_{\text{aff}}, S)$. From the definitions it follows immediately that

$$(\mathfrak{o} \bullet u) \bullet w = (\mathfrak{o} \bullet (uwu^{-1})) \bullet u$$

and hence by the universal property of the semidirect product $\Omega \ltimes W_{\text{aff}}$ the two actions give rise to an action of W on $\mathcal{O}(W_{\text{aff}}, S)$.

1.5.17 Remark. There exists a natural intrinsic right action of an extended Coxeter group W on the set $\mathcal{O}(W)$ of its orientations. If $\widetilde{\mathfrak{o}}$ is an orientation of W associated to an orientation $\mathfrak{o}$ of W_{aff}, then for any $w \in W$ the map $\widetilde{\mathfrak{o}} \bullet w$ defined by

$$(\widetilde{\mathfrak{o}} \bullet w)(w', s) := \widetilde{\mathfrak{o}}(ww', s)$$

is again an orientation. In fact, if we write $w = w_0 u$ and $w' = w_0' u'$ with $w_0, w_0' \in W_{\text{aff}}$ and $u, u' \in \Omega$ then

$$\begin{aligned}
\widetilde{\mathfrak{o}}(ww', s) &= \widetilde{\mathfrak{o}}(w_0 u w_0' u^{-1} u u', s) = \mathfrak{o}(w_0 u w_0' u^{-1}, (uu')(s)) \\
&= (\mathfrak{o} \bullet w_0)(u w_0' u^{-1}, u(u'(s))) = ((\mathfrak{o} \bullet w_0) \bullet u)(w_0', u'(s)) \\
&= (\mathfrak{o} \bullet w)(w_0', u'(s))
\end{aligned}$$

Hence, $\widetilde{\mathfrak{o}} \bullet w$ is associated to $\mathfrak{o} \bullet w$. This computation also shows that the natural bijective map

$$\mathcal{O}(W_{\text{aff}}, S) \xrightarrow{\sim} \mathcal{O}(W)$$

is W-equivariant with respect to the two actions described.

1.5.18 Remark. The set $\mathcal{O}(W)$ of orientations of an extended Coxeter group W also carries a natural topology, namely the subspace topology induced by the space $\{\pm\}^{W \times S}$ and its compact-open topology. The above bijection then is actually a homeomorphism. This follows immediately from the fact that the extension map

$$\begin{aligned}
\{\pm\}^{W_{\text{aff}} \times S} &\hookrightarrow \{\pm\}^{W \times S} \\
f &\longmapsto ((wu, s) \mapsto f(w, u(s)))
\end{aligned}$$

is a homeomorphism onto the subspace

$$\{f \in \{\pm\}^{W \times S} : f(wu, s) = f(w, u(s)) \quad \forall w \in W,\ u \in \Omega,\ s \in S\}$$

Since this subspace is closed, it follows that also the set of orientations of (W, S) is a closed subspace of $\{\pm\}^{W \times S}$.

1.5.19 Definition. Let $W^{(1)}$ be a pro-p Coxeter group and $\mathfrak{o}$ be an orientation of the underlying extended Coxeter group W. The map $\widetilde{\mathfrak{o}} : W^{(1)} \times S \longrightarrow \{\pm 1\}$ defined by

$$\widetilde{\mathfrak{o}}(w, s) := \mathfrak{o}(\pi(w), s)$$

is called the **orientation of $W^{(1)}$ associated to $\mathfrak{o}$**.

An **orientation** of $W^{(1)}$ is a map $W^{(1)} \times S \longrightarrow \{\pm 1\}$ associated to an orientation of W in the above sense, and the set of all such orientations is denoted by $\mathcal{O}(W^{(1)})$, or simply by $\mathcal{O}$ if $W^{(1)}$ is understood.

1.5.20 Remark. There exists a natural right action of $W^{(1)}$ on the set of all orientations of $W^{(1)}$ again by the formula $(\mathfrak{o} \bullet w)(w', s) := \mathfrak{o}(ww', s)$. There also exists an action of $W^{(1)}$ on the set of all orientations of W and W_{aff} respectively via pulling back the W-actions along $\pi : W^{(1)} \to W$. The natural bijection

$$\mathcal{O}(W) \xrightarrow{\sim} \mathcal{O}(W^{(1)})$$

is then equivariant with respect to these $W^{(1)}$-actions.

By remark 1.5.17, we may therefore identify $\mathcal{O}(W^{(1)})$ and $\mathcal{O}(W_{\text{aff}}, S)$ as $W^{(1)}$-sets, and may consider the former as a topological space through identification with the latter.

1.6 Bernstein maps

In this section, we will introduce the first of three related families of functions $\theta_{\mathfrak{o}}, \widehat{\theta}_{\mathfrak{o}}, \widetilde{\theta}_{\mathfrak{o}}$ which we loosely refer to as "Bernstein maps", as they are related to Bernstein's presentation of Iwahori-Hecke algebras. We fix a pro-p Coxeter group $W^{(1)}$ throughout and denote by $\mathcal{O} = \mathcal{O}(W^{(1)})$ the set of orientations of $W^{(1)}$.

The following theorem is essentially the transposition of ([Gör07, Thm 1.1.1]) into our context. We first phrase it in terms of the braid group $\mathfrak{A}(W^{(1)})$ (see definition 1.4.1).

1.6.1 Theorem. *There exists a unique map*

$$\theta : W^{(1)} \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, \mathfrak{A}(W^{(1)})), \quad w \mapsto (\mathfrak{o} \mapsto \theta_{\mathfrak{o}}(w))$$

satisfying the cocycle rule

$$\theta_{\mathfrak{o}}(ww') = \theta_{\mathfrak{o}}(w)\theta_{\mathfrak{o}\bullet w}(w') \quad \forall w, w' \in W^{(1)} \tag{1.6.1}$$

such that for $s \in S, \mathfrak{o} \in \mathcal{O}$

$$\theta_{\mathfrak{o}}(n_s) = T^{\varepsilon}_{n_s^{\varepsilon}} \quad \textit{where} \quad \varepsilon = \mathfrak{o}(1, s) \in \{\pm 1\}$$

and for $u \in \Omega^{(1)}, \mathfrak{o} \in \mathcal{O}$

$$\theta_{\mathfrak{o}}(u) = T_u$$

Proof. We apply lemma 1.2.1 to the $W^{(1)}$-module $M = \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, \mathfrak{A}(W^{(1)}))$ and the pair (σ, ρ), where

$$\sigma(s) = \left(\mathfrak{o} \mapsto T^{\varepsilon}_{n_s^{\varepsilon}}\right), \quad \varepsilon = \mathfrak{o}(1, s)$$

and ρ is the 'trivial' cocycle

$$\rho(u) = (\mathfrak{o} \mapsto T_u)$$

Here, the monoid structure on M is given by pointwise multiplication and the left $W^{(1)}$-action is induced by the right action on $\mathcal{O}$ of remark 1.5.20. It then only remains to verify conditions (i)-(iii) of lemma 1.2.1. Bearing in mind the defining property (**OR1**) of an orientation, condition (i) amounts to showing that for all $s \in S$ and $\mathfrak{o} \in \mathcal{O}$

$$T^{\varepsilon}_{n_s^{\varepsilon}} T^{-\varepsilon}_{n_s^{-\varepsilon}} = T_{n_s^2}$$

where $\varepsilon = \mathfrak{o}(1, s)$. First of all, note that $T_{n_s^2}$ commutes with T_{n_s} since

$$T_{n_s} = T_{n_s^2 n_s n_s^{-2}} \overset{(!)}{=} T_{n_s^2} T_{n_s} T^{-1}_{n_s^2}$$

where we used that $n_s^2 \in T \subseteq \Omega^{(1)}$. Therefore T_{n_s} commutes also with

$$T_{n_s^{-1}} = T_{n_s^{-2} n_s} = T^{-1}_{n_s^2} T_{n_s}$$

Given $\varepsilon \in \{\pm 1\}$, we have

$$T_{n_s^{\varepsilon}} = T_{n_s^{2\varepsilon} n_s^{-\varepsilon}} = T^{\varepsilon}_{n_s^2} T_{n_s^{-\varepsilon}}$$

and hence

$$T_{n_s^{\varepsilon}} T^{-1}_{n_s^{-\varepsilon}} = T^{\varepsilon}_{n_s^2}$$

Since T_{n_s} and $T_{n_s^{-1}}$ commute, we can raise the last equation to the power ε to get

$$T^{\varepsilon}_{n_s^{\varepsilon}} T^{-\varepsilon}_{n_s^{-\varepsilon}} = T_{n_s^2}$$

We now turn to the verification of condition (ii). Unwinding the definitions and observing that the values of ρ lie in the invariants $M^{W^{(1)}}$, we see that condition (ii) amounts to showing that for $\mathfrak{o} \in \mathcal{O}$, $s \in S$ and $u \in \Omega^{(1)}$ we have

$$T_u T^{\varepsilon}_{n_s^{\varepsilon}} = T^{\varepsilon}_{n^{\varepsilon}_{u(s)}} T_{ut_{s,u}}$$

where we abbreviated $\varepsilon = \mathfrak{o}(1, u(s))$. When $\varepsilon = 1$, this reduces immediately to the defining equation $un_s = n_{u(s)} ut_{s,u}$ of $t_{s,u}$. When $\varepsilon = -1$, we first compute

$$T_{n^{-1}_{u(s)}} T_u = T_{n^{-1}_{u(s)} u} = T_{ut_{s,u} n_s^{-1}} = T_{ut_{s,u}} T_{n_s^{-1}}$$

Rearranging then gives the desired equation. Finally, let us verify condition (iii). Given $s, t \in S$ with $m(s,t) < \infty$, we have to show that for every orientation $\mathfrak{o}$ we have

$$T^{\varepsilon(1)}_{n_s^{\varepsilon(1)}} T^{\varepsilon(2)}_{n_t^{\varepsilon(2)}} T^{\varepsilon(3)}_{n_s^{\varepsilon(3)}} \cdots = T^{\varepsilon'(1)}_{n_t^{\varepsilon'(1)}} T^{\varepsilon'(2)}_{n_s^{\varepsilon'(2)}} T^{\varepsilon'(3)}_{n_t^{\varepsilon'(3)}} \cdots \tag{1.6.2}$$

where

$$\varepsilon(1) = \mathfrak{o}(1, s), \quad \varepsilon(2) = \mathfrak{o}(s, t), \quad \varepsilon(3) = \mathfrak{o}(st, s), \quad \varepsilon(4) = \mathfrak{o}(sts, t), \quad \ldots$$

and

$$\varepsilon'(1) = \mathfrak{o}(1, t), \quad \varepsilon'(2) = \mathfrak{o}(t, s), \quad \varepsilon'(3) = \mathfrak{o}(ts, t), \quad \varepsilon'(4) = \mathfrak{o}(tst, s), \quad \ldots$$

are precisely the sign sequences appearing in condition (**OR2**) for $w = 1$ in definition 1.5.1. By condition (**OR2**), these sequences are in one of two forms. Without loss of generality we may assume that they are in the first form, i.e.

$$(\varepsilon(1), \varepsilon(2), \ldots) = (\underbrace{+, \ldots, +}_{k}, \underbrace{-, \ldots, -}_{m(s,t)-k})$$

and

$$(\varepsilon'(1), \varepsilon'(2), \ldots) = (\underbrace{-, \ldots, -}_{m(s,t)-k}, \underbrace{+, \ldots, +}_{k})$$

Writing

$$s_1 = n_s, \ s_2 = n_t, \ s_3 = n_s, \ \ldots \qquad s'_1 = n_t, \ s'_2 = n_s, \ s'_3 = n_t, \ \ldots$$

eq. (1.6.2) is thus of the form

$$T_{s_1} \ldots T_{s_k} T^{-1}_{s^{-1}_{k+1}} \ldots T^{-1}_{s^{-1}_{m(s,t)}} = T^{-1}_{s'^{-1}_1} \ldots T^{-1}_{s'^{-1}_{m(s,t)-k}} T_{s'_{m(s,t)-k+1}} \ldots T_{s'_{m(s,t)}}$$

Rearranging the last equation slightly, we see that it is equivalent to

$$T_{s'^{-1}_{m(s,t)-k}} \ldots T_{s'^{-1}_1} T_{s_1} \ldots T_{s_k} = T_{s'_{m(s,t)-k+1}} \ldots T_{s'_{m(s,t)}} T_{s^{-1}_{m(s,t)}} \ldots T_{s^{-1}_{k+1}}$$

Both sides of this equation are words $T_{w_1} \ldots T_{w_{m(s,t)}}$ of length $m(s,t)$ in the distinguished generators T_w of $\mathfrak{A}(W^{(1)})$. Moreover, the words $w_1 \ldots w_{m(s,t)}$ in the elements of $W^{(1)}$ corresponding to them define reduced expressions, since under $W^{(1)} \twoheadrightarrow W$ they project to alternating words of length $m(s,t)$ in s and t. Therefore, we can simplify both sides of the above equation to get

$$T_{s'^{-1}_{m(s,t)-k} \ldots s'^{-1}_1 s_1 \ldots s_k} = T_{s'_{m(s,t)-k+1} \ldots s'_{m(s,t)} s^{-1}_{m(s,t)} \ldots s^{-1}_{k+1}}$$

The validity of this equation now follows from the equation

$$s'^{-1}_{m(s,t)-k} \ldots s'^{-1}_1 s_1 \ldots s_k = s'_{m(s,t)-k+1} \ldots s'_{m(s,t)} s^{-1}_{m(s,t)} \ldots s^{-1}_{k+1}$$

in $W^{(1)}$, which by backtransforming is seen to be equivalent to the braid relation (1.1.1)

$$s_1 \ldots s_{m(s,t)} = s'_1 \ldots s'_{m(s,t)}$$

which holds by assumption. □

1.6.2 Definition. The map θ defined in the previous theorem is called the (unnormalized) **Bernstein map**. Given a generic pro-p Hecke algebra $\mathcal{H}^{(1)} = \mathcal{H}^{(1)}(a, b)$ associated to parameters $a = (a_s)_{s \in S}$ and $b = (b_s)_{s \in S}$ with a_s invertible, we have (by proposition 1.4.3) a morphism of monoids

$$\mathfrak{A}(W^{(1)}) \longrightarrow R[\mathfrak{A}(W^{(1)})] \longrightarrow \mathcal{H}^{(1)}$$

Using this map we can push θ forward to obtain a map

$$W^{(1)} \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, \mathcal{H}^{(1)})$$

still satisfying the 1-cocycle rule. This map will also be denoted by θ and referred to as the (unnormalized) Bernstein map.

Let now $\mathcal{H}^{(1)}$ be a generic pro-p Hecke-algebra with invertible parameters a_s as above. Fixing an orientation $\mathfrak{o} \in \mathcal{O}$, we thus have a family $\{\theta_{\mathfrak{o}}(w)\}_{w\in W^{(1)}}$ of elements in $\mathcal{H}^{(1)}$.

The crucial point is that this family forms an R-basis of $\mathcal{H}^{(1)}$. This is the content of the next proposition, which shows that in fact the change of basis matrix between $\{\theta_{\mathfrak{o}}(w)\}_{w\in W^{(1)}}$ and $\{T_w\}_{w\in W^{(1)}}$ is 'upper triangular'. We will see later (equation (2.2.1)) that for a certain orientation $\mathfrak{o}$ the restriction of $\theta_{\mathfrak{o}}$ to the subgroup $X \leq W$ of translations recovers the map θ of Lusztig [Lus89]. This motivates the terminology 'Bernstein map'.

1.6.3 Proposition. *In $\mathcal{H}^{(1)}$ one has an expansion of the form*

$$\theta_{\mathfrak{o}}(w) = c_{w,w}T_w + \sum_{w'<w} c_{w,w'}T_{w'}$$

where $c_{w,w} \in R^\times$ and $c_{w,w'} \in R$ zero for almost all w', for every $w \in W$. Here $<$ denotes the strong Bruhat order on $W^{(1)}$ (see definition 1.1.17). In particular $\{\theta_{\mathfrak{o}}(w)\}_{w\in W^{(1)}}$ is an R-basis of $\mathcal{H}^{(1)}$.

Proof. The first claim follows by taking an expression $w = n_{s_1}\dots n_{s_r}u$ with $\ell(w) = r$ and expanding

$$\theta_{\mathfrak{o}}(w) = T^{\varepsilon_1}_{n^{\varepsilon_1}_{s_1}} \dots T^{\varepsilon_r}_{n^{\varepsilon_r}_{s_r}} T_u$$

using (cf. eq. (1.3.8))

$$T^{-1}_{n_s^{-1}} = a_s^{-1}(T_{n_s} - b_s) \tag{1.6.3}$$

and the commutation rule (1.3.5). Here one also uses that $T_{n_{s_1}}\dots T_{n_{s_r}} = T_{n_{s_1}\dots n_{s_r}}$ and that for every $w \in W^{(1)}$ one either has

$$T_{n_s}T_w = T_{n_sw}$$

or

$$T_{n_s}T_w = a_sT_{n_sw} + b_sT_w$$

according to whether $\ell(n_sw) = 1 + \ell(w)$ or $\ell(n_sw) = \ell(w) - 1$. The second claim is a formal consequence of the first and the irreflexivity and transitivity of the relation $<$. □

1.7 A 2-coboundary $\mathbb{X}$ appearing in Coxeter geometry

The purpose of this section is to pave the way for introducing an integral $\widehat{\theta}$ and a normalized version $\widetilde{\theta}$ of the Bernstein map θ defined in the previous section.

The map θ has the 'defect' that it is only defined when the parameters a_s are invertible. In view of the study of mod p representations of pro-p-Iwahori Hecke algebras (where $a_s = 0$), it is important to have an integral version which is defined for *all* parameters. Such variants of the classical Bernstein-Lusztig basis have been first introduced by Vignéras [Vig05], [Vig06]. The construction of $\widehat{\theta}$ is based on the following relation (see eq. (1.3.8))

$$a_sT^{-1}_{n_s^{-1}} = T_{n_s} - b_s$$

which is an immediate consequence of the quadratic relations. It suggests to formally multiply

$$\theta_{\mathfrak{o}}(w) = T^{\varepsilon_1}_{n^{\varepsilon_1}_{s_1}} \dots T^{\varepsilon_r}_{n^{\varepsilon_r}_{s_r}} T_u$$

by the product

$$\overline{\gamma}_{\mathfrak{o}}(w) = \prod_{i:\varepsilon_i=-1} a_{s_i}$$

to get an integral expression in the generators T_w, and to define $\widehat{\theta}_{\mathfrak{o}}(w)$ as the resulting element. However, a priori the factor $\overline{\gamma}_{\mathfrak{o}}(w)$ and therefore $\widehat{\theta}_{\mathfrak{o}}(w)$ depends on the chosen expression $w = n_{s_1}\dots n_{s_r}u$ for w as a product in the distinguished generators. The first goal of this section is therefore to establish the independence of $\overline{\gamma}_{\mathfrak{o}}(w)$ from the chosen expression for w. As this is a purely combinatorial question, it will be useful to work with formal products of hyperplanes instead of products of the parameters a_s, and to replace $\overline{\gamma}$ by a purely combinatorially defined map γ.

The second goal of this section is to determine the multiplicative properties of γ, as these determine the multiplicative properties of $\widehat{\theta}$ and the usefulness of θ wholly depends on the fact that it satisfies the cocycle rule. We will achieve this by identifying the coboundary of γ (viewed as a map $w \mapsto (\mathfrak{o} \mapsto \gamma_{\mathfrak{o}}(w))$ in one parameter) with another explicitly and combinatorially defined map $\mathbb{X}$.

We will then give a second characterization of $\mathbb{X}$ as a coboundary of a 'generalized length function' $\sqrt{\mathbb{L}}$, which is needed in order to introduce and prove the multiplicative properties of a normalized variant $\widetilde{\theta}$ of θ. This normalized version is closely related to the classical Bernstein-Lusztig basis of the Iwahori-Hecke algebra (see section 2.2.1).

Since everything in this section only involves the combinatorics of extended Coxeter groups, we need only to fix an extended Coxeter group $W = (W, W_{\text{aff}}, \Omega, S)$.

Let us start by defining the 'coboundary' mentioned in the title of this section.

1.7.1 Definition. Given $w, w' \in W$ let

$$\mathbb{X}(w, w') := \prod_H \mathbf{a}_H \in \mathbb{N}[\mathfrak{H}]$$

where $\mathbb{N}[\mathfrak{H}]$ denotes the free abelian monoid on the set $\mathfrak{H}$ of hyperplanes and the product is taken over all hyperplanes $H \in \mathfrak{H}$ which both separate 1 from w and w from ww'.

In other words, $\mathbb{X}(w, w')$ is the product over all hyperplanes which are crossed twice by any gallery that is the concatenation of a minimal gallery from 1 to w and a minimal gallery from w to ww'. In particular we have the following observation, which we record separately.

1.7.2 Remark. For all $w, w' \in W$

$$\ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \quad \mathbb{X}(w, w') = 1$$

1.7.3 Remark. From the definition of $\mathbb{X}$, it also follows directly that

$$\mathbb{X}(w, w') = 1$$

whenever $w \in \Omega$ or $w' \in \Omega$.

Next, we will show that the sign attached by an orientation to crossing a hyperplane H at a chamber $w \in W_{\text{aff}}$ does not depend upon the chamber itself but only upon which half-space with respect to H this chamber lies in.

1.7.4 Lemma. *Let $\mathfrak{o} \in \mathcal{O}(W_{\text{aff}}, S)$ be any orientation of W_{aff}. If $w, \widetilde{w} \in W_{\text{aff}}$ and $s, \widetilde{s} \in S$ are such that*

$$wsw^{-1} = \widetilde{w}\widetilde{s}\widetilde{w}^{-1} \quad \textit{and} \quad \ell(sw^{-1}\widetilde{w}) = 1 + \ell(w^{-1}\widetilde{w})$$

that is, if w, ws and $\widetilde{w}, \widetilde{w}\widetilde{s}$ are separated by the same wall $H = wsw^{-1} = \widetilde{w}\widetilde{s}\widetilde{w}^{-1}$ and $w, \widetilde{w}$ lie on the same side with respect to H, then

$$\mathfrak{o}(w, s) = \mathfrak{o}(\widetilde{w}, \widetilde{s})$$

Proof. After replacing $\mathfrak{o}$ by $\mathfrak{o} \bullet w$, we may assume that $w = 1$. Then $s\widetilde{w} = \widetilde{w}\widetilde{s}$ and $\ell(s\widetilde{w}) = \ell(\widetilde{w}\widetilde{s}) = \ell(\widetilde{w}) + 1$. Therefore, if we take any reduced expression $\widetilde{w} = s_1 \dots s_r$, then

$$ss_1 \dots s_r = s_1 \dots s_r\widetilde{s}$$

will be two *reduced* expressions of the same element in W_{aff} and $\mathfrak{o}(1, s), \mathfrak{o}(\widetilde{w}, \widetilde{s})$ are the signs which appear in these galleries when crossing the wall H. It therefore suffices to show that for any two reduced expressions of the same element in W_{aff} and any hyperplane H the signs which appear when crossing H are the same for both expressions. By Tits' solution of the word problem [Bro89, II.3C], two such reduced expressions can be transformed into each other by a finite sequence of transformations of type (III) (cf. proof of theorem 1.6.1)

$$t_1 \dots t_i \underbrace{sts\dots}_{m(s,t)<\infty} t_{i+1} \dots t_m \longmapsto t_1 \dots t_i \underbrace{tst\dots}_{m(s,t)<\infty} t_{i+1} \dots t_m$$

If H is crossed before or after the part where these two galleries differ, the signs are equal for trivial reasons. It therefore suffices to show that for $s, t \in S$ with $m(s,t) < \infty$ the signs of all the walls crossed by the two galleries corresponding to the reduced expressions

$$\underbrace{sts\dots}_{m(s,t)} = \underbrace{tst\dots}_{m(s,t)}$$

are equal. But by remark 1.5.4, the signs are determined by the distance to some reference chamber in $\langle s, t \rangle$. In particular the sign $\mathfrak{o}(w, s)$ only depends on which half-space with respect to $H = wsw^{-1}$ the fundamental chamber lies in. $\square$

1.7.5 Notation. Thanks to the previous lemma, we may extend any orientation $\mathfrak{o}$ canonically to a map

$$\mathfrak{o} : W \times \mathfrak{H} \to \{\pm\}$$

by letting

$$\mathfrak{o}(w, H) := \mathfrak{o}(ww_0, s), \quad w \in W,\ H \in \mathfrak{H}$$

where $w_0 \in W$ and $s \in S$ are such that

$$w_0 s w_0^{-1} = H$$

and $1, w_0$ lie in the same half-space with respect to H. It follows quite easily that this does indeed give rise to a well-defined map $W \times \mathfrak{H} \to \{\pm\}$ that extends $\mathfrak{o}$. In the terminology of 1.1.3 and 1.5.2, the sign $\mathfrak{o}(w, H)$ has the geometric interpretation as being the sign that is attached to crossing the hyperplane wHw^{-1} at any chamber that lies in the same half-space with respect to wHw^{-1} as w. In particular $\mathfrak{o}(1, H)$ is the sign attached by $\mathfrak{o}$ to crossing H at any chamber that lies in the same half-space with respect to H as the fundamental chamber.

1.7.6 Corollary. *Given an orientation $\mathfrak{o}$ of W, there exists a unique map $\gamma_{\mathfrak{o}}$ from W into the free commutative monoid $\mathbb{N}[\mathfrak{H}]$ with generators $\mathbf{a}_H$ corresponding to the hyperplanes $H \in \mathfrak{H}$ such that if $w = s_1 \dots s_r u$, $s_i \in S$, $u \in \Omega$ is a reduced expression for w, then $\gamma_{\mathfrak{o}}(w)$ equals the product of the hyperplanes crossed in the negative direction by the gallery corresponding to this reduced expression. In other words*

$$\gamma_{\mathfrak{o}}(w) = \prod_{i\ :\ \varepsilon_i = -1} \mathbf{a}_{H_i} \tag{1.7.1}$$

where $\varepsilon_i = \mathfrak{o}(s_1 \dots s_{i-1}, s_i)$ and $H_i = (s_1 \dots s_{i-1}) s_i (s_1 \dots s_{i-1})^{-1}$.

Proof. We need to verify the independence of the right-hand side of equation (1.7.1) from the choice of the reduced expression. Since $s_1 \dots s_r$ is a reduced expression of $wu^{-1} \in W_{\mathrm{aff}}$, the walls H_i appearing are pairwise distinct and are equal to the walls separating 1 and w. On the other hand, by the previous lemma the sign ε_i only depends on which half-space with respect to H_i the fundamental chamber lies in. Therefore, the H_i with $\varepsilon_i = -1$ only depend on w and $\mathfrak{o}$. ☐

1.7.7 Remark. As promised, we will now explicitly determine the 'coboundary' of the map γ defined above. More precisely, let us view the collection of all elements $\gamma_{\mathfrak{o}}(w)$ as the map

$$\gamma : W \longrightarrow M, \quad w \mapsto (\mathfrak{o} \mapsto \gamma_{\mathfrak{o}}(w))$$

taking values in the W-module $M = \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, \mathbb{Z}[\mathfrak{H}])$. The structure of an abelian group on M is 'pointwise', and $\mathbb{Z}[\mathfrak{H}] \supseteq \mathbb{N}[\mathfrak{H}]$ denotes the free commutative *group* on $\mathfrak{H}$. The W-action on M is induced by the canonical right action on $\mathcal{O}$ and the canonical left action on $\mathfrak{H}$, i.e.

$$(w \bullet \phi)(\mathfrak{o}) = w \bullet \phi(\mathfrak{o} \bullet w) \quad \forall w \in W,\ \phi \in M,\ \mathfrak{o} \in \mathcal{O}$$

Finally, let us view $\mathbb{X}$ as a map

$$\mathbb{X} : W \times W \longrightarrow M, \quad (w, w') \mapsto (\mathfrak{o} \mapsto \mathbb{X}(w, w'))$$

The statement of the next lemma is then equivalent to the coboundary equation

$$d\gamma = \mathbb{X}$$

of the inhomogeneous standard cochain complex on M.

1.7.8 Lemma. *For all $w, w' \in W$, one has*

$$\gamma_{\mathfrak{o}}(w) w \left(\gamma_{\mathfrak{o} \bullet w}(w')\right) = \mathbb{X}(w, w') \gamma_{\mathfrak{o}}(ww')$$

Proof. Write $w = w_0 u$ and $w' = w_0' u'$ with $w_0, w_0' \in W_{\mathrm{aff}}$ and $u, u' \in \Omega$. Then by definition

$$\gamma_{\mathfrak{o}}(w) = \gamma_{\mathfrak{o}}(w_0) \quad \gamma_{\mathfrak{o} \bullet w}(w') = \gamma_{\mathfrak{o} \bullet w}(w_0') \quad \gamma_{\mathfrak{o}}(ww') = \gamma_{\mathfrak{o}}(w_0 u(w_0'))$$

and

$$\mathbb{X}(w, w') = \mathbb{X}(w_0, w_0 u(w_0'))$$

Moreover, it follows from the definitions that

$$\gamma_{(\mathfrak{o}\bullet w_0)\bullet u}(w_0') = u^{-1}\left(\gamma_{\mathfrak{o}\bullet w_0}(u(w_0'))\right)$$

It therefore suffices to prove the formula for $w, w' \in W_{\text{aff}}$. Taking reduced expressions $w = s_1 \ldots s_r$ and $w' = s_{r+1} \ldots s_{r+m}$, one has

$$\gamma_{\mathfrak{o}}(w) w\left(\gamma_{\mathfrak{o}\bullet w}(w')\right) = \prod_H \mathbf{a}_H$$

where the product extends over all walls H which are crossed with a negative sign by the gallery corresponding to the possibly nonreduced expression $s_1 \ldots s_{r+m}$. A wall H will be crossed by this gallery if and only if it separates 1 from w or w from ww'. A wall H is crossed twice iff it separates both 1 from w and w from ww', otherwise it is crossed only once. The walls that are crossed once are exactly the walls that separate 1 from ww' and they are crossed with the same sign as in a minimal gallery from 1 to ww'. The walls that are crossed twice are crossed once with a positive and once with a negative sign. It therefore follows immediately that

$$\gamma_{\mathfrak{o}}(w) w\left(\gamma_{\mathfrak{o}\bullet w}(w')\right) = \mathbb{X}(w, w')\gamma_{\mathfrak{o}}(ww')$$

□

The length $\ell(w)$ of an element $w \in W$ is given by the *number* of walls separating 1 and w. Replacing numbers by formal products of walls we get the notion of the *generalized length* $\mathbb{L}(w)$ of an element, which leads to another characterization of $\mathbb{X}$ as a coboundary.

1.7.9 Definition. The **generalized length** $\mathbb{L}(w)$ of $w \in W$ is the element of $\mathbb{N}[\mathfrak{H}]$ given by

$$\mathbb{L}(w) := \prod_H \mathbf{a}_H$$

where the product is taken over all $H \in \mathfrak{H}$ separating 1 and w.

1.7.10 Lemma. *For all $w, w' \in W$ we have*

$$\mathbb{L}(w) w\left(\mathbb{L}(w')\right) = \mathbb{X}(w, w')^2 \mathbb{L}(ww')$$

Proof. This follows from the same arguments given in the proof of lemma 1.7.8. The only difference is that here every wall that is crossed twice also appears twice. □

1.7.11 Remark. (i) The length $\ell(w)$ of an element $w \in W$ and its generalized length $\mathbb{L}(w)$ are related via the 'cardinality morphism'

$$\# : (\mathbb{N}[\mathfrak{H}], \cdot) \longrightarrow (\mathbb{N}, +), \quad \mathbf{a}_H \mapsto 1$$

by the equation

$$\ell(w) = \#\mathbb{L}(w)$$

The lemma above therefore gives the formula

$$\#\mathbb{X}(w, w') = \ell(w) + \ell(w') - \ell(ww')$$

which reproves and generalizes remark 1.7.2. The lemma also shows that

$$\ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \quad \mathbb{L}(ww') = \mathbb{L}(w) w(\mathbb{L}(w'))$$

(ii) The above lemma says that $\mathbb{X}$ is the coboundary of the formal square root $\sqrt{\mathbb{L}}$ of $\mathbb{L}$. More precisely, letting $\mathbb{Z}[\sqrt{\mathfrak{H}}]$ denote the free abelian group on the symbols $\sqrt{\mathbf{a}_H}$, $H \in \mathfrak{H}$ we can view $\mathbb{Z}[\mathfrak{H}]$ as a subset of $\mathbb{Z}[\sqrt{\mathfrak{H}}]$ via the embedding given by $\mathbf{a}_H \mapsto \left(\sqrt{\mathbf{a}_H}\right)^2$. Pushing $\mathbb{L} : W \to \mathbb{Z}[\sqrt{\mathfrak{H}}]$ forward via this embedding, it has a unique square root $\sqrt{\mathbb{L}} : W \to \mathbb{Z}[\sqrt{\mathfrak{H}}]$. Viewing $\mathbb{X}$ as a map $W \times W \to \mathbb{Z}[\sqrt{\mathfrak{H}}]$, the formula of the above lemma is equivalent to the coboundary equation

$$d\sqrt{\mathbb{L}} = \mathbb{X}$$

of the inhomogeneous cochain complex on $\mathbb{Z}[\sqrt{\mathfrak{H}}]$.

(iii) The construction of the integral Bernstein-Lusztig basis in [Vig06] heavily depends on the 'lemme fondamental' [Vig06, 1.2]. There it is proven that a certain expression $q_{wv}q_w^{-1}q_v$ $(w, v \in W)$ which is a product of formal parameters is a square of an element $c_{w,v}$. This relates to the previous lemma as follows. Consider the orbit map $\mathfrak{H} \to W\backslash\mathfrak{H}$ of the canonical action of W on $\mathfrak{H}$. Pushing $\mathbb{L}$ and $\mathbb{X}$ forward along the induced map $\mathbb{Z}[\mathfrak{H}] \to \mathbb{Z}[W\backslash\mathfrak{H}]$, we get maps $\overline{\mathbb{L}}$ and $\overline{\mathbb{X}}$ with values in $\mathbb{Z}[W\backslash\mathfrak{H}]$. The formula proven in the above lemma then simplifies to

$$\overline{\mathbb{L}}(w)\overline{\mathbb{L}}(w')\overline{\mathbb{L}}(ww')^{-1} = \overline{\mathbb{X}}(w, w')^2 \tag{1.7.2}$$

Identifying the formal parameter ('poid générique') q_s $(s \in S)$ of [Vig06] with the generator $\mathbf{a}_{[s]} \in \mathbb{Z}[W\backslash\mathfrak{H}]$ corresponding to the class $[s] \in W\backslash\mathfrak{H}$, the element q_w $(w \in W)$ defined in loc. cit. identifies with $\overline{\mathbb{L}}(w)$. In this notation the above formula reads

$$q_w q_{w'} q_{ww'}^{-1} = \overline{\mathbb{X}}(w, w')^2$$

In particular we find that

$$q_{ww'}q_w^{-1}q_{w'} = q_{ww'}q_w^{-1}q_{w'}^{-1}q_{w'}^2 = \overline{\mathbb{X}}(w, w')^{-2}q_{w'}^2$$

and therefore that the element $c_{w,w'}$ defined in [Vig06, 1.2] is given by

$$c_{w,w'} = \overline{\mathbb{X}}(w, w')^{-1}q_{w'}$$

This element is more explicitly given as the product

$$c_{w,w'} = \prod_H \mathbf{a}_{[H]}$$

where the product runs over all hyperplanes H which separate 1 from w' *but don't* separate 1 from w.

1.8 A characterization of pro-p Coxeter groups in terms of $\mathbb{X}$

Throughout this section, we will fix an extended Coxeter group $W = (W, W_{\text{aff}}, \Omega, S)$. Our goal here is to characterize some (all, if $W = W_{\text{aff}}$) pro-p Coxeter groups $W^{(1)}$ whose underlying extended Coxeter group equals W, using the '2-coboundary' $\mathbb{X}$ of the previous section. Even though this result will not be used in the rest of the text, we choose to present it because we think it is of independent interest.

By definition, a pro-p Coxeter group $W^{(1)}$ is given by a group extension

$$1 \longrightarrow T \longrightarrow W^{(1)} \longrightarrow W \longrightarrow 1$$

of W by an *abelian* group T, together with a choice of lifts $(n_s)_{s\in S}$ of the distinguished generators which satisfy the braid relations. In the case $W = W_{\text{aff}}$, such groups have been studied by Tits [Tit66] under the name[16] of 'extended Coxeter groups'. Among the many interesting results obtained in [Tit66] is a characterization ([Tit66, 3.4 Proposition]) of such extensions in terms of data related to W and T, and the construction and explicit description of a 'universal' extension V. Implicit in this (see especially [Tit66, 3.4 Proposition]) is that the 2-cocycle

$$\phi : W \times W \to T, \quad \phi(w, w') = n(w)n(w')n(ww')^{-1}$$

associated to the extension and the *canonical* set-theoretic section

$$n : W \to W^{(1)}$$

determined by $n(s) = n_s$ and

$$\ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \quad n(ww') = n(w)n(w')$$

can be explicitly computed. However in [Tit66] an explicit expression for this 2-cocycle was not given. We shall therefore explicitly compute these cocycles in terms of $\mathbb{X}$, and deduce the existence of a universal extension (without reference to [Tit66]) whose corresponding 2-cocycle identifies with $\mathbb{X}$.

Let us begin with a definition.

[16]We apologize for not following the terminology of [Tit66], because in our contexts we have to consider extensions of groups which are themselves (split) extensions of Coxeter groups.

1.8.1 Definition. The category $\mathcal{W}^{(1)}_{/W}$ is the category whose objects consist of extensions

$$1 \longrightarrow T \longrightarrow G \longrightarrow W \longrightarrow 1$$

of W by an abelian group, together with a set-theoretic section $n : W \to G$ of the map $G \to W$ satisfying

$$\ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \quad n(ww') = n(w)n(w') \quad \forall w, w' \in W$$

A morphism $f : (G, T, n) \to (G', T', n')$ is given by a morphism $f : G \to G'$ which makes the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f"] & G \arrow[r] \arrow[d, "f"] & W \arrow[d, "\mathrm{id}"] \\ T' \arrow[r] & G' \arrow[r] & W \end{tikzcd}$$

commute and which satisfies $f \circ n = n'$.

1.8.2 Remark. Essentially, the objects of the category $\mathcal{W}^{(1)}_{/W}$ are pro-p Coxeter groups whose underlying extended Coxeter group equals W. However, not all pro-p Coxeter groups give rise to objects of this category. More precisely, an object $(W^{(1)}, T, n)$ of $\mathcal{W}^{(1)}_{/W}$ corresponds to a pro-p Coxeter group $W^{(1)}$ together with a section *of groups*

$$\widetilde{n} : \Omega \to \pi^{-1}(\Omega)$$

of the restriction of $\pi : W^{(1)} \to W$ to Ω, such that we have the relation

$$\widetilde{n}(u) n_s \widetilde{n}(u)^{-1} = n_{u(s)} \quad \forall u \in \Omega, \ s \in S$$

The map $n : W \to W^{(1)}$ is then uniquely determined by $\widetilde{n}$ and $(n_s)_{s \in S}$ by requiring

$$n(u) = \widetilde{n}(u), \ n(s) = n_s \quad \forall u \in \Omega, \ s \in S$$

Note that given a pro-p Coxeter group $W^{(1)}$, such a section $\widetilde{n}$ might not exist. And even if it does, the relation $\widetilde{n}(u) n_s \widetilde{n}(u)^{-1} = n_{u(s)}$ might not be fulfilled. However, when $W = W_{\mathrm{aff}}$, the set of pro-p Coxeter groups with underlying extended Coxeter group W and the set of objects $\mathcal{W}^{(1)}_{/W}$ are canonically identified.

1.8.3 Lemma. *Given an object (G, T, n) of $\mathcal{W}^{(1)}_{/W}$, the 2-cocycle $\phi : W \times W \to T$ determined by the section n via*

$$\phi(w, w') = n(w)n(w')n(ww')^{-1}$$

satisfies

$$\phi(w, w') = h(\mathbb{X}(w, w'))$$

Here

$$h : \mathbb{Z}[\mathfrak{H}] \longrightarrow T$$

denotes the unique W-equivariant homomorphism of abelian groups satisfying

$$h(s) = n(s)^2 \quad \forall s \in S$$

Proof. First, note that h is obviously unique if it exists since we have

$$h(wsw^{-1}) = wn(s)^2 w^{-1} \quad \forall w \in W, \ s \in S$$

by assumption. Therefore, such a map exists if and only if for $w \in W$ and $s, t \in S$ we have

$$wsw^{-1} = t \quad \Rightarrow \quad wn(s)^2 w^{-1} = n(t)^2$$

But replacing w by ws if necessary, we may assume $\ell(ws) = \ell(w) + 1$. Hence, we also have $\ell(tw) = \ell(w) + 1$ and therefore

$$n(w)n(s) = n(ws) = n(tw) = n(t)n(w)$$

implying

$$wn(s)^2 w^{-1} = (n(w)n(s)n(w)^{-1})^2 = n(t)^2$$

Now, both ϕ and $h \circ \mathbb{X}$ fulfill the 2-cocycle relation

$$\phi(w_1, w_2)\phi(w_1w_2, w_3) = w_1(\phi(w_2, w_3))\phi(w_1, w_2w_3)$$

Moreover, both of these maps vanish whenever one of their arguments lies in Ω. For ϕ, this follows from the relation

$$\ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \quad n(ww') = n(w)n(w')$$

For $h \circ \mathbb{X}$, this follows from remark 1.7.3. Therefore, in order to show that $\phi = h \circ \mathbb{X}$, it suffices to prove that

$$\phi(s, w) = h(\mathbb{X}(s, w)) \quad \forall s \in S,\ w \in W$$

Since both maps vanish on pairs (w, w') satisfying $\ell(ww') = \ell(w) + \ell(w')$ (cf. remark 1.7.2), it suffices to treat the case $\ell(sw) = \ell(w) - 1$. Take a reduced expression

$$sw = s_1 \dots s_r u, \quad s_i \in S,\ u \in \Omega,\ r = \ell(sw)$$

of sw, then

$$w = ss_1 \dots s_r u$$

is a reduced expression of w. Hence

$$n(w) = n(s)n(s_1) \dots n(s_r)n(u)$$

and

$$n(sw) = n(s_1) \dots n(s_r)n(u)$$

and therefore

$$\phi(s, w) = n(s)^2 = h(\mathbb{X}(s, w))$$

because $\mathbb{X}(s, w) = s$ is the unique hyperplane crossed twice by the gallery $(s, s, s_1, \dots, s_r)$. □

1.8.4 Definition. $\mathcal{T}_W$ is the category whose objects are given by pairs (T, h) consisting of an abelian group W endowed with a $\mathbb{Z}$-linear W-action and a W-equivariant map

$$h : \mathfrak{H} \longrightarrow T$$

(identified with its linear extension $h : \mathbb{Z}[\mathfrak{H}] \to T$) and whose morphisms $f : (T, h) \to (T', h')$ are given by W-equivariant group homomorphisms $f : T \to T'$ satisfying $f \circ h = h'$.

With the above definition, we have the following immediate corollary of the above lemma.

1.8.5 Corollary. *The functor*

$$\mathcal{W}^{(1)}_{/W} \longrightarrow \mathcal{T}_W$$

given on morphisms in the obvious way and on objects by

$$(G, T, n) \longmapsto (T, h), \quad h(s) = n(s)^2$$

is an equivalence of categories, with quasi-inverse associating to an object (T, h) the object $(T \times W, T, \iota)$, where the set $T \times W$ is endowed with the group law

$$(t, w) \cdot (t', w') = (tw(t')h(\mathbb{X}(w, w')), ww')$$

and $\iota : W \to T \times W$ is given by $\iota(w) = (1, w)$.

The following corollary essentially recovers Tits' description of the group V (cf. [Tit66, 2.5 Théorème]; see also [DW05, 3.3]).

1.8.6 Corollary. *The category $\mathcal{W}^{(1)}_{/W}$ has an initial object V given by*

$$V = (V, T, n) = (\mathbb{Z}[\mathfrak{H}] \times W, \mathbb{Z}[\mathfrak{H}], \iota)$$

where $V = \mathbb{Z}[\mathfrak{H}] \times W$ is endowed with a group law via

$$(t, w) \cdot (t', w') = (tw(t')\mathbb{X}(w, w'), ww')$$

and $\iota : W \to V$ is given by $\iota(w) = (1, w)$.

Proof. Immediate from the above corollary, since the pair (T, h) with $T = \mathbb{Z}[\mathfrak{H}]$ and $h = \mathrm{id}$ obviously forms an initial object of the category $\mathcal{T}_W$. □

1.9 Relation of Bernstein maps to Cherednik's cocycle

In this optional section, independent from the rest of the text, we discuss the work of Ivan Cherednik on Hecke algebras and its connection to Bernstein maps. This connection arises through the cocycle rule eq. (1.6.1).

Motivated by problems in quantum physics, Cherednik has constructed various 1-cocycles of Coxeter groups with values in (localizations of) affine Hecke algebras and their degenerate (i.e. graded) versions, viewing these cocycles as generalized 'R-matrices'.

By definition, *R-matrices* are solutions of the *Yang-Baxter equation.* This remarkable equation—connecting low-dimensional topology, representation theory, category theory and physics—was discovered independently by C. N. Yang [Yan67] and R. J. Baxter [Bax72], who worked on finding *exact solutions* of certain physical models from quantum and statistical mechanics respectively. Its simplest and most recognizable form is

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1.9.1}$$

with the R_{ij} being elements of some monoid (usually an algebra, although the case where the R_{ij} are endomorphisms of a set is of considerable interest too; see 'set-theoretical solutions of the Yang-Baxter-equation'). Assuming the existence of an action of the symmetric group S_3 on the monoid in which the R_{ij} take values, and assuming 'W-invariance' of the R-matrix, i.e.

$$^{\sigma}R_{ij} = R_{\sigma(i)\sigma(j)}$$

for all σ and i, j for which both sides are defined, the Yang-Baxter equation (1.9.1) can be rewritten equivalently as

$$R_s{}^sR_t{}^{st}R_s = R_t{}^tR_s{}^{ts}R_t \tag{1.9.2}$$

where $s = (12)$, $t = (23)$ and indices ij are identified with transpositions (ij). This equation in turn is nothing but the self-consistency condition necessary for the existence of a 1-cocycle $\sigma \mapsto R_\sigma$ that results from the braid relation $sts = tst$ in the symmetric group. This relation is almost sufficient for the existence of such a cocycle; necessary and sufficient is the above relation together with the 'unitarity condition'

$$R_s{}^sR_s = R_t{}^tR_t = 1$$

resulting from $s^2 = t^2 = 1$ (cf. [Che84, Prop. 4]). Thus, *unitary invariant* R-matrices are identified with 1-cocycles of the group S_3.

Cherednik used this observation to define a general notion of 'R-matrices' attached to root systems as cocycles of Weyl groups [Che92b, Sect. 2], and has constructed examples given by the Demazure-Lusztig operators [Che92b, Prop. 3.5] and the standard intertwining operators [Che92b, Prop. 3.8] (cf. [Che91, Prop. 1.2]) familiar from the representation theory of reductive groups. The latter are directly connected to Bernstein maps, realizing them as a limit. In order to make this precise, let us recall the definition of the standard intertwiners. In the following, all algebras will be over $\mathbb{C}$.

Given a root datum $(X, \Phi, X^\vee, \Phi^\vee)$ with basis $\Delta \subseteq \Phi$ and extended affine Weyl group $W = X \rtimes W_0$, the standard intertwiners are elements F_w $(w \in W_0)$ of the localization

$$H_{\text{gen}} := H_q(W, S) \otimes_Z \operatorname{Frac}(Z)$$

of the affine Hecke algebra at its center Z, determined by (cf. [Che91, Prop. 1.2]; also [HKP10, Lem. 1.13.1])

$$F_{ww'} = F_wF_{w'} \text{ if } \ell(ww') = \ell(w) + \ell(w') \tag{1.9.3}$$

$$F_{s_\alpha} = T_{s_\alpha} + (q_{s_\alpha} - 1)(Y_\alpha - 1)^{-1}, \quad \alpha \in \Delta \tag{1.9.4}$$

where we write

$$Y_\alpha := \widetilde{\theta}(\alpha)$$

for easier comparison with [Che91]. The F_w now constitute an R-matrix in the following sense. The basis property of the Bernstein-Lusztig basis $\{\widetilde{\theta}(x)T_w\}_{x\in X,\ w\in W_0}$ implies that we have *linear* isomorphisms

$$\mathbb{C}[X] \otimes H_0 \xrightarrow{\sim} H_q(W, S) \quad \text{and} \quad \mathbb{C}(X) \otimes H_0 \xrightarrow{\sim} H_{\text{gen}}$$

Here $\mathbb{C}(X) = \operatorname{Frac}(\mathbb{C}[X])$ and H_0 denotes the finite Hecke subalgebra spanned by T_w, $w \in W_0$. Note that the group W_0 acts on $\mathbb{C}(X) \otimes H_0$ via its canonical action on X. Now, if we consider $\mathbb{C}(X) \otimes H_0$ with its canonical (tensor) algebra structure, then the map

$$\phi : W_0 \longrightarrow \mathbb{C}(X) \otimes H_0$$

defined by $w \mapsto F_w$ satisfies the partial cocycle relation

$$\phi(ww') = \phi(w)w(\phi(w')) \quad \text{if} \quad \ell(ww') = \ell(w) + \ell(w')$$

i.e. defines a (non-unitary) R-matrix in the sense of Cherednik (cf. [Che92b, Thm. 2.3 a)]). This follows by easy calculations from relation (1.9.3) and the intertwining property

$$F_w a = w(a)F_w, \quad a \in \mathcal{A}$$

The intertwiners F_w can be normalized so that one gets a proper cocycle (unitary R-matrix) instead: let (cf. [HKP10, 2.2], [Lus89, Prop. 5.2]; also [Opd09, p. 146])

$$K_{s_\alpha} := q_{s_\alpha}^{-1} \frac{1 - Y_\alpha}{1 - q_{s_\alpha}^{-1} Y_\alpha} F_{s_\alpha}$$

for a simple root α and extend to elements K_w for all $w \in W_0$ using (1.9.3) as before. It can be shown that these normalized intertwiners satisfy $K_w K_{w'} = K_{ww'}$ for *all* w, w' and therefore define a cocycle $\psi : W_0 \longrightarrow \mathbb{C}(X) \otimes H_0$ in the usual sense.

This cocycle ψ partially recovers the Bernstein map $\theta : W \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, H_q(W,S)^\times)$ as follows (cf. [Opd09]). Elements of $\mathbb{C}(X) \otimes H_0$ can be viewed as meromorphic function on the complex torus $\mathbb{T} = X^\vee \otimes \mathbb{C}^\times$ with values in H_0. For every Weyl chamber D in $V^\vee = X^\vee \otimes \mathbb{R}$, given as an intersection

$$D = \bigcap_i \{x \in V : \alpha_i(x) > 0\}$$

of half-spaces, one can add a point ξ_D at infinity to $\mathbb{T}$, such that

$$\lim_{n \to \infty} t_n = \xi_D \quad \Leftrightarrow \quad \lim_{n \to \infty} \alpha_i(t) = 0 \ \forall i$$

for every sequence $(t_n)_n$ in $\mathbb{T}$. Then θ is partially recovered as the pointwise limit

(1.9.5) $$\theta_{\mathfrak{o}_D}(w) = \lim_{t \to \xi_D} \psi(w)(t), \quad \forall w \in W_0$$

with respect to the natural topology on $H_0 = \bigoplus_w \mathbb{C} T_w$. Note that $\theta_{\mathfrak{o}_D}(w)$ lies in $H_0 \subseteq H_q(W,S)$ for all $w \in W_0$ a priori; indeed, the restriction of $\mathfrak{o}_D$ to $W_0 \subseteq W$ is nothing but the chamber orientation (definition 1.5.7) towards the element $w_D \in W_0$ corresponding to D via $w_D(C) = D$, where C denotes the fundamental Weyl chamber, and $\theta_{\mathfrak{o}_D}(w)$ identifies with the image under the Bernstein map $\theta_{\mathfrak{o}_{w_D}} : W_0 \to H_0^\times$ of the finite Hecke algebra. Thus, eq. (1.9.5) can also be seen as recovering the cocycle $\theta : W_0 \to \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}(W_0), H_0^\times)$ of the finite Hecke algebra.

Because of the cocycle rule, eq. (1.9.5) needs only to be checked in the case $w = s_\alpha$, where it follows from easy computations. Indeed

$$\lim_{t \to \xi_D} Y_\alpha = 0 \quad \text{or} \quad \lim_{t \to \xi_D} Y_\alpha^{-1} = 0$$

depending on whether D lies in the positive $\{x : \alpha(x) > 0\}$ or negative half-space $\{x : \alpha(x) < 0\}$ defined by α, respectively. Moreover, from the expression defining F_{s_α} it is immediate that

$$F_{s_\alpha}(Y_\alpha = 0) = T_{s_\alpha}$$

and

$$F_{s_\alpha}(Y_\alpha^{-1} = 0) = T_{s_\alpha} - (q_{s_\alpha} - 1) = q_{s_\alpha} T_{s_\alpha}^{-1}$$

where the second equality follows from the quadratic relation $T_{s_\alpha}^2 = q_{s_\alpha} T_{s_\alpha} + (q_{s_\alpha} - 1)$. Hence

$$K_{s_\alpha}(Y_\alpha = 0) = T_{s_\alpha} \quad \text{and} \quad K_{s_\alpha}(Y_\alpha^{-1} = 0) = T_{s_\alpha}^{-1}$$

which proves (1.9.5) for $w = s_\alpha$, taking into account the definition of θ and $\mathfrak{o}_D$ (see definitions 1.6.2 and 2.4.1 resp.)[17].

Thus one notices a curious fact: to construct R-matrices (cocycles) in the *finite* Hecke algebra H_0, one should study intertwiners of the *affine* Hecke algebra $H_q(W,S)$, which contains the former as a subalgebra.

[17]Note: the expression for F_{s_α} can be interpreted as defining a 'Yang-Baxterization' of the element $T_{s_\alpha} \in H_0$, i.e. a parametric deformation $F_{s_\alpha} = F_{s_\alpha}(Y_\alpha)$ that satisfies the Yang-Baxter equation with spectral parameter Y_α. This deformation interpolates between $T_{s_\alpha} = F_{s_\alpha}(Y_\alpha = 0)$ and $q_{s_\alpha} T_{s_\alpha}^{-1} = F_{s_\alpha}(Y_\alpha = \infty)$.

Does this pattern continue? Cherednik has shown that it does (at least for affine Hecke algebras). His *double affine Hecke algebras* $\ddot{H}_q(W,S)$ contain the affine Hecke algebras $H_q(W,S)$ as subalgebras, and one can define elements $\widehat{F}_w \in \ddot{H}_q(W,S)$ for all elements $w \in W$ of the affine Weyl group (cf. [Che92a, Theorem 3.3]), satisfying

$$\widehat{F}_w \widehat{F}_{w'} = \widehat{F}_{ww'} \quad \text{if } \ell(w) + \ell(w') = \ell(ww')$$

and defining R-matrices (with spectral parameters) with values in $H_q(W,S)$ that recover the whole Bernstein map $\theta : W \to \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, H_q(W,S)^\times)$ as a limit.

1.10 Integral and normalized Bernstein maps

We now apply the results of section 1.7 to the construction of an integral and a normalized version of the Bernstein map. Throughout this section we fix a pro-p Coxeter group $W^{(1)}$, a coefficient ring R, and a generic pro-p Hecke algebra $\mathcal{H}^{(1)} = \mathcal{H}^{(1)}(a,b)$ with arbitrary parameters.

Let us begin by constructing the integral Bernstein map.

1.10.1 Theorem. *For every orientation $\mathfrak{o}$ of $W^{(1)}$, there exists a unique map*

$$\widehat{\theta}_{\mathfrak{o}} : W^{(1)} \longrightarrow \mathcal{H}^{(1)}$$

such that if $w = n_{s_1} \dots n_{s_r} u$ with $u \in \Omega^{(1)}$ and $\ell(w) = r$, then

$$\widehat{\theta}_{\mathfrak{o}}(w) = T_1 \dots T_r T_u$$

where

$$T_i := \begin{cases} T_{n_{s_i}} & : \varepsilon_i = +1 \\ T_{n_{s_i}} - b_{s_i} & : \varepsilon_i = -1 \end{cases}$$

and $\varepsilon_i = \mathfrak{o}(s_1 \dots s_{i-1}, s_i)$. Moreover, whenever the a_s are units in R we have the equality

$$\widehat{\theta}_{\mathfrak{o}}(w) = \overline{\gamma}_{\mathfrak{o}}(\pi(w)) \theta_{\mathfrak{o}}(w) \tag{1.10.1}$$

where $\overline{\gamma}_{\mathfrak{o}} : W \to R$ is the composition of $\gamma_{\mathfrak{o}}$ with the specialization map $\mathbb{N}[\mathfrak{H}] \to R$ sending $\mathbf{a}_H$ to a_H (see remark 1.3.5 for the definition of the elements $a_H \in R$).

Proof. Because of the relation (1.6.3), we have

$$a_s T_{n_s^{-1}}^{-1} = T_{n_s} - b_s$$

whenever $a_s \in R^\times$. The second claim therefore follows immediately from the definitions provided the existence of $\widehat{\theta}_{\mathfrak{o}}$. We are therefore left to show that the expression $T_1 \dots T_r T_u$ does not depend on the choice of the expression $w = n_{s_1} \dots n_{s_r} u$. If this independence result is true for the generic pro-p Hecke algebra $\mathcal{H}^{(1)}$ over R, then it is also true for the generic pro-p Hecke algebra $\mathcal{H}^{(1)}((\phi(a_s))_s, (\phi(b_s))_s) \simeq \mathcal{H}^{(1)} \otimes_R R'$ over R' for every $\phi : R \to R'$. We may therefore replace the parameters $a_s \in R$ by indeterminates $\mathbf{a}_s$ which satisfy $\mathbf{a}_s = \mathbf{a}_t$ whenever s,t are conjugate via W and replace R by the polynomial ring $R[\mathbf{a}_s]$ and prove the claim for $\mathcal{H}^{(1)}((\mathbf{a}_s)_s, (b_s)_{s \in S})$. Because $\mathcal{H}^{(1)}$ is a free $R[\mathbf{a}_s]$-module, the localization map

$$\mathcal{H}^{(1)} \longrightarrow \mathcal{H}^{(1)} \otimes_{R[\mathbf{a}_s]} R[\mathbf{a}_s, \mathbf{a}_s^{-1}]$$

is injective. It therefore suffices to prove the independence in the localization, that is it suffices to prove it in the case the a_s are invertible. In this case we may use (1.10.1) as a definition of $\widehat{\theta}_{\mathfrak{o}}$. From the definition of $\theta_{\mathfrak{o}}$ it follows immediately that the map defined this way satisfies $\widehat{\theta}_{\mathfrak{o}}(w) = T_1 \dots T_r T_u$ for every reduced expression $w = n_{s_1} \dots n_{s_r} u$. $\square$

1.10.2 Definition. The map

$$\widehat{\theta} : W^{(1)} \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, \mathcal{H}^{(1)}), \quad w \longmapsto (\mathfrak{o} \mapsto \widehat{\theta}_{\mathfrak{o}}(w))$$

defined in the above theorem is called the **integral Bernstein map**.

1.10.3 Remark. The above technique of establishing a certain identity for generic pro-p Hecke algebras with arbitrary parameters by reducing it to the case where the a_s are invertible is the main advantage of considering Hecke algebras with two formal parameters over considering only one-parameter Hecke algebras or Hecke algebras with fixed parameters.

We will use this argument over and over again, and will therefore often refer to it simply as the '**specialization argument**'.

1.10.4 Notation. In remark 1.7.11 we considered the composition $\overline{\mathbb{X}}$ of the '2-coboundary' $\mathbb{X} : W \times W \to \mathbb{N}[\mathfrak{H}]$ with the quotient map $\mathbb{N}[\mathfrak{H}] \to \mathbb{N}[W\backslash\mathfrak{H}]$. Let us, by abuse of notation, write $\overline{\mathbb{X}}$ to also denote the composition of $\overline{\mathbb{X}}$ with the evaluation map $\mathbb{N}[W\backslash\mathfrak{H}] \to R$ sending $\mathbf{a}_{[s]}$ ($s \in S$) to a_s. Let us further denote by $\overline{\mathbb{X}}$ the composition of $\overline{\mathbb{X}} : W \times W \to R$ with $\pi \times \pi : W^{(1)} \times W^{(1)} \to W \times W$. With these conventions, we have the following corollary of the previous theorem and of lemma 1.7.8.

1.10.5 Corollary. *For every $w, w' \in W^{(1)}$*

$$\widehat{\theta}_{\mathfrak{o}}(w)\widehat{\theta}_{\mathfrak{o}\bullet w}(w') = \overline{\mathbb{X}}(w,w')\widehat{\theta}_{\mathfrak{o}}(ww')$$

Proof. By the specialization argument, it suffices to prove this when the a_s are invertible. In this case, the claim follows by combining the identity $\widehat{\theta}_{\mathfrak{o}}(w) = \overline{\gamma}_{\mathfrak{o}}(\pi(w))\theta_{\mathfrak{o}}(w)$ with the cocycle property of θ and the equation

$$\overline{\gamma}_{\mathfrak{o}}(\pi(w))\overline{\gamma}_{\mathfrak{o}\bullet w}(\pi(w')) = \overline{\mathbb{X}}(w,w')\overline{\gamma}_{\mathfrak{o}}(\pi(ww'))$$

following immediately from lemma 1.7.8. □

1.10.6 Remark. Let us record a few relations that will be useful later. First of all, we have that for any $u \in \Omega^{(1)}$ and any orientation $\mathfrak{o} \in \mathcal{O}$

$$\widehat{\theta}_{\mathfrak{o}}(u) = T_u$$

by construction. This together with remark 1.7.3 and the formula proven in the previous corollary shows that

$$\widehat{\theta}_{\mathfrak{o}}(w)T_u = \widehat{\theta}_{\mathfrak{o}}(wu)$$

and that

$$T_u\widehat{\theta}_{\mathfrak{o}\bullet u}(w) = \widehat{\theta}_{\mathfrak{o}}(uw)$$

for any $w \in W^{(1)}$. In particular, we get that

$$T_u\widehat{\theta}_{\mathfrak{o}}(w)T_u^{-1} = \widehat{\theta}_{\mathfrak{o}\bullet u^{-1}}(uwu^{-1})$$

Moreover, since the group T acts trivially on orientations by definition, for $u = t \in T$ these relations simplify to

$$\widehat{\theta}_{\mathfrak{o}}(w)T_t = \widehat{\theta}_{\mathfrak{o}}(wt)$$

and

$$T_t\widehat{\theta}_{\mathfrak{o}}(w) = \widehat{\theta}_{\mathfrak{o}}(tw)$$

respectively. Using the conjugation action $w(t) = wtw^{-1}$ of $W^{(1)}$ on T, these relations combine to give

$$\widehat{\theta}_{\mathfrak{o}}(w)T_t = T_{w(t)}\widehat{\theta}_{\mathfrak{o}}(w)$$

and more generally

$$\widehat{\theta}_{\mathfrak{o}}(w)b = w(b)\widehat{\theta}_{\mathfrak{o}}(w) \tag{1.10.2}$$

for any $b \in R[T] \subseteq \mathcal{H}^{(1)}$.

1.10.7 Corollary.

$$\widehat{\theta}_{\mathfrak{o}}(w) = T_w + \sum_{w'<w} c_{w,w'}T_{w'}$$

for some $c_{w,w'} \in R$, almost all of them being zero. In particular, $(\widehat{\theta}_{\mathfrak{o}}(w))_{w\in W^{(1)}}$ is an R-basis of $\mathcal{H}^{(1)}$.

Proof. The proof is the same as for proposition 1.6.3. □

1.10.8 Remark. Consider an orientation $\mathfrak{o}$ and a submonoid $U \leq \mathrm{Stab}_{W^{(1)}}(\mathfrak{o})$. By corollary 1.10.7, the R-submodule $\mathcal{A}_{\mathfrak{o}}^{(1)}(U)$ of $\mathcal{H}^{(1)}$ spanned by $\widehat{\theta}_{\mathfrak{o}}(x)$, $x \in U$ is in fact a free R-module on $\{\widehat{\theta}_{\mathfrak{o}}(x)\}_{x\in U}$. By corollary 1.10.5, this submodule $\mathcal{A}_{\mathfrak{o}}^{(1)}(U) \subseteq \mathcal{H}_{\mathfrak{o}}^{(1)}$ is also an R-subalgebra. When the a_s are units in R, the $\theta_{\mathfrak{o}}(x)$, $x \in U$ provide a different basis of $\mathcal{A}_{\mathfrak{o}}^{(1)}(U)$ inducing an isomorphism of the monoid algebra $R[U]$ with $\mathcal{A}_{\mathfrak{o}}^{(1)}(U)$. In particular, $\mathcal{A}_{\mathfrak{o}}^{(1)}(U)$ is commutative if U is commutative. From the specialization argument it follows that this last statement is true even if the a_s are not invertible. In fact, this also follows directly from the product formula (corollary 1.10.5) and the fact that

$$\overline{\mathbb{X}}(w,w') = \overline{\mathbb{X}}(w',w)$$

whenever $ww' = w'w$, which itself follows immediately from formula (1.7.2).

The statement of corollary 1.10.5 says informally that $d\widehat{\theta} = \overline{\mathbb{X}}$. The fact that $d\sqrt{\mathbb{L}} = \mathbb{X}$ suggests that we can restore the cocycle property of $\widehat{\theta}$ by formally twisting it with $\sqrt{\mathbb{L}}^{-1}$. This is made precise in the following definition.

1.10.9 Definition. Assume that the parameters $a_s \in R$ are units and squares in R. Recall from remark 1.3.5 that a_s only depends on the class $[s]$ it defines in $S/_\sim \simeq W\backslash\mathfrak{H}$ (see remark 1.1.8 for notation), and that a_H for $H \in \mathfrak{H}$ is by definition equal to a_s, for any $s \in S$ which is W-conjugate to H. We may therefore also write

$$a_c := a_H, \quad H \in c \text{ arbitrary}$$

for a class $c \in W\backslash\mathfrak{H}$. For every class $c \in W\backslash\mathfrak{H}$, choose now a square root $\sqrt{a_c}$ of a_c, and write

$$\sqrt{a_H} := \sqrt{a_c} \quad \forall H \in c$$

Then let

$$\widetilde{\theta}_{\mathfrak{o}}(w) := \overline{\sqrt{\mathbb{L}}}(w)^{-1}\widehat{\theta}_{\mathfrak{o}}(w) \quad \forall \mathfrak{o} \in \mathcal{O},\ w \in W^{(1)}$$

where $\overline{\sqrt{\mathbb{L}}}$ denotes the composition of maps

$$W^{(1)} \xrightarrow{\pi} W \xrightarrow{\sqrt{\mathbb{L}}} \mathbb{Z}[\sqrt{\mathfrak{H}}] \longrightarrow R^\times$$

with $\sqrt{\mathbb{L}} : W \to \mathbb{Z}[\sqrt{\mathfrak{H}}]$ the formal square root of the generalized length function $\mathbb{L}$ defined in remark 1.7.11 and

$$\mathbb{Z}[\sqrt{\mathfrak{H}}] \longrightarrow R^\times$$

the group homomorphism sending a formal square $\sqrt{\mathbf{a}_H}$ to $\sqrt{a_H}$.

The map

$$\widetilde{\theta} : W^{(1)} \longrightarrow \mathrm{Hom}_{\mathrm{Set}}(\mathcal{O}, \mathcal{H}^{(1)}), \quad w \longmapsto (\mathfrak{o} \mapsto \widetilde{\theta}_{\mathfrak{o}}(w))$$

is called the **normalized Bernstein map** (with respect to the chosen square roots $\sqrt{a_c}$).

In the situation of the above definition, we have the following immediate corollary of corollary 1.10.5 and remark 1.7.11.

1.10.10 Corollary. *For all $w, w' \in W^{(1)}$ and $\mathfrak{o} \in \mathcal{O}$*

$$\widetilde{\theta}_{\mathfrak{o}}(ww') = \widetilde{\theta}_{\mathfrak{o}}(w)\widetilde{\theta}_{\mathfrak{o}\bullet w}(w')$$

1.10.11 Remark. For our purposes the main reason for introducing the normalized Bernstein map lies in the fact that it gets transformed into the integral Bernstein map under a certain isomorphism of Hecke algebras. More precisely, in the situation of the above definition we have an isomorphism

$$\varphi : \mathcal{H}^{(1)}(a_s, b_s) \xrightarrow{\sim} \mathcal{H}^{(1)}(1, \sqrt{a_s}^{-1}b_s)$$

of R-modules determined by $T_w \mapsto \overline{\sqrt{\mathbb{L}}}(w)T_w$. Note that $\mathcal{H}^{(1)}(1, \sqrt{a_s}^{-1}b_s)$ is well-defined as the parameters again satisfy the conditions of theorem 1.3.1. This isomorphism is also an isomorphism of R-algebras, which follows easily by combining the presentation of $\mathcal{H}^{(1)}(a_s, b_s)$ given in section 1.4 with remarks 1.7.2 and 1.7.11 and verifying the following quadratic relation

$$(\sqrt{a}_s T_{n_s})^2 = a_s T_{n_s^2} + (\sqrt{a}_s T_{n_s})b_s$$

in the Hecke algebra $\mathcal{H}^{(1)}(1, \sqrt{a_s}^{-1}b_s)$.

The normalized Bernstein map $\widetilde{\theta}$ of $\mathcal{H}^{(1)}(a_s, b_s)$ and the integral Bernstein map $\widehat{\theta}$ of $\mathcal{H}^{(1)}(1, \sqrt{a_s}^{-1}b_s)$ are now related as follows

$$\widehat{\theta}_{\mathfrak{o}}(w) = \varphi(\widetilde{\theta}_{\mathfrak{o}}(w)) \quad \forall \mathfrak{o} \in \mathcal{O},\ w \in W^{(1)} \tag{1.10.3}$$

1.11 Bernstein relations

In this section we again fix a generic pro-p Hecke algebra $\mathcal{H}^{(1)} = \mathcal{H}^{(1)}(a_s, b_s)$ and assume that the parameters $a_s \in R$ are units and squares in R, and that a choice of square roots $\sqrt{a_s}$ and consequently of a normalized Bernstein map $\widetilde{\theta}$ has been made according to the previous section.

The goal of this section is to compute the difference

$$\widetilde{\theta}_{\mathfrak{o}}(w) - \widetilde{\theta}_{\mathfrak{o}'}(w) \tag{1.11.1}$$

as a sum over certain hyperplanes, for two orientations $\mathfrak{o}, \mathfrak{o}' \in \mathcal{O}$ that are 'adjacent'. This computation will be crucial in section 2, where we will use it to show that certain elements $z_{\mathfrak{o}}(\gamma)$ of an affine pro-p Hecke algebra lie in the center. In the classical case ($W^{(1)} = W$) this computation is essentially equivalent to Bernstein's relations for the Iwahori-Hecke algebra.

We remind the reader that (see 1.1.3)

$$\mathfrak{H} = \{wsw^{-1} : s \in S,\ w \in W_{\text{aff}}\} = \{wsw^{-1} : s \in S,\ w \in W\} \subseteq W_{\text{aff}}$$

denotes the set of *hyperplanes* of the underlying Coxeter group W_{aff} of $W^{(1)}$.

The next proposition introduces some canonical elements in the generic pro-p Hecke algebra, which will appear in the sum expansion of expression (1.11.1).

1.11.1 Proposition/Definition. *For any hyperplane $H \in \mathfrak{H}$ and any orientation $\mathfrak{o} \in \mathcal{O}$, there exists a unique element $\Xi_{\mathfrak{o}}(H) \in \mathcal{H}^{(1)}$, such that if $s \in S$, $w \in W^{(1)}$ with*

$$\pi(wn_sw^{-1}) = H$$

then

$$\Xi_{\mathfrak{o}}(H) = \sqrt{a_s}^{-1} w(b_s) \cdot \widetilde{\theta}_{\mathfrak{o}}(wn_s^{-1}w^{-1}) = \sqrt{a_s}^{-1} \widetilde{\theta}_{\mathfrak{o}}(wn_s^{-1}w^{-1}) \cdot w(b_s)$$

Proof. Applying the isomorphism φ of remark 1.10.11, we may assume that $a_s = 1$ for all $s \in S$ and that $\widetilde{\theta} = \widehat{\theta}$. Moreover, we observe that $w(b_s)$ and $\widehat{\theta}_{\mathfrak{o}}(wn_s^{-1}w^{-1})$ commute with each other. Indeed, by applying the commutation relation (1.10.2) this is easily reduced to show the basic identity

$$n_s^{-1}(b_s) = b_s$$

which was already seen to be true in (1.3.6). Therefore, it only remains to show that the expression

$$w(b_s) \cdot \widehat{\theta}_{\mathfrak{o}}(wn_s^{-1}w^{-1}) = \widehat{\theta}_{\mathfrak{o}}(wn_s^{-1}w^{-1}) \cdot w(b_s)$$

only depends on the element

$$\pi(wn_sw^{-1}) = H \in \mathfrak{H}$$

and not on the choice of $w \in W^{(1)}$ and $s \in S$. So let $w_1, w_2 \in W^{(1)}$ and $s, t \in S$ with

$$\pi(w_1n_sw_1^{-1}) = \pi(w_2n_tw_2^{-1})$$

By the above equation, we may apply condition (1.3.1) of theorem 1.3.1 on the existence of generic pro-p Hecke algebras to $w = w_1^{-1}w_2$ (in the notation of said theorem). Condition (1.3.1) then states that

$$(n_swn_t^{-1}w^{-1}) \cdot w(b_t) = b_s$$

as an equality in $R[T]$. Acting on both sides with w_1, we get the formula

$$w_1(b_s) = (w_1n_sw_1^{-1}w_2n_t^{-1}w_2^{-1}) \cdot w_2(b_t)$$

Bearing in mind that $w_1n_sw_1^{-1}w_2n_t^{-1}w_2^{-1} \in T$, we can use the relations proved in remark 1.10.6 to compute

$$\begin{aligned}
\widehat{\theta}_{\mathfrak{o}}(w_1n_s^{-1}w_1^{-1}) \cdot w_1(b_s) &= \widehat{\theta}_{\mathfrak{o}}(w_1n_s^{-1}w_1^{-1}) \cdot (w_1n_sw_1^{-1}w_2n_t^{-1}w_2^{-1}) \cdot w_2(b_t) \\
&= \widehat{\theta}_{\mathfrak{o}}(w_1n_s^{-1}w_1^{-1}w_1n_sw_1^{-1}w_2n_t^{-1}w_2^{-1}) \cdot w_2(b_t) \\
&= \widehat{\theta}_{\mathfrak{o}}(w_2n_t^{-1}w_2^{-1}) \cdot w_2(b_t)
\end{aligned}$$

□

The classical Bernstein relations compute the difference (1.11.1) when $\mathfrak{o}' = \mathfrak{o} \bullet s_\alpha$ for a 'simple root' α of a root system and the orientation $\mathfrak{o}$ is 'spherical' (cf. definition 2.4.1). The following definition allows us to state the Bernstein relations in a more general context.

Recall that by lemma 1.7.4, an orientation $\mathfrak{o}$ of a Coxeter group is given by defining for every hyperplane $H \in \mathfrak{H}$ a notion of *positive/negative crossing* for passing from one half-space (with respect to H) into the other. It therefore makes sense to say that two orientations agree (or disagree) at a hyperplane H if the signs attached by the orientations to passing from one half-space with respect to H into the other are equal (or unequal).

1.11.2 Definition. Two orientations $\mathfrak{o}, \mathfrak{o}' \in \mathcal{O}$ of W are said to be **adjacent** if for every wall $H \in \mathfrak{H}$ at which $\mathfrak{o}$ and $\mathfrak{o}'$ disagree, we have

$$\mathfrak{o} \bullet s_H = \mathfrak{o}'$$

Note that the notion of adjacency is symmetric in $\mathfrak{o}$ and $\mathfrak{o}'$. We are now ready to give the 'Bernstein relation'.

1.11.3 Theorem. *Let $w \in W^{(1)}$ and $\mathfrak{o}, \mathfrak{o}' \in \mathcal{O}$ be adjacent. Then*

$$\widetilde{\theta}_{\mathfrak{o}}(w) - \widetilde{\theta}_{\mathfrak{o}'}(w) = \left(\sum_H \mathfrak{o}(1, H) \Xi_{\mathfrak{o}'}(H) \right) \widetilde{\theta}_{\mathfrak{o}}(w) \tag{1.11.2}$$

where the sum is taken over all hyperplanes $H \in \mathfrak{H}$ which separate 1 *and w, and at which $\mathfrak{o}$ and $\mathfrak{o}'$ disagree.*

Proof. We may again invoke remark 1.10.11 to reduce to the case $a_s = 1$ and $\widetilde{\theta} = \widehat{\theta} = \theta$. Now take *any* (not necessarily reduced) expression

$$w = n_{s_1} \dots n_{s_r} u, \quad s_i \in S, \ u \in \Omega^{(1)}$$

Using this expression, the cocycle rule and the definition of the Bernstein map together give the following explicit expressions

$$\widehat{\theta}_{\mathfrak{o}}(w) = T^{\varepsilon_1}_{n^{\varepsilon_1}_{s_1}} \dots T^{\varepsilon_r}_{n^{\varepsilon_r}_{s_r}} T_u, \quad \widehat{\theta}_{\mathfrak{o}'}(w) = T^{\varepsilon'_1}_{n^{\varepsilon'_1}_{s_1}} \dots T^{\varepsilon'_r}_{n^{\varepsilon'_r}_{s_r}} T_u$$

where

$$\varepsilon_i = \mathfrak{o}(s_1 \dots s_{i-1}, s_i), \quad \varepsilon'_i = \mathfrak{o}'(s_1 \dots s_{i-1}, s_i)$$

We expand the difference $\widehat{\theta}_{\mathfrak{o}}(w) - \widehat{\theta}_{\mathfrak{o}'}(w)$ now as a telescopic sum

$$\widehat{\theta}_{\mathfrak{o}}(w) - \widehat{\theta}_{\mathfrak{o}'}(w) = \sum_{i=1}^{r} T^{\varepsilon'_1}_{n^{\varepsilon'_1}_{s_1}} \dots T^{\varepsilon'_{i-1}}_{n^{\varepsilon'_{i-1}}_{s_{i-1}}} \left(T^{\varepsilon_i}_{n^{\varepsilon_i}_{s_i}} - T^{\varepsilon'_i}_{n^{\varepsilon'_i}_{s_i}} \right) T^{\varepsilon_{i+1}}_{n^{\varepsilon_{i+1}}_{s_{i+1}}} \dots T^{\varepsilon_r}_{n^{\varepsilon_r}_{s_r}} T_u$$

In this sum the i-th summand vanishes unless $\varepsilon_i \neq \varepsilon'_i$, so let us fix an index i where $\varepsilon_i \neq \varepsilon'_i$. Observing that (cf. eq. (1.3.8))

$$T^{\varepsilon}_{n^{\varepsilon}_s} - T^{-\varepsilon}_{n^{-\varepsilon}_s} = \varepsilon b_s \quad \forall s \in S, \ \varepsilon \in \{\pm\}$$

and using the commutation rule (cf. (1.10.2))

$$\widehat{\theta}_{\mathfrak{o}}(w) b = w(b) \widehat{\theta}_{\mathfrak{o}}(w) \quad \forall w \in W^{(1)}, \ b \in R[T]$$

we see that the i-th summand can be rewritten as

$$\varepsilon_i \widetilde{w}(b_{s_i}) T^{\varepsilon'_1}_{n^{\varepsilon'_1}_{s_1}} \dots T^{\varepsilon'_{i-1}}_{n^{\varepsilon'_{i-1}}_{s_{i-1}}} T^{\varepsilon_{i+1}}_{n^{\varepsilon_{i+1}}_{s_{i+1}}} \dots T^{\varepsilon_r}_{n^{\varepsilon_r}_{s_r}} T_u$$

where we have put

$$\widetilde{w} := n_{s_1} \dots n_{s_{i-1}}$$

Since $\mathfrak{o}$ and $\mathfrak{o}'$ disagree at

$$s_H = H := \pi(\widetilde{w} n_{s_i} \widetilde{w}^{-1}) = (s_1 \dots s_{i-1}) s_i (s_1 \dots s_{i-1})^{-1}$$

and $\mathfrak{o}$ and $\mathfrak{o}'$ are adjacent, we have

$$\mathfrak{o}' = \mathfrak{o} \bullet s_H$$

In particular, for $j > i$ we have

$$\begin{aligned} \varepsilon_j &= \mathfrak{o}(s_1 \dots s_{j-1}, s_j) \\ &= \mathfrak{o}(s_H s_1 \dots \widehat{s_i} \dots s_{j-1}, s_j) \\ &= (\mathfrak{o} \bullet s_H)(s_1 \dots \widehat{s_i} \dots s_{j-1}, s_j) \\ &= \mathfrak{o}'(s_1 \dots \widehat{s_i} \dots s_{j-1}, s_j) \end{aligned}$$

This implies that

$$\begin{aligned} T^{\varepsilon'_1}_{n^{\varepsilon'_1}_{s_1}} \dots T^{\varepsilon'_{i-1}}_{n^{\varepsilon'_{i-1}}_{s_{i-1}}} T^{\varepsilon_{i+1}}_{n^{\varepsilon_{i+1}}_{s_{i+1}}} \dots T^{\varepsilon_r}_{n^{\varepsilon_r}_{s_r}} T_u &\overset{(!)}{=} \widehat{\theta}_{\mathfrak{o}'}(n_{s_1} \dots \widehat{n_{s_i}} \dots n_{s_r} u) \\ &= \widehat{\theta}_{\mathfrak{o}'}(\widetilde{w} n_{s_i}^{-1} \widetilde{w}^{-1} w) \\ &= \widehat{\theta}_{\mathfrak{o}'}(\widetilde{w} n_{s_i}^{-1} \widetilde{w}^{-1}) \widehat{\theta}_{\mathfrak{o}' \bullet s_H}(w) \\ &= \widehat{\theta}_{\mathfrak{o}'}(\widetilde{w} n_{s_i}^{-1} \widetilde{w}^{-1}) \widehat{\theta}_{\mathfrak{o}}(w) \end{aligned}$$

Recalling proposition/definition 1.11.1, we see that

$$\widetilde{w}(b_{s_i}) \widehat{\theta}_{\mathfrak{o}'}(\widetilde{w} n_{s_i}^{-1} \widetilde{w}^{-1}) = \Xi_{\mathfrak{o}'}(H)$$

and therefore

$$\widehat{\theta}_{\mathfrak{o}}(w) - \widehat{\theta}_{\mathfrak{o}'}(w) = \sum_{\substack{i \in \{1, \dots, r\} \\ \varepsilon_i \neq \varepsilon'_i}} \varepsilon_i \Xi_{\mathfrak{o}'}(H_i) \widehat{\theta}_{\mathfrak{o}}(w)$$

where

$$H_i := (s_1 \dots s_{i-1}) s_i (s_1 \dots s_{i-1})^{-1}$$

is the hyperplane crossed by the gallery $(s_1, \dots, s_r)$ in the i-th step. Until now we have not assumed this gallery, i.e. the expression

$$w = n_{s_1} \dots n_{s_r} u$$

to be reduced. Assume now that this is the case. Then the hyperplanes crossed by the gallery $(s_1, \dots, s_r)$ are exactly the hyperplanes separating 1 and w. Moreover, in this case we have

$$\varepsilon_i = \mathfrak{o}(1, H_i)$$

and hence the theorem follows. □

As already mentioned, the 'Bernstein relation' proven above will be used to show that certain elements of affine pro-p Hecke algebras lie in the center. This application of the Bernstein relation will involve showing that

$$\widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o} \bullet s_\alpha}(x) = -\left(\widehat{\theta}_{\mathfrak{o}}(s_\alpha(x)) - \widehat{\theta}_{\mathfrak{o} \bullet s_\alpha}(s_\alpha(x)) \right)$$

for x an element of a certain subgroup $X^{(1)} \subseteq W^{(1)}$ and $s_\alpha \in W$ a reflection associated to a simple root α. This will follow from the above theorem and the following elementary property of the elements $\Xi_{\mathfrak{o}}(H)$.

1.11.4 Lemma. *Let $H \in \mathfrak{H}$, $\mathfrak{o}$ an orientation and $x \in C_{W^{(1)}}(T)$ an element of the centralizer of T in $W^{(1)}$. Then we have that*

$$\Xi_{\mathfrak{o}}(H) \cdot \widetilde{\theta}_{\mathfrak{o} \bullet s_H}(s_H(x) x^{-1}) = \Xi_{\mathfrak{o}}(\pi(x) H \pi(x)^{-1})$$

where $s_H(x)$ denotes the induced action of W on $C_{W^{(1)}}(T)$ by conjugation.

Proof. Letting $w \in W^{(1)}$ and $s \in S$ be such that

$$H = \pi(w n_s w^{-1})$$

we have by definition that

$$\Xi_{\mathfrak{o}}(H) = \sqrt{a_s}^{-1} w(b_s) \widetilde{\theta}_{\mathfrak{o}}(w n_s^{-1} w^{-1})$$

Since $w n_s^{-1} w^{-1}$ acts both on $\mathfrak{o}$ and on $s_H(x)$ via s_H, we have that

$$\widetilde{\theta}_{\mathfrak{o}}(w n_s^{-1} w^{-1}) \widetilde{\theta}_{\mathfrak{o} \bullet s_H}(s_H(x) x^{-1}) = \widetilde{\theta}_{\mathfrak{o}}(w n_s^{-1} w^{-1} s_H(x) x^{-1}) \overset{(!)}{=} \widetilde{\theta}_{\mathfrak{o}}(x w n_s^{-1} w^{-1} x^{-1})$$

Therefore

$$\begin{aligned} \Xi_{\mathfrak{o}}(H) \cdot \widetilde{\theta}_{\mathfrak{o} \bullet s_H}(s_H(x) x^{-1}) &= \sqrt{a_s}^{-1} w(b_s) \widetilde{\theta}_{\mathfrak{o}}(x w n_s^{-1} w^{-1} x^{-1}) \\ &= \sqrt{a_s}^{-1} (xw)(b_s) \widetilde{\theta}_{\mathfrak{o}}(x w n_s^{-1} w^{-1} x^{-1}) \\ &= \Xi_{\mathfrak{o}}(\pi(x) H \pi(x)^{-1}) \end{aligned}$$

where we have used the fact that x acts trivially on T on the second line, and the definition of Ξ on the third line. □

2 Affine pro-p Hecke algebras

In this section, we want to apply the general theory developed so far to the study of a special class of generic pro-p Hecke algebras, the '*affine* pro-p Hecke algebras'. We will give a description of the center of these algebras and prove that there they are module-finite over their center in section 2.7, recovering classical results of Bernstein-Zelevinsky in the case of $W = W^{(1)}$.

In order to obtain these results, we need to assume that the group W is of a special form. Basically we need W to be a semi-direct product $W = X \rtimes W_0$ of a finitely generated commutative group X and a finite reflection group W_0. Moreover, we need to assume that there exists a representation of W as a group of isometries preserving a locally finite affine hyperplane arrangement which is compatible with the abstract decompositions $W = W_{\text{aff}} \rtimes \Omega$ and $W = X \rtimes W_0$. Finally we need to assume that X is 'large enough' with respect to this representation. This will be made precise in the next section.

2.1 Affine extended Coxeter groups and affine pro-p Hecke algebras

Before we give the definition of an *affine* extended Coxeter group, let us introduce some notations and recall some basic facts from the theory of affine reflection groups (see for instance [Bou07, Ch. V, §1]).

Given a finite dimensional euclidean vector space V and a hyperplane[18] $H \leq V$, there exists a unique element $s_H \in \operatorname{Aut}_{\text{Euclid}}(V)$ of the group of euclidean motions such that $s_H \neq \text{id}$ and s_H operates on H as the identity. This element s_H is called the *orthogonal reflection* with respect to H. More generally, an affine endomorphism $s \in \operatorname{End}_{\text{aff}}(V)$ is a called a *reflection* if $s^2 = \text{id}$ and if the linear part s_0 of s is a linear reflection in the sense that (cf. [Bou07, Ch. V, §2.2])

$$s_0^2 = \text{id} \quad \text{and} \quad \text{id} - s_0 \text{ is of rank 1}$$

Note here that $s_0^2 = \text{id}$ follows already from $s^2 = \text{id}$. The set $H := \{x : s(x) = x\}$ of fix points of a reflection s is an affine hyperplane, and it is therefore the unique hyperplane fixed by s. Of course, a reflection s is not determined by the affine hyperplane H that it fixes, but if s also happens to be an element of $\operatorname{Aut}_{\text{Euclid}}(V)$, then it must coincides with the orthogonal reflection s_H with respect to H.

For a given affine hyperplane Given a set $\mathfrak{H}$ of hyperplanes in V, we let

$$W(\mathfrak{H}) = \langle s_H : H \in \mathfrak{H} \rangle \leq \operatorname{Aut}_{\text{Euclid}}(V)$$

denote the group generated by the reflections with respect to the hyperplanes in $\mathfrak{H}$. If $\alpha \in V^\vee$ is a non-zero functional and $k \in \mathbb{R}$, we write

$$H_{\alpha,k} := \{x \in V : \alpha(x) + k = 0\}$$

and $s_{\alpha,k} := s_{H_{\alpha,k}}$, $s_\alpha := s_{\alpha,0}$.

A point $x \in V$ is called *special* with respect to $\mathfrak{H}$ if for every $H \in \mathfrak{H}$ there exists a hyperplane $H' \in \mathfrak{H}$ parallel to H with $x \in H'$. A set $\mathfrak{H}$ of hyperplanes in V is called *locally finite* if for every $x \in X$ there exists a neighbourhood U of x such that $\{H \in \mathfrak{H} : H \cap U \neq \emptyset\}$ is finite.

Assume that a locally finite set $\mathfrak{H}$ of hyperplanes on V is given. The elements of the set

$$\mathfrak{C} := \pi_0(V - \bigcup_{H \in \mathfrak{H}} H)$$

of connected components of the complement of all hyperplanes are called *chambers*[19]. A hyperplane $H \in \mathfrak{H}$ is called a *wall* of a chamber C if $H \cap \overline{C}$ has non-empty interior as a subset of H, or equivalently, if the affine span of $H \cap \overline{C}$ equals H. We let

$$S(C) := \{H \in \mathfrak{H} : H \text{ wall of } C\}$$

denote the set of all walls of C. If the group $W(\mathfrak{H})$ leaves the set $\mathfrak{H}$ invariant, then it follows[20] that for every chamber C the pair $(W(\mathfrak{H}), \{s_H : H \in S(C)\})$ is a Coxeter group (cf. [Bou07, Ch. V, §3.2, Théorème 1]) and that $S(C)$ is finite (cf. [Bou07, Ch. V, §3.6, Théorème 3]).

We will now give the definition of 'affine' extended Coxeter groups.

[18] Unless specified otherwise, hyperplane means affine hyperplane.

[19] It is common to use the term *alcove* instead of chamber if the hyperplanes $H \in \mathfrak{H}$ aren't all linear, but we will not make this distinction.

[20] In [Bou07] it is assumed that the group $W(\mathfrak{H})$ acts properly discontinuously, and the local finiteness is deduced as a consequence. However, it is enough to only assume that $\mathfrak{H}$ is locally finite and $W(\mathfrak{H})$ preserves $\mathfrak{H}$, as these assumptions already imply that $W(\mathfrak{H})$ acts properly discontinuously.

2.1.1 Definition. An **affine extended Coxeter group** W consists of a group W together with a homomorphism

$$\rho : W \longrightarrow \mathrm{Aut}_{\mathrm{aff}}(V)$$

of W into the group of affine automorphisms of a finite-dimensional real vector space V, a locally finite set $\mathfrak{H}$ of (affine) hyperplanes in V, a chamber $C_0 \in \pi_0(V - \bigcup_{H\in\mathfrak{H}} H)$ and for every $H \in \mathfrak{H}$ an element $\widetilde{s}_H \in W$ such that the following hold.

(**ACI**) W leaves $\mathfrak{H}$ invariant, i.e. $\rho(w)(H) \in \mathfrak{H}$ for all $w \in W$ and $H \in \mathfrak{H}$.

(**ACII**) For every $H \in \mathfrak{H}$, $\rho(\widetilde{s}_H)$ is a *reflection* fixing H.

(**ACIII**) Letting ρ_0 denote the composition of ρ with the projection

$$\mathrm{Aut}_{\mathrm{aff}}(V) = V \rtimes \mathrm{GL}(V) \longrightarrow \mathrm{GL}(V)$$

onto the linear part, the group

$$W_0 := \rho_0(W)$$

is *finite*.

(**ACIV**) $0 \in V$ is a special point of $\mathfrak{H}$.

(**ACV**) The subgroup $\rho(W) \cap V$ of translations in $\rho(W)$ generates the quotient V/L as an $\mathbb{R}$-vector space, where

$$L = \bigcap_{H\in\mathfrak{H},\ 0\in H} H$$

(**ACVI**) For every $H \in \mathfrak{H}$ and $w \in W$ we have

$$w\widetilde{s}_H w^{-1} = \widetilde{s}_{w(H)}$$

where we abbreviate $w(H) = \rho(w)(H)$.

(**ACVII**) For every pair $H_1, H_2 \in S(C_0)$ of walls of C_0 such that $\rho(\widetilde{s}_{H_1}\widetilde{s}_{H_2})$ is of finite order $m_{1,2}$, we have the relation

$$(\widetilde{s}_{H_1}\widetilde{s}_{H_2})^{m_{1,2}} = 1$$

in W.

(**ACVIII**) The group W_0 is generated by the images of the $\widetilde{s}_H$, $H \in \mathfrak{H}$ under the natural map $W \to W_0$.

(**ACIX**) $0 \in \overline{C_0}$

(**ACX**) Let

$$X = \rho^{-1}(V) \leq W$$

denote the subgroup of all elements of W which are mapped to a translation under ρ. Then X is finitely generated and commutative.

Note that given the remaining axioms, (**ACIV**) and (**ACIX**) are always satisfied up to a translation and only serve to fix notation. The rationale behind the above definition of an *affine extended Coxeter group* is to have a set of axioms which are easy to verify in examples. However, as it stands the definition does not even mention extended Coxeter groups. Our first task will therefore be to 'unpack' this definition.

2.1.2 Lemma. *Let $W = (W, V, \rho, \mathfrak{H}, C_0, (\widetilde{s}_H)_H)$ be an affine extended Coxeter group. Let*

$$W_{\mathrm{aff}} := \langle \widetilde{s}_H : H \in \mathfrak{H} \rangle\,, \quad S := \{\widetilde{s}_H : H \in S(C_0)\}$$

and

$$\Omega := \mathrm{Stab}_W(C_0)$$

Then the following holds.

(i) There exists a positive definite scalar product on V invariant with respect to W_0, i.e. such that W acts by euclidean motions.

(ii) (W_{aff}, S) is a Coxeter group and for any choice of an invariant scalar product, ρ induces an isomorphism

$$(W_{\mathrm{aff}}, S) \xrightarrow{\sim} (W(\mathfrak{H}), \{s_H : H \in S(C_0)\})$$

of Coxeter groups, where s_H denotes the orthogonal reflection with respect to H and $W(\mathfrak{H})$ denotes the group generated by s_H for $H \in \mathfrak{H}$. In particular, $W(\mathfrak{H})$ and the s_H do not depend on the choice of the scalar product.

(iii) $(W, W_{\mathrm{aff}}, S, \Omega)$ is an extended Coxeter group.

*(iv) The group W_0 is equal to the special subgroup of $(W(\mathfrak{H}), \{s_H : H \in S(C_0)\})$ generated by the s_H with $0 \in H$. In particular, $(W_0, \{s_H : H \in S(C_0),\ 0 \in H\})$ is a Coxeter group. Moreover, the subspace $L \leq V$ of (**ACV**) is given by*

$$L = \bigcap_{H \in \mathfrak{H},\ 0 \in H} H = V^{W_0}$$

(v) Let

$$\Phi := \{\alpha \in V^\vee : \forall k \in \mathbb{R} \quad H_{\alpha,k} \in \mathfrak{H} \Leftrightarrow k \in \mathbb{Z}\}$$

Then $(\mathbb{R}\Phi, \Phi)$ is a reduced root system and

$$\mathfrak{H} = \{H_{\alpha,k} : \alpha \in \Phi,\ k \in \mathbb{Z}\}$$

Moreover, $V \xrightarrow{\sim} V^{\vee\vee}$ induces an isomorphism

$$V/L \xrightarrow{\sim} (\mathbb{R}\Phi)^\vee$$

(vi) The map

$$W_0 \longrightarrow \mathrm{GL}(V/L) \simeq \mathrm{GL}((\mathbb{R}\Phi)^\vee)$$

induced by $\rho_0 : W_0 \to \mathrm{GL}(V)$ is injective and identifies W_0 with the Weyl group $W(\Phi^\vee)$ of the dual root system $((\mathbb{R}\Phi)^\vee, \Phi^\vee)$. Moreover, this is an identification of Coxeter groups if we endow $W(\Phi^\vee)$ with the generating set $\{s_\alpha^\vee : \alpha \in \Delta\}$ corresponding to the basis

$$\Delta = \{\alpha \in \Phi : H_\alpha \in S(C_0),\ \alpha|_{C_0} > 0\}$$

The basis Δ corresponds to the positive root system $\Phi^+ \subseteq \Phi$ given by

$$\Phi^+ = \{\alpha \in \Phi : \alpha(x) > 0\}$$

where $x \in C_0$ is arbitrary.

(vii) The exact sequence

$$0 \longrightarrow X \longrightarrow W \xrightarrow{\rho_0} W_0 \longrightarrow 1$$

splits via the map $W_0 \to W$ given by the composition

$$W_0 \subseteq W(\mathfrak{H}) \xrightarrow{\rho^{-1}} W_{\mathrm{aff}} \subseteq W$$

Viewing W_0 as a subgroup of W_{aff} via this splitting, W_0 equals the special subgroup of (W_{aff}, S) generated by

$$S_0 := \{s_\alpha : \alpha \in \Delta\} = \{\widetilde{s}_H : H \in S(C_0),\ 0 \in H\} \subseteq S$$

where $s_\alpha := \widetilde{s}_{H_{\alpha,0}}$.

Proof. Point (i) follows immediately from the finiteness of W_0, since given any positive definite scalar product $B : V \times V \to \mathbb{R}$, the expression

$$(x, y) := \sum_{w \in W_0} B(w(x), w(y)), \quad x, y \in V$$

defines a W_0-invariant positive definite scalar product. To prove (ii) we may assume a W_0-invariant scalar product has been fixed. In this case we may invoke [Bou07, Ch. V, §3.2, Théorème 1] to conclude that $(W(\mathfrak{H}), \{s_H : H \in S(C_0)\})$ is a Coxeter group. Since $\rho(\widetilde{s}_H)$ is a reflection fixing H by (**ACII**) and W acts by euclidean motions with respect to the chosen scalar product, we must have $\rho(\widetilde{s}_H) = s_H$ for every $H \in \mathfrak{H}$. Since W_{aff} is generated by the $\widetilde{s}_H$, this shows that we have a well-defined group homomorphism

$$\rho : W_{\mathrm{aff}} \longrightarrow W(\mathfrak{H})$$

that moreover maps S into $S(C_0)$. Since $W(\mathfrak{H}, S(C_0))$ is a Coxeter group, by one of the various characterizations ([Bou07, Ch. IV, §1.3, Définition 3]) of Coxeter groups, $W(\mathfrak{H})$ has a presentation

$$W(\mathfrak{H}) = \langle s_H,\ H \in S(C_0) \mid (s_H s_{H'})^m = 1 \text{ if } m = \mathrm{ord}(s_H s_{H'}) < \infty \rangle$$

and hence by property (**ACVII**) there exists a unique homomorphism

$$\varphi : W(\mathfrak{H}) \longrightarrow W_{\mathrm{aff}}$$

of groups with $\varphi(s_H) = \widetilde{s}_H$ for every $H \in S(C_0)$. Since $\rho \circ \varphi = \mathrm{id}$, it follows that φ is injective. We claim that φ is also surjective, or equivalently that S generates W_{aff}. From the theory of affine reflection groups it follows (cf. [Bou07, Ch. V, §3.2, Corollaire]) that for every $H \in \mathfrak{H}$ there exists an element $w \in W(\mathfrak{H})$ and a wall $H' \in S(C_0)$ such that $w s_{H'} w^{-1} = s_H$ or equivalently $w(H') = H$. Writing

$$w = s_{H_1} \dots s_{H_r}, \quad H_i \in S(C_0)$$

and putting

$$\widetilde{w} := \widetilde{s}_{H_1} \dots \widetilde{s}_{H_r} \in \langle S \rangle \subseteq W_{\mathrm{aff}}$$

we have $\rho(\widetilde{w}) = w$ and hence by (**ACVI**)

$$\widetilde{w} \widetilde{s}_{H'} \widetilde{w}^{-1} = \widetilde{s}_{\rho(\widetilde{w})(H')} = \widetilde{s}_H$$

lies in the subgroup of W_{aff} generated by S. Since W_{aff} is generated by the $\widetilde{s}_H$, it follows that $\langle S \rangle = W_{\mathrm{aff}}$ and hence that φ is an isomorphism of groups. Since $\rho \circ \varphi = \mathrm{id}$, also ρ must be an isomorphism of groups. Moreover, as ρ preserves the distinguished sets of generators, it is also an isomorphism of Coxeter groups.

Now to prove (iii), we only need to verify that Ω preserves the subset $S \subseteq W_{\mathrm{aff}}$ under conjugation and that every element $w \in W$ can be written as a product $w = w'u$ with $w' \in W_{\mathrm{aff}}$ and $u \in \Omega$. But the invariance of S follows immediately from (**ACVI**) and the fact Ω permutes the walls of C_0 (as it preserves C_0 and therefore also $\overline{C_0}$ setwise). Because $W(\mathfrak{H})$ acts transitively on the set $\pi_0(V - \bigcup_{H \in \mathfrak{H}} H)$ of chambers (see [Bou07, Ch. V, §3.2, Théorème 1]), we can find $w'' \in W(\mathfrak{H})$ with $\rho(w)(C_0) = w''(C_0)$. Since $\rho(W_{\mathrm{aff}}) = W(\mathfrak{H})$, we can find $w' \in W_{\mathrm{aff}}$ with $\rho(w') = w''$. It follows that $u := w'^{-1} w \in \Omega$.

Next, we show that (iv) holds. Observe that by (**ACVIII**), the group W_0 is generated by the set of linear parts of the s_H with $H \in \mathfrak{H}$. By (**ACIV**), the point $0 \in V$ is special and hence the aforementioned set coincides with $\{s_H : H \in \mathfrak{H},\ 0 \in H\}$. In particular, $W_0 \subseteq W(\mathfrak{H})$ and the formula $L = V^{W_0}$ holds. Let $F \subseteq V$ be the unique facet of $(V, \mathfrak{H})$ containing 0. By (**ACIX**), F is a face of C_0. From [Bou07, Ch. V, §3.3, Proposition 1] it therefore follows that W_0 must be contained in the subgroup of $W(\mathfrak{H})$ generated by the s_H with $H \in S(C_0)$ and $F \subseteq H$. So we have the inclusion

$$W_0 = \langle s_H : H \in \mathfrak{H},\ 0 \in H \rangle \subseteq \langle s_H : H \in S(C_0),\ 0 \in H \rangle$$

and hence equality holds.

Claim (v) follows from (**ACIV**), (**ACV**) and a slight modification of the arguments in [Bou07, Ch. VI, §2.5, Proposition 8]. Fix an invariant positive definite scalar product $(-,-)$ on V. Given $H \in \mathfrak{H}$ with $0 \in H$, let $\alpha \in V^\vee$ be any element with $\ker(\alpha) = H$. Consider

$$\Lambda_\alpha := \{k \in \mathbb{R} : H_{\alpha,k} \in \mathfrak{H}\}$$

Then $k \mapsto H_{\alpha,k}$ gives a bijection between Λ_α and the $H' \in \mathfrak{H}$ parallel to H. Then Λ must contain a positive element, for we have $0 \in \Lambda_\alpha$ and by (**ACV**) there exists an element $w \in W$ such that $\rho(w)$ equals the translation by a vector $v \in V$ with $\alpha(v) \neq 0$. Replacing w by w^{-1} if necessary, we may assume that $\alpha(v) < 0$. Since W preserves $\mathfrak{H}$, it follows that

$$\rho(w)(H_{\alpha,0}) = v + H_{\alpha,0} = H_{\alpha,-\alpha(v)} \in \mathfrak{H}$$

and hence $-\alpha(v) \in \Lambda$. Let now $\delta > 0$ be the smallest positive element of Λ_α. This element exists because $\mathfrak{H}$ is locally finite. We claim that $\Lambda_\alpha = \mathbb{Z}\delta$. To see this, first note that given any two parallel hyperplanes H' and H'' the product $s_{H''}s_{H'}$ of the associated orthogonal reflections equals the translation by $2t$, where t is the unique vector orthogonal to H' with $H'' = t + H'$. Let now $H' = H_{\alpha,k}$, $H'' = H_{\alpha,\ell}$ with $k, \ell \in \Lambda_\alpha$ and let $n \in V$ be the unique vector orthogonal to H satisfying $\alpha(n) = 1$. Then $H_{\alpha,k} = -kn + H_\alpha$ and $H_{\alpha,\ell} = -\ell n + H_\alpha$. Since $W(\mathfrak{H})$ leaves $\mathfrak{H}$ invariant, it follows that

$$(s_{H''}s_{H'})(H') = 2(k-\ell)n + H' = H_{\alpha,2\ell-k}$$

must again be a member of $\mathfrak{H}$, i.e. $2\ell - k \in \Lambda_\alpha$. Taking $\ell = 0$ it follows that Λ_α is stable under inversion. Taking $\ell = \delta$ it follows that Λ_α is stable under translation by $\pm 2\delta$. Every element $k \in \Lambda_\alpha$ can therefore be written in the form $k = x + n\delta$ with $n \in \mathbb{Z}$ and $0 \leq x < 2\delta$. If $x \leq \delta$, it follows that $x = \delta$ by minimality of δ. If $\delta < x \leq 2\delta$, it follows by the above that $0 \leq 2\delta - x < \delta$ lies in Δ_α and hence $2\delta - x = \delta$ by minimality. In both cases it follows that $k \in \mathbb{Z}\delta$.

From the above discussion it is now clear that given $H \in \mathfrak{H}$ with $0 \in H$ there exists $\alpha \in V^\vee$ uniquely determined up to $\pm$ such that

$$\{H' \in \mathfrak{H} : H' \text{ parallel to } H\} = \{H_{\alpha,k} : k \in \mathbb{Z}\}$$

Then

$$\Phi = \{\alpha \in V^\vee : \forall k \in \mathbb{R} \quad H_{\alpha,k} \in \mathfrak{H} \Leftrightarrow k \in \mathbb{Z}\}$$

is just the set of these α. Obviously $(\mathbb{R}\Phi, \Phi)$ is reduced if it is a root system, so it suffices to verify the root system axioms (RSI)-(RSIII) (see [Bou07, Ch. VI, §1.1]). There is only a finite number of $H \in \mathfrak{H}$ with $0 \in H$ by the local finiteness of $\mathfrak{H}$ and hence it follows readily that Φ is finite. Moreover, $0 \notin \Phi$ by construction, and hence (RSI) is verified.

Now we prove (RSII). First, we remark that $\mathbb{R}\Phi$ equals the image of the dual of the projection $V \twoheadrightarrow V/L$. This is equivalent to the claim that $V \xrightarrow{\sim} V^{\vee\vee}$ induces an isomorphism $(V/L) \xrightarrow{\sim} (\mathbb{R}\Phi)^\vee$ and follows from

$$L = \bigcap_{H \in \mathfrak{H},\ 0 \in H} H = \bigcap_{\alpha \in \Phi} \ker(\alpha)$$

Given $\alpha \in \Phi$, the associated reflection $s_\alpha \in O(\mathbb{R}\Phi)$ is given by the restriction

$$s_\alpha = s_H^\vee|_{\mathbb{R}\Phi}$$

of the transpose of the orthogonal reflection $s_H \in O(V)$ with respect to $H = \ker(\alpha)$. This holds since both are elements of $O(\mathbb{R}\Phi)$ having as fix-point set the hyperplane

$$\alpha^\perp = \mathbb{R}\Phi \cap \{\omega \in V^\vee : \omega(v) = 0\}$$

where $v \in V$ is any vector $\neq 0$ orthogonal to H. Since s_H leaves $\mathfrak{H}$ invariant, it follows that s_α leaves Φ invariant; thus $s_{\alpha,\alpha^\vee} = s_\alpha$ for $\alpha^\vee := 2\frac{(\alpha,\cdot)}{(\alpha,\alpha)} \in (\mathbb{R}\Phi)^{\vee\vee}$ leaves Φ invariant, and (RSII) is verified. Lastly to prove (RSIII), let $\alpha, \beta \in \Phi$ be given. Identifying $(\mathbb{R}\Phi)^\vee$ with the subspace $L^\perp \leq V$, the dual root $\alpha^\vee$ is the unique element of V orthogonal to H_α satisfying $\alpha(\alpha^\vee) = 2$. In particular letting $H' = H_{\alpha,0}$ and $H'' = H_{\alpha,1} = -\frac{1}{2}\alpha^\vee + H'$ we have that

$$(s_{H''}s_{H'})(H_{\beta,0}) = -\alpha^\vee + H_{\beta,0} = H_{\beta,\beta(\alpha^\vee)} \in \mathfrak{H}$$

and hence $\beta(\alpha^\vee) \in \mathbb{Z}$ since $\beta \in \Phi$.

Next, we prove (vi) keeping the choice of an invariant scalar product on V. The injectivity of the map $W_0 \to \mathrm{GL}(V/L)$ follows from the fact W_0 is finite and hence acts by semi-simple transformations on V. Indeed since $L = V^{W_0}$, any $w \in W_0$ lying in the kernel of $W_0 \to \mathrm{GL}(V/L)$ acts trivially on L and V/L and hence must act trivially on V by semi-simplicity. In the proof of (iv) we have already seen that W_0 is generated by the s_H with $H \in \mathfrak{H}$ and $0 \in H$. By (v) we know that H is of the form $H = \ker(\alpha)$ with $\alpha \in \Phi$. Moreover, we have already seen that the image of s_H under $W_0 \to \mathrm{GL}(V/L) \simeq \mathrm{GL}((\mathbb{R}\Phi)^\vee)$ equals the transpose $s_\alpha^\vee$ of the reflection associated to α. This shows that the image of $W_0 \hookrightarrow \mathrm{GL}((\mathbb{R}\Phi)^\vee)$ is given by $W(\Phi)^\vee = W(\Phi^\vee)$. Moreover it's clear by the previous remarks that under $W_0 \xrightarrow{\sim} W(\Phi^\vee)$ the generating set $\{s_H : H \in S(C_0)\}$ corresponds to $\{s_\alpha^\vee : \alpha \in \Delta\}$. Now to see that Δ is a basis of the root system Φ, let $D_0 \in \pi_0(V - \bigcup_{\alpha \in \Phi} \ker(\alpha))$ be the unique chamber of the spherical arrangement containing C_0. The image $\pi(D_0)$ of D_0 under $\pi : V \twoheadrightarrow V/L \simeq (\mathbb{R}\Phi)^\vee$ then is a chamber of the linear arrangement on $(\mathbb{R}\Phi)^\vee$ induced by Φ. Moreover, for $\alpha \in \Phi$ the hyperplane $\ker(\alpha)$ is a wall of C_0 if and only if the hyperplane in $(\mathbb{R}\Phi)^\vee$ associated to α is a wall of $\pi(D_0)$, and α is positive on C_0 if and only if α is positive on $\pi(D_0)$. By the theory of root systems, it then follows that Δ is a basis of Φ, in fact Δ is the basis of Φ associated to the dual chamber $\pi(D_0)^\vee \subseteq \mathbb{R}\Phi$ (see [Bou07, Ch. VI, §1.5, Rémarque 5]). Moreover, it is obvious that Φ^+ consists of the roots which take positive values on $\pi(D_0)$. It hence follows (see [Bou07, Ch. VI, §1.6]) that Φ coincides with the set of positive roots associated to $\pi(D_0)^\vee$. Since we have $\Delta \subseteq \Phi^+$, it follows that Δ is the root basis associated to Φ^+.

Finally (vii) follows immediately from (iv)-(vi) and the fact that for $\alpha \in \Phi$ we have $\rho^{-1}(s_\alpha) = \widetilde{s}_{H_{\alpha,0}}$. □

2.1.3 Example. (i) Let $(X, \Phi, X^\vee, \Phi^\vee)$ be a root datum (in the sense of [DG70, Exposé XXI]) and $\Delta \subseteq \Phi^\vee$ a root basis. In particular, $\Phi \subseteq X$ and $\Phi^\vee \subseteq X^\vee$ are finite subsets that are in bijection via a *given* pair of inverse bijections

$$\Phi \overset{\sim}{\leftrightarrow} \Phi^\vee$$

both denoted by $\alpha \mapsto \alpha^\vee$, and $X, X^\vee$ are free abelian groups of finite rank in duality via a *given* pairing

$$\langle\cdot,\cdot\rangle : X^\vee \times X \longrightarrow \mathbb{Z}$$

Let $W_0 := W(\Phi)$ be the finite Weyl group, i.e. the subgroup of $\mathrm{GL}_{\mathbb{Z}}(X)$ generated by the reflections s_α, $\alpha \in \Phi$ given by

$$s_\alpha(x) = x - \langle\alpha^\vee, x\rangle\, \alpha$$

Let $W := X \rtimes W_0$ be the *extended affine Weyl group*. Let us now see that W carries a canonical structure of an affine extended Coxeter group in the sense of definition 2.1.1, and therefore also a canonical structure of an extended Coxeter group via lemma 2.1.2.

We let $V := X \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\rho : W \longrightarrow \mathrm{GL}_{\mathrm{aff}}(V)$ be the inclusion

$$W = X \rtimes W_0 \subseteq V \rtimes \mathrm{GL}(V) \simeq \mathrm{GL}_{\mathrm{aff}}(V)$$

This action leaves invariant the collection $\mathfrak{H}$ of hyperplanes given by $H_{\alpha,k}$, $\alpha \in \Phi^\vee$, $k \in \mathbb{Z}$ where

$$H_{\alpha,k} = \{x \in V : \langle\alpha, x\rangle + k = 0\}$$

Since ρ is injective, the choice of the $\widetilde{s}_H$ is unique in this case. Moreover, it is clear that for any choice of a chamber C_0 with $0 \in \overline{C_0}$, the axioms (**ACI**)-(**ACX**) are satisfied, in particular if we let C_0 be chamber corresponding to Δ determined by the conditions

$$0 \in \overline{C_0} \quad \text{and} \quad C_0 \subseteq \{x \in V : \langle\alpha, x\rangle > 0 \quad \forall \alpha \in \Delta\}$$

Moreover, the groups W_0 and X of definition 2.1.1 coincide with the groups denoted by the same letters here. The root system Φ and the basis Δ constructed in the lemma above coincide with $\Phi^\vee$ and Δ respectively. The structure $W = (W, W_{\mathrm{aff}}, S, \Omega)$ of an extended Coxeter group induced on W by the above lemma can be made more explicit as follows. Let $Q := \mathbb{Z}\Phi \leq X$ be the root lattice. Then elementary arguments (see [Bou07, Ch. VI, §1.2, Proposition 1]) show that the affine Weyl group $W_{\mathrm{aff}} \leq W$ is the semi-direct product $W_{\mathrm{aff}} = Q \rtimes W_0$. Hence, there is an isomorphism

$$\Omega \simeq X/Q$$

By definition, the generating set S of W_{aff} consists of the reflections s_H for all walls H of C_0. Using the theory of root systems it can be seen that the walls of C_0 are either of the form $H = H_{\alpha,0}$ with $\alpha \in \Delta$ or $H = H_{-\alpha,1}$ with α a *highest coroot*, i.e. a maximal element of $\Phi^\vee$ with respect to the partial order

$$\alpha \leq \beta \quad \Leftrightarrow \quad \langle\alpha, x\rangle \leq \langle\beta, x\rangle \quad \forall x \in C_0$$

Hence

$$S = \{s_\alpha : \alpha \in \Delta\} \cup \{s_{-\alpha,1} : \alpha \in \Phi^\vee \text{ maximal}\}$$

where (by slight abuse of notation) s_α and $s_{\alpha,k}$ for $\alpha \in \Phi^\vee$, $k \in \mathbb{Z}$ denote the elements of W_{aff} given by

$$s_\alpha(x) = x - \langle\alpha, x\rangle\, \alpha^\vee \quad \text{and} \quad s_{\alpha,k}(x) = x - (\langle\alpha, x\rangle + k)\alpha^\vee$$

(ii) We specialize the above example now to the root datum of the group GL_n. In this case we have

$$X = X^\vee = \mathbb{Z}^n$$

with the pairing between X and $X^\vee$ being the canonical one. Moreover

$$\Phi = \Phi^\vee = \{e_i - e_j : 1 \leq i, j \leq n,\ i \neq j\}$$

and the correspondence $\alpha \leftrightarrow \alpha^\vee$ between roots and coroots is the identity. The finite Weyl group W_0 identifies with the symmetric group S_n on n letters. The choice of the (co-)root basis

$$\Delta = \{e_2 - e_1, \dots, e_n - e_{n-1}\}$$

makes $W_0 = S_n$ into a Coxeter group with generators $s_1, \dots, s_{n-1}$, where

$$s_i = s_{e_{i+1}-e_i} = (i\ i+1)$$

is the transposition permuting the i-th and $i+1$-th coordinate. The chamber determined by Δ is given by

$$C_0 = \{x \in \mathbb{R}^n : x_1 < \ldots < x_n < x_1 + 1\}$$

The root sublattice $Q = \mathbb{Z}\Phi \leq \mathbb{Z}^n$ is the kernel of the 'augmentation map'

$$\mathbb{Z}^n \longrightarrow \mathbb{Z}, \quad e_i \mapsto 1$$

hence the group $\Omega \simeq X/Q$ (which as a subgroup of W depends on the choice of C_0!) is canonically isomorphic to $\mathbb{Z}$, with canonical generator u given by

$$u = \tau^{e_n}(n\ n-1\ \ldots\ 1)$$

Here, in order to avoid confusion arising from mixing the additive group notation on X and the multiplicative group notation on $W = X \rtimes W_0$, we use the exponential expression τ^x instead of x when we want to view an element $x \in X$ as an element of the group W. Thus, $\tau^x\tau^y = \tau^{x+y}$ for all $x, y \in X$ in this notation.

The highest (co-)root is unique and given by $\alpha = e_n - e_1$. Hence, the generating set S of W_{aff} is given by

$$S = \{s_1, \dots, s_{n-1}, s_{-\alpha,1}\}$$

with

$$s_{-\alpha,1} = \tau^{e_n - e_1}(1\ n)$$

Writing $s_0 = s_{-\alpha,1}$ and viewing $\{0, 1, \dots, n-1\}$ as the group $\mathbb{Z}/n\mathbb{Z}$, the action of Ω on S is determined by

$$us_iu^{-1} = s_{i-1} \tag{2.1.1}$$

We are now in the position to define the principal object of study of this article, the class of *affine* generic pro-p Hecke algebras (or simply *affine pro-p Hecke algebras*) as those algebras whose underlying extended Coxeter group W arises as in the above lemma from an affine extended Coxeter group. Since the description of the structure of these algebras will depend on the decomposition $W = X \rtimes W_0$, it makes sense to make the affine extended Coxeter group part of the datum.

2.1.4 Definition. An **affine pro-p Hecke algebra** $\mathcal{H}^{(1)}$ over a ring R consists of a generic pro-p Hecke algebra $\mathcal{H}^{(1)}$ over R and an affine extended Coxeter group W such that the extended Coxeter group underlying the pro-p Coxeter group $W^{(1)}$ associated with $\mathcal{H}^{(1)}$ coincides with the extended Coxeter group associated to W by lemma 2.1.2.

2.1.5 Terminology. Following tradition and to prevent confusion with the chambers $C \in \pi_0(V - \bigcup_{H\in\mathfrak{H}} H)$, the connected components of the complement of the finite linear hyperplane arrangement $\{H \in \mathfrak{H} : 0 \in H\}$ will be called *Weyl chambers*. They will usually denoted by the letter 'D', while 'C' will be used to denote the chambers of the affine hyperplane arrangement $\mathfrak{H}$.

The main goal of this article will be to describe the center of affine pro-p Hecke algebras using the Bernstein maps introduced in the previous section. As in the classical work of Bernstein and Lusztig, this involves constructing big (almost) commutative subalgebras of $\mathcal{H}^{(1)}$. In view of remark 1.10.8, this amounts to constructing orientations with big stabilizers, which we will do later in section 2.4.

2.2 Main examples of affine pro-p Hecke algebras

In this section we want to consider the main examples of affine pro-p Hecke algebras, the classical *affine Hecke algebras* and two 'new' examples, the *pro-p-Iwahori Hecke* algebras and the *affine Yokonuma-Hecke* algebras. We point out that the last two examples slightly overlap.

2.2.1 Affine Hecke algebras and Iwahori-Hecke algebras

The *affine Hecke algebras* are (cf. [Mac03, 4.1]) the generic pro-p Hecke algebras for the *extended affine Weyl groups*, i.e. for pro-p Coxeter groups $W^{(1)}$ of the form $W^{(1)} = W$, $T = 1$, $W = X \rtimes W_0$ for a root datum $(X, \Phi, X^\vee, \Phi^\vee)$ with chosen basis $\Delta \subseteq \Phi^\vee$ as in example 2.1.3. As was explained there, the group W carries a canonical structure of an affine extended Coxeter group in the sense of definition 2.1.1, hence these algebras are affine pro-p Hecke algebras in the sense of definition 2.1.4.

Affine Hecke algebras play an important role in various different but related subjects, including the representation theory of reductive groups over local fields, the theory of orthogonal polynomials [Mac03], the theory of knot invariants, and in physics in the study of certain exactly solvable systems (see [Mar91]). Historically, affine Hecke algebras made their debut in the first of the subjects mentioned, namely in the 1965 paper of Iwahori and Matsumoto [IM65] that elucidated the structure of double coset algebras $H_R(G, I)$ (cf. section 2.2.3) attached to pairs (G, I), where $G = \mathbf{G}(F)$ is the group of rational points of a split, connected, semisimple reductive group (Chevalley group) $\mathbf{G}$ over a nonarchimedean local field F, and $I \leq G$ is a certain open compact subgroup nowadays referred to as 'Iwahori subgroup'.

One of the main results (Propositions 3.5, 3.7 and 3.8) of [IM65] was the description of a presentation of $H_R(G, I)$ in terms of the extended affine Weyl group $W = X \rtimes W_0$ of the root datum corresponding to $\mathbf{G}$, i.e. an isomorphism of $H_R(G, I)$ with an affine Hecke algebra. More precisely, they showed that $H_R(G, I)$ is isomorphic to the R-algebra generated by symbols T_w, $w \in W$ subject to the relations

$$T_w T_{w'} = T_{ww'} \qquad \text{if } \ell(w) + \ell(w') = \ell(ww') \tag{1}$$

$$T_s^2 = q + (q-1)T_s \qquad s \in S \tag{2}$$

where q denotes the cardinality of the residue field of F. Hence, $H_R(G, I)$ identifies with the generic pro-p Hecke algebra $\mathcal{H}^{(1)}(a_s, b_s)$ for $W^{(1)} = W$ and constant parameters $a_s = q$, $b_s = q - 1$ by proposition 1.4.2, which is an affine Hecke algebra.

The algebras of the form $H_R(G, I)$ are commonly referred to as *Iwahori-Hecke algebras*. Sometimes the terms 'affine Hecke algebra' and 'Iwahori-Hecke algebra' are used synonymously, but here we will distinguish between the two. The notion of Iwahori subgroup is defined in great generality for any connected reductive group $\mathbf{G}$ over a local field [Tit79, 3.7], and one can consider the corresponding algebras $H_R(G, I)$. These more general Iwahori-Hecke algebras have a similar presentation in terms of a certain group $W = X \rtimes W_0$ which admits the structure of an affine extended Coxeter group but where the constant coefficients q and $q - 1$ are replaced by coefficients q_s and $q_s - 1$ that can depend on s (cf. lemma 2.2.4).

This was first proved by Vignéras [Vig16, Proposition 4.1, 4.4], although it has long been a part of mathematical folklore that 'Iwahori-Hecke algebras for non-split groups are affine Hecke algebras for unequal parameters'. The latter is in fact not true. The algebra $H_R(G, I)$ is isomorphic to the generic pro-p Hecke algebra $\mathcal{H}^{(1)}(q_s, q_s - 1)$ associated to the affine extended Coxeter group W and hence is an affine pro-p Hecke algebra, but it is not always an affine Hecke algebra (in our sense) as the group $W = X \rtimes W_0$ does not necessarily arise from a root datum. In fact, X is a finitely generated abelian group with nontrivial torsion part in general. However, when the group $\mathbf{G}$ is split, this subtlety disappears and the corresponding Iwahori-Hecke algebras are affine Hecke algebras with constant coefficients $a_s = q$, $b_s = q - 1$ for the extended affine Weyl group corresponding to the root datum of $\mathbf{G}$.

The most important structural results concerning affine Hecke algebras in general are the 'Bernstein relations', the 'Bernstein presentation' and the computation of the center in terms of invariants of certain commutative subalgebras. These results were obtained by Bernstein and Zelevinsky in an unpublished work for the special case of constant parameters $a_s = q, b_s = q - 1$. Lusztig later published a generalized version of these results in [Lus89], where he took the parameters to be of the form $a_s = q_s$, $b_s = q_s - 1$ with $q_s = v^{2n_s}$ for some integers n_s and an invertible formal variable $v \in R = \mathbb{C}[v, v^{-1}]$. Lusztig obtained these results using a group homomorphism θ from the group X of translations into the group of units of the affine Hecke algebra. We will see below (in (2.2.1)) that this map coincides with the restriction of our map $\widetilde{\theta}_{\mathfrak{o}}$ (see definition 1.10.9) to $X \leq W$, where $\mathfrak{o} = \mathfrak{o}_D$ denotes the spherical orientation (see definition 2.4.1) corresponding to the dominant Weyl chamber D. These results of Bernstein, Zelevinsky and Lusztig were further generalized by Vignéras in [Vig06] to allow for parameters of the form $a_s = q_s$, $b_s = q_s - 1$ with q_s not necessarily invertible or admitting a square root.

The Bernstein relations and the description of the center in all of the above cases are recovered here in theorems 1.11.3 and 2.7.1. Note that the results of theorem 2.7.1 hold unconditionally in these cases since $T = 1$ (cf. remark 2.7.3). For the readers convenience we will quote the construction of the Bernstein-Zelevinsky subalgebra and the description of the center from theorem 2.7.1 for our special case. For every spherical orientation $\mathfrak{o} = \mathfrak{o}_D$ of W, associated to a Weyl chamber D (see definition 2.4.1), the integral Bernstein map $\widehat{\theta}_{\mathfrak{o}} : W \to \mathcal{H}^{(1)}$ (see definition 1.10.2) gives rise to a commutative subalgebra

$$\mathcal{A}_{\mathfrak{o}} := \bigoplus_{x \in X} R\widehat{\theta}_{\mathfrak{o}}(x) \subseteq \mathcal{H}^{(1)}$$

whose multiplicative structure is determined by the product rule (corollary 1.10.5)

$$\widehat{\theta}_{\mathfrak{o}}(x)\widehat{\theta}_{\mathfrak{o}}(y) = \overline{\mathbb{X}}(x, y)\widehat{\theta}_{\mathfrak{o}}(x + y)$$

If the parameters $a_s \in R$ are units and squares, then we can also consider the normalized Bernstein map $\widetilde{\theta}_{\mathfrak{o}} : W \to \mathcal{H}^{(1)}$ whose restriction to X is determined by the fact that it is multiplicative and satisfies the following relation (see definition 1.10.9 for details)

$$\widetilde{\theta}_{\mathfrak{o}}(x) = \overline{\sqrt{\mathbb{L}}}(x)^{-1} T_x \quad \forall x \in X \cap \overline{D} \tag{2.2.1}$$

These properties together imply that $\widetilde{\theta}_{\mathfrak{o}}\Big|_X$ coincides with the map denoted by θ by Lusztig [Lus89], which appears in the classical Bernstein-Lusztig basis $\{\theta_x T_w\}_{x \in X, w \in W_0}$. Moreover, $\widetilde{\theta}_{\mathfrak{o}}$ is related to the integral Bernstein map via $\widetilde{\theta}_{\mathfrak{o}}(w) = \overline{\sqrt{\mathbb{L}}}(L)^{-1}(w)\widehat{\theta}_{\mathfrak{o}}(w)$ (see theorem 1.10.1). It follows from this that $\mathcal{A}_{\mathfrak{o}}$ can also be expressed as

$$\mathcal{A}_{\mathfrak{o}} = \bigoplus_{x \in X} R\widetilde{\theta}_{\mathfrak{o}}(x)$$

and that $\widetilde{\theta}_{\mathfrak{o}}$ induces an isomorphism of the group algebra $R[X]$ with $\mathcal{A}_{\mathfrak{o}}$. In any case, the group W_0 acts on $\mathcal{A}_{\mathfrak{o}}$ by permuting the basis elements $w(\widehat{\theta}_{\mathfrak{o}}(x)) = \widehat{\theta}_{\mathfrak{o}}(w(x))$ and the center of $\mathcal{H}^{(1)}$ is given by the W_0-invariants

$$Z(\mathcal{H}^{(1)}) = \mathcal{A}_{\mathfrak{o}}^{W_0} = \bigoplus_{\gamma \in W_0 \backslash X} Rz_\gamma$$

with

$$z_\gamma = \sum_{x \in \gamma} \widehat{\theta}_{\mathfrak{o}}(x)$$

independent of the orientation (Weyl chamber) chosen.

2.2.2 The affine Hecke algebra of GL_n

We now specialize our discussion to the case where the root datum defining W is the root datum of GL_n. The affine Hecke algebra of W with parameters a_s, b_s will be denoted by $H_n^{\mathrm{aff}}(a_s, b_s)$ or simply by H_n^{aff}. The generalized braid groups $\mathcal{A}(W)$, $\mathcal{A}(W_0)$ (see definition 1.4.1) associated to $W = X \rtimes W_0$ and W_0 will be denoted by $\mathfrak{A}_n^{\mathrm{aff}}$ and $\mathfrak{A}_n$ respectively. From example 2.1.3(ii) we recall that

$$W = \mathbb{Z}^n \rtimes S_n = W_{\mathrm{aff}} \rtimes \Omega, \quad W_{\mathrm{aff}} = \langle S \rangle, \quad S = \{s_0, \dots, s_n\}, \quad \Omega = \langle u \rangle$$

where

$$s_0 = \tau^{e_n - e_1}(1\ n), \quad s_i = (i\ i+1) \text{ for } i > 0 \quad \text{and} \quad u = \tau^{e_n}(n\ n-1\ \dots 1)$$

with u acting on S as

$$us_iu^{-1} = s_{i-1}$$

In particular, all the generators $s \in S$ are conjugate under W and condition (1.3.1) on the parameters a_s, b_s is equivalent to

$$a_s = a_t, \ b_s = b_t \quad \forall s, t \in S$$

Hence, we can write $H_n^{\mathrm{aff}}(a, b) = H_n^{\mathrm{aff}}(a_s, b_s)$ with parameters $a, b \in R$ subject to no further constraint. The 'affine braid group' $\mathfrak{A}_n^{\mathrm{aff}}$ can be interpreted ([Dũn83],[Lek83]) topologically as a group of braids as follows. Consider the real affine hyperplane arrangement

$$\mathfrak{H} = \{H_\alpha : \alpha \in \Phi_{\mathrm{af}}\}, \quad H_\alpha = \{x \in A : \alpha(x) = 0\}, \quad \Phi_{\mathrm{af}} = \{\alpha + k : \alpha \in \Phi, \ k \in \mathbb{Z}\} \subset \mathrm{Hom}(A, \mathbb{R})$$

in $A = \mathbb{R}^n$ induced by the root datum

$$(X, \Phi, X^\vee, \Phi^\vee) = (\mathbb{Z}^n, \{e_i - e_j : i \neq j\}, \mathbb{Z}^n, \{e_i - e_j : i \neq j\})$$

of GL_n (cf. example 2.1.3). The complement $X := A - \bigcup_{H \in \mathfrak{H}} H$ is disconnected, the connected components being in bijection with the infinite group W_{aff}, but the complement $Y := A_{\mathbb{C}} - \bigcup_{H \in \mathfrak{H}} H_{\mathbb{C}}$ of the complexified arrangement is connected. The fundamental groupoid $\pi_1(Y)$ of Y can be described as follows (see [Dũn83],[Lek83]). For any two points $x, y \in X \subseteq Y$ let $\mathcal{P}_{y,x}$ be the subspace of the space of all paths $\gamma : [0,1] \to Y$ consisting of those γ which satisfy

(i) $\gamma(0) = x, \gamma(1) = y$

(ii) $\forall t \in [0,1],\ \alpha \in \Phi_{\mathrm{af}} \quad \Re(\alpha_{\mathbb{C}}(\gamma(t))) = 0 \quad \Rightarrow \quad \alpha(x)\alpha(y) < 0$

(iii) $\forall t \in [0,1],\ \alpha \in \Phi_{\mathrm{af}} \quad \alpha(x)\alpha(y) < 0 \quad \Rightarrow \quad \Im(\alpha_{\mathbb{C}}(\gamma(t))) \cdot (\alpha(x) - \alpha(y)) \geq 0$

In words the second condition says that the real part of γ should only cross these hyperplanes $H \in \mathfrak{H}$ which separate x and y, while the third condition means that for every hyperplane $H_\alpha \in \mathfrak{H}$, $\alpha \in \Phi_{\mathrm{af}}$ separating x and y, the path $\alpha_{\mathbb{C}} \circ \gamma : [0,1] \to \mathbb{C}$ should wind around the origin counter-clockwise and should stay completely in either the upper or lower half-plane. It is easy to see that $\mathcal{P}_{y,x}$ is contractible, hence giving rise to a well-defined homotopy-class

$$\gamma_{y,x} \in \mathrm{Hom}_{\pi_1(Y)}(x, y)$$

It is even easier to see that

$$w(\mathcal{P}_{y,x}) = \mathcal{P}_{w(y),w(x)}$$

and hence

$$w(\gamma_{y,x}) = \gamma_{w(y),w(x)}$$

for $w \in W$. Moreover, for any three points $x, y, z \in X$ it holds true that

$$\mathcal{P}_{z,y} \circ \mathcal{P}_{y,x} \subseteq \mathcal{P}_{z,x}$$

and therefore that

$$\gamma_{z,y} \circ \gamma_{y,x} = \gamma_{z,x}$$

if the set of hyperplanes separating x and y is disjoint from the set of hyperplanes separating y and z, i.e. if

$$d(C_x, C_y) + d(C_y, C_Z) = d(C_x, C_z)$$

where C_p denotes the connected component of X (chamber) containing p and $d(C, C')$ denotes the distance between two chambers. One can now show ([Dũn83],[Lek83]) that the full subgroupoid of $\pi_1(Y)$ corresponding to $X \subseteq Y$ is described algebraically as the free groupoid on symbols $\gamma_{y,x}$ subject to the relation

$$d(C_x, C_y) + d(C_y, C_z) = d(C_x, C_z) \quad \Rightarrow \quad \gamma_{z,y} \circ \gamma_{y,x} = \gamma_{z,x}$$

From this one deduces a description of the fundamental group of the quotient space $W \backslash Y$, where the action of W is naturally extended to $A_{\mathbb{C}}$. Indeed, W acts properly discontinuously and without fix points on Y, therefore $Y \to W \backslash Y$ is a covering map with Galois group W. Fixing a base point $x_0 \in X$ and letting

$$T_w := p_*(\gamma^{-1}_{x_0, w(x_0)}) \in \pi_1(W \backslash Y, p(x_0)), \quad w \in W$$

it follows easily from the above that

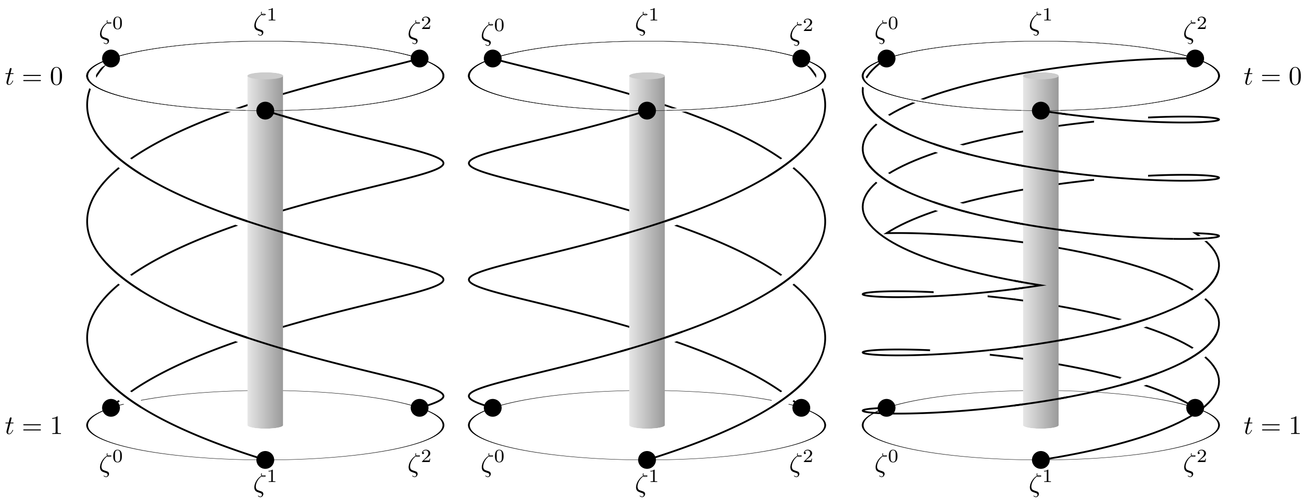
$$T_w T_{w'} = T_{ww'} \quad \text{if} \quad \ell(w) + \ell(w') = \ell(ww')$$

and that the elements T_w together with the above relation define a presentation of $\pi_1(W \backslash Y, p(x_0))$ and hence an isomorphism of this group with $\mathfrak{A}_n^{\mathrm{aff}}$. The interpretation of $\pi_1(W \backslash Y, p(x_0))$ as a group of 'affine braids' arises by viewing $W \backslash Y$ as the iterated quotient

$$W \backslash Y \simeq S_n \backslash (\mathbb{Z}^n \backslash Y) \simeq S_n \backslash ((\mathbb{C}^\times)^n - \Delta)$$

where $\Delta = \bigcup_{i \neq j} \{z_i = z_j\}$ is the diagonal and $\mathbb{Z}^n \backslash Y$ is identified with $(\mathbb{C}^\times)^n - \Delta$ via $z \mapsto \exp(2\pi i z)$. A loop γ in $S_n \backslash ((\mathbb{C}^\times)^n - \Delta)$ around $p(x_0)$ can be identified with the braid

$$\bigcup_{i=1}^{n} \{(t, \widehat{\gamma}(t)_i) : t \in [0,1]\} \subseteq [0,1] \times \mathbb{C}^\times$$



(a) The braid T given by the path $\widehat{\gamma}(t) = (e^{2\pi it}, \zeta e^{2\pi it}, \zeta^2 e^{2\pi it})$. (b) The inverse T^{-1} of the braid T. (c) The 'composite' TT^{-1}.

Figure 3: Loops in $S_n \backslash ((\mathbb{C}^\times)^n - \Delta)$ based at $\big[(1, \zeta, \zeta^2, \dots, \zeta^{n-1})\big]$ ($\zeta := \exp \frac{2\pi i}{n}$) can be identified with braids in $[0,1] \times \mathbb{C}^\times$ with endpoints $\{(0, \zeta^i) : i = 0, \dots, n-1\}$ and $\{(1, \zeta^i) : i = 0, \dots, n-1\}$, by lifting a loop γ to a path $\widehat{\gamma}$ in $(\mathbb{C}^\times)^n - \Delta$ and associating to it the braid $\bigcup_{i=1}^n \{(t, \widehat{\gamma}(t)_i) : t \in [0,1]\}$.

where $\widehat{\gamma}$ denotes any lift of γ to a path in $(\mathbb{C}^\times)^n - \Delta$. Under this bijection, composition of paths corresponds to 'stacking' of braids (rescaling the t-coordinate by $\frac{1}{2}$), and the inverse of a braid is given by its reflection along the $t = \frac{1}{2}$-plane. This is illustrated in figure 3 (for the base point $x_0 = \big(0, \frac{1}{n}, \dots, \frac{n-1}{n}\big)$ and $n = 3$), where the 'missing' central line $[0,1] \times \{0\} \subseteq [0,1] \times \mathbb{C}$ has been enlarged to a flagpole for better visibility. Figure 4 depicts the braids corresponding to some representatives of the generators $T_i = T_{s_i}$, $X_1 = T_{-e_1}^{-1}$ of the group $\mathfrak{A}_3^{\mathrm{aff}}$ appearing in lemma 2.2.1 below.

The classical Artin braid group $\mathfrak{A}_n$ can be interpreted similarly either as the fundamental group of $S_n \backslash (\mathbb{C}^n - \Delta)$ or as a group braids (without a flagpole). From the topological picture it is therefore clear that there should be a canonical map

$$\mathfrak{A}_n^{\mathrm{aff}} \longrightarrow \mathfrak{A}_n$$

induced by the inclusion $S_n \backslash ((\mathbb{C}^\times)^n - \Delta) \subseteq S_n \backslash (\mathbb{C}^n - \Delta)$, corresponding to 'removing the flagpole' on the level of braids. However, this map is *not* simply given by $T_w \mapsto T_{p(w)}$, where p denotes the canonical projection $W = X \rtimes W_0 \to W_0$. To describe it we need another presentation of the group $\mathfrak{A}(W)$.

2.2.1 Lemma. *Let $\widetilde{\mathfrak{A}}_n^{\mathrm{aff}}$ be the group generated by elements*

$$T_1, \dots T_{n-1}, X_1$$

subject to the relations

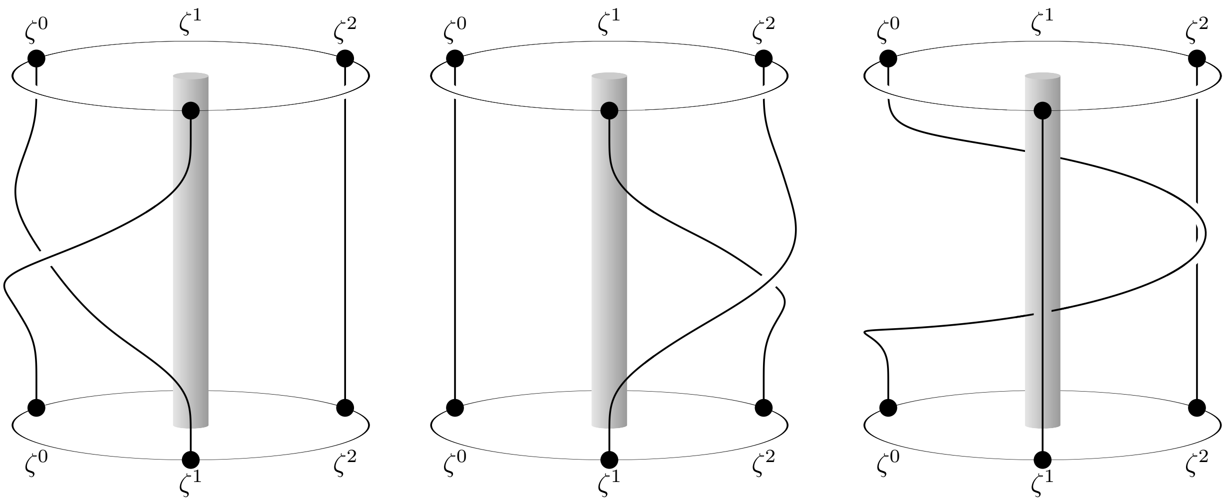
$$
\begin{array}{lrll}
(1) & T_i T_j &= T_j T_i & \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 1 \\
(2) & T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & \text{for all } i = 1, \dots, n-2 \\
(3) & X_1 T_1 X_1 T_1 &= T_1 X_1 T_1 X_1 & \\
(4) & X_1 T_i &= T_i X_1 & \text{for all } i = 2, \dots, n-1
\end{array}
$$

Then there are inverse isomorphisms $\Phi : \widetilde{\mathfrak{A}}_n^{\mathrm{aff}} \to \mathfrak{A}_n^{\mathrm{aff}}$, $\Psi : \mathfrak{A}_n^{\mathrm{aff}} \to \widetilde{\mathfrak{A}}_n^{\mathrm{aff}}$ of groups determined by

$$
\begin{aligned}
\Phi(T_i) &= T_{s_i} \quad i = 1, \dots, n-1 \\
\Phi(X_1) &= T_{-e_1}^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi(T_{s_i}) &= T_i \quad i = 1, \dots, n-1 \\
\Psi(T_{s_0}) &= \Psi(T_u) T_1 \Psi(T_u)^{-1} \\
\Psi(T_u) &= T_{n-1} \dots T_1 X_1
\end{aligned}
$$



(a) $T_1 = T_{s_1}$ (b) $T_2 = T_{s_2}$ (c) $X_1 = T_{-e_1}^{-1}$

Figure 4: The generators of $\widetilde{\mathfrak{A}}_3^{\mathrm{aff}}$ viewed as braids in $[0,1] \times \mathbb{C}^\times$.

Proof. We only give some brief indications as the proof consists mostly of straightforward computations. First of all, for every extended Coxeter group W the decomposition $W = W_{\mathrm{aff}} \ltimes \Omega$ induces an isomorphism

$$\mathfrak{A}(W_{\mathrm{aff}} \ltimes \Omega) \simeq \mathfrak{A}(W_{\mathrm{aff}}) \ltimes \Omega$$

where the action of Ω on $\mathfrak{A}(W_{\mathrm{aff}})$ is determined by $u(T_w) = T_{u(w)}$. Moreover, one sees easily that there is an isomorphism

$$\mathfrak{A}(W_{\mathrm{aff}}) \simeq \left\langle \{T_s\}_{s\in S} : \underbrace{T_sT_tT_s\ldots}_{m \text{ factors}} = \underbrace{T_tT_sT_t\ldots}_{m \text{ factors}}, \quad s,t \in S,\ \mathrm{ord}(st) = m < \infty \right\rangle$$

To see that Ψ is well-defined it is therefore enough to check that

$$\begin{aligned} \Psi(T_{s_i})\Psi(T_{s_j}) &= \Psi(T_{s_j})\Psi(T_{s_i}), \quad i,j = 1,\ldots,n-1,\ |i-j| > 1 \\ \Psi(T_{s_i})\Psi(T_{s_{i+1}})\Phi(T_{s_i}) &= \Psi(T_{s_{i+1}})\Psi(T_{s_i})\Psi(T_{s_{i+1}}), \quad i = 1,\ldots,n-2 \\ \Psi(T_{u^{-1}})\Psi(T_{s_i})\Psi(T_{u^{-1}})^{-1} &= \Psi(T_{s_{i-1}}), \quad i \in \{0,\ldots,n-1\} = \mathbb{Z}/n\mathbb{Z} \end{aligned}$$

The first two relations are immediate and the last one follows from a lengthy computation. By definition, the well-definedness of Φ amounts to checking relations (1)-(4). Again relations (1) and (2) are immediate, while (3) and (4) follow from (1), (2) and

$$T_{u^{-1}}T_{s_i}T_{u^{-1}}^{-1} = T_{s_{i-1}}$$

Finally, more straightforward and lengthy computations show that Φ and Ψ are inverse to each other. $\square$

In terms of this description, the map $\mathfrak{A}_n^{\mathrm{aff}} \longrightarrow \mathfrak{A}_n$ is then given by

$$\widetilde{\mathfrak{A}}_n^{\mathrm{aff}} \longrightarrow \mathfrak{A}_n, \quad T_i \longmapsto T_{s_i},\ X_1 \mapsto 1$$

Writing $H_n = H_n(a,b)$ for the generic pro-p Hecke algebra of W_0 with constant parameters a, b, the above morphism of groups induces a morphism of algebras

$$\pi : H_n^{\mathrm{aff}} \longrightarrow H_n$$

by proposition 1.4.3 (one easily checks that the quadratic relations are preserved). Explicitly, this map is the identity on H_n (viewing it as a subalgebra of H_n^{aff}) and sends the generator T_{s_0} to the elements

$$T_{s_{n-1}}\ldots T_{s_1}T_{s_1}T_{s_{n-1}}^{-1}\ldots T_{s_1}^{-1}$$

The map π is very important because it gives a description of the center of H_n in terms of the center of H_n^{aff}. Namely, it turns out that π maps the center of H_n^{aff} surjectively onto the center of H_n. This should be contrasted with the fact that

$$Z(H_n^{\mathrm{aff}}) \cap H_0 = R$$

More explicitly, the center of H_n is the algebra of symmetric polynomials in the pairwise commutative *Jucys-Murphy elements* $J_1, \ldots, J_n$ given recursively by

$$J_1 := 1, \quad J_{i+1} = T_{s_i} J_i T_{s_i}$$

The following lifts of the J_i under π are also called Jucys-Murphy elements

$$J_1^{\mathrm{aff}} := X_1, \quad J_{i+1}^{\mathrm{aff}} = T_{s_i} J_i^{\mathrm{aff}} T_{s_i}$$

The elements $J_1^{\mathrm{aff}}, \ldots, J_n^{\mathrm{aff}}$ also commute pairwise. In fact, they are nothing else but the images of the standard basis vectors $e_i \in \mathbb{Z}^n$ under the unnormalized Bernstein map.

2.2.2 Lemma. *Let $\mathfrak{o} = \mathfrak{o}_D$ be the spherical orientation (see definition 2.4.1) of W associated to the dominant Weyl chamber*

$$D = \{x \in \mathbb{R}^n : x_1 < \ldots < x_n\}$$

Then

$$J_i^{\mathrm{aff}} = \theta_{\mathfrak{o}}(e_i), \quad i = 1, \ldots, n$$

as an equality in H_n^{aff} (in fact already in $\mathfrak{A}_n^{\mathrm{aff}}$).

Proof. By induction. For $i = 1$ the statement follows immediately from the definitions. Indeed, $-e_1 \in \overline{D}$ and therefore

$$\theta_{\mathfrak{o}}(e_1) = \theta_{\mathfrak{o}}(-e_1)^{-1} = T_{-e_1}^{-1} = X_1 = J_1^{\mathrm{aff}}$$

For the induction step we need to prove that

$$T_{s_i} \theta_{\mathfrak{o}}(e_i) T_{s_i} = \theta_{\mathfrak{o}}(e_{i+1})$$

But this is shown in [Mac03, 3.2.4], where the notation Y^x is used instead of $\theta_{\mathfrak{o}}(x)$. □

2.2.3 Pro-p-Iwahori Hecke algebras

Let F be a nonarchimedean local field, i.e. a field endowed with a nontrivial discrete valuation $\nu_F : F \to \mathbb{Z} \cup \{+\infty\}$ whose residue field k is a finite field with cardinality q a power of some prime p. Let $\mathbf{G}$ be a connected reductive group over F, $G = \mathbf{G}(F)$ the group of rational points, $I \leq G$ an Iwahori subgroup in the sense of [Tit79, 3.7] and $I(1) \leq I$ its pro-p radical. Recall that the pro-p radical of a profinite group containing an open pro-p subgroup is by definition (see [HV15, 3.6]) its largest open normal pro-p subgroup. Finally, let R be a commutative ring.

To this data one associates an R-algebra, the pro-p-Iwahori Hecke algebra, as follows. Let

$$\mathcal{H}^{(1)} := H_R(G, I(1)) = \mathrm{End}_G(\mathrm{ind}_{I(1)}^G \mathbb{1}_R)$$

be the ring of endomorphisms of the G-representation induced from the trivial representation of $I(1)$. This R-algebra is canonically identified with the convolution algebra $R[I(1)\backslash G/I(1)]$ of $I(1)$-double cosets, where the product of the basis elements $T_t, T_{t'}$ corresponding to double cosets $t, t' \in I(1)\backslash G/I(1)$ is given by

$$T_t T_{t'} = \sum_{t''} m(t, t'; t'') T_{t''}$$

Here the sum runs over all double cosets and $m(t, t'; t'')$ denotes the number of $I(1)$-left cosets of $t \cap g t'^{-1}$ for $g \in t''$ arbitrary.

Vignéras [Vig16] has shown that the set $I(1)\backslash G/I(1)$ is in bijection with a certain group $W^{(1)}$ (which can be given the structure of a pro-p Coxeter group) and that the basis elements T_w of $\mathcal{H}^{(1)}$ satisfy relations of Iwahori-Matsumoto type

$$T_w T_{w'} = T_{ww'}, \quad \text{if } \ell(w) + \ell(w') = \ell(ww')$$
$$T_s^2 = a_s T_{s^2} + b_s T_s, \quad \text{if } \ell(s) = 1$$

Given a suitable structure of a pro-p Coxeter group on $W^{(1)}$ and an affine extended Coxeter group on W, the above presentation implies that $\mathcal{H}^{(1)}$ is an affine pro-p Hecke algebra in the sense of definition 2.1.4. Our goal now is to explicitly construct these structures.

Let C be the chamber of the building of $\mathbf{G}$ which corresponds to the Iwahori subgroup $I = I_C$ and let $\mathbf{S} \leq \mathbf{G}$ denote the maximal split torus corresponding to an apartment containing C. Let $\mathbf{Z} \leq \mathbf{N} \leq \mathbf{G}$ respectively denote the centralizer and normalizer of $\mathbf{S}$ in $\mathbf{G}$, and let $Z = \mathbf{Z}(F)$ and $N = \mathbf{N}(F)$ denote their groups of rational points. Let

$$Z_0 := Z \cap I, \quad Z_0(1) := Z \cap I(1), \quad Z_k := Z_0/Z_0(1)$$

The groups Z_0 and $Z_0(1)$ are normal in N (cf. [Vig16, 3.7]) and thus we may form the quotient groups

$$W := N/Z_0, \quad W^{(1)} := N/Z_0(1)$$

The inclusion $N \subseteq G$ induces a bijection ([Vig16, Proposition 3.35])

$$W^{(1)} \xrightarrow{\sim} I(1)\backslash G/I(1), \quad [n] \longmapsto I(1)nI(1)$$

and therefore $\mathcal{H}^{(1)}$ has a canonical basis T_w indexed by elements $w \in W^{(1)}$. Moreover, we have an exact sequence

$$1 \longrightarrow Z_k \longrightarrow W^{(1)} \longrightarrow W \longrightarrow 1 \tag{2.2.2}$$

with Z_k finite abelian (in fact Z_k identifies with the rational points of a torus over k, cf. [Vig16, 3.7]).

Let us now see how W can be given the structure of an affine extended Coxeter group. The theory of buildings associates to the triple $(\mathbf{G}, \mathbf{S}, F)$ an apartment $A = A(\mathbf{G}, \mathbf{S}, F)$ (see [Tit79, 1.2]), which is an affine space over the vector space $V = X_*(\mathbf{S}) \otimes \mathbb{R}$ endowed with a homomorphism

$$\nu : N \longrightarrow \mathrm{Aut}_{\mathrm{aff}}(A)$$

into the group of affine automorphisms of A, such that we have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & Z \arrow[r] \arrow[d, "\nu"] & N \arrow[r] \arrow[d, "\nu"] & W_0 \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & V \arrow[r] & \mathrm{Aut}_{\mathrm{aff}}(A) \arrow[r] & \mathrm{GL}(V) \arrow[r] & 1 \end{tikzcd} \tag{2.2.3}$$

Here the rightmost vertical map is the canonical (faithful) representation of the finite Weyl group W_0 as a reflection group in V, and the leftmost vertical map is uniquely determined by the condition

$$\chi(\nu(z)) = -\nu_F(\chi(z)) \quad \forall z \in Z, \ \chi \in X^*(\mathbf{Z})$$

This condition implies that $\nu(Z) \leq V$ is a discrete subgroup of rank $\dim(V)$, i.e. a lattice in V. As I is compact so is $Z_0 = I \cap Z$, and hence $Z_0 \leq \ker(\nu)$ since ν is continuous. Therefore, ν factors to a map

$$\nu : W \longrightarrow \mathrm{Aut}_{\mathrm{aff}}(A)$$

which after an appropriate choice of an origin in A and hence an identification $A \simeq V$ will define a map $\rho : W \to \mathrm{Aut}_{\mathrm{aff}}(V)$. Regardless of this choice the above diagram shows that the subgroup $\rho_0(W)$ defined in (**ACIII**) is equal to the image of W_0 in $\mathrm{GL}(V)$, and hence (**ACIII**) is verified. Moreover, the injectivity of $W_0 \to \mathrm{GL}(V)$ and the commutativity of the above diagram imply that

$$\rho(W) \cap V = \nu(Z)$$

regardless of the choice of an origin. In particular (**ACV**) holds, since $\nu(Z)$ is a lattice in V. Now in order to choose an origin, we first need to define the locally finite set $\mathfrak{H}$ of hyperplanes.

For this we need to recall a few more facts from the theory of buildings. Let

$$\Phi := \Phi(\mathbf{G}, \mathbf{S}) \subseteq X^*(\mathbf{S}) \subseteq V^\vee$$

denote the root system[21] of the pair $(\mathbf{G}, \mathbf{S})$.

Let us for a root $a \in \Phi$ denote by $\mathbf{U}_a \leq \mathbf{G}$ the root subgroup corresponding to a, let $U_a := \mathbf{U}_a(F)$ and $U_a^* := U_a - \{1\}$. For every $u \in U_a^*$, the set $U_{-a}uU_{-a} \cap N$ consists of a single element, denoted $m(u)$ in [Tit79, 1.4]. The linear part of the image $\nu(m(u)) \in \mathrm{Aut}_{\mathrm{aff}}(A)$ is the reflection $s_a \in \mathrm{GL}(V)$ associated to a, which implies that $\nu(m(u))$ is an affine reflection. Let $\alpha(a, u)$ denote the affine function whose linear part is a and

[21] Note that in general, Φ is not reduced and so in particular it will *not* be the reduced root system Φ attached to the affine extended Coxeter group W by lemma 2.1.2.

whose vanishing set is the hyperplane fixed by $\nu(m(u))$. For any affine function $\alpha : A \to \mathbb{R}$ with linear part a, let

$$X_\alpha := \{u \in U_a^* : \alpha(a,u) \geq \alpha\} \cup \{1\}$$

It is a major result of the theory of buildings that X_α is in fact a subgroup of U_a. However, from the definition it is immediately clear that the X_α with α running over all affine functions with linear part a form an exhaustive and separated filtration of U_a. Moreover, any two elements $m(u), m(u')$ with $u, u' \in U_a^*$ differ only by an element of Z. Since $\nu(Z) \leq V$ is a discrete subgroup, it follows that the filtration $\{X_\alpha\}_\alpha$ is locally constant and that

$$\mathfrak{H} := \{\{\alpha(a,u) = 0\} : a \in \Phi,\ u \in U_a^*\}$$

is a locally finite set of hyperplanes. The set $\mathfrak{H}$ is left invariant under the action of W, thus verifying (**ACI**), which follows from

$$nU_a^* n^{-1} = U_{n(a)}^*, \quad n \in N,\ a \in \Phi$$

and

$$nm(u)n^{-1} = m(nun^{-1}), \quad n \in N,\ a \in \Phi,\ u \in U_a^* \tag{2.2.4}$$

The last equation also shows that

$$n^{-1} X_\alpha n = X_{\alpha \circ \nu(n)}$$

Let $W(\mathfrak{H})$ denote the subgroup of $\mathrm{Aut}_{\mathrm{aff}}(A)$ generated by all $\nu(m(u))$ with $a \in \Phi'$, $u \in U_a^*$. Then $W(\mathfrak{H})$ leaves $\mathfrak{H}$ invariant as $\mathfrak{H}$ is already invariant under W. Fixing a W_0-invariant positive definite scalar product on V, the group $W(\mathfrak{H})$ becomes the affine reflection group generated by the orthogonal reflections s_H with $H \in \mathfrak{H}$. By [Bou07, Ch. V, §3.10, Proposition 10], there exists a special point $p \in A$. As $W(\mathfrak{H})$ maps special points to special points and acts transitively on the set of chambers, we may assume that p lies in the closure of the chamber C which corresponds to I.

Identifying A and V via p, we will assume from now on that $A = V$ and $p = 0$ and we will put $\rho := \nu$. Letting $C_0 := C$, we fulfill (**ACIV**) and (**ACIX**). This gives $W(\mathfrak{H})$ the structure of a Coxeter group (cf. section 2.1) by letting the set of distinguished generators be the set $S(C_0)$ of reflections with respect to the walls of C_0.

We now want to construct lifts $\widetilde{s}_H \in W$ of the reflections $s_H \in W(\mathfrak{H})$, $H \in \mathfrak{H}$ satisfying (**ACVI**) and (**ACVII**). Note that (**ACII**) and (**ACVIII**) are then satisfied automatically. Consider the subgroup $N_{\mathrm{aff}} \leq N$ generated by Z_0 and all $m(u)$, $u \in U_a^*$, $a \in \Phi$. From relation (2.2.4) it follows that it is a normal subgroup of N. Moreover, by construction the map ν restricts to a surjection

$$N_{\mathrm{aff}} \twoheadrightarrow W(\mathfrak{H})$$

which we claim has kernel Z_0 (cf. [Vig16, 3.9]). Admitting this claim, it follows that the subgroup

$$W_{\mathrm{aff}} = N_{\mathrm{aff}}/Z_0 \subseteq N/Z_0 = W$$

maps isomorphically onto $W(\mathfrak{H})$ under ν and hence that (**ACVI**) and (**ACVII**) are fulfilled by letting $\widetilde{s}_H$ be the unique preimage in W_{aff} of s_H under ν.

Let us now show that $\ker(\nu) \cap N_{\mathrm{aff}} = Z_0$. We already saw that Z_0 is contained in the kernel. The reverse inclusion follows from the following characterization of the Iwahori subgroup given by Haines and Rapoport (Def. 1, Prop. 3 and Lemma 17 in [HR08])

$$I_C = \mathrm{Fix}_G(\overline{C}) \cap G_{\mathrm{aff}}$$

Here $\overline{C}$ denotes the chamber in the reduced building of $\mathbf{G}$ corresponding to C and $\mathrm{Fix}_G(\overline{C})$ denotes the subgroup of all elements of G fixing $\overline{C}$ pointwise. Note that $\mathrm{Fix}_G(C) \subseteq \mathrm{Fix}_G(\overline{C})$. Moreover, G_{aff} denotes the subgroup of G generated by all parahoric subgroups, or equivalently, the subgroup generated by Z_0 and the root subgroups U_a, $a \in \Phi$. It follows that $N_{\mathrm{aff}} \subseteq G_{\mathrm{aff}}$ and therefore that

$$\ker(\nu) \cap N_{\mathrm{aff}} \subseteq \mathrm{Fix}_G(\overline{C}) \cap G_{\mathrm{aff}} = I$$

Since $\ker(\nu) \subseteq Z$, this implies

$$\ker(\nu) \cap N_{\mathrm{aff}} \subseteq Z \cap I = Z_0$$

We have therefore now verified conditions (**ACI**)–(**ACIX**). It remains to show that (**ACX**) holds, i.e. that the subgroup

$$X = \rho^{-1}(V) \overset{(!)}{=} Z/Z_0 \leq W$$

is finitely generated and commutative. But this is shown in [HR10, Theorem 1.0.1], where are Z,Z_0 are denoted by $M(F)$, $M(F)_1$. To apply this theorem one has to note that Z_0 is the unique parahoric subgroup of Z (see the discussion before [Vig16, Proposition 3.15]).

We have therefore given $W = N/Z_0$ the structure of an affine extended Coxeter group. By lemma 2.1.2, this induces on W the structure $W = (W, W_{\text{aff}}, S, \Omega)$ of an extended Coxeter group. The exact sequence (2.2.2) therefore makes $W^{(1)} = N/Z_0(1)$ into a pro-p Coxeter group, provided we specify lifts of the generators $\widetilde{s}_H \in W_{\text{aff}}$, $H \in S(C_0)$ that satisfy the braid relations. This is the content of the next lemma. In order to state it, we need to recall a few more things from [Tit79].

Let $\{\overline{X}_\alpha\}_\alpha$ denote the family of quotients of the descending filtration $\{X_\alpha\}_\alpha$, i.e.

$$\overline{X}_\alpha = X_\alpha / X_{\alpha+\varepsilon}$$

for $\varepsilon > 0$ sufficiently small. If $a \in \Phi$ with $2a \in \Phi$, then the inclusion $U_{2a} \subseteq U_a$ induces an inclusion $\overline{X}_{2\alpha} \subseteq \overline{X}_\alpha$ for every α with $\alpha_0 = a$. The set Φ_{af} of affine roots is then defined to be (cf. [Tit79, 1.6])

$$\Phi_{\text{af}} = \{\alpha : \alpha_0 \in \Phi,\ \overline{X}_{2\alpha} \neq \overline{X}_\alpha\}$$

where $\overline{X}_{2\alpha} = \{1\}$ by convention if $2\alpha_0 \notin \Phi$. Note that if $\overline{X}_\alpha \neq 1$ but $\alpha \notin \Phi_{\text{af}}$, then necessarily $2\alpha \in \Phi_{\text{af}}$. Hence, every $H \in \mathfrak{H}$ is of the form $H = \{\alpha = 0\}$ for some $\alpha \in \Phi_{\text{af}}$.

2.2.3 Lemma. *In the situation of the above example, the following holds. Given a wall $H \in S(C_0)$ let $\alpha \in \Phi_{\text{af}}$ denote the unique affine root with*

$$H = \{\alpha = 0\}, \quad C_0 \subseteq \{\alpha > 0\} \quad \textit{and} \quad \frac{1}{2}\alpha \notin \Phi_{\text{af}}$$

and put $n_H = m(u)$ for some arbitrary $u \in X_\alpha$ with nonzero image under $X_\alpha \twoheadrightarrow \overline{X}_\alpha$. Then for all $H, H' \in S(C_0)$ with $\operatorname{ord}(s_H s_{H'}) < \infty$ we have the relation

$$n_H n_{H'} n_H \ldots \equiv n_{H'} n_H n_{H'} \ldots \mod Z_0(1)$$

in N, where the number of factors on both sides equals $\operatorname{ord}(s_H s_{H'})$.

Proof. If H, H' are parallel, then either $H = H'$ or $\operatorname{ord}(s_H s_{H'}) = \infty$, in which case there is nothing to prove. So we may assume that H, H' are not parallel, and hence that the intersection $H \cap H'$ contains a non-empty face of the fundamental chamber C_0. To every face F of C_0 is associated a subgroup (parahoric) $K_F \leq G$ as follows (see also [Vig16, 3.7]). Every face F of C_0 corresponds to a face in the apartment $A^\natural$ corresponding to $\mathbf{S}$ in the reduced building of $\mathbf{G}$. To every *nonempty* bounded subset $\Omega \subseteq A^\natural$ is attached ([BT84, 4.6.26] and [BT84, 5.1.9]) a smooth affine group scheme $\mathfrak{G}^0_\Omega$ over the ring of integers of the local field F ($\mathcal{O}^\natural$ in the notation of [BT84]) with generic fiber G. In the notation of [BT84] the parahoric K_F corresponding to F is then defined to be (see [BT84, 5.2.6] and the remark before [BT84, 5.2.9])

$$K_F = \mathfrak{G}^0_F(\mathcal{O}) \cap G(K^\natural) = \mathfrak{G}^0_F(\mathcal{O}^\natural)$$

From [BT84, 5.2.4] it follows that the group K_F is also characterized as the subgroup generated by Z_0 and the X_α for all α with $\alpha_0 \in \Phi$ and $F \subseteq \{\alpha > 0\}$.

For $F = C_0$ one has $K_F = I$ and for any two faces F, F' of C_0 (see [Vig16, Corollary 3.21])

$$F \subseteq \overline{F'} \quad \Rightarrow \quad K_F \supseteq K_{F'} \ \text{ and } \ K_F(1) \subseteq K_{F'}(1)$$

Here $K_F(1)$ denotes the pro-p radical of K_F. In particular

$$K_F(1) \subseteq I(1) \tag{2.2.5}$$

for all faces F of C_0. Let now $F \neq \emptyset$ be a face of C_0 contained in $H \cap H'$. The subset

$$\Phi_F := \{\alpha_0 : \alpha \in \Phi_{\text{af}},\ F \subseteq \{\alpha = 0\}\} \subseteq \Phi$$

is a sub root system of Φ. Moreover, elementary arguments show that $\alpha_0, \alpha'_0 \in \Phi_F$ are part of a basis of Φ_F. Here it is used that $\frac{1}{2}\alpha, \frac{1}{2}\alpha' \notin \Phi_{\text{af}}$.

Let $\overline{G}_F = \mathfrak{G}^0_F \times_{\operatorname{Spec}(\mathcal{O}_F)} \operatorname{Spec}(k)$ be the reduction of the group scheme $\mathfrak{G}^0_F$ and let $\overline{G}_F^{\text{red}}$ denote the quotient of $\overline{G}_F$ by its unipotent radical. Identifying $\overline{G}_F^{\text{red}}$ with the unique Levi subgroup of $\overline{G}_F$ containing the reduction

$\overline{S}$ of the canonical model of S over $\mathcal{O}_F$, the group $\overline{G}_F^{\mathrm{red}}$ coincides with the group denoted by the same symbol in [Tit79, 3.5]. The canonical map

$$K_F = \mathfrak{G}_F^0(\mathcal{O}_F) \longrightarrow \overline{G}_F^{\mathrm{red}}(k)$$

is surjective and its kernel is equal to the pro-p-radical $K_F(1)$ by [HV15, 3.7]. The group $\overline{G}_F^{\mathrm{red}}$ is a connected reductive group over k and its root system with respect to the maximal split subtorus $\overline{S}$, as a subset of $X^*(\overline{S}) = X^*(S)$, is equal to Φ_F (see [Tit79, 3.5.1]). Moreover, for any $\alpha \in \Phi_{\mathrm{af}}$ with $F \subseteq \{\alpha = 0\}$ we have

$$X_\alpha \subseteq K_F \tag{2.2.6}$$

and (see [Tit79, 3.5.1])

$$\overline{X}_\alpha = \overline{U}_{\alpha_0}(k) \tag{2.2.7}$$

as an equality of subgroups of $\overline{G}_F^{\mathrm{red}}(k) = K_F/K_F(1)$. Here $\overline{U}_{\alpha_0}$ denotes the root subgroup of $\overline{G}_F^{\mathrm{red}}$ corresponding to $\alpha_0 \in \Phi_F$.

Let now $\alpha, \alpha' \in \Phi_{\mathrm{af}}$ and $u \in X_\alpha$, $u' \in X_{\alpha'}$ with $n_H = m(u)$, $n_{H'} = m(u')$ be as in the statement of this lemma. Denote by $\overline{u}, \overline{u'}$ the images of u, u' under $K_F \twoheadrightarrow \overline{G}_F^{\mathrm{red}}(k)$. By (2.2.7) and the choice of u, u', the elements $\overline{u}, \overline{u'}$ are not reduced to the neutral element. Applying (2.2.7) to the reductions of the elements appearing in the decomposition of $m(u)$ and $m(u')$ respectively, it follows that

$$\overline{m(u)} = m(\overline{u}), \quad \overline{m(u')} = m(\overline{u'}) \tag{2.2.8}$$

by uniqueness, where $\overline{m(u)}$, $\overline{m(u')}$ denote the images of $m(u), m(u') \in K_F$ under $K_F \twoheadrightarrow \overline{G}_F^{\mathrm{red}}(k)$ and $m(\overline{u})$, $m(\overline{u'})$ are associated to $\overline{u}, \overline{u'}$ in the same way as $m(u), m(u')$ are associated to u, u'. In fact, $m(\overline{u}), m(\overline{u'})$ are the elements canonically associated to the elements $\overline{u}, \overline{u'}$ and the root datum $(Z(\overline{S})(k), (\overline{U}_a(k))_{a\in\Phi_F})$ (in the sense of [BT72, 6.1.1]) by [BT72, 6.1.2 (2)]. Applying Proposition [BT72, 6.1.8] to the root datum given by restricting $(Z(\overline{S})(k), (\overline{U}_a(k))_{a\in\Phi_F})$ to the rank two sub root system $(\mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_0') \cap \Phi_F$, it follows that

$$m(\overline{u})m(\overline{u'})m(\overline{u})\ldots = m(\overline{u'})m(\overline{u})m(\overline{u'})\ldots \tag{2.2.9}$$

where the number of factors on both sides equals the order of $s_{\alpha_0}s_{\alpha_0'} \in W_0(\Phi_F) \subseteq W_0(\Phi)$. Let x be an arbitrary point of the face F. As s_H and $s_{H'}$ both lie in the stabilizer $\mathrm{Aut}_{\mathrm{aff}}(V)_x$ and the map $\mathrm{Aut}_{\mathrm{aff}}(V) \to \mathrm{GL}(V)$ restricts to an injection $\mathrm{Aut}_{\mathrm{aff}}(V)_x \hookrightarrow \mathrm{GL}(V)$, the order of the image $s_{\alpha_0}s_{\alpha_0'}$ of $s_H s_H' \in \mathrm{Aut}_{\mathrm{aff}}(V)$ is equal to the order of $s_H s_H'$. Hence, it follows from (2.2.9) and (2.2.8) that $ab^{-1} \in K_F(1)$ where

$$a = m(u)m(u')m(u)\ldots \qquad b = m(u')m(u)m(u')\ldots$$

and the number of factors on the right hand side of each equation equals the order of $s_H s_H'$. On the other hand we have $ab^{-1} \in Z$ since Z equals the kernel of the composition $N \to W(\mathfrak{H}) \to W_0$, and a, b are mapped to the same element under $N \to W(\mathfrak{H})$. Hence

$$ab^{-1} \in Z \cap I(1) = Z_0(1)$$

and the claim follows. □

Now, for a generator $s = \widetilde{s}_H$, $H \in S(C_0)$ of W_{aff} let $n_s \in W^{(1)} = N/Z_0(1)$ be the class of an element $n_H \in N$ as chosen according to the lemma. Then the lemma states that $W^{(1)}$ together with the choice of these lifts becomes a pro-p Coxeter group in the sense of definition 1.1.13.

We can now finally state the relation between pro-p-Iwahori Hecke algebras and generic pro-p Hecke algebras.

2.2.4 Lemma. *Given a generator $s = \widetilde{s}_H \in W_{\mathrm{aff}}$, $H \in S(C_0)$, let $\alpha \in \Phi_{\mathrm{af}}$ be the unique affine root with $H = \{\alpha = 0\}$ and $\frac{1}{2}\alpha \notin \Phi_{\mathrm{af}}$. Then the following holds.*

(i) We have

$$q_s := \#In_sI/I = \#I(1)n_sI(1)/I(1) = \#\overline{X}_\alpha = q^{d(v)}$$

where $d(v) \in \mathbb{N}$ denotes the integer associated to the vertex v of the local Dynkin diagram $\Delta(\Phi_{\mathrm{af}})$ (see [Tit79, 1.8]) corresponding to H.

(ii) Let F be any face of C_0 contained in H, let $\overline{G}_{F,s}$ be the subgroup of $\overline{G}_F^{\mathrm{red}}$ generated by $\overline{X}_\alpha$ and $\overline{X}_{-\alpha}$ (cf. proof of lemma 2.2.3) and let

$$Z_{k,s} := \overline{G}_{F,s} \cap Z_k \leq Z_k$$

Then $Z_{k,s}$ is independent of the choice of F.

(iii) Let

$$c_s := \sum_{t \in Z_{k,s}} c_s(t) t \in R[Z_k]$$

with

$$c_s(t) := \# \left(n_s \overline{X}_\alpha n_s \cap \overline{X}_\alpha n_s t \overline{X}_\alpha \right)$$

where the intersection is taken inside $\overline{G}_F^{\text{red}}(k)$ *(for any* F *as in (ii)), and* n_s *denotes (by abuse of notation) the image of the element* $n_H = m(u) \in X_\alpha$ *under* $X_\alpha \twoheadrightarrow \overline{X}_\alpha \subseteq \overline{G}_F^{\text{red}}(k)$*.*

Then the families $(q_s)_s$*,* $(c_s)_s$ *fulfill condition* (1.3.1) *with respect to the pro-p Coxeter group* $W^{(1)}$ *defined above, and the R-linear isomorphism*

$$\mathcal{H}_R^{(1)}(q_s, c_s) \xrightarrow{\sim} H_R(G, I(1)), \quad T_w \longmapsto T_w$$

is a morphism of R-algebras. Moreover, the integers $c_s(t)$ *satisfy*

$$\forall t \in Z_{k,s} \quad c_s(t) > 0 \quad \text{and} \quad \sum_{t \in Z_{k,s}} c_s(t) = q_s - 1$$

In particular, the integers $c_s(t)$ *are all equal to* 1 *if and only if the order of* $Z_{k,s}$ *equals* $q_s - 1$*. For example, this is the case if* $\mathbf{G}$ *is split and simply connected. More generally, if* $\mathbf{G}$ *is only split (but not necessarily simply connected), it holds that*

$$\forall t, t' \in Z_{k,s} \quad c_s(t) = c_s(t')$$

Proof. ad (i): For the first equality and second equality we refer to [Vig16, Corollary 3.30], recalling that (see (2.2.7)) $\overline{X}_\alpha$ is naturally identified with the group $U_{\alpha_0}(k)$ (denoted $U_{\alpha,\mathfrak{F},k}$ in [Vig16]). The last equality follows directly from the definition of the integer $d(v)$ as the sum of the dimensions of k-vector spaces (cf. [Tit79, 1.6]) $\overline{X}_\alpha / \overline{X}_{2\alpha}$ and $X_{2\alpha}$.

ad (ii): The independence of $Z_{k,s}$ from the choice of F is implicit in the proof of [Vig16, Proposition 4.4]. It can also be seen as follows (cf. [Vig16, Proposition 3.26]). Given two (nonempty) faces F, F' of C_0 with $F' \subseteq \overline{F}$, we have an inclusion $K_F \subseteq K_{F'}$. The image of K_F in $K_{F',k}$ under the natural map $K_{F'} \twoheadrightarrow K_{F',k} = K_{F'}/K_{F'}(1)$ is equal to the subgroup M_F generated by Z_k and the groups $\overline{U}_a(k)$, $a \in \Phi_F \subseteq \Phi_{F'}$. Moreover, M_F appears as a Levi subgroup of a parabolic subgroup $Q_F = M_F \ltimes U_F$, such that the inverse image of the unipotent radical U_F under $K_F \to K_{F',k}$ equals $K_F(1)$. Hence, we have an induced injective map

$$K_{F,k} \xrightarrow{\sim} M_F \subseteq K_{F',k}$$

which is the identity on Z_k and the $\overline{U}_a(k)$, $a \in \Phi_F$. In particular, the subgroup $\overline{G}_{F',s}$ of $\overline{G}_{F'}^{\text{red}} = K_{F',k}$ generated by $\overline{X}_\alpha = \overline{U}_{\alpha_0}(k)$ and $\overline{X}_{-\alpha} = \overline{U}_{-\alpha_0}(k)$ equals the image of $\overline{G}_{F,s}$ under the embedding $K_{F,k} \hookrightarrow K_{F',k}$. As this embedding is the identity on Z_k, it follows that

$$\overline{G}_{F,s} \cap Z_k = \overline{G}_{F',s} \cap Z_k$$

ad (iii): As Vignéras has observed [Vig16, Theorem 4.7], the condition (1.3.1) is not only sufficient but also necessary for the existence of an algebra structure on the free R-module over $W^{(1)}$ satisfying (1.3.2) and (1.3.3). As the latter two conditions are satisfied for $H_R(G, I(1))$ by [Vig16, Proposition 4.1] and [Vig16, Proposition 4.4], it follows that (1.3.1) is satisfied. Moreover, the fact that the relations (1.3.2) and (1.3.3) hold in $H_R(G, I(1))$ implies that $\mathcal{H}_R^{(1)}(q_s, c_s) \xrightarrow{\sim} H_R(G, I(1))$ is a morphism of algebras.

Finally, for the properties of the integers $c_s(t)$ we refer to step 3 of the proof of [Vig16, Proposition 4.4]; however note, that the proof contains some errors as it is incorrectly[22] claimed there that $\#Z_{k,s} = q_s - 1$ always holds when $\mathbf{G}$ is split. ☐

We have therefore now recognized $H_R(G, I(1))$ as an affine pro-p Hecke algebra. Since in this case the abelian group $T = Z_k$ underlying $W^{(1)}$ is finite, all the structure results of theorem 2.7.1 hold unconditionally for $H_R(G, I(1))$ (cf. remark 2.7.3). In particular the center of $H_R(G, I(1))$ is finitely generated as an R-algebra, and $H_R(G, I(1))$ is module-finite over its center.

[22] for example, for $\mathbf{G} = \mathrm{PGL}_2$ one has $\#Z_{k,s} = \frac{q_s - 1}{2}$

2.2.4 Affine Yokonuma-Hecke algebras

The Yokonuma-Hecke algebras $Y_{d,n}$ of example 1.3.8 have a natural variant, the *affine Yokonuma-Hecke algebras* $Y^{\mathrm{aff}}_{d,n}$. According to [CS15, Introduction] these algebras have first been introduced by Juyumaya and Lambropoulou [JL] under the name of '*d-th framization of the Iwahori-Hecke algebra of B-type*'. Later they were studied by Chlouveraki and Poulain d'Andecy [Cd14] under the name '*affine Yokonuma-Hecke algebra*'. The different terminologies reflect the two different ways in which $Y^{\mathrm{aff}}_{d,n}$ can be seen as modifications of other algebras. This is visualized in the following commutative diagram (defined down below)

(2.2.10)
$$\begin{tikzcd} Y^{\mathrm{aff}}_{d,n} \arrow[r] \arrow[d] & H^{\mathrm{aff}}_n \arrow[d, "\pi"] \\ Y_{d,n} \arrow[r] & H_n \end{tikzcd}$$

were the left column is the 'framization' of the right column, and the upper row is the 'affinization' of the lower one.

Chlouveraki and Sécherre have recognized [CS15] the algebra $Y^{\mathrm{aff}}_{d,n}$ as (in our terminology) generic pro-p Hecke algebras for the split pro-p Coxeter group $W^{(1)} = T \rtimes W$, $T = (\mathbb{Z}/d\mathbb{Z})^n$, $W = \mathbb{Z}^n \rtimes S_n$. In fact, we will see in a moment that they are affine pro-p Hecke algebras in the sense of definition 2.1.4.

Let us first recall the definition (cf. [Cd14, 3.1]) of the affine Yokonuma-Hecke algebras. For integers $d, n \geq 1$, the algebra $Y^{\mathrm{aff}}_{d,n}$ is the algebra over $R = \mathbb{C}[u^{\pm 1}, v]$ generated by elements

$$g_1, \ldots, g_{n-1}, t_1, \ldots, t_n, X_1, X_1^{-1}$$

subject to the relations

(1)	$g_i g_j = g_j g_i$	for all $i, j = 1, \ldots, n-1$ such that $\lvert i-j \rvert > 1$
(2)	$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$	for all $i = 1, \ldots, n-2$
(3)	$t_i t_j = t_j t_i$	for all $i, j = 1, \ldots, n$
(4)	$g_i t_j = t_{s_i(j)} g_i$	for all $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$
(5)	$t_j^d = 1$	for all $j = 1, \ldots, n$
(6)	$X_1 X_1^{-1} = X_1^{-1} X_1 = 1$	
(7)	$X_1 g_1 X_1 g_1 = g_1 X_1 g_1 X_1$	
(8)	$X_1 g_i = g_i X_1$	for all $i = 2, \ldots, n-1$
(9)	$X_1 t_j = t_j X_1$	for all $j = 1, \ldots, n$
(10)	$g_i^2 = u^2 + v e_i g_i$	for all $i = 1, \ldots, n-1$

where as in example 1.3.8 we let

$$e_i = \frac{1}{d} \sum_{0 \leq s < d} (t_i / t_{i+1})^s$$

Note that this definition of the Yokonuma-Hecke algebra slightly differs from the one given in [Cd14], as we are considering $Y^{\mathrm{aff}}_{d,n}$ as an algebra over the ring $\mathbb{C}[u^{\pm}, v]$ in two formal variables. The algebra of [Cd14] is obtained by specializing $Y^{\mathrm{aff}}_{d,n}$ along the ring homomorphism $\mathbb{C}[u^{\pm 1}, v] \to \mathbb{C}[q^{\pm 1}]$ sending $u \mapsto 1$ and $v \mapsto q - q^{-1}$.

Let us now recognize $Y^{\mathrm{aff}}_{d,n}$ as an affine pro-p Hecke algebra. More precisely, let us show that $Y^{\mathrm{aff}}_{d,n}$ is isomorphic to a generic pro-p Hecke algebra $\mathcal{H}^{(1)}$ for the pro-p Coxeter group $W^{(1)} = (\mathbb{Z}/d\mathbb{Z})^n \rtimes W$, where $W = \mathbb{Z}^n \rtimes S_n$ is the affine extended Coxeter group of example 2.1.3(ii), (cf. also section 2.2.2) acting on $(\mathbb{Z}/d\mathbb{Z})^n$ by permuting the coordinates via the projection $W \twoheadrightarrow W_0 = S_n$. By proposition 1.4.3, we have an isomorphism $\mathcal{H}^{(1)} \simeq R[\mathfrak{A}(W^{(1)})]/I$, where I is the ideal generated by the elements $T_{n_s}^2 - a_s T_{n_s^2} - b_s T_{n_s}$. It therefore suffices to see that $Y^{\mathrm{aff}}_{d,n}$ is a quotient of $R[\mathfrak{A}(W^{(1)})]$ by the same ideal I.

For this we need the following 'framed version' of lemma 2.2.1, providing two descriptions of the **d-modular framed affine braid group**.

2.2.5 Lemma. *Let $\widetilde{\mathfrak{A}}^{\mathrm{aff},(1)}_{d,n}$ denote the group generated by elements*

$$g_1, \ldots, g_{n-1}, t_1, \ldots, t_n, X_1$$

subject to the relations (1)-(9) above, and let $\mathfrak{A}^{\mathrm{aff},(1)}_{d,n} = \mathfrak{A}(W^{(1)})$ *with* $W^{(1)}$ *as above. Then there are inverse isomorphisms* $\Phi : \widetilde{\mathfrak{A}}^{\mathrm{aff},(1)}_{d,n} \to \mathfrak{A}^{\mathrm{aff},(1)}_{d,n}$ *and* $\Psi : \mathfrak{A}^{\mathrm{aff},(1)}_{d,n} \to \widetilde{\mathfrak{A}}^{\mathrm{aff},(1)}_{d,n}$ *determined by*

$$\begin{aligned} \Phi(g_i) &= T_{s_i} \quad i = 1, \ldots, n-1 \\ \Phi(t_i) &= T_{t_i} \quad i = 1, \ldots, n \\ \Phi(X_1) &= T_{-e_1}^{-1} \end{aligned}$$

and

$$\begin{aligned} \Psi(T_{s_i}) &= g_i \quad i = 1, \ldots, n-1 \\ \Psi(T_{s_0}) &= \Psi(T_u) g_1 \Psi(T_u)^{-1} \\ \Psi(T_u) &= g_{n-1} \ldots g_1 X_1 \\ \Psi(T_{t_i}) &= t_i \quad i = 1, \ldots, n \end{aligned}$$

where $t_1, \ldots, t_n$ *denote the canonical generators of* $(\mathbb{Z}/d\mathbb{Z})^n$*.*

Proof. Follows immediately from $\mathfrak{A}(W^{(1)}) = T \rtimes \mathfrak{A}(W)$, $\widetilde{\mathfrak{A}}^{\mathrm{aff},(1)}_{d,n} = T \rtimes \widetilde{\mathfrak{A}}^{\mathrm{aff}}_n$ and lemma 2.2.1 (where the T_i have to be replaced by g_i). □

From the above lemma, it follows readily that the affine Yokonuma-Hecke algebra $Y^{\mathrm{aff}}_{d,n}$ is the quotient of the group algebra $R[\mathfrak{A}^{\mathrm{aff},(1)}_{d,n}]$ by the ideal generated by the relations in (10). This doesn't yet prove that $Y^{\mathrm{aff}}_{d,n}$ is isomorphic to the generic pro-p Hecke algebra $\mathcal{H}^{(1)}$ (with the obvious parameters a_s, b_s), since we still need to show that the latter exists. Moreover, carefully comparing the relations in (10) with the generators of the ideal I realizing the isomorphism $\mathcal{H}^{(1)} \simeq R[\mathfrak{A}(W^{(1)})]/I$ of proposition 1.4.3, one notes that I is generated by one extra relation not appearing in (10). However, as we will see now, this extra relation is redundant.

2.2.6 Theorem. *Let* $W^{(1)} = T \rtimes W$, $T = (\mathbb{Z}/d\mathbb{Z})^n$, $n_s = s$, $W = \mathbb{Z}^n \rtimes S_n$ *be the split pro-p Coxeter group constructed above. For* $i = 0, \ldots, n-1$ *put*

$$a_{s_i} := u^2 \in R, \quad b_{s_i} := \frac{v}{d} \sum_{s \in \mathbb{Z}/d\mathbb{Z}} (t_i/t_{i+1})^s \in R[T]$$

where $t_0 := t_n$ *by convention and the group* T *is written multiplicatively. Then the following holds.*

(i) The parameter families $(a_s)_{s\in S}$*,* $(b_s)_{s\in S}$ *defined above satisfy condition* (1.3.1) *of theorem 1.3.1, and hence the generic pro-p Hecke algebra* $\mathcal{H}^{(1)}_{d,n} := \mathcal{H}^{(1)}(a_s, b_s)$ *for these parameters exists.*

(ii) There is an isomorphism of R-algebras

$$Y^{\mathrm{aff}}_{d,n} \xrightarrow{\sim} \mathcal{H}^{(1)}_{d,n}$$

determined by

$$\begin{aligned} g_i &\mapsto T_{s_i} \\ t_i &\mapsto T_{t_i} \\ X_1 &\mapsto T_{-e_1}^{-1} \end{aligned}$$

where the element $T_{-e_1}^{-1} \in \mathcal{H}^{(1)}_{d,n}$ *is well-defined since the parameters* a_s *are invertible in R (cf. section 1.4).*

(iii) Via the structure of an affine extended Coxeter group on W from example 2.1.3, $\mathcal{H}^{(1)}_{d,n}$ *(and hence* $Y^{\mathrm{aff}}_{d,n}$*) becomes an affine pro-p Hecke algebra. Moreover, as the group T is finite, all results of theorem 2.7.1 apply without restriction (cf. remark 2.7.3). In particular* $\mathcal{H}^{(1)}_{d,n}$ *is finite as a module over its center, and the latter is given by the invariants*

$$Z(\mathcal{H}^{(1)}_{d,n}) = \left(\mathcal{A}^{(1)}_{\mathfrak{o}}\right)^W$$

of the subalgebra

$$\mathcal{A}^{(1)}_{\mathfrak{o}} = \bigoplus_{x \in X^{(1)}} R\widehat{\theta}_{\mathfrak{o}}(x) \subseteq \mathcal{H}^{(1)}_{d,n}$$

where $\mathfrak{o}$ is any spherical orientation of W. Since the parameters a_s are units in R, the unnormalized Bernstein map $\theta_{\mathfrak{o}}$ exists and provides an isomorphism

$$R[X^{(1)}] \xrightarrow{\sim} \mathcal{A}_{\mathfrak{o}}^{(1)}, \quad x \mapsto \widehat{\theta}_{\mathfrak{o}}(x) \quad (x \in X^{(1)})$$

Since the group $X^{(1)} = T \times \mathbb{Z}^n$ is commutative, the algebra $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is also commutative and hence by theorem 2.7.1 is equal to its centralizer in $\mathcal{H}_{d,n}^{(1)}$.

Proof. We begin by showing (i) using the equivalent reformulation of condition (1.3.1) given in remark 1.3.6. From example 2.1.3(ii) recall that

$$us_iu^{-1} = s_{i-1}$$

for all $i \in \{0, \ldots, n-1\} = \mathbb{Z}/n\mathbb{Z}$. In particular all elements of S are conjugate. The first of the two conditions of remark 1.3.6 therefore follows from the equation

$$u(b_{s_i}) = b_{s_{i-1}} \tag{2.2.11}$$

Recall here that $u = \tau^{e_n} s_{n-1} \ldots s_1$ and hence u acts on $T = (\mathbb{Z}/d\mathbb{Z})^n$ via the cycle $s_{n-1} \ldots s_1 = (n-1 \ \ldots 1)$. Now in order to see that the second condition of remark 1.3.6 holds true, first note that by a general result on reflection groups (see [Bou07, Ch. V, §3.3, Proposition 2 (I)]), it follows that for every $s \in S$

$$\{w \in W : wsw^{-1} = s\} = \{1, s\} \cdot \{v \in \Omega : vsv^{-1} = s\} = \{1, s\} \cdot \{u^{nk} : k \in \mathbb{Z}\}$$

Here

$$u^n = \tau^{e_1 + \ldots + e_n} \in X$$

Taking $\widetilde{w} = w$ to be the canonical lift for the split pro-p group $W^{(1)} = T \rtimes W$ for every w as above, condition (ii) of remark 1.3.6 follows then from

$$s(b_s) = b_s, \quad s \in S$$

and

$$s(t)t^{-1}b_s = b_s, \quad s \in S, \ t \in T$$

The latter two equations follow by a simple computation, for instance

$$s_i(b_{s_i}) = \frac{v}{d} \sum_{s \in \mathbb{Z}/d\mathbb{Z}} s_i(t_i/t_{i+1})^s = \frac{v}{d} \sum_{s \in \mathbb{Z}/d\mathbb{Z}} (t_{i+1}/t_i)^s = \frac{v}{d} \sum_{s \in \mathbb{Z}/d\mathbb{Z}} (t_i/t_{i+1})^{-s} = b_{s_i}$$

and writing $t = t_1^{k_1} \ldots t_n^{k_n}$ we have

$$s_i(t)t^{-1}b_{s_i} = t_i^{k_{i+1}-k_i} t_{i+1}^{k_i - k_{i+1}} b_{s_i} = (t_i/t_{i+1})^{k_{i+1}-k_i} b_{s_i} = b_{s_i}$$

Now claim (ii) is an almost immediate consequence of (i), lemma 2.2.5 and proposition 1.4.3, since $Y_{d,n}^{\mathrm{aff}}$ and $\mathcal{H}_{d,n}^{(1)}$ both are quotients of the group algebra $R[\mathfrak{A}_{d,n}^{(1)}]$ by ideals I and I' respectively, generated by the elements

$$T_{n_{s_i}}^2 - a_{s_i} T_{n_{s_i}^2} - b_{s_i} T_{n_{s_i}} = T_{s_i}^2 - a_{s_i} - b_{s_i} T_{s_i}$$

However, for I' the index i ranges from 0 to n whereas for I it only ranges from 1 to n. But by equation (2.2.11) we have

$$T_u \left(T_{s_1}^2 - a_{s_1} - b_{s_1} T_{s_1} \right) T_{u^{-1}} = T_{s_0}^2 - a_{s_0} - b_{s_0} T_{s_0}$$

and therefore $I = I'$. In fact, this argument shows that

$$I = I' = (T_{s_1}^2 - a_{s_1} - b_{s_1} T_{s_1})$$

Finally for (iii), there is nothing to prove. □

2.2.7 Remark. By definition, the algebras $Y_{d,n}^{\mathrm{aff}}$ and $\mathcal{H}_{d,n}^{(1)}$ are algebras over the ring $R = \mathbb{C}[u^{\pm 1}, v]$. However, the definition of $Y_{d,n}^{\mathrm{aff}}$ and the verification of condition (1.3.1) did not make use of the invertibility of u, i.e. both algebras can already be defined over $\mathbb{C}[u, v]$. In contrast, the above isomorphism between $Y_{d,n}^{\mathrm{aff}}$ and $\mathcal{H}_{d,n}^{(1)}$ *does* make explicit use of the invertibility of u. This poses the question whether both algebras are isomorphic over $\mathbb{C}[u, v]$.

2.2.8 Remark. In the beginning of section 2.2 we remarked that the examples of pro-p-Iwahori Hecke algebras and affine Yokonuma-Hecke algebras overlap. Let us now make this more precise. It is not hard to see that whenever d is of the form

$$d = q - 1, \quad q = p^r \text{ a prime-power}$$

the pro-p Coxeter group $W^{(1)} = T \rtimes W$ considered above (together with the structure of an affine extended Coxeter group on W !) can be identified with the pro-p Coxeter group $W^{(1)} = N/Z_0(1)$ associated to the reductive group GL_n, the diagonal subtorus in GL_n and the Iwahori subgroup

$$I = \begin{pmatrix} \mathcal{O}_F^\times & \mathfrak{p}_F & \dots & \mathfrak{p}_F \\ \mathcal{O}_F & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \mathfrak{p}_F \\ \mathcal{O}_F & \dots & \mathcal{O}_F & \mathcal{O}_F^\times \end{pmatrix} \leq \mathrm{GL}_n(F)$$

by section 2.2.3, where F denotes any nonarchimedean local field with residue field $k = \mathcal{O}_F/\mathfrak{p}_F$ of cardinality q. Explicitly, the choice of a uniformizing element $\pi \in \mathcal{O}_F$ provides a splitting of the exact sequence

$$1 \longrightarrow Z_k \longrightarrow W^{(1)} \longrightarrow W \longrightarrow 1$$

by identifying an element $w = \tau^x\sigma \in W = \mathbb{Z}^n \rtimes S_n$ with the class of the monomial matrix $(\pi^{-x_i}\delta_{\sigma(i),j})_{i,j}$. Moreover, the choice of a primitive d-th root of unity in k provides an isomorphism of the group $Z_k = (k^\times)^n$ with $(\mathbb{Z}/d\mathbb{Z})^n$. Unwinding the definition of the groups $Z_{k,s}$ in lemma 2.2.4, one sees that for $s = s_i$, $i \in \{1, \dots, n-1\}$

$$Z_{k,s} = \{(t_i/t_{i+1})^j : j \in \mathbb{Z}/d\mathbb{Z}\}$$

as subgroups of $Z_k = T$, where we recall that the t_i denote the standard generators of $T = (\mathbb{Z}/d\mathbb{Z})^n$ and that this group is written multiplicatively. From this and lemma 2.2.4, it then follows immediately that we have an isomorphism

$$Y_{d,n}^{\mathrm{aff}} \otimes_R \mathbb{C} \simeq \mathcal{H}_{d,n}^{(1)} \otimes_R \mathbb{C} \simeq H_{\mathbb{C}}(\mathrm{GL}_n(F), I(1))$$

where the base change $- \otimes_R \mathbb{C}$ is with respect to the homomorphism $R = \mathbb{C}[u^{\pm 1}, v] \to \mathbb{C}$ sending u to $\sqrt{q}$ and v to $q - 1$.

We now come back to the commutative diagram (2.2.10)

$$\begin{tikzcd} Y_{d,n}^{\mathrm{aff}} \arrow[r] \arrow[d] & H_n^{\mathrm{aff}} \arrow[d, "\pi"] \\ Y_{d,n} \arrow[r] & H_n \end{tikzcd}$$

that was mentioned in the beginning, and will define all the maps involved. The right vertical arrow is the quotient map π constructed in section 2.2.2. We recall that π was induced by the map

$$\begin{aligned} \mathfrak{A}_n^{\mathrm{aff}} \simeq \widetilde{\mathfrak{A}}_n^{\mathrm{aff}} &\longrightarrow \mathfrak{A}_n \\ T_i &\longmapsto T_{s_i}, \quad i = 1, \dots, n-1 \\ X_1 &\longmapsto 1 \end{aligned}$$

between braid groups. In fact, the whole diagram (2.2.10) is induced by a diagram

$$\begin{tikzcd} \mathfrak{A}_n^{\mathrm{aff},(1)} \arrow[r] \arrow[d] & \mathfrak{A}_n^{\mathrm{aff}} \arrow[d, "\pi"] \\ \mathfrak{A}_n^{(1)} \arrow[r] & \mathfrak{A}_n \end{tikzcd} \tag{2.2.12}$$

of braid groups, where $\mathfrak{A}_n^{(1)} = \mathfrak{A}(T \rtimes W_0)$ denotes the d-modular framed braid group. The horizontal arrows are given by the projection onto the second factor, with respect to the isomorphisms (cf. example 1.4.4)

$$\mathfrak{A}_n^{\mathrm{aff},(1)} \simeq T \rtimes \mathfrak{A}_n^{\mathrm{aff}}, \quad \mathfrak{A}_n^{(1)} \simeq T \rtimes \mathfrak{A}_n$$

Finally the left vertical arrow is given by

$$\mathfrak{A}_n^{\mathrm{aff},(1)} \simeq T \rtimes \mathfrak{A}_n^{\mathrm{aff}} \xrightarrow{\mathrm{id} \times \pi} T \rtimes \mathfrak{A}_n \simeq \mathfrak{A}_n^{(1)}$$

It is easy to see that (2.2.12) respects the quadratic relations and hence induces a diagram between Hecke algebras.

2.3 Some finiteness properties of affine extended Coxeter groups

In this section, we will collect some properties of affine extended Coxeter groups and their associated affine hyperplane arrangements that will be needed in the structure theorem of affine pro-p Hecke algebras. In particular, we will prove some finiteness properties which will directly imply corresponding finiteness properties for affine pro-p Hecke algebras.

Throughout this section, we will fix an affine extended Coxeter group W. The notations introduced in definition 2.1.1 and lemma 2.1.2 will be used freely.

2.3.1 Remark. Let us begin by relating the abstract geometric terminology introduced in 1.1.3 to the concrete geometry of the hyperplane arrangement $(V, \mathfrak{H})$. It is a basic result (see. [Bou07, Ch. V, §3.2, Théorème 1]) of the theory of affine reflection groups that $W(\mathfrak{H})$ acts simply transitively on the set of chambers $\pi_0(V - \bigcup_{H\in\mathfrak{H}} H)$. Since ρ induces an isomorphism $W_{\text{aff}} \xrightarrow{\sim} W(\mathfrak{H})$, also W_{aff} acts simply transitively on the set of chambers. Via the map $w \mapsto w \bullet C_0$ we can therefore identify the set of 'abstract chambers' (in the sense of 1.1.3) with the chambers in V. Moreover, under this identification the 'abstract orbit map' $W \longrightarrow W_{\text{aff}}$ of 1.1.3 coincides with the actual orbit map given by $w \mapsto w \bullet C_0$. The identification of abstract and concrete chambers also extends to hyperplanes such that the notion of 'separation' is preserved. More precisely, the map

$$\begin{aligned} \mathfrak{H} &\longrightarrow \{wsw^{-1} : w \in W_{\text{aff}},\ s \in S\} \\ H &\longmapsto \widetilde{s}_H \end{aligned}$$

is a bijection, and for $H \in \mathfrak{H}$ and $w, w' \in W_{\text{aff}}$ it holds true that H separates $w(C_0)$ from $w'(C_0)$ if and only if the abstract hyperplane $\widetilde{s}_H$ separates the abstract chambers w, w' in the sense of 1.1.3. The bijectivity follows easily from the fact that ρ gives an isomorphism $W_{\text{aff}} \xrightarrow{\sim} W(\mathfrak{H})$ that satisfies $\rho(\widetilde{s}_H) = s_H$ and maps the set $S \subseteq W_{\text{aff}}$ bijectively onto the set of reflections with respect to the walls of the fundamental chamber C_0. That the notion of 'separation' is preserved follows from the fact that the set of abstract hyperplanes separating $1, w$ and the set of concrete hyperplanes separating $C_0, w \bullet C_0$ respectively can both be read off from the choice of a reduced expression $w = s_1 \dots s_r$. We may therefore identify concrete and abstract hyperplanes without harm and write

$$H = \widetilde{s}_H$$

Using the formal notation $s_H = H$ of 1.1.3, we therefore have

$$\widetilde{s}_H = s_H \in W_{\text{aff}}$$

and the compatibility $\rho(\widetilde{s}_H) = s_H$ can be written as

$$\rho(s_H) = s_H \in W(\mathfrak{H})$$

Whenever it matters, it will either be stated explicitly or it will be clear from the context whether we view s_H as an element of W_{aff} or of $W(\mathfrak{H})$, so that no confusion will arise.

As we just saw, the abstract geometry of an affine extended Coxeter group W is faithfully reflected (no pun intended) in the geometry of the affine hyperplane arrangement $(V, \mathfrak{H})$. Using the extra structure available on $(V, \mathfrak{H})$, this dictionary between abstract and concrete geometry makes some questions concerning W very transparent.

Consider for instance the following basic problem of Coxeter geometry. Given chambers C, C' and C'', when does

$$d(C, C'') = d(C, C') + d(C', C'')$$

hold true? This problem can be made more transparent with the help of the following 'vector-valued' distance.

2.3.2 Definition. Given chambers $C, C' \in \pi_0(V - \bigcup_{H\in\mathfrak{H}} H)$ the element

$$\vec{d}(C, C') \in \mathbb{Z}^{\Phi^+}$$

defined component-wise via

$$\vec{d}(C, C')_\alpha = \pi_0(-\alpha)(C') - \pi_0(-\alpha)(C) \in \mathbb{Z}$$

is called the **vector-valued distance** between C and C'. Here $\pi_0(-\alpha)$ denotes the map induced on connected components by

$$-\alpha : V - \bigcup_{H\in\mathfrak{H}} H \longrightarrow \mathbb{R} - \mathbb{Z}$$

and the difference $\pi_0(-\alpha)(C') - \pi_0(-\alpha)(C)$ is to be understood in the sense of affine spaces over $\mathbb{Z}$, with $\pi_0(\mathbb{R} - \mathbb{Z})$ carrying the obvious affine structure. In other words

$$\vec{d}(C, C')_\alpha = k' - k$$

if $k, k' \in \mathbb{Z}$ are such that

$$-\alpha(C) \subseteq]k, k+1[\quad \text{and} \quad -\alpha(C') \subseteq]k', k'+1[$$

2.3.3 Remark. From the definition it is quite obvious that $|\vec{d}(C, C')_\alpha|$ equals the number of hyperplanes of the form $H_{\alpha,k}$, $k \in \mathbb{Z}$ separating C from C'. In particular the vector-valued and the normal distance are related by the formula

$$d(C, C') = |\vec{d}(C, C')| := \sum_{\alpha \in \Phi^+} |\vec{d}(C, C')_\alpha| \tag{2.3.1}$$

which justifies the terminology. In particular, using remark 2.3.1, the length ℓ on W can be expressed in terms of $\vec{d}$ as

$$\ell(w) = d(C_0, w(C_0)) = \sum_{\alpha \in \Phi^+} |\vec{d}(C_0, w(C_0))_\alpha|$$

where C_0 denotes the fundamental chamber and $w(C_0) = \rho(w)(C_0)$ the action of W via $\rho : W \to \mathrm{Aut}_{\mathrm{aff}}(V)$. By definition, an element $x \in X$ acts by translation by $\rho(x) \in V$ on V. It is therefore easy to see that

$$\vec{d}(C_0, \rho(x)(C_0))_\alpha = -\alpha(\rho(x))$$

leading to the more useful formula

$$\ell(x) = \sum_{\alpha \in \Phi^+} |\alpha(\rho(x))|, \quad x \in X \tag{2.3.2}$$

2.3.4 Remark. Let us now return to the problem posed above, to determine when three chambers C, C', C'' fulfill the relation

$$d(C, C'') = d(C, C') + d(C', C'')$$

and let us see how the vector-valued distance helps in making this problem more transparent. Immediately from the definition it follows that

$$\vec{d}(C, C'') = \vec{d}(C, C') + \vec{d}(C', C'')$$

By equation (2.3.1), we are therefore reduced to determine for which $x, y \in \mathbb{Z}^{\Phi^+}$ we have

$$|x + y| = |x| + |y|$$

where $|x| = \sum_\alpha |x_\alpha|$. Let $\preceq$ denote relation on $\mathbb{Z}^{\Phi^+}$ defined by

$$\begin{aligned} x \preceq y :&\Leftrightarrow |y| = |x| + |y - x| \\ &\Leftrightarrow x_\alpha(y_\alpha - x_\alpha) \geq 0 \quad \forall \alpha \\ &\Leftrightarrow x_\alpha = 0 \ \vee \ (x_\alpha y_\alpha > 0 \ \wedge \ |x_\alpha| \leq |y_\alpha|) \quad \forall \alpha \end{aligned} \tag{2.3.3}$$

It is easy to see that $\preceq$ is a partial order. Moreover, the above problem can now be phrased equivalently in terms of $\preceq$ as

$$d(C, C'') = d(C, C') + d(C', C'') \quad \Leftrightarrow \quad \vec{d}(C, C') \preceq \vec{d}(C, C'') \tag{2.3.4}$$

2.3.5 Remark. In particular, fixing a chamber C, the relation

$$\begin{aligned} C' \preceq_C C'' \quad &:\Leftrightarrow \quad d(C, C'') = d(C, C') + d(C', C'') \\ &\Leftrightarrow \quad \vec{d}(C, C') \preceq \vec{d}(C, C'') \end{aligned}$$

defines a partial order on chambers. For $C = C_0$ this is just the *weak Bruhat order*, i.e. the partial order induced on W_{aff} via

$$\begin{aligned} w' \preceq w'' \quad &:\Leftrightarrow \quad w'(C_0) \preceq_{C_0} w''(C_0) \\ &\Leftrightarrow d(C_0, w''(C_0)) = d(C_0, w'(C_0)) + d(w'(C_0), w''(C_0)) \\ &\Leftrightarrow \quad \ell(w'') = \ell(w') + \ell((w')^{-1} w'') \end{aligned}$$

is the weak Bruhat order.

It is known that the weak Bruhat order on an affine Coxeter group is a *well partial order*, in fact the affine Coxeter groups are characterized among the infinite Coxeter group as those for which this property holds (see [Hul07]). In the next lemma we will prove that $\preceq$ defines a well partial order on $\mathbb{Z}^{\Phi^+}$, thus recovering the first statement about the weak Bruhat order, as the proof is not difficult and moreover the result is crucial for the structure theory of affine pro-p Hecke algebras. In fact, the well partial order property guarantees that $\mathcal{H}^{(1)}$ is finitely generated as a left module over a certain subalgebra $\mathcal{A}_{\mathfrak{o}}^{(1)} \subseteq \mathcal{H}^{(1)}$ (see the proof of theorem 2.7.1), which is an important step in showing that $\mathcal{H}^{(1)}$ is finitely generated as a module over its center.

Let us recall the notion of a well partial order (cf. [Kru72]).

2.3.6 Definition. A partial order $\leq$ on a set X is said to be a **well partial order** if for every nonempty subset $\Lambda \subseteq X$ the set $\min(\Lambda)$ of minimal elements of Λ is *nonempty* and *finite*.

Obviously this generalizes the notion of a well ordering from total orders to partial orders, hence the name. Let us now show that $\preceq$ defines a well partial order on $\mathbb{Z}^{\Phi^+}$.

2.3.7 Lemma ("Dickson's lemma"). *$(\mathbb{Z}^{\Phi^+}, \preceq)$ is a well partial order.*

Proof. Let $\Lambda \subseteq X$ be a nonempty subset and assume that $\min(\Lambda)$ was infinite. We would then find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of pairwise distinct elements $\lambda_n \in \min(\Lambda)$, which would necessarily be also pairwise incomparable. Choose a numbering $\Phi^+ = \{\alpha_1, \ldots, \alpha_m\}$ of the positive roots, and look at the sequence $(\lambda_n(\alpha_1))_{n\in\mathbb{N}}$ of 'first coordinates'.

There are two possibilities, either this sequence is finite or infinite. In the first case we may (after possibly replacing $(\lambda_n)_{n\in\mathbb{N}}$ by a subsequence) assume that the sequence $(\lambda_n(\alpha_1))_{n\in\mathbb{N}}$ is constant. In the second case we can assume (again replacing $(\lambda_n)_{n\in\mathbb{N}}$ by a subsequence if necessary) that the sequence $(\lambda_n(\alpha_1))_{n\in\mathbb{N}}$ is strictly increasing or decreasing with respect to the usual total order on $\mathbb{Z}$, i.e. strictly increasing with respect to the well partial order (!) $x \preceq y :\Leftrightarrow x(y-x) \geq 0$ on $\mathbb{Z}$.

Repeating this procedure with $\alpha_2, \alpha_3, \ldots, \alpha_m$, we may therefore assume that for every $\alpha \in \Phi^+$ the sequence $(\lambda_n(\alpha))_{n\in\mathbb{N}}$ is either constant or strictly increasing with respect to the well partial order $\preceq$ on $\mathbb{Z}$. In particular, since the order $(\mathbb{Z}^{\Phi^+}, \preceq)$ is just the power of the order $(\mathbb{Z}, \preceq)$, we would have $\lambda_1 \preceq \lambda_2$, contradicting the fact that the λ_n are pairwise incomparable. $\square$

2.3.8 Corollary. *For every chamber C, the relation $\preceq_C$ on the set of chambers defined in remark 2.3.5 is a well partial order.*

2.3.9 Remark. Obviously the above proof holds verbatim with $(\mathbb{Z}, \preceq)$ replaced by any well partial order, and the argument recovers the basic fact that finite products of well partial orders are again well partial orders (cf. [Kru72]).

As already mentioned, the well partial order property of $\preceq$ is crucial for proving the finiteness of $\mathcal{H}^{(1)}$ as a left module over $\mathcal{A}_{\mathfrak{o}}^{(1)}$. But it is also crucial for proving yet another finiteness property, namely it ensures that $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is finitely generated as an algebra (see theorem 2.7.1). This rests on the finiteness property of the submonoids $X_D \leq X$ defined below, which we will prove in the next lemma.

2.3.10 Definition. Given a Weyl chamber $D \in \pi_0\left(V - \bigcup_{\alpha\in\Phi} H_\alpha\right)$, we let

$$X_D := \{x \in X \ : \ \rho(x) \in \overline{D}\}$$

be the submonoid of X consisting of all elements which act by translation by an element of the closure of $D \subseteq V$ under $\rho : W \to \mathrm{Aut}_{\mathrm{aff}}(V)$.

2.3.11 Lemma ("Gordan's lemma"). *X_D is finitely generated as a monoid.*

Proof. Consider the evaluation map

$$\begin{aligned} \nu : V &\longrightarrow \mathbb{R}^{\Phi^+} \\ v &\longmapsto (\alpha \mapsto \alpha(v)) \end{aligned}$$

Since the action of X preserves the set $\mathfrak{H}$ of affine hyperplanes and by definition of Φ we have for every $\alpha \in \Phi$

$$\forall k \in \mathbb{R} \quad H_{\alpha,k} \in \mathfrak{H} \Leftrightarrow k \in \mathbb{Z}$$

it follows from $\rho(x)(H_{\alpha,0}) = H_{\alpha,-\alpha(\rho(x))} \in \mathfrak{H}$ that $\alpha(\rho(x)) \in \mathbb{Z}$ for every $x \in X$. Hence, $\nu(\rho(x))$ lies in $\mathbb{Z}^{\Phi^+}$ and we may consider the image

$$\Xi_D := \nu(\rho(X_D)) \subseteq \mathbb{Z}^{\Phi^+}$$

As ν is a group homomorphism, $\Xi_D \subseteq \mathbb{Z}^{\Phi^+}$ is a submonoid. Moreover, the partial order $\preceq$ restricted to Ξ_D is compatible with the monoid structure in the sense that

$$a \preceq a+b \quad \forall a,b \in \Xi_D \tag{2.3.5}$$

To see this, let $\varepsilon_{D,\alpha}$ denote the sign of $\alpha(q)$ for $q \in D$ arbitrary. We then have the following equivalence for an element $v \in V$

$$v \in \overline{D} \quad \Leftrightarrow \quad \forall \alpha \in \Phi^+ \;\; \varepsilon_{D,\alpha}\alpha(v) \geq 0$$

The implication '$\Rightarrow$' is obvious and the reverse implication follows by choosing a point $q \in D$ and noting that v lies in the closure of the half-open line segment

$$\{(1-\lambda)q + \lambda v \;:\; 0 \leq \lambda < 1\} \subseteq D$$

This implies that Ξ_D is characterized as

$$\Xi_D = \{a \in \nu(\rho(X)) \;:\; \forall \alpha \in \Phi^+ \;\; \varepsilon_{D,\alpha}a(\alpha) \geq 0\} \tag{2.3.6}$$

In particular, for $a, b \in \Xi_D$ and every $\alpha \in \Phi^+$ we have $a(\alpha)b(\alpha) \geq 0$ and hence $a \preceq a+b$ by definition of $\preceq$.

Let us now call an element $a \in \Xi_D$ *irreducible* if $a \neq 0$ and a cannot be written as a sum $a = b + c$ with $b, c \in \Xi_D$ and $b, c \neq 0$. Since $\preceq$ is a well partial order, it is in particular a well-founded relation. This implies that every element $a \in \Xi_D$ can be written as a (possibly empty) sum of irreducible elements. Indeed, if this was not the case, we repeatedly expand a as a sum

$$a = a_1 + b_1 = a_2 + b_2 + b_1 = a_3 + b_3 + b_2 + b_1 = \dots$$

with $a_i, b_i \neq 0$ and it would follow from property (2.3.5) that we would have an infinite strictly descending chain

$$\dots \prec a_2 \prec a_1 \prec a$$

contradicting the fact that $\preceq$ is well-founded.

Hence, every element $a \in \Xi_D$ can be written as a sum of irreducible elements. Because of property (2.3.5), every element of $\Xi_D - \{0\}$ minimal with respect to $\preceq$ is irreducible. But the converse also holds. Indeed, by (2.3.6) and the fact that $\nu(\rho(X)) \subseteq \mathbb{Z}^{\Phi^+}$ is a subgroup, it follows that for $a, b \in \Xi_D$ we have the implication

$$a \preceq b \quad \Rightarrow \quad b - a \in \Xi_D$$

Namely if $b-a$ would not lie in Ξ_D, it would follow from (2.3.6) that $\varepsilon_{D,\alpha}(b-a)(\alpha) < 0$ for some $\alpha \in \Phi^+$. If $a(\alpha) = 0$, this would imply that $\varepsilon_{D,\alpha}b(\alpha) < 0$ and hence $b \notin \Xi_D$ by (2.3.6) again. If $a(\alpha) \neq 0$, it would follow from $a \preceq b$ that $\operatorname{sgn}(a(\alpha)) = \operatorname{sgn}((b-a)(\alpha))$ and hence $\varepsilon_{D,\alpha}a(\alpha) < 0$ implying $a \notin \Xi_D$ by (2.3.6).

Therefore, the irreducible elements are precisely the minimal elements of $\Xi_D - \{0\}$ with respect to $\preceq$. Since $\preceq$ is a well partial order, this set is finite. Hence, there exist finitely many elements $x_1, \dots, x_r \in X_D$ such that every element $x \in X$ can be written as

$$x = \sum_{i=1}^{r} n_i x_i + y$$

with $n_i \in \mathbb{Z}_{\geq 0}$ and

$$y \in \ker(\nu \circ \rho)$$

By (**ACX**) X, is a finitely generated abelian group. Hence, the subgroup $\ker(\nu \circ \rho) \leq X$ is also finitely generated as a group, say by $y_1, \dots, y_s$. Hence

$$\{x_1, \dots, x_r, y_1, -y_1, \dots, y_s, -y_s\}$$

forms a set of generators of X_D as a monoid. □

For later reference we need to record another property of X_D.

2.3.12 Lemma. *The submonoid $X_D \leq X$ generates X as a group, i.e. every element $x \in X$ can be written as*

$$x = y - z$$

with $y, z \in X_D$.

Proof. It suffices to show that the subset (using the notation of the proof of the previous lemma)

$$\{x \in X \ : \ \rho(x) \in D\} = \{x \in X \ : \ \forall \alpha \in \Phi^+ \ \ \varepsilon_{D,\alpha}\alpha(\rho(x)) > 0\} \subseteq X_D$$

is non-empty, since if y denotes an element of this set, then for $n \in N$ sufficiently large we have

$$x + ny \in X_D$$

and hence

$$x = (x + ny) - ny$$

with $x + ny, ny \in X_D$. Let $p : V \twoheadrightarrow V/L$ denote the projection, where

$$L = \bigcap_{\alpha \in \Phi^+} H_\alpha$$

is the common kernel of the $\alpha \in \Phi^+$. Denoting $\overline{\alpha} : V/L \to \mathbb{R}$ the functional induced by $\alpha \in \Phi^+$, we have

$$\{x \in X \ : \ \forall \alpha \in \Phi^+ \ \ \varepsilon_{D,\alpha}\alpha(\rho(x))\} = \{x \in X \ : \ \forall \alpha \in \Phi^+ \ \ \varepsilon_{D,\alpha}\overline{\alpha}(p(\rho(x))) > 0\}$$

Thus it suffices to show that the subgroup $p(\rho(X)) \leq V/L$ has non-empty intersection with the image

$$(V \twoheadrightarrow V/L)(D) = \{x \in V/L \ : \ \forall \alpha \in \Phi^+ \ \ \varepsilon_{D,\alpha}\overline{\alpha}(v) > 0\} \subseteq V/L$$

of the chamber D under $V \twoheadrightarrow V/L$. But since by (**ACV**) this subgroup generates V/L as an $\mathbb{R}$-vector space, it contains a basis and hence a full sublattice of V/L. And since the image of D is a non-empty open cone in V/L, it has non-empty intersection with every full sublattice of V/L. □

2.4 Spherical orientations

In this section we fix an affine extended Coxeter group W.

Our goal (in view of remark 1.10.8) is to construct for every Weyl chamber D an orientation $\mathfrak{o}_D$ satisfying

$$\mathfrak{o}_D \bullet x = \mathfrak{o}_D \quad \forall x \in X$$

and

$$\mathfrak{o}_D \bullet w = \mathfrak{o}_{w^{-1}(D)} \quad \forall w \in W_0$$

The construction of $\mathfrak{o}_D$ can be seen as a variant of the orientations $\mathfrak{o}_{w_0}$ defined in definition 1.5.7. Instead of ‘orienting towards’ a chamber $w \in W_{\mathrm{aff}}$ of the affine chamber complex corresponding to W_{aff}, we orient towards the chamber induced by D in the ‘spherical chamber complex at infinity’. In fact, we will show that $\mathfrak{o}_D$ is the limit

$$\mathfrak{o}_D = \lim_{w_0} \mathfrak{o}_{w_0}$$

in the sense of *nets*, where the limit is taken over the directed set of chambers endowed with the *dominance order* induced by D (defined below).

Let us now define these orientations.

2.4.1 Definition. Given a Weyl chamber $D \in \pi_0\left(V - \bigcup_{\alpha \in \Phi} H_\alpha\right)$, the associated **spherical orientation $\mathfrak{o}_D$** of W is the map

$$\mathfrak{o}_D : W \times S \longrightarrow \{\pm\}$$

defined as follows. Given $w \in W$ and $s \in S$, let $(\alpha, k) \in \Phi \times \mathbb{Z}$ be the unique pair such that α is *D-positive*, i.e.

$$D \subseteq \{v \in V \ : \ \alpha(v) > 0\}$$

and such that $H_{\alpha,k}$ is the hyperplane separating $w(C_0)$ and $ws(C_0)$. Then let

$$\mathfrak{o}_D(w, s) := \mathrm{sgn}\left(\pi_0(\alpha)(ws(C_0)) - \pi_0(\alpha)(w(C_0))\right)$$

where

$$\pi_0(\alpha) : \pi_0\left(V - \bigcup_{H \in \mathfrak{H}} H\right) \longrightarrow \pi_0(\mathbb{R} - \mathbb{Z})$$

is the map induced on connected components by the restriction of α, and the difference is to be understood with respect to the structure on $\pi_0(\mathbb{R} - \mathbb{Z})$ of an affine space over $\mathbb{Z}$.

2.4.2 Remark. From the defining formula of $\mathfrak{o}_D$ it follows immediately that

$$\mathfrak{o}_D \bullet w = \mathfrak{o}_{\rho_0(w)^{-1}(D)} \quad \forall w \in X$$

In particular

$$\mathfrak{o}_D \bullet x = \mathfrak{o}_D \quad \forall x \in X$$

and

$$\mathfrak{o}_D \bullet w = \mathfrak{o}_{w^{-1}(D)} \quad \forall w \in W_0$$

However, we still have to show that $\mathfrak{o}_D$ actually is an orientation.

2.4.3 Remark. In lemma 1.7.4 we have seen that orientations are given by singling out for every hyperplane $H \in \mathfrak{H}$ one of the two half-spaces bounded by H as positive, such that $\mathfrak{o}(w,s) = +$ iff ws lies in the positive half-space bounded by $H = wsw^{-1}$, where the notions of hyperplane and half-space are to be understood in the sense of abstract Coxeter geometry. Unwinding the above definition, one sees that under the dictionary between the abstract geometry of W and the concrete geometry of the hyperplane arrangement $(V, \mathfrak{H})$, the orientation $\mathfrak{o}_D$ is given by letting

$$U_H^+ = \{v \in V \ : \ \alpha(v) + k > 0\}$$

be the positive half-space bounded by $H = H_{\alpha,k}$ if α is D-positive.

2.4.4 Definition. Given a Weyl chamber $D \in \pi_0\left(V - \bigcup_{\alpha\in\Phi} H_\alpha\right)$ the **dominance order $\preccurlyeq_D$ associated to** D is the partial order on the set of chambers given by

$$C \preccurlyeq_D C' \quad :\Leftrightarrow \quad \pi_0(\alpha)(C) \leq \pi_0(\alpha)(C') \quad \forall \alpha \ D\text{-positive}$$

where $\pi_0(\mathbb{R} - \mathbb{Z})$ is endowed with the total order $\leq$ induced from $\mathbb{R}$.

2.4.5 Remark. Obviously $\preccurlyeq$ is a partial order. Moreover, any two chambers C, C' are dominated $C, C' \preccurlyeq_D C''$ by a third, thus making the set of chambers endowed with $\preccurlyeq_D$ into a directed set.

Indeed, for a D-positive root α let

$$r_\alpha := \max(\sup \pi_0(\alpha)(C), \sup \pi_0(\alpha)(C')) \in \mathbb{Z}$$

Then any chamber C'' contained in

$$U := \{v \in V : \alpha(v) > r_\alpha \quad \forall \alpha \ D\text{-positive}\}$$

satisfies $C, C' \preccurlyeq_D C''$. It's easy to see that such a chamber always exists. Since

$$D = \{v \in V : \alpha(v) > 0 \quad \forall \alpha \ D\text{-positive}\} \neq \emptyset$$

it follows that U must also be non-empty, hence it (as an open non-empty subset in V) must meet some chamber C'', which then must already be contained in U.

Let us now show that 'spherical orientations' are indeed orientations.

2.4.6 Proposition. *The map $\mathfrak{o}_D$, considered as an element of the mapping space $\{\pm\}^{W\times S}$ with its compact-open topology (cf. remarks 1.5.6 and 1.5.18), is the limit*

$$\mathfrak{o}_D = \lim_C \mathfrak{o}_C$$

in the sense of nets, where the limit is taken over the directed set of chambers endowed with the dominance order $\preccurlyeq_D$, and where $\mathfrak{o}_C = \mathfrak{o}_w$ denotes the orientation towards the 'chamber' w in the sense of definition 1.5.7 and $w \in W_{\text{aff}}$ is the unique abstract chamber corresponding to C via $w(C_0) = C$.

In particular, $\mathfrak{o}_D$ lies in the closure of

$$\{\mathfrak{o}_w : w \in W_{\text{aff}}\} \subseteq \{\pm\}^{W\times S}$$

and hence by remark 1.5.18 it also lies in the subset of orientations.

Proof. To show that

$$\mathfrak{o}_D = \lim_C \mathfrak{o}_C$$

means concretely to show that for every $w \in W$ and $s \in S$ we have

$$\mathfrak{o}_D(w,s) = \mathfrak{o}_C(w,s)$$

for C sufficiently large with respect to $\preccurlyeq_D$. Recall that we have $\mathfrak{o}_C(w,s) = +$ iff $ws(C_0)$ is closer to C than $w(C_0)$, i.e. iff the hyperplane H separating $w(C_0)$ and $ws(C_0)$ also separates $w(C_0)$ from C, i.e. if C and $ws(C_0)$ lie in the same half-space with respect to H. Let $H = H_{\alpha,k}$ with α D-positive. Then on the other hand we have $\mathfrak{o}_D(w,s) = +$ iff $\pi_0(\alpha)(w(C_0)) < \pi_0(\alpha)(ws(C_0))$, i.e. if $ws(C_0)$ lies in the positive half-space U_H^+ determined by $\mathfrak{o}_D$. Therefore, $\mathfrak{o}_D(w,s) = \mathfrak{o}_C(w,s)$ iff C lies in the positive half-space U_H^+. Moreover, if C, C' are chambers with $C \subseteq U_H^+$ and $C \preccurlyeq_D C'$ then C' also lies in U_H^+. Letting C denote an arbitrary chamber contained in U_H^+, we therefore have

$$\mathfrak{o}_D(w,s) = \mathfrak{o}_{C'}(w,s)$$

for every chamber C' with $C \preccurlyeq_D C'$. □

2.5 Some (almost) commutative subalgebras

In this section, we let $\mathfrak{o}$ denote an arbitrary spherical orientation (see definition 2.4.1) of W. In remark 1.10.8, we saw that every submonoid $U \leq \mathrm{Stab}_{W^{(1)}}(\mathfrak{o})$ gives rise to a subalgebra $\mathcal{A}_{\mathfrak{o}}^{(1)}(U) \subseteq \mathcal{H}^{(1)}$ that has a canonical R-basis $\{\widehat{\theta}_{\mathfrak{o}}(x)\}_{x\in U}$ indexed by the elements of U. By remark 2.4.2, we may take $U = X^{(1)}$.

2.5.1 Definition.

$$\mathcal{A}_{\mathfrak{o}}^{(1)} := \mathcal{A}_{\mathfrak{o}}^{(1)}(X^{(1)}) = \bigoplus_{x\in X^{(1)}} R\widehat{\theta}_{\mathfrak{o}}(x)$$

2.5.2 Remark. Recall from remark 1.10.8 that the subalgebra $\mathcal{A}_{\mathfrak{o}}^{(1)}(X^{(1)})$ is commutative if the subgroup $X^{(1)}$ is. Since $X^{(1)}$ is an extension

$$1 \longrightarrow T \longrightarrow X^{(1)} \longrightarrow X \longrightarrow 0$$

of abelian groups, certainly $X^{(1)}$ is commutative if this sequence splits (the reverse doesn't need to hold unless X is projective). For example, when $W^{(1)}$ is the pro-p Coxeter group associated to a connected reductive group $\mathbf{G}$ over a nonarchimedean local field F as in section 2.2.3, the sequence above splits when the group $\mathbf{G}$ is split because

$$X^{(1)} = Z/Z_0(1)$$

, where Z is the group of F-rational points of the centralizer $\mathbf{Z} = \mathbf{Z}(\mathbf{T})$ of the chosen maximal split torus $\mathbf{T} \leq \mathbf{G}$, and because (see [Mil17, 17.61])

$$\mathbf{G} \text{ split} \quad \Leftrightarrow \quad \mathbf{Z} = \mathbf{T}$$

As a first step towards the computation of the center of $\mathcal{H}^{(1)}$ in theorem 2.6.3, we will now determine the centralizer of the subalgebra $\mathcal{A}_{\mathfrak{o}}^{(1)}$ of $\mathcal{H}^{(1)}$. Here and in theorem 2.6.3, we will make use of the following auxiliary notion.

2.5.3 Definition. Given an element

$$z = \sum_{w\in W^{(1)}} c_w \widehat{\theta}_{\mathfrak{o}}(w) \in \mathcal{H}^{(1)}, \quad c_w \in R$$

and an orientation $\mathfrak{o}$ of $W^{(1)}$, the set

$$\mathrm{supp}_{\mathfrak{o}}(z) := \{w \in W^{(1)} : c_w \neq 0\}$$

is called the **support** of z (with respect to $\mathfrak{o}$).

2.5.4 Proposition. *The centralizer $C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)})$ of the R-subalgebra*

$$\mathcal{A}_{\mathfrak{o}}^{(1)} \subseteq \mathcal{H}^{(1)}$$

is given by the $X^{(1)}$-invariants

$$C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)}) = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$$

with respect to the R-linear $X^{(1)}$-action on $\mathcal{A}^{(1)}$ determined by

$$x(\widehat{\theta}_{\mathfrak{o}}(y)) = \widehat{\theta}_{\mathfrak{o}}(xyx^{-1})$$

In particular $\mathrm{Z}(\mathcal{H}^{(1)}) \subseteq \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$.

Proof. First, we show that $C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)}) \subseteq \mathcal{A}_{\mathfrak{o}}^{(1)}$. For this, consider an arbitrary element z of the centralizer of $\mathcal{A}_{\mathfrak{o}}^{(1)}$ in $\mathcal{H}^{(1)}$. Write

$$z = \sum_{w \in W^{(1)}} c_w \widehat{\theta}_{\mathfrak{o}}(w), \quad c_w \in R$$

We need to show that $\mathrm{supp}_{\mathfrak{o}}(z) \subseteq X^{(1)}$. Assume this is not the case and choose $w \in \mathrm{supp}_{\mathfrak{o}}(z) - X^{(1)}$ with $\ell(w)$ maximal. Fix an element $x \in X^{(1)}$ such that $\pi(x) \in \Xi$, where $\Xi \subseteq X$ is the set associated to w by lemma 2.5.6 below. Consider now the elements $\widehat{\theta}_{\mathfrak{o}}(x)z$ and $z\widehat{\theta}_{\mathfrak{o}}(x)$. Using the product formula (corollary 1.10.5) and the fact that $\mathfrak{o}$ is invariant under X, we see that on the one hand we have

$$\widehat{\theta}_{\mathfrak{o}}(x)z = \sum_{w' \in W^{(1)}} c_{w'} \widehat{\theta}_{\mathfrak{o}}(x)\widehat{\theta}_{\mathfrak{o}}(w') = \sum_{w' \in W^{(1)}} c_{w'} \overline{\mathbb{X}}(x, w')\widehat{\theta}_{\mathfrak{o}}(\tau^x w')$$

On the other hand we have (again using the product formula)

$$\begin{aligned} z\widehat{\theta}_{\mathfrak{o}}(x) &= \sum_{w' \in W^{(1)}} c_{w'} \widehat{\theta}_{\mathfrak{o}}(w')(\widehat{\theta}_{\mathfrak{o} \bullet w'}(x) + \widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o} \bullet w'}(x)) \\ &= \sum_{w' \in W^{(1)}} c_{w'} \overline{\mathbb{X}}(w', x)\widehat{\theta}_{\mathfrak{o}}(w'\tau^x) + \sum_{w' \in W^{(1)}} c_{w'} \widehat{\theta}_{\mathfrak{o}}(w')(\widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o} \bullet w'}(x)) \end{aligned}$$

By the change of basis formula (corollary 1.10.7), the expansions of the two elements $\widehat{\theta}_{\mathfrak{o}}(x)$ and $\widehat{\theta}_{\mathfrak{o} \bullet w'}(x)$ in the Iwahori-Matsumoto basis $\{T_{w''}\}_{w'' \in W^{(1)}}$ have the same leading term T_x with respect to the Bruhat order on $W^{(1)}$. Therefore, $\widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o} \bullet w'}(x)$ is an R-linear combination of terms $T_{w''}$ with $w'' < \tau^x$ and hence $\ell(w'') < \ell(x)$. It follows that in the expansion of $\widehat{\theta}_{\mathfrak{o}}(w')(\widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o} \bullet w'}(x))$ in the Iwahori-Matsumoto basis only terms $T_{w''}$ with

$$\ell(w'') < \ell(w') + \ell(x) \leq \ell(w) + \ell(x) = \ell(\tau^x w)$$

appear. Using corollary 1.10.7 again, it follows that the same is true for the expansion of this expression in the basis $\{\widehat{\theta}_{\mathfrak{o}}(w'')\}_{w'' \in W^{(1)}}$. In particular, the coefficient of $\widehat{\theta}_{\mathfrak{o}}(\tau^x w)$ vanishes. Comparing the coefficients of $\widehat{\theta}_{\mathfrak{o}}(\tau^x w)$ on both sides of the equation $\widehat{\theta}_{\mathfrak{o}}(x)z = z\widehat{\theta}_{\mathfrak{o}}(x)$, we see that there exists $w' \in W^{(1)}$ such that $\tau^x w = w'\tau^x$ and

$$c_w \overline{\mathbb{X}}(x, w) = c_{w'} \overline{\mathbb{X}}(w', x)$$

Since $\pi(x) \in \Xi$ we have $\ell(\tau^x w) = \ell(x) + \ell(w)$ by definition of Ξ and hence $\overline{\mathbb{X}}(x, w) = 1$ by remark 1.7.2. Since $c_w \neq 0$ by assumption, it follows from the above equation that $c_{w'} \neq 0$ and hence $w' \in \mathrm{supp}_{\mathfrak{o}}(z)$. Moreover, we have

$$w' = \tau^x w \tau^{-x} = \tau^{x - w(x)} w$$

By lemma 2.5.6 below, we can assume that x has been chosen such that $\ell(w(x) - x) > 2\ell(w)$. But then

$$\ell(w') = \ell(\tau^{x-w(x)} w) \geq \ell(\tau^{x-w(x)}) - \ell(w) > \ell(w)$$

But this is a contradiction to the choice of w, and hence we have shown that

$$C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)}) \subseteq \mathcal{A}_{\mathfrak{o}}^{(1)}$$

Now in order to show that

$$C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)}) \subseteq \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$$

we have to show that the coefficients of z satisfy

$$c_x = c_{yxy^{-1}} \quad \forall x, y \in X^{(1)}$$

By lemma 2.5.7 below, it suffices to show this for $y \in X^{(1)}$ satisfying $\ell(xy) = \ell(x) + \ell(y)$. From

$$\widehat{\theta}_{\mathfrak{o}}(y)z = z\widehat{\theta}_{\mathfrak{o}}(y)$$

and the product formula it follows immediately that

$$\overline{\mathbb{X}}(y,x)c_x = \overline{\mathbb{X}}(yxy^{-1},y)c_{yxy^{-1}}$$

Since the image of $X^{(1)}$ under $\pi : W^{(1)} \to W$ is commutative, we have

$$\overline{\mathbb{X}}(yxy^{-1},y) = \overline{\mathbb{X}}(x,y)$$

by definition of $\overline{\mathbb{X}}$. Moreover, from $\ell(xy) = \ell(x) + \ell(y)$ and remark 1.7.2 it follows that

$$\overline{\mathbb{X}}(x,y) = \overline{\mathbb{X}}(y,x) = 1$$

Therefore

$$c_x = c_{yxy^{-1}}$$

Thus it only remains to show the reverse inclusion

$$\left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}} \subseteq C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)})$$

So let

$$z = \sum_{x\in X^{(1)}} c_x \widehat{\theta}_{\mathfrak{o}}(x)$$

be an element of the invariants, i.e.

$$c_x = c_{yxy^{-1}}, \quad \forall x,y \in X^{(1)}$$

We need to show that

$$z\widehat{\theta}_{\mathfrak{o}}(y) = \widehat{\theta}_{\mathfrak{o}}(y)z \quad \forall y \in X^{(1)}$$

This amounts to showing that

$$\overline{\mathbb{X}}(y,x)c_x = \overline{\mathbb{X}}(yxy^{-1},y)c_{yxy^{-1}}$$

for *all* $x,y \in X^{(1)}$. But since

$$\overline{\mathbb{X}}(yxy^{-1},y) = \overline{\mathbb{X}}(x,y)$$

this follows from

$$\overline{\mathbb{X}}(x,y) = \overline{\mathbb{X}}(y,x)$$

□

2.5.5 Remark. The action of $X^{(1)}$ on itself is trivial if and only if $X^{(1)}$ is commutative, and when this is the case, it follows from proposition 2.5.4 that the subalgebra

$$\mathcal{A}_{\mathfrak{o}}^{(1)} = \mathcal{A}_{\mathfrak{o}}^{(1)}(X^{(1)}) \subseteq \mathcal{H}^{(1)}$$

(which is then also commutative; see remark 1.10.8) equals its own centralizer. In particular, this is the case when $W^{(1)}$ arises from a *split* reductive group $\mathbf{G}$ (see remark 2.5.2), and one should see the equality

$$\mathcal{A}_{\mathfrak{o}}^{(1)} = C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)})$$

as a reflection of the equality

$$\mathbf{T} = \mathbf{Z}_{\mathbf{G}}(\mathbf{T})$$

between the split maximal torus $\mathbf{T}$ and its centralizer $\mathbf{Z}_{\mathbf{G}}(\mathbf{T})$ in $\mathbf{G}$.

2.5.6 Lemma. *Let $W = X \rtimes W_0$ be an affine extended Coxeter group (see definition 2.1.1 and lemma 2.1.2 for notation). Let $w \in W$ with $w \notin X$. Then the set*

$$\Xi := \{x \in X : \ell(\tau^x w) = \ell(x) + \ell(w)\}$$

satisfies

$$\sup\{\ell(w(x) - x) : x \in \Xi\} = \infty$$

Proof. By remark 2.3.5 we know that

$$\ell(\tau^x w) = \ell(x) + \ell(w) \quad \Leftrightarrow \quad \vec{d}(C_0, \tau^x(C_0)) \preceq \vec{d}(C_0, (\tau^x w)(C_0)) \tag{2.5.1}$$

where C_0 denotes the fundamental chamber and $\vec{d}$ is the 'vector-valued distance' with values in $\mathbb{Z}^{\Phi^+}$ and $\preceq$ the partial order on $\mathbb{Z}^{\Phi^+}$ defined in remark 2.3.4. Moreover, from the definition of $\vec{d}$ it follows immediately that

$$\vec{d}(C_0, \tau^x(C_0)) = -\nu(\rho(x)) \quad \text{and} \quad \vec{d}(C_0, (\tau^x w)(C_0)) = -\nu(\rho(x)) + \vec{d}(C_0, w(C_0))$$

where ν is the evaluation

$$\nu : V \longrightarrow \mathbb{R}^{\Phi^+}, \quad v \longmapsto (\alpha \mapsto \alpha(v))$$

map. Note that $\nu(\rho(x)) \in \mathbb{Z}^{\Phi^+}$, as we verified in the proof of lemma 2.3.11. Let $D \in \pi_0\left(V - \bigcup_{\alpha\in\Phi} H_\alpha\right)$ be the Weyl chamber containing $w(C_0)$. For $\alpha \in \Phi^+$ let

$$\varepsilon_{D,\alpha} := \operatorname{sgn}(\alpha(p))$$

where $p \in D$ is any point. Then

$$D = \{x \in V \ : \ \forall\alpha \in \Phi^+ \ \ \varepsilon_{D,\alpha}\alpha(x) > 0\}$$

and hence the closure of D is given by (cf. proof of lemma 2.3.11)

$$\overline{D} = \{x \in V \ : \ \forall\alpha \in \Phi^+ \ \ \varepsilon_{D,\alpha}\alpha(x) \geq 0\}$$

Moreover, by choosing p to lie in $w(C_0)$ it follows easily from the definition of $\vec{d}$ (remembering that $0 \in \overline{C_0}$) that

$$-\varepsilon_{D,\alpha}\vec{d}(C_0, w(C_0))_\alpha \geq 0 \quad \forall\alpha \in \Phi^+$$

From the above and the definition of $\preceq$ it follows that

$$-\nu(\rho(x)) \preceq -\nu(\rho(x)) + \vec{d}(C_0, w(C_0))$$

for all $x \in X_D$, where

$$X_D = \{x \in X \ : \ \rho(x) \in \overline{D}\}$$

From (2.5.1) it therefore follows that

$$X_D \subseteq \Xi$$

Since

$$\ell(x) = |\vec{d}(C_0, \tau^x(C_0))| = |-\nu(\rho(x))| = \sum_{\alpha\in\Phi^+} |\alpha(\rho(x))|$$

it follows from the definition of X_D that

$$\ell(x+y) = \ell(x) + \ell(y) \quad \forall x, y \in X_D$$

In particular we have $\ell(nx) = n\ell(x)$ for $n \in \mathbb{N}$, so in order to prove the claim it suffices to show that

$$\{\ell(w(x) - x) \ : \ x \in X_D\}$$

contains a nonzero element. If this was not the case, we would have

$$\rho(w(x) - x) = \rho_0(w)(\rho(x)) - \rho(x) \in L = \bigcap_{\alpha\in\Phi^+} H_\alpha$$

for all $x \in X_D$, where we recall that $\rho_0 : W \to \mathrm{GL}(V)$ denotes the composition of $\rho : W \to \mathrm{Aut}_{\mathrm{aff}} = V \rtimes \mathrm{GL}(V)$ with the projection onto the linear part. But every $x \in X$ can be written as a difference $x = y - z$ with $y, z \in X_D$ by lemma 2.3.12, hence we would have

$$\rho_0(w)(v) - v \in L$$

for all $v \in \rho(X)$. Since the image of $\rho(X) \subseteq V$ under $V \twoheadrightarrow V/L$ generates the vector space V/L by (**ACV**), it would follow that $\rho_0(w)$ acts trivially on the quotient V/L. But by lemma 2.1.2 the group $W_0 = \rho_0(W)$ acts faithfully on V/L, hence

$$w \in \ker(\rho_0) = X$$

contradicting the assumption. $\square$

2.5.7 Lemma. *For all $x \in X^{(1)}$*

$$\{yxy^{-1} : y \in X^{(1)}\} = \{yxy^{-1} : y \in X^{(1)},\ \ell(xy) = \ell(x) + \ell(y)\}$$

Proof. From remark 2.3.3 recall equation (2.3.2)

$$\ell(x) = \ell(\pi(x)) = \sum_{\alpha \in \Phi^+} |\alpha(\rho(\pi(x)))|$$

Now given any $x, y \in X^{(1)}$ we have

$$\widetilde{x} := yxy^{-1} = \widetilde{x}^k \widetilde{x} \widetilde{x}^{-k} = (\widetilde{x}^k y) x (\widetilde{x}^k y)^{-1}$$

for all $k \in \mathbb{Z}$. It therefore suffices to show that

$$\ell(x\widetilde{x}^k y) = \ell(x) + \ell(\widetilde{x}^k y)$$

for $k > 0$ sufficiently large. Since X is commutative we have $\pi(\widetilde{x}) = \pi(x)$ and hence

$$\alpha(\rho(\pi(\widetilde{x}^k y))) = \alpha(\rho(\pi(x)^k \pi(y))) = k\alpha(\rho(\pi(x))) + \alpha(\rho(\pi(y)))$$

If $\alpha(\rho(\pi(x))) \neq 0$, we can therefore always choose k big enough such that $\alpha(\rho(\pi(\widetilde{x}^k y)))$ and $\alpha(\rho(\pi(x)))$ have the same sign and hence

$$|\alpha(\rho(\pi(x\widetilde{x}^k y)))| = |\alpha(\rho(\pi(x))) + \alpha(\rho(\pi(\widetilde{x}^k y)))| = |\alpha(\rho(\pi(x)))| + |\alpha(\rho(\pi(\widetilde{x}^k y)))|$$

For those α for which $\alpha(\rho(\pi(x))) = 0$ the equation

$$|\alpha(\rho(\pi(x))) + \alpha(\rho(\pi(\widetilde{x}^k y)))| = |\alpha(\rho(\pi(x)))| + |\alpha(\rho(\pi(\widetilde{x}^k y)))|$$

holds true for trivial reasons. Hence, for k sufficiently large we have

$$|\alpha(\rho(\pi(x\widetilde{x}^k y)))| = |\alpha(\rho(\pi(x)))| + |\alpha(\rho(\pi(\widetilde{x}^k y)))|$$

for every $\alpha \in \Phi^+$, and hence

$$\ell(x\widetilde{x}^k y) = \ell(x) + \ell(\widetilde{x}^k y)$$

□

2.6 The center of affine pro-p Hecke algebras

In this section, let $\mathcal{H}^{(1)}$ be an arbitrary affine pro-p Hecke algebra. Our goal is to show that, for *any* orientation $\mathfrak{o}$, the center of $\mathcal{H}^{(1)}$ is given by the invariants

$$Z(\mathcal{H}^{(1)}) = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}}$$

of the R-linear action of $W^{(1)}$ on $\mathcal{A}_{\mathfrak{o}}^{(1)}$ by permutation of the basis elements $\widehat{\theta}_{\mathfrak{o}}(x), x \in X^{(1)}$. Note that the action of $W^{(1)}$ is by *algebra automorphisms*, since we have

$$\overline{\mathbb{X}}(w(x), w(y)) = \overline{\mathbb{X}}(x, y) \quad \forall w \in W^{(1)},\ x, y \in X^{(1)}$$

which follows immediately from formula (1.7.2) and the W_0-invariance (see lemma 2.6.2) of $\mathbb{L}$ on elements of $X \subseteq W$. In particular, the invariants form a subalgebra.

Let us now show one inclusion.

2.6.1 Proposition. *Let $W^{(1)}\backslash X^{(1)}$ denote the set of orbits with respect to the natural conjugation action of $W^{(1)}$ on $X^{(1)}$ and let $(W^{(1)}\backslash X^{(1)})_{\text{fin}}$ denote the subset of finite orbits. For every $\gamma \in (W^{(1)}\backslash X^{(1)})_{\text{fin}}$, the element*

$$z_\gamma := \sum_{x \in \gamma} \widehat{\theta}_{\mathfrak{o}}(x), \quad \mathfrak{o} \text{ spherical orientation}$$

is well defined independent of the choice of a spherical orientation $\mathfrak{o}$ of $W^{(1)}$. Moreover, the element z_γ lies in the center of $Z(\mathcal{H}^{(1)})$, and hence the subalgebra of $W^{(1)}$-invariants

$$\left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}} \subseteq Z(\mathcal{H}^{(1)})$$

is contained in the center and independent of $\mathfrak{o}$, with distinguished R-basis $\{z_\gamma\}, \gamma \in (W^{(1)}\backslash X^{(1)})_{\text{fin}}$.

Proof. Using the specialization argument (see remark 1.10.3), it suffices to prove the statement in the case when the $a_s \in R$ are invertible and admit square roots. In this case we have by definition 1.10.9 (for some fixed choice of square roots $\sqrt{a_s}$)

$$\widehat{\theta}_{\mathfrak{o}}(w) = \sqrt{\mathbb{L}}(w)\widetilde{\theta}_{\mathfrak{o}}(w)$$

From the definition of $\sqrt{\mathbb{L}} : W \to R$ and lemma 2.6.2 below, it follows that we have

$$\sqrt{\mathbb{L}}(w(x)) = \sqrt{\mathbb{L}}(x) \quad \forall w \in W^{(1)},\ x \in X^{(1)}$$

Therefore, it follows that

$$\sum_{x\in\gamma} \widehat{\theta}_{\mathfrak{o}}(x) = \sqrt{\mathbb{L}}(x_0) \sum_{x\in\gamma} \widetilde{\theta}_{\mathfrak{o}}(x)$$

for any $x_0 \in \gamma$. We may therefore prove the claim with $\widehat{\theta}$ replaced by $\widetilde{\theta}$, i.e. using the isomorphism of remark 1.10.11 we may assume that $a_s = 1$. In this case the independence of the element

$$\sum_{x\in\gamma} \widehat{\theta}_{\mathfrak{o}}(x) \in \mathcal{H}^{(1)}$$

from the choice of $\mathfrak{o}$ is equivalent to this element lying in the center since W_0 acts transitively on spherical orientations and because of the formula

$$\widehat{\theta}_{\mathfrak{o}}(w)\widehat{\theta}_{\mathfrak{o}\bullet w}(x)\widehat{\theta}_{\mathfrak{o}}(w)^{-1} = \widehat{\theta}_{\mathfrak{o}}(w(x)) \quad \forall w \in W^{(1)},\ x \in X^{(1)}$$

So it suffices to show the well-definedness of z_γ. Since spherical orientations are in bijection with Weyl chambers and any two Weyl chambers are connected by a gallery, it suffices to show that

$$\sum_{x\in\gamma} \widehat{\theta}_{\mathfrak{o}}(x) = \sum_{x\in\gamma} \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(x)$$

where $\mathfrak{o}$ is any spherical orientation and $s_\alpha \in W_0$ is associated to a root $\alpha \in \Phi$ that is simple with respect to the Weyl chamber $D_{\mathfrak{o}}$ to which the orientation $\mathfrak{o}$ corresponds. In this situation, $\mathfrak{o}$ and $\mathfrak{o} \bullet s_\alpha$ are *adjacent* in the sense of definition 1.11.2 since s_α permutes the positive roots with respect to $D_{\mathfrak{o}}$ that are not parallel to α among themselves.

The decomposition $W = W_0 \ltimes X$ induces an identification $W^{(1)}/X^{(1)} \simeq W_0$, and therefore the $W^{(1)}$-orbit γ decomposes into a disjoint union of $X^{(1)}$-orbits that are permuted amongst themselves by W_0. Considering the action of the subgroup $\{1, s_\alpha\} \leq W_0$, we can therefore write

$$\gamma = \coprod_{i\in I} \xi_i \cup s_\alpha(\xi_i)$$

where[23] $\xi_i \in X^{(1)}\backslash X^{(1)}$ and either $s_\alpha(\xi_i) = \xi_i$ or $s_\alpha(\xi_i) \cap \xi_i = \emptyset$. Accordingly, if $J \subseteq I$ denotes the indices i where $s_\alpha(\xi_i) = \xi_i$ and $\sigma \in W^{(1)}$ denotes any lift of s_α, we have that

$$\sum_{x\in\gamma} \widehat{\theta}_{\mathfrak{o}}(x) = \sum_{i\in J}\sum_{x\in\xi_i} \widehat{\theta}_{\mathfrak{o}}(x) + \sum_{i\in I-J}\sum_{x\in\xi_i} \widehat{\theta}_{\mathfrak{o}}(x) + \widehat{\theta}_{\mathfrak{o}}(\sigma(x))$$

Whence it suffices to show that for all $x \in \xi_i$ with $s_\alpha(\xi_i) = \xi_i$ we have that

$$\widehat{\theta}_{\mathfrak{o}}(x) = \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(x)$$

and that for *any* $x \in X^{(1)}$ we have that

$$\widehat{\theta}_{\mathfrak{o}}(x) + \widehat{\theta}_{\mathfrak{o}}(\sigma(x)) = \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(x) + \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(\sigma(x))$$

Let us begin by proving the first statement, and assume that $s_\alpha(\xi_i) = \xi_i$. Note that because X is commutative, $\pi : W^{(1)} \longrightarrow W$ maps $X^{(1)}$-orbits to singletons; in particular, $\pi(\sigma(x)) = \pi(x)$. Therefore

$$s_\alpha(\pi(x)) = \pi(\sigma(x)) = \pi(x)$$

[23] $X^{(1)}\backslash X^{(1)}$ denotes the set of orbits of $X^{(1)}$ acting on itself via conjugation

Further, recall that the abstract geometry of the extended Coxeter group W and the concrete geometry of the affine hyperplane arrangement $(V, \mathfrak{H})$ are compatible via ρ (see remark 2.3.1). By definition of the $s_\alpha \in W_0$ (cf. lemma 2.1.2), we have $\rho(s_\alpha) = s_\alpha$, where $s_\alpha \in \mathrm{GL}(V)$ is given by the formula

$$s_\alpha(\rho(\pi(x))) = \rho(\pi(x)) - \alpha(\rho(\pi(x)))\alpha^\vee$$

Therefore, it follows from applying ρ to the equality $s_\alpha(\pi(x)) = \pi(x)$ that $\alpha(\rho(\pi(x))) = 0$. This means that $1, x$ are not separated by any hyperplane of type α, where we agree to call H a hyperplane of type α if $H = H_{\alpha,k}$ for some $k \in \mathbb{Z}$. Since $\mathfrak{o}$ and $\mathfrak{o} \bullet s_\alpha$ agree except at the hyperplanes of type α, it follows that

$$\widehat{\theta}_{\mathfrak{o}}(x) = \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(x)$$

Let us now prove the second statement and let $x \in X^{(1)}$ be arbitrary. Since $\mathfrak{o}$ and $\mathfrak{o} \bullet s_\alpha$ are adjacent, we may apply the Bernstein relation (theorem 1.11.3) to conclude that (remembering that $\widehat{\theta} = \widetilde{\theta}$ in our case)

$$\widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(x) = \left(\sum_{\widetilde{H}} \mathfrak{o}(1, \widetilde{H}) \Xi_{\mathfrak{o}\bullet s_\alpha}(\widetilde{H}) \right) \widehat{\theta}_{\mathfrak{o}}(x)$$

where the sum runs over all hyperplanes $\widetilde{H}$ of type α which separate 1 and x. On the other hand applying theorem 1.11.3 to $\sigma(x)$ instead of x gives

$$\begin{aligned}
\widehat{\theta}_{\mathfrak{o}}(\sigma(x)) - \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(\sigma(x)) &= \left(\sum_{H} \mathfrak{o}(1, H) \Xi_{\mathfrak{o}\bullet s_\alpha}(H) \right) \widehat{\theta}_{\mathfrak{o}}(\sigma(x)) \\
&= \left(\sum_{H} \mathfrak{o}(1, H) \Xi_{\mathfrak{o}\bullet s_\alpha}(H) \widehat{\theta}_{\mathfrak{o}}(\sigma(x)x^{-1}) \right) \widehat{\theta}_{\mathfrak{o}}(x)
\end{aligned}$$

where the sum runs over all hyperplanes H of type α separating 1 and $\sigma(x)$. By lemma 1.11.4 we have

$$\Xi_{\mathfrak{o}\bullet s_\alpha}(H) \widehat{\theta}_{\mathfrak{o}}(\sigma(x)x^{-1}) = \Xi_{\mathfrak{o}\bullet s_\alpha}(\pi(x)H\pi(x)^{-1})$$

The result follows if we can show that

$$H \longmapsto \widetilde{H} := \pi(x)H\pi(x)^{-1}$$

gives a bijection between the hyperplanes H of type α separating 1 and $\sigma(x)$ and the hyperplanes $\widetilde{H}$ of type α that separate 1 and x, and that

$$\mathfrak{o}(1, \pi(x)H\pi(x)^{-1}) = -\mathfrak{o}(1, H)$$

since then

$$\begin{aligned}
\widehat{\theta}_{\mathfrak{o}}(\sigma(x)) - \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(\sigma(x)) &= \left(\sum_{H} \mathfrak{o}(1, H) \Xi_{\mathfrak{o}\bullet s_\alpha}(\pi(x)H\pi(x)^{-1}) \right) \widehat{\theta}_{\mathfrak{o}}(x) \\
&= -\left(\sum_{\widetilde{H}} \mathfrak{o}(1, \widetilde{H}) \Xi_{\mathfrak{o}\bullet s_\alpha}(\widetilde{H}) \right) \widehat{\theta}_{\mathfrak{o}}(x) \\
&= -\left(\widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o}\bullet s_\alpha}(x) \right)
\end{aligned}$$

Let $H = H_{\alpha,k}$ be a hyperplane of type α and $y \in C_0$ an arbitrary point. Then H separates two elements $w, w' \in W$ if and only if $w(y)$ and $w'(y)$ lie in different connected components of $V - H_{\alpha,k}$, i.e. if and only if $\alpha(w(y)) + k$ and $\alpha(w'(y)) + k$ have different signs. Moreover, for a hyperplane H we have $H = H_{\alpha,k}$ if and only if $\rho(s_H) = s_{\alpha,k}$ where

$$s_{\alpha,k}(y) = y - (\alpha(y) + k)\alpha^\vee$$

Denoting by $\tau^v \in \mathrm{Aut}_{\mathrm{aff}}(V)$ the translation by a vector $v \in V$, we have the formula

$$\tau^y s_{\alpha,k} \tau^{-y} = s_{\alpha,k-\alpha(y)}$$

Let now $H = H_{\alpha,k}$ be a hyperplane of type α. Since $X \subseteq W$ gets mapped into the subgroup $V \leq \mathrm{Aut}_{\mathrm{aff}}(V)$ of translations under ρ, it hence follows that

$$\rho(\pi(x)H\pi(x)^{-1}) = \tau^{\rho(\pi(x))} s_{\alpha,k} \tau^{-\rho(\pi(x))} = s_{\alpha,k-\alpha(\rho(\pi(x)))}$$

Hence, $\pi(x)H\pi(x)^{-1}$ separates $1, x$ if and only if $\alpha(y) + k - \alpha(\rho(\pi(x)))$ and

$$\alpha(y + \rho(\pi(x))) + k - \alpha(\rho(\pi(x))) = \alpha(y) + k$$

have different signs. On the other hand H separates $1, \sigma(x)$ if and only if $\alpha(y) + k$ and

$$\alpha(y + \rho(s_\alpha(\pi(x)))) + k = \alpha(y) + k - \alpha(\rho(\pi(x)))$$

have different signs. Hence, $H \mapsto \widetilde{H}$ gives a bijection as desired. Moreover

$$\mathfrak{o}(1, \pi(x)H\pi(x)^{-1}) = -\mathfrak{o}(1, H)$$

By notation 1.7.5, $\mathfrak{o}(1, H)$ is the sign attached by $\mathfrak{o}$ to crossing H at any chamber lying in the same half-space as the fundamental chamber. Letting $\varepsilon \in \{\pm\}$ be such that $\varepsilon\alpha$ is positive with respect to the Weyl chamber $D_{\mathfrak{o}}$ corresponding to $\mathfrak{o}$, it then follows that

$$\mathfrak{o}(1, H) = -\varepsilon \operatorname{sgn}(\alpha(y) + k)$$

and

$$\mathfrak{o}(1, \pi(x)H\pi(x)^{-1}) = -\varepsilon \operatorname{sgn}(\alpha(y) + k - \alpha(\pi(x)))$$

As we saw above, H separates $1, x$ if and only if $\alpha(y) + k$ and $\alpha(y) + k - \alpha(\pi(x))$ have different signs. Hence, the claim follows. $\square$

2.6.2 Lemma. *The length function of definition 1.7.9*

$$\mathbb{L} : W \longrightarrow \mathbb{N}[\mathfrak{H}]$$

satisfies

$$\mathbb{L}(w(x)) = \rho_0(w)(\mathbb{L}(x)) \quad \forall w \in W,\ x \in X$$

where $\rho_0 : W \to W_0$ denotes the projection.

Proof. Recall from remark 2.3.3 that the *number* of hyperplanes of the form $H_{\alpha,k}$, $k \in \mathbb{Z}$ separating the fundamental chamber C_0 from $\rho(x)(C_0)$ is given by

$$|\vec{d}(C_0, \rho(x)(C_0))_\alpha| = |-\alpha(\rho(x))|$$

With a bit more notation, we can be more precise and specify the *set* of these hyperplanes. For $k \in \mathbb{Z}$ let

$$[0, k[:= \begin{cases} \{0, 1, \ldots, k-1\} & : k > 0 \\ \emptyset & : k = 0 \\ \{-k+1, \ldots, -1, 0\} & : k < 0 \end{cases}$$

Using that $0 \in \overline{C_0}$ by (**ACIX**) and that

$$C_0 \subseteq \{v \in V \ : \ \forall \alpha \in \Phi^+ \ \ \alpha(v) > 0\}$$

by definition of Φ^+, it is easy to see that the set of hyperplanes of the form $H_{\alpha,k}$ which separate C_0 and $\rho(x)(C_0)$ is in fact given by

$$\{H_{\alpha,k} \ : \ k \in [0, -\alpha(\rho(x))[\}$$

Hence

$$\mathbb{L}(x) = \prod_{\alpha \in \Phi^+} \prod_{k \in [0, -\alpha(\rho(x))[} H_{\alpha,k}$$

Moreover

$$\rho(w(x)) = \rho_0(w)(\rho(x))$$

and hence

$$\alpha(\rho(w(x))) = \alpha(\rho_0(w)(\rho(x))) = (\rho_0(w)^{-1} \bullet \alpha)(\rho(x))$$

Since Φ is the disjoint union of Φ^+ and $-\Phi^+$, we have

$$\rho_0(w)^{-1} \bullet \alpha = \varepsilon_\alpha \phi(\alpha)$$

for some uniquely determined $\varepsilon_\alpha \in \{\pm\}$ and $\phi(\alpha) \in \Phi^+$. Using that

$$w(H_{\alpha,k}) = H_{w\bullet\alpha,k} \quad \forall w \in W_0,\ \alpha \in \Phi,\ k \in \mathbb{Z}$$

that

$$H_{\alpha,k} = H_{-\alpha,-k} \quad \forall \alpha \in \Phi,\ k \in \mathbb{Z}$$

and that $\phi : \Phi^+ \to \Phi^+$ is a bijection, we now simply compute

$$\begin{aligned}
\mathbb{L}(w(x)) &= \prod_{\alpha\in\Phi^+} \prod_{k\in[0,-\alpha(\rho(w(x)))[} H_{\alpha,k} \\
&= \prod_{\alpha\in\Phi^+} \prod_{k\in[0,-\varepsilon_\alpha\phi(\alpha)(\rho(x))[} H_{\alpha,k} \\
&= \prod_{\alpha\in\Phi^+} \prod_{k\in[0,-\phi(\alpha)(\rho(x))[} H_{\alpha,\varepsilon_\alpha k} \\
&= \prod_{\alpha\in\Phi^+} \prod_{k\in[0,-\phi(\alpha)(\rho(x))[} H_{\varepsilon_\alpha \alpha,k} \\
&= \prod_{\alpha\in\Phi^+} \prod_{k\in[0,-\phi(\alpha)(\rho(x))[} \rho_0(w)(H_{\phi(\alpha),k}) \\
&= \rho_0(w)\left(\prod_{\alpha\in\Phi^+} \prod_{k\in[0,-\phi(\alpha)(\rho(x))[} H_{\phi(\alpha),k}\right) \\
&= \rho_0(w)(\mathbb{L}(x))
\end{aligned}$$

□

We will now show that the center $Z(\mathcal{H}^{(1)})$ is in fact equal to $\left(\mathfrak{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}}$, via induction on the support (see definition 2.5.3) of an element.

2.6.3 Theorem. *The center $Z(\mathcal{H}^{(1)})$ of the affine pro-p Hecke algebra $\mathcal{H}^{(1)}$ is given by*

$$Z(\mathcal{H}^{(1)}) = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W(1)}$$

for every spherical orientation $\mathfrak{o}$ of $W^{(1)}$. It is a free R-module with distinguished basis $\{z_\gamma\}_\gamma$ indexed by the finite orbits $\gamma \in (W^{(1)}\backslash X^{(1)})_{fin}$ of $W^{(1)}$ in $X^{(1)}$, where

$$z_\gamma = \sum_{x\in\gamma} \widehat{\theta}_{\mathfrak{o}}(x)$$

for every spherical orientation $\mathfrak{o}$.

Proof. It only remains to prove that

$$Z(\mathcal{H}^{(1)}) \subseteq \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}}$$

In view of the computation of the centralizer of $\mathcal{A}_{\mathfrak{o}}^{(1)}$ in proposition 2.5.4 we already know that

$$Z(\mathcal{H}^{(1)}) \subseteq \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$$

Therefore, it only remains to show that for an element

$$z = \sum_{x\in X^{(1)}} c_x \widehat{\theta}_{\mathfrak{o}}(x) = \sum_{\xi\in(X^{(1)}\backslash X^{(1)})_{\text{fin}}} c_\xi \sum_{x\in\xi} \widehat{\theta}_{\mathfrak{o}}(x) \in Z(\mathcal{H}^{(1)}), \quad c_\xi \in R$$

we have

$$c_\xi = c_{w(\xi)}$$

for every $w \in W_0$. We will prove this by induction on $\operatorname{supp}_{\mathfrak{o}}(z)$ using proposition 2.6.1. If $\operatorname{supp}_{\mathfrak{o}}(z) = \emptyset$, then $z = 0$ and the claim is clear. So let's assume that $\operatorname{supp}_{\mathfrak{o}}(z) \neq \emptyset$. Choose $x \in \operatorname{supp}_{\mathfrak{o}}(z)$ with $\ell(x)$ maximal and let $\xi = X^{(1)} \bullet x$ be the (finite) $X^{(1)}$-orbit associated to it. We now want to show that

$$c_\xi = c_{w(\xi)} \quad \forall w \in W_0$$

in order to apply induction. For this, recall that W_0 is generated by the reflections s_α for roots α that are simple with respect to the Weyl chamber D_0 containing C_0 (see lemma 2.1.2) and let $s = s_\alpha \in S$ for any such α. We have $\ell(n_s x) = \ell(x) \pm 1$. Moreover, we claim that

$$\ell(n_s x) = \ell(x) - 1 \quad \Rightarrow \quad \ell(x n_s) = \ell(x) + 1$$

To see this, let $x_0 \in C_0$ be arbitrary. We have $\ell(n_s x) = \ell(x) + 1$ if and only if

$$d(C_0, n_s x C_0) = d(C_0, n_s C_0) + d(n_s C_0, n_s x C_0)$$

that is, if and only if the set of hyperplanes separating C_0 and $n_s C_0$ and the set of hyperplanes separating $n_s C_0$ and $n_s x C_0 = n_s(x) n_s C_0 = n_s(\pi(x)) + n_s C_0$ are disjoint. Since C_0 and $n_s C_0$ are separated only by $H_\alpha = \ker(\alpha)$ and this hyperplanes separates $n_s C_0$ and $n_s(\pi(x)) + n_s C_0$ if and only if

$$\operatorname{sgn}(\alpha(n_s(\pi(x)) + n_s(x_0))) = -\operatorname{sgn}(\alpha(n_s(x_0)))$$

we see that

$$\ell(n_s x) = \ell(x) + 1 \quad \text{or} \quad \ell(n_s x) = \ell(x) - 1$$

depending on whether

$$\operatorname{sgn}(\alpha(\pi(x)) + \alpha(x_0)) = \operatorname{sgn}(\alpha(x_0)) \quad \text{or} \quad \operatorname{sgn}(\alpha(\pi(x)) + \alpha(x_0)) = -\operatorname{sgn}(\alpha(x_0))$$

respectively. Using that $x n_s = n_s s(x)$ and $\ell(s(x)) = \ell(x)$ it follows from the above that

$$\ell(x n_s) = \ell(x) + 1 \quad \text{or} \quad \ell(x n_s) = \ell(x) - 1$$

depending on whether

$$\operatorname{sgn}(-\alpha(\pi(x)) + \alpha(x_0)) = \operatorname{sgn}(\alpha(x_0)) \quad \text{or} \quad \operatorname{sgn}(-\alpha(\pi(x)) + \alpha(x_0)) = -\operatorname{sgn}(\alpha(x_0))$$

In particular it follows that

$$\ell(n_s x) = \ell(x) - 1 \quad \Rightarrow \quad \ell(x n_s) = \ell(x) + 1$$

We now distinguish two cases. First, assume that $\ell(n_s x) = \ell(x) + 1$. We have

$$\widehat{\theta}_{\mathfrak{o}}(n_s) z = \sum_{y \in X^{(1)}} c_y \widehat{\theta}_{\mathfrak{o}}(n_s)(\widehat{\theta}_{\mathfrak{o}}(y) - \widehat{\theta}_{\mathfrak{o} \bullet s}(y)) + \sum_{y \in X^{(1)}} \overline{\mathbb{X}}(s, \pi(y)) \widehat{\theta}_{\mathfrak{o}}(n_s y)$$

We claim that the $n_s x$ does not appear in the support of the first big sum. In fact, by the change of basis formula (corollary 1.10.7) we have

$$\operatorname{supp}_{\mathfrak{o}}(\widehat{\theta}_{\mathfrak{o}}(y) - \widehat{\theta}_{\mathfrak{o} \bullet s}(y)) \subseteq \{w \in W^{(1)} : w < y\}$$

In particular each w appearing in the support of $\widehat{\theta}_{\mathfrak{o}}(y) - \widehat{\theta}_{\mathfrak{o} \bullet s}(y)$ is of length

$$\ell(w) \leq \ell(y) - 1$$

Hence, for all

$$w \in \operatorname{supp}_{\mathfrak{o}}(\widehat{\theta}_{\mathfrak{o}}(n_s)(\widehat{\theta}_{\mathfrak{o}}(y) - \widehat{\theta}_{\mathfrak{o} \bullet s}(y)))$$

we have

$$\ell(w) \leq \ell(n_s) + (\ell(y) - 1) = \ell(y) \leq \ell(x) < \ell(x) + 1 = \ell(n_s x)$$

The coefficient of $\widehat{\theta}_{\mathfrak{o}}(n_s x)$ in $\widehat{\theta}_{\mathfrak{o}}(n_s) z$ is therefore given by

$$c_x \overline{\mathbb{X}}(s, \pi(x)) = c_x = c_\xi$$

Here we have used that $\overline{\mathbb{X}}(s,\pi(x)) = 1$, which follows from $\ell(n_s x) = \ell(n_s) + \ell(x)$ and remark 1.7.2. On the other hand we have

$$z\widehat{\theta}_{\mathfrak{o}}(n_s) = \sum_{y\in X^{(1)}} c_y \overline{\mathbb{X}}(\pi(y), s)\widehat{\theta}_{\mathfrak{o}}(yn_s)$$

and hence the coefficient of $n_s x = n_s(x)n_s$ in $\widehat{\theta}_{\mathfrak{o}}(n_s)z$ is given by

$$c_{n_s(x)}\overline{\mathbb{X}}(\pi(s(x)), s) = c_{n_s(x)}$$

where again we have used remark 1.7.2 and the fact that

$$\ell(s(x)n_s) = \ell(n_s x) = \ell(x) + \ell(n_s) = \ell(n_s(x)) + \ell(n_s)$$

Since $\widehat{\theta}_{\mathfrak{o}}(x)z = z\widehat{\theta}_{\mathfrak{o}}(x)$ it follows that

$$c_x = c_{n_s(x)}$$

Consider now the case $\ell(n_s x) = \ell(x) - 1$. As already observed, in this situation we must have $\ell(n_s s(x)) = \ell(xn_s) = \ell(x) + 1$. Replacing x by $s(x)$, we are therefore reduced to the first case. Thus we have shown that for any $x \in \operatorname{supp}_{\mathfrak{o}}(z)$ with $\ell(x)$ maximal we have

$$c_\xi = c_x = c_{n_s(x)} = c_{s(\xi)}$$

for all simple reflections $s = s_\alpha \in W_0$, $\alpha \in \Delta$, where $\xi = X^{(1)}$ denotes the (finite) $X^{(1)}$-orbit of x. In particular $n_s(x)$ is again an element of $\operatorname{supp}_{\mathfrak{o}}(z)$, and since $\ell(x) = \ell(n_s(x))$ it is also of maximal length. Inductively it therefore follows that

$$c_\xi = c_{w(\xi)}$$

for all $w \in W_0$. Letting $\gamma = W^{(1)} \bullet x = \bigcup_{w\in W_0} w \bullet \xi$, the element

$$z - c_\xi z_\gamma \in Z(\mathcal{H}^{(1)})$$

therefore has support strictly contained in $\operatorname{supp}_{\mathfrak{o}}(z)$, and by induction we conclude that

$$z \in \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}}$$

□

2.7 The structure of affine pro-p Hecke algebras

In this section we will give the main theorem on the structure of affine pro-p Hecke algebras.

2.7.1 Theorem. *Let $\mathcal{H}^{(1)}$ be an affine pro-p Hecke algebra over a ring R in the sense of definition 2.1.4 and let $\mathfrak{o}$ a spherical orientation of $W^{(1)}$ in the sense of definition 2.4.1.*

Let $W^{(1)}\backslash X^{(1)}$ denote the set of orbits of the conjugation actions of $W^{(1)}$ on $X^{(1)}$, let $(W^{(1)}\backslash X^{(1)})_{\text{fin}}$ denote the subset of finite orbits, and consider the condition

(**HeckeFin**) $\quad (W^{(1)}\backslash X^{(1)})_{\text{fin}} = W^{(1)}\backslash X^{(1)} \ \wedge\ (\ T$ *finite* $\vee\ T$ *finitely generated and* R *noetherian*)

Then the following holds.

(i) There exists an R-subalgebra

$$\mathcal{A}_{\mathfrak{o}}^{(1)} \subseteq \mathcal{H}^{(1)}$$

with R-basis $\{\widehat{\theta}_{\mathfrak{o}}(x)\}_{x\in X^{(1)}}$ defined in theorem 1.10.1. The product of two basis elements is given by

$$\widehat{\theta}_{\mathfrak{o}}(x)\widehat{\theta}_{\mathfrak{o}}(y) = \overline{\mathbb{X}}(\pi(x), \pi(y))\widehat{\theta}_{\mathfrak{o}}(xy)$$

where $\overline{\mathbb{X}} : W \times W \to R$ denotes the '2-coboundary' defined in notation 1.10.4.

(ii) The 'conjugation action' of $W^{(1)}$ on $X^{(1)}$ induces an action on $\mathcal{A}_{\mathfrak{o}}^{(1)}$ by R-algebra automorphisms via

$$w(\widehat{\theta}_{\mathfrak{o}}(x)) = \widehat{\theta}_{\mathfrak{o}}(w(x))$$

The centralizer $C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)})$ of $\mathcal{A}_{\mathfrak{o}}^{(1)}$ in $\mathcal{H}^{(1)}$ is given by the subalgebra of $X^{(1)}$-invariants in $\mathcal{A}_{\mathfrak{o}}^{(1)}$. In particular, the centralizer is contained in $\mathcal{A}_{\mathfrak{o}}^{(1)}$ and hence equals the center of $\mathcal{A}_{\mathfrak{o}}^{(1)}$:

$$Z(\mathcal{A}_{\mathfrak{o}}^{(1)}) = C_{\mathcal{H}^{(1)}}(\mathcal{A}_{\mathfrak{o}}^{(1)}) = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$$

(iii) *If the abelian group T is finitely generated, the R-algebra $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is finitely generated. More precisely, the following holds. The algebra $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is a finite sum*

$$\mathcal{A}_{\mathfrak{o}}^{(1)} = \sum_D \iota_D(R[X_D^{(1)}])$$

of subalgebras, where the sum ranges over the Weyl chambers $D \in \pi_0(V - \bigcup_{\alpha\in\Phi} H_\alpha)$. Here $R[X_D^{(1)}]$ denotes the monoid algebra over the submonoid

$$X_D^{(1)} = \{x \in X^{(1)} : \rho(\pi(x)) \in \overline{D}\}$$

of $X^{(1)}$ consisting of those elements which act through $\rho : W \to \mathrm{Aut}_{\mathrm{aff}}(V)$ by translation by an element lying in the closure of $D \subseteq V$, and ι_D denotes the algebra embedding

$$\iota_D : R[X_D^{(1)}] \hookrightarrow \mathcal{A}_{\mathfrak{o}}^{(1)}$$

determined by $\iota_D(x) = \widehat{\theta}_{\mathfrak{o}}(x)$ for all $x \in X_D^{(1)}$. Moreover, if T is finitely generated, then the submonoid $X_D^{(1)}$ and hence the algebra $R[X_D^{(1)}]$ are finitely generated, and thus $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is finitely generated in this case too.

(iv) *If* (**HeckeFin**) *holds, then $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is a finitely generated $\left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$-module.*

(v) *If* (**HeckeFin**) *holds, then $\left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$ is a finitely generated R-algebra.*

(vi) *The center $Z(\mathcal{H}^{(1)})$ of $\mathcal{H}^{(1)}$ is given by the subalgebra of $W^{(1)}$-invariants in $\mathcal{A}_{\mathfrak{o}}^{(1)}$*

$$Z(\mathcal{H}^{(1)}) = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}}$$

It has a distinguished R-basis $\{z_\gamma\}, \gamma \in (W^{(1)}\backslash X^{(1)})_{\mathrm{fin}}$ with

$$z_\gamma = \sum_{x\in\gamma} \widehat{\theta}_{\mathfrak{o}}(x)$$

independent of the choice of the spherical orientation $\mathfrak{o}$.

(vii) *$\mathcal{H}^{(1)}$ is a finitely generated left $\mathcal{A}_{\mathfrak{o}}^{(1)}$-module. More precisely, a finite set of generators is given as follows. For $w \in W_0$ let $X \bullet w^{-1}(C_0)$ denote the set of X-translates of the chamber $w^{-1}(C_0)$. Let Λ_w denote the set of minimal elements of $X \bullet w^{-1}(C_0)$ with respect to the partial order $\preceq_{C_0}$ defined in remark 2.3.5. By corollary 2.3.8, the set Λ_w is finite.*

Choose for each $C \in \Lambda_w$ an element $\widetilde{w} \in W^{(1)}$ with

$$\pi(\widetilde{w}) \in Xw \subseteq W \quad \text{and} \quad \widetilde{w}^{-1}(C_0) = C$$

and let $\widetilde{\Lambda_w} \subseteq W^{(1)}$ denote the set of these elements. Then

$$\{\widehat{\theta}_{\mathfrak{o}}(\widetilde{w}) : w \in W_0,\ \widetilde{w} \in \widetilde{\Lambda_w}\}$$

is a set of generators of $\mathcal{H}^{(1)}$ as a left module over $\mathcal{A}_{\mathfrak{o}}^{(1)}$.

(viii) *If* (**HeckeFin**) *holds, then $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is a finitely generated $Z(\mathcal{H}^{(1)})$-module.*

(ix) *If* (**HeckeFin**) *holds, then $Z(\mathcal{H}^{(1)})$ is a finitely generated R-algebra.*

(x) *If* (**HeckeFin**) *holds, then $\mathcal{H}^{(1)}$ is a finitely generated $Z(\mathcal{H}^{(1)})$-module.*

(xi) *If* (**HeckeFin**) *holds and R is noetherian, then the R-algebras $Z(\mathcal{H}^{(1)})$, $\mathcal{A}_{\mathfrak{o}}^{(1)}$, and $\mathcal{H}^{(1)}$ are noetherian (i.e. left and right noetherian).*

Proof. (xi): Follows directly from (viii), (ix), and (x). (i): Was proven in section 1.10. (ii): Was shown in proposition 2.5.4. (vi): Was shown in theorem 2.6.3. (vii): By corollary 1.10.5 and remark 1.7.2 we know that

$$\ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \quad \widehat{\theta}_{\mathfrak{o}}(ww') = \widehat{\theta}_{\mathfrak{o}}(w)\widehat{\theta}_{\mathfrak{o}\bullet w}(w') \quad \forall\, w, w' \in W^{(1)}$$

Since $\mathfrak{o}$ is assumed to be a spherical orientation, we have $\mathfrak{o} \bullet x = \mathfrak{o}$ for every $x \in X^{(1)}$ and hence

$$\ell(xw) = \ell(x) + \ell(w) \quad \Rightarrow \quad \widehat{\theta}_{\mathfrak{o}}(xw) = \widehat{\theta}_{\mathfrak{o}}(x)\widehat{\theta}_{\mathfrak{o}}(w) \quad \forall\, x \in X^{(1)},\ w \in W^{(1)} \tag{2.7.1}$$

We have the equivalences

$$\begin{aligned} \ell(xw) = \ell(x) + \ell(w) &\Leftrightarrow \ell(w^{-1}x^{-1}) = \ell(x^{-1}) + \ell(w^{-1}) \\ &\Leftrightarrow d(C_0, (w^{-1}x^{-1})(C_0)) = d(C_0, w^{-1}(C_0)) + d(C_0, x^{-1}(C_0)) \\ &\Leftrightarrow d(C_0, (w^{-1}x^{-1})(C_0)) = d(C_0, w^{-1}(C_0)) \\ &+ d(w^{-1}(C_0), (w^{-1}x^{-1})(C_0)) \\ &\Leftrightarrow w^{-1}(C_0) \preceq_{C_0} (w^{-1}x^{-1})(C_0) \end{aligned} \tag{2.7.2}$$

Here the first equivalence is simply the invariance $\ell(w) = \ell(w^{-1})$ of the length under inverses, the third equivalence is the W-invariance of the distance d and the last equivalence is by definition (see remark 2.3.5).

Let now $w' \in W^{(1)}$ be arbitrary. Because $W = X \rtimes W_0$, we have $\pi(w') \in Xw$ for some $w \in W_0$. In particular $(w')^{-1}(C_0) \in X \bullet w^{-1}(C_0)$ and hence by definition of $\widetilde{\Lambda_w}$ we can find $\widetilde{w} \in \widetilde{\Lambda_w}$ with $\pi(\widetilde{w}) \in Xw$ and $\widetilde{w}^{-1}(C_0) \preceq_{C_0} (w')^{-1}(C_0)$. Hence, for some $x \in X^{(1)}$ we have

$$w' = x\widetilde{w}$$

From

$$\widetilde{w}^{-1}(C_0) \preceq (\widetilde{w}^{-1}x^{-1})(C_0)$$

and (2.7.2) above it therefore follows that

$$\ell(x\widetilde{w}) = \ell(x) + \ell(\widetilde{w})$$

Hence, by equation (2.7.1) above we have

$$\widehat{\theta}_{\mathfrak{o}}(w') = \widehat{\theta}_{\mathfrak{o}}(x)\widehat{\theta}_{\mathfrak{o}}(\widetilde{w})$$

and hence $\widehat{\theta}_{\mathfrak{o}}(w')$ lies in the $\mathcal{A}_{\mathfrak{o}}^{(1)}$-submodule generated by the set we claim to be a set of generators. Since $w' \in W^{(1)}$ was arbitrary, (vii) follows.

Next, we prove (iii). First, we need to show that ι_D is well-defined. Recall that it was shown in the proof of lemma 2.5.6 that on the submonoid

$$X_D = \pi(X_D^{(1)}) = \{x \in X \ :\ \rho(x) \in \overline{D}\} \leq X$$

the length function is additive

$$\ell(x + y) = \ell(x) + \ell(y) \quad \forall x, y \in X_D$$

Since the length function on $W^{(1)}$ arises by pullback along $\pi : W^{(1)} \to W$ of the length function on W, it follows that $\ell(xy) = \ell(x) + \ell(y)$ and hence $\overline{\mathbb{X}}(x, y) = 1$ for all $x, y \in X_D^{(1)}$. The product formula (corollary 1.10.5) therefore implies that $x \mapsto \widehat{\theta}_{\mathfrak{o}}(x)$ defines a morphism of monoids

$$X_D^{(1)} \longrightarrow \mathcal{A}_{\mathfrak{o}}^{(1)}$$

inducing ι_D. Moreover, since $V = \bigcup_D \overline{D}$, we have $X^{(1)} = \bigcup_D X_D^{(1)}$ and therefore $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is the R-module sum of the subalgebras $\iota_D(R[X_D^{(1)}])$ as claimed. Lastly we need to show that the monoid $X_D^{(1)}$ is finitely generated, assuming that T is finitely generated as an abelian group (and hence as a monoid). But since $T = \ker(\pi : W^{(1)} \twoheadrightarrow W)$ is the kernel of π, it suffices to show that the image $X_D = \pi(X_D^{(1)})$ is finitely generated as a monoid. But this was shown in lemma 2.3.11.

It remains to show claims (iv),(v),(viii) and (ix). Since the subalgebras

$$Z(\mathcal{H}^{(1)}) = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}} \subseteq \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}} \subseteq \mathcal{A}_{\mathfrak{o}}^{(1)} \subseteq \mathcal{H}^{(1)}$$

have explicit R-bases, it is easy to see that they are preserved under base change. If we are in the situation where T is finite, we may therefore reduce to the case $R = R^{\mathrm{univ}}$ of the universal coefficient ring, which exists and is noetherian by remark 1.3.9. Therefore, it suffices to prove the claim in the case where T is finitely generated and R is noetherian. In this case, claims (iv),(v) and (viii),(ix) each follow from lemma 2.7.2 (which follows after this theorem). To get (iv) and (v), we apply lemma 2.7.2 with

$$C = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}} \subseteq \mathcal{A}_{\mathfrak{o}}^{(1)} = B$$

$$\Lambda = \{\widehat{\theta}_{\mathfrak{o}}(x) : x \in \Lambda_D,\ D \text{ Weyl chamber}\} \subseteq \mathcal{A}_{\mathfrak{o}}^{(1)}$$

and

$$\Pi = \{xyx^{-1}y^{-1} : x, y \in \Lambda\} \subseteq T \subseteq Z(\mathcal{A}_{\mathfrak{o}}^{(1)}) = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$$

where Λ_D denotes any finite set of generators of the monoid $X_D^{(1)}$ whose existence is guaranteed by (iii). Let us verify that assumptions (a)-(d) of the lemma are satisfied. In the discussion of claim (iii) we have seen that (a) holds. To see that (d) holds, note that an element $t \in T$ is annihilated by the monic polynomial

$$\prod_{t' \in W \bullet t} (X - t')$$

with coefficients in

$$R[T]^W \subseteq \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}} = C$$

Assumption (c) follows by a formal computation from the product formula, the fact that (cf. remark 1.10.6)

$$\widehat{\theta}_{\mathfrak{o}}(tw) = t\widehat{\theta}_{\mathfrak{o}}(w) \quad \forall w \in W^{(1)}$$

and the fact that (cf. remark 1.10.8)

$$\forall w, w' \in W^{(1)} \quad ww' = w'w \quad \Rightarrow \quad \overline{\mathbb{X}}(w, w') = \overline{\mathbb{X}}(w', w)$$

Indeed, for any $x, y \in X^{(1)}$ we have

$$\begin{aligned}
\widehat{\theta}_{\mathfrak{o}}(x)\widehat{\theta}_{\mathfrak{o}}(y) &= \overline{\mathbb{X}}(x, y)\widehat{\theta}_{\mathfrak{o}}(xy) \\
&= \overline{\mathbb{X}}(y, x)\widehat{\theta}_{\mathfrak{o}}(\underbrace{xyx^{-1}y^{-1}}_{=:t \in \Pi} yx) \\
&= t\overline{\mathbb{X}}(y, x)\widehat{\theta}_{\mathfrak{o}}(yx) \\
&= t\widehat{\theta}_{\mathfrak{o}}(y)\widehat{\theta}_{\mathfrak{o}}(x)
\end{aligned}$$

Finally we need to verify (b), i.e. we need to provide monic polynomials $f_x(Z) \in \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}[Z]$ with $f_x(\widehat{\theta}_{\mathfrak{o}}(x)) = 0$. Even though $\mathcal{A}_{\mathfrak{o}}^{(1)}$ is possibly non-commutative, it still makes sense to form the polynomial ring $\mathcal{A}_{\mathfrak{o}}^{(1)}[Z]$ in one variable Z that commutes with $\mathcal{A}_{\mathfrak{o}}^{(1)}$. For $x \in X^{(1)}$ arbitrary, let $\xi = X^{(1)} \bullet x$ be the orbit of x under the (conjugation) action of $X^{(1)}$ and let

$$f_x(Z) := f_\xi(Z) := \prod_{y \in \xi}(Z - \widehat{\theta}_{\mathfrak{o}}(y)) \in \mathcal{A}_{\mathfrak{o}}^{(1)}[Z]$$

Note that ξ is finite since $W^{(1)}$ (and therefore $X^{(1)}$) acts on $X^{(1)}$ with finite orbits by assumption. However, a priori the above expression is still ill-defined, as it depends on the choice of an ordering of the factors. However, the elements $\widehat{\theta}_{\mathfrak{o}}(y)$ with $y \in \xi$ in fact commute with each other pairwise. This follows from remark 1.10.8 and the fact that the elements of the orbit ξ themselves commute with each other pairwise, as an easy computation shows. The expression $f_\xi(Z)$ therefore is well-defined. Moreover, the R-algebra action of $W^{(1)}$ on $\mathcal{A}_{\mathfrak{o}}^{(1)}$ extends to $\mathcal{A}_{\mathfrak{o}}^{(1)}[Z]$ by acting on coefficients. A formal computation shows that $f_\xi(Z)$ is invariant under $X^{(1)}$ with respect to this action, hence we have a well-defined element

$$f_\xi(Z) \in \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}[Z]$$

Moreover, $f_\xi(Z)$ annihilates $\widehat{\theta}_{\mathfrak{o}}(x)$ which can be seen as follows. Let A denote the subalgebra of $\mathcal{A}_{\mathfrak{o}}^{(1)}$ generated by the $\widehat{\theta}_{\mathfrak{o}}(y)$, $y \in \xi$ over the center $\left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}$ of $\mathcal{A}_{\mathfrak{o}}^{(1)}$. From the previous remarks it follows that A is commutative. Moreover, we have

$$f_\xi(Z) = \prod_{y\in\xi}(Z - \widehat{\theta}_{\mathfrak{o}}(y)) \in A[Z]$$

as an equation in $A[Z]$. Using the evaluation homomorphism

$$\begin{aligned} \mathrm{ev} : A[Z] &\longrightarrow A \\ f(Z) &\longmapsto f(\widehat{\theta}_{\mathfrak{o}}(x)) \end{aligned}$$

it follows that

$$f_\xi(\widehat{\theta}_{\mathfrak{o}}(x)) = \mathrm{ev}(f_\xi) = \prod_{y\in\xi}(\widehat{\theta}_{\mathfrak{o}}(x) - \widehat{\theta}_{\mathfrak{o}}(y)) = 0$$

Thus the assumptions of the next lemma are satisfied and (iv) and (v) follow. In order to get claims (viii) and (ix), we apply the lemma with B,Λ and Π as before but with

$$C = \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}}$$

In order to see that assumption (b) of the lemma is satisfied, it suffices to show that

$$g_x(Z) := \prod_{\eta\in W_0\bullet\xi} f_\eta(Z) \in \left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{X^{(1)}}[Z]$$

has coefficients in $\left(\mathcal{A}_{\mathfrak{o}}^{(1)}\right)^{W^{(1)}}$, i.e. that it is invariant under W_0. But, considering the expression of the coefficients of f_ξ as symmetric polynomials in the $\widehat{\theta}_{\mathfrak{o}}(y)$, $y \in \xi$, it follows that

$$w(f_\xi) = f_{w(\xi)} \quad \forall w \in W_0$$

Hence, it follows that g_x is invariant under W_0 by a formal computation. $\square$

2.7.2 Lemma. *Let R be a commutative ring, B a not necessarily commutative R-algebra and $C \subseteq Z(B)$ an R-subalgebra of the center of B. Assume that there exist finite subsets*

$$\Lambda = \{x_1, \ldots, x_n\} \subseteq B$$

and

$$\Pi \subseteq Z(B)$$

such that

(a) B is generated as an R-algebra by Λ.

(b) Every $x_i \in \Lambda$ satisfies a monic equation

$$f_i(x_i) = 0, \quad f_i(X) = X^{n_i} + a_{i,1}X^{n_i-1} + \ldots + a_{i,n_i} \in C[X]$$

with coefficients in C.

(c) The generators commute up to elements of Π, i.e.

$$\forall x, y \in \Lambda\ \exists t \in \Pi \quad xy = tyx$$

(d) Every $t \in \Pi$ satisfies a monic equation with coefficients in C, i.e. the C-subalgebra $C[\Pi] \subseteq Z(B)$ generated by Π is finitely generated as a C-module.

Then

(i) B is generated as a C-module by

$$\{tx_1^{\nu_1} \ldots x_n^{\nu_n} : t \in C[\Pi],\ 0 \leq \nu_i < n_i\ \forall i\}$$

In particular B is a finitely generated C-module.

(ii) If R is noetherian, then C is a finitely generated R-algebra.

Proof. Claim (i) follows immediately by combining assumptions (a)-(c). Claim (ii) follows as in the classical commutative case by dévissage. Let C' be the R-subalgebra of C generated by the coefficients $a_{i,j}$ and the coefficients of the monic equations satisfied by the elements of Π, and let $C'\langle x_1, \dots, x_n\rangle$ be the C'-subalgebra of B generated by $x_1, \dots, x_n$. This situation is summarized in the following diagram

$$C' \subseteq C \subseteq C'\langle x_1, \dots, x_n\rangle \subseteq B$$

The assumptions of this lemma are still satisfied if one replaces C by C' and B by $C'\langle x_1, \dots, x_n\rangle$. From part (i) it therefore follows that $C'\langle x_1, \dots, x_n\rangle$ is a finite C'-module. Since C' is the homomorphic image of a polynomial ring over R in a finite number of variables it is noetherian, hence it follows that the submodule $C \subseteq C'\langle x_1, \dots, x_n\rangle$ is also finitely generated. In particular C is a finitely generated C'-algebra. Since C' is a finitely generated R-algebra, it follows by transitivity that C is a finitely generated R-algebra. □

2.7.3 Remark. In some of the finiteness results proved in the main theorem we had to assume that $W^{(1)}$ acts with finite orbits on $X^{(1)}$. Let us see what this condition amounts to. Since $W_0 \simeq W^{(1)}/X^{(1)}$ is finite, the group $W^{(1)}$ acts by finite orbits if and only if the subgroup $X^{(1)}$ acts by finite orbits. But, if $x, y \in X^{(1)}$ then by definition

$$\pi(x) \bullet y = xyx^{-1} = \underbrace{xyx^{-1}y^{-1}}_{=:[x,y]} y$$

Since X is commutative by assumption, the commutator $[x, y]$ lies in T. Thus if T is finite, the group $W^{(1)}$ always acts with finite orbits.

Let us now consider the case when T is contained in the center of $X^{(1)}$ (but possibly infinite). This means that $X^{(1)}$ is a central extension

$$1 \longrightarrow T \longrightarrow X^{(1)} \longrightarrow X \longrightarrow 0$$

of abelian groups, and therefore the commutator $[x, y]$ only depends on $\pi(x)$ and $\pi(y)$ and gives rise to an alternating bilinear pairing

$$[-,-] : X \times X \longrightarrow T$$

By the above computation the orbit of an element $y \in X^{(1)}$ under X is given by the coset

$$X^{(1)} \bullet y = [X, \pi(y)]y$$

under the subgroup

$$[X, \pi(y)] \leq T$$

This subgroup is always finitely generated, since X is finitely generated by assumption. It is therefore finite if and only if it lies in the torsion subgroup $T_{\mathrm{tors}} \leq T$. Thus, when T is contained in the center of $X^{(1)}$, the group $W^{(1)}$ (actually W) acts with finite orbits if and only if the pairing $[-,-]$ takes values in T_{tors}. This is for instance the case if $X^{(1)}$ is abelian or T is finite.

3 The Hecke algebra of $\mathrm{PGL}_2(\mathbb{Z})$

In the following, we will study the generic Hecke algebra $\mathcal{H}$ associated to the Coxeter group $(\mathrm{PGL}_2(\mathbb{Z}), \{s_1, s_2, s_3\})$ over a coefficient ring R. Here, $\mathrm{PGL}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z})/\{\pm\}$ denotes *extended modular group* of invertible integer 2×2-matrices modulo center. The image of a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

under the map $\mathrm{GL}_2(\mathbb{Z}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{Z})$ will be denoted by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{Z})$$

The distinguished generators s_i are given by

$$s_1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad s_2 = \begin{bmatrix} -1 & 1 \\ & 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$$

The orders $m(s,t) = \mathrm{ord}(st)$ are given by

$$m(s_1, s_2) = 3, \quad m(s_1, s_3) = 2, \quad m(s_2, s_3) = \infty$$

We will write $T_i := T_{s_i}$ for the generators of $\mathcal{H}$, and $a_i := a_{s_i} \in R$ and $b_i := b_{s_i} \in R$ for the parameters of $\mathcal{H}$. Note that

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2$$

as s_1 and s_2 are conjugate ($(s_1 s_2) s_1 (s_1 s_2)^{-1} = s_2$), but otherwise the parameters are unconstrained.

3.1 Boundary orientations of $\mathrm{PGL}_2(\mathbb{Z})$

Let us now investigate the set $\mathcal{O}_{\text{boundary}}$ of the hyperbolic Coxeter group $W = \mathrm{PGL}_2(\mathbb{Z})$. It is indeed hyperbolic in every sense of the word, as it's a hyperbolic reflection group in the sense of Vinberg (cf. [Vin85, Introduction]), i.e. it's a discrete subgroup of the group of isometries of the hyperbolic plane $\mathbb{H}^2$ generated by reflections at hyperplanes (totally geodesic codimension one submanifolds) in $\mathbb{H}^2$. In fact, such a representation of $\mathrm{PGL}_2(\mathbb{Z})$ is afforded by its canonical action on the upper half-plane

$$\mathfrak{H} := \{z \in \mathbb{C} : \Im z > 0\}$$

considered as a model of $\mathbb{H}^2$ with metric $g(x+iy) = \frac{1}{y} dx \otimes dy$, via fractional linear transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \bullet z = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } ad - bc = 1 \\[2ex] \dfrac{a\overline{z}+b}{c\overline{z}+d} & \text{if } ad - bc = -1 \end{cases}$$

The generators s_1, s_2, s_3 act as the reflections at the hyperplanes

$$H_1 = \{z \in \mathfrak{H} : |z| = 1\}, \quad H_2 = \{z \in \mathfrak{H} : \Re z = \frac{1}{2}\}, \quad H_3 = \{z \in \mathfrak{H} : \Re z = 0\}$$

bounding the fundamental polytope

$$C := \{z \in \mathfrak{H} : |z| > 1, \ 0 < \Re z < \frac{1}{2}\}$$

To describe the boundary representations of $\mathrm{PGL}_2(\mathbb{Z})$, it is useful to extend the hyperbolic plane by its natural boundary, replacing the upper half-plane by the extended upper half-plane

$$\overline{\mathfrak{H}} := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{R}) = \{z \in \mathbb{C} : \Im z \geq 0\} \cup \{\infty\}$$

considered as a closed subset of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. The boundary orientations $\mathfrak{o} \in \mathcal{O}_{\text{boundary}}$ we want to describe are attached to actual boundary points $x \in \mathbb{P}^1(\mathbb{R})$, but to certain points corresponds more than one orientation. A precise statement is that there is a W-equivariant correspondence

$$\begin{tikzcd} & \mathfrak{F} \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1(\mathbb{R}) & & \mathcal{O}_{\text{boundary}} \end{tikzcd}$$

defined as follows. Since this construction is in part completely general, let (W,S) be an arbitrary Coxeter group for the moment. The set $\mathfrak{F}$ is the quotient $\Gamma/_\sim$ of the set

$$\Gamma := \{(w_n)_{n\in\mathbb{N}} : \forall n \ w_n \in W, \ w_n^{-1} w_{n+1} \in S, \ \ell(w_0^{-1} w_n) = n\}$$

of (semi-)infinite reduced galleries (carrying a natural W-action via $w \bullet (w_n)_{n\in\mathbb{N}} = (ww_n)_{n\in\mathbb{N}}$) by the ($W$-invariant) equivalence relation $\sim$ on Γ characterized uniquely by requiring

(3.1.1) $$w_0 = w_0' \quad \Rightarrow \quad (w_n)_{n\in\mathbb{N}} \sim (w_n')_{n\in\mathbb{N}} \ \Leftrightarrow \ \forall m \exists n \ n \geq m \wedge w_n = w_n'$$

and

(3.1.2) $$\forall m, m' \in \mathbb{N} \quad (w_n)_{n\in\mathbb{N}} \sim (w_n')_{n\in\mathbb{N}} \Leftrightarrow (w_{n+m})_{n\in\mathbb{N}} \sim (w_{n+m'}')_{n\in\mathbb{N}}$$

for all $(w_n)_{n\in\mathbb{N}}, (w'_n)_{n\in\mathbb{N}} \in \Gamma$. Moreover, there is a natural W-equivariant map

$$\mathfrak{F} \longrightarrow \mathcal{O}_{\text{boundary}}, \quad [(w_n)_{n\in\mathbb{N}}] \mapsto \lim_{n\in\mathbb{N}} \mathfrak{o}_{w_n}$$

where the left action on $\mathcal{O}_{\text{boundary}}$ is given in terms of the natural right action as $w \bullet \mathfrak{o} := \mathfrak{o} \bullet w^{-1}$. The equivariance then follows from the formula $\mathfrak{o}_{w'} \bullet w^{-1} = \mathfrak{o}_{ww'}$ (cf. remark 1.5.8). The limit $\lim_{n\in\mathbb{N}} \mathfrak{o}_{w_n}$ exists (a priori only as an element of $\{\pm\}^{W\times S}$, but by remark 1.5.6 also as an element of $\mathcal{O}$), as it does exist for any infinite gallery $(w_n)_{n\in\mathbb{N}}$ that crosses every hyperplane only finitely many times. The limit orientation $\mathfrak{o} = \lim_{n\in\mathbb{N}} \mathfrak{o}_{w_n}$ must lie in $\mathcal{O}_{\text{boundary}}$ because

$$\mathfrak{o}(w_n, w_n^{-1}w_{n+1}) = +1$$

for all n by construction, which would be impossible if $\mathfrak{o}$ were of the form $\mathfrak{o} = \mathfrak{o}_w$ or $\mathfrak{o} = \mathfrak{o}_w^{\text{op}}$.

The set $\mathfrak{F}$ can be described a little more explicitly (at the price of making the W-action more complicated) as follows. The embedding $\Gamma_0 \subseteq \Gamma$ of the subset of infinite reduced galleries starting in $w_0 = 1$ induces a bijection $\Gamma_0/_\sim \simeq \Gamma/_\sim$ of the quotient of Γ_0 by the equivalence relation defined by eq. (3.1.1) with $\mathfrak{F}$. Indeed, given any $(w_n)_{n\in\mathbb{N}} \in \Gamma$, we can choose $m \in \mathbb{N}$ such that the subgallery $w_m, w_{m+1}, \ldots$ does not cross any of the (finitely many) hyperplanes separating w_0 and 1, and then $(w'_n)_{n\in\mathbb{N}}$ defined by

$$w'_n := \begin{cases} w''_n & \text{if } n <= r \\ w_{n-r+m} & \text{if } n > r \end{cases}$$

is an element of Γ_0 equivalent to $(w_n)_{n\in\mathbb{N}}$, for any reduced gallery $w_0 = w''_0, \ldots, w''_r = 1$ from w_0 to 1.

Let now be $(W, S) = (\mathrm{PGL}_2(\mathbb{Z}), \{s_1, s_2, s_3\})$ again, then one can make $\mathfrak{F}$ even more explicit. Indeed in this case, a complete system of representatives for $\Gamma_0/_\sim$ is given by galleries corresponding to the infinite formal words in the generators of the form

$$(s_2s_3)^{a_0}s_1(s_2s_3)^{a_1}s_1(s_2s_3)^{a_2}s_1\ldots \quad \text{and} \quad (s_2s_3)^{a_0}s_1\ldots(s_1s_2)^{a_r}s_1(s_2s_3)^{\pm\infty}$$

where $a_i \in \mathbb{Z}$, subject to the condition that $\forall i\ a_i \geq 0$ or $\forall i\ a_i \leq 0$, and $a_i \neq 0$ for all $i > 0$. Here, the expressions $(s_2s_3)^{-\infty}$ and $(s_2s_3)^{+\infty}$ are to be understood as $s_3s_2s_3s_2\ldots$ and $s_2s_3s_2s_3\ldots$ respectively. We can identify these expressions with formal continued fraction as

$$[a_0, a_1, a_2, \ldots] \quad \text{and} \quad [a_0, \ldots, a_r, \pm\infty]$$

The map

$$\mathfrak{F} \longrightarrow \mathbb{P}^1(\mathbb{R})$$

is then given by evaluation of formal fractions, sending $[a_0, a_1, a_2, \ldots]$ to

$$[a_0, a_1, \ldots] := \lim_{n\to\infty} [a_0, \ldots, a_n]$$

where $[a_0, \ldots, a_n] \in \mathbb{Q}$ is defined recursively by

$$[a_0, \ldots, a_n, a_{n+1}] := [a_0, \ldots, a_n + \frac{1}{a_{n+1}}], \quad [a_0] := \frac{1}{a_0}$$

as usual. The W-equivariance of the map $\mathfrak{F} \longrightarrow \mathbb{P}^1(\mathbb{R})$ can most easily be verified by establishing that the value of a class $[(w_n)_{n\in\mathbb{N}}]$ is given by the limit $\lim_{n\to\infty} w_n \bullet z$, independent of the choice of a $z \in \overline{\mathfrak{H}}$. From this it also follows that the point $x \in \mathbb{P}^1(\mathbb{R})$ and the orientation $\mathfrak{o} \in \mathcal{O}_{\text{boundary}}$ defined by an element of $\mathfrak{F}$ satisfy

$$\text{(3.1.3)} \qquad \mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Z})}(\mathfrak{o}) \subseteq \mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Z})}(x)$$

From the theory of continued fractions it follows that the map $\mathfrak{F} \to \mathbb{P}^1(\mathbb{R})$ is surjective and that the infinite formal continued fractions $[a_0, a_1, a_2, \ldots]$ map bijectively onto $\mathbb{R} - \mathbb{Q}$, while the finite ones map many to one to $\mathbb{P}^1(\mathbb{Q})$, with $[a_0, \ldots, a_r, -\infty], [a_0, \ldots, a_r, +\infty]$ both mapping to $[a_0, \ldots, a_r] \in \mathbb{P}^1(\mathbb{Q})$. More precisely, $\infty \in \mathbb{P}^1(\mathbb{Q})$ has the two preimages $[-\infty], [+\infty]$ ($r = -1$), whereas every $x \in \mathbb{Q}$ has four preimages because it can be expressed as two distinct continued fractions, due to the identity

$$[a_0, \ldots, a_r, 1] = [a_0, \ldots, a_r + 1]$$

The orientation $\mathfrak{o} = \lim_n \mathfrak{o}_{w_n}$ defined by a $(w_n)_{n\in\mathbb{N}} \in \Gamma$ can be described concretely in terms of the corresponding point $x \in \mathbb{P}^1(\mathbb{R})$ as follows. The set $\mathfrak{H} = \{wsw^{-1} : w \in W,\ s \in S\}$ of formal hyperplanes

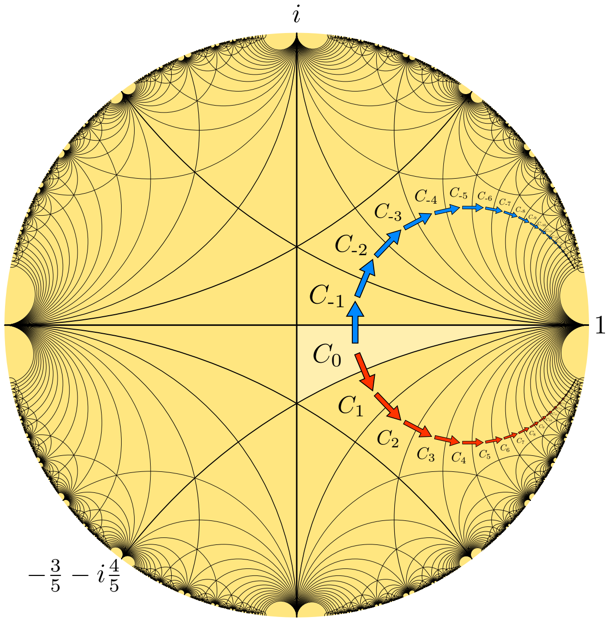


Figure 5: The canonical hyperplane arrangement realizing the Coxeter group $W = \mathrm{PGL}_2(\mathbb{Z})$ as a hyperbolic reflection group, viewed in the disk model (isometric to the upper half-plane $\mathfrak{H}$ via the Cayley transform $q = \frac{z-i}{z+i}$) of the hyperbolic plane. The two orientations of W attached to the 'point at infinity' $z = \infty$ ($q = 1 \in \overline{\Delta}$) are the limits $\lim_{n\to-\infty} \mathfrak{o}_{C_n}$, $\lim_{n\to+\infty} \mathfrak{o}_{C_n}$ attached to the two semi-infinite galleries contained in the 'horocycle' $(C_n)_{n\in\mathbb{Z}}$ and starting in the fundamental polytope C_0.

can be identified as a W-set with the set $\{w \bullet H_1, w \bullet H_2, w \bullet H_3 : w \in W\}$ of W-conjugates of the hyperplanes in $\mathbb{H}^2 = \mathfrak{H}$ bounding the fundamental polytope. If $H = wsw^{-1}$ corresponds to a hyperplane $H \subseteq \mathfrak{H}$ such that $x \notin \overline{H}$, then

$$\text{(3.1.4)} \qquad \mathfrak{o}(w,s) = +1 \quad \Leftrightarrow \quad w \bullet C \text{ and } x \text{ lie in different connected components of } \overline{\mathfrak{H}} - \overline{H}$$

The condition $x \notin \overline{H}$ is always satisfied if $x \in \mathbb{R} - \mathbb{Q}$, since the endpoints of the hyperplanes $H \in \mathfrak{H}$ on $\mathbb{P}^1(\mathbb{R})$ lie in $\mathbb{P}^1(\mathbb{Q})$, and the orientation attached to x is then uniquely and explicitly determined by eq. (3.1.4). It follows easily that in this case the inclusion in eq. (3.1.3) is an equality. Since the stabilizer $\mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Z})}(x)$ is non-trivial precisely when $x \in \mathbb{P}^1(\mathbb{Q})$ or x is a quadratic irrational number (by a classical exercise), the orientation $\mathfrak{o}_x$ attached to an irrational number $x \in \mathbb{R} - \mathbb{Q}$ has a non-trivial stabilizer if and only if it is quadratic irrational.

The two orientations attached to rational boundary points $x \in \mathbb{P}^1(\mathbb{Q})$ also have a non-trivial stabilizer, but the inclusion in eq. (3.1.3) is proper in this case. In fact, since

$$x \in \overline{H} \quad \Leftrightarrow \quad s_H = wsw^{-1} \in \mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Z})}(x)$$

we have that $s_2 = s_{H_2}, s_3 = s_{H_3} \in \mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Z})}(\infty)$, but s_2 and s_3 both interchange the two orientations attached to $x = \infty$ (cf. figure 5). The stabilizers of these orientations are instead equal to the subgroup

$$\langle s_2 s_3 \rangle = \left\{ \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix} : z \in \mathbb{Z} \right\}$$

3.2 The geometry of $\mathrm{PGL}_2(\mathbb{Z})$

3.2.1 Lemma. *There exists a unique subset $\mathfrak{F} \subseteq W$ such that every element of W is of the form $w = xw_0$ with $x \in X_\infty$ and $w_0 \in \mathfrak{F}$, and*

$$\text{(3.2.1)} \qquad \forall x \in X_\infty,\ w \in \mathfrak{F} \quad \ell(xw) = \ell(x) + \ell(w)$$

Moreover, this set $\mathfrak{F}$ is given by

$$\begin{aligned} \mathfrak{F} &= \{w \in W : 1 \text{ and } w \text{ lie in the same half-spaces with respect to } s_2 \text{ and } s_3\} \\ &= \{w \in W : \text{no reduced expression of } w \text{ starts with } s_2 \text{ or } s_3\} \\ &= \{w \in W : \ell(s_i w) > \ell(w) \text{ for } i = 2,3\} \end{aligned}$$

Proof. We leave it to the reader to verify that the three expressions given for $\mathfrak{F}$ define the same set (cf. figure 6). Let us begin by showing that the elements of $\mathfrak{F}$ constitute a complete system of representatives for $X_\infty \backslash \mathrm{PGL}_2(\mathbb{Z})$. If $w \in \mathrm{PGL}_2(\mathbb{Z})$ does not already lie in $\mathfrak{F}$, then there exists a reduced expression $w = s_{i_1} w'$, $\ell(w) = 1 + \ell(w')$ with $i_1 = 2$ or $i_1 = 3$. Continuing this reasoning, we can write

$$w = s_{i_1} s_{i_2} \dots s_{i_r} w''$$

with $\ell(w) = r + \ell(w'')$ and $i_j \in \{2, 3\}$, and where w'' does not admit a reduced expression starting in s_2 or s_3, i.e. $w'' \in \mathfrak{F}$. Thus

$$w = xw'', \quad \ell(w) = \ell(x) + \ell(w''), \quad x \in X_\infty, w'' \in \mathfrak{F}$$

where $x := s_{i_1} \dots s_{i_r}$. This proves in particular that every element of $\mathrm{PGL}_2(\mathbb{Z})$ is of the form xf, and it also proves that $\ell(xf) = \ell(x) + \ell(f)$ once we've proven that such an expression is unique. So assume that $w = xf = x'f'$ with $x, x' \in X_\infty$, $f, f' \in \mathfrak{F}$ and $\ell(x) + \ell(f) = \ell(x') + \ell(f')$. Without loss of generality, we may assume that $\ell(x) \le \ell(x')$. By taking reduced expressions of $x^{-1}x$ and f', inserting them into the equality

$$f = x^{-1}x'f'$$

and reducing the resulting expression using the deletion condition, it follows that there is an expression

$$f = x''f'', \quad \ell(f) = \ell(x'') + \ell(f'')$$

where $x'' \le x^{-1}x'$ and $f'' \le f'$ in the strong Bruhat order. Since

$$\ell(f) \ge \ell(f') \ge \ell(f'') = \ell(f) - \ell(x'')$$

it follows that either these inequalities are equalities and therefore $x'' = 1$, $f'' = f$ and hence $f' = f$ (since $f'' \le f'$) or that these inequalities are strict and that $x' \ne 1$, in which case f would have a reduced expression starting in s_2 or s_3 contradicting $f \in \mathfrak{F}$.

Thus, the set $\mathfrak{F}$ as defined satisfies the claimed properties. The uniqueness of $\mathfrak{F}$ is clear, since if $f \in \mathfrak{F}$, then every other element of the Orbit $X_\infty \bullet f$ is of the form $w = xf$ with $\ell(w) = \ell(x) + \ell(f) > \ell(f)$. But then

$$\ell(x^{-1}w) = \ell(f) < \ell(x) + \ell(f) = \ell(x^{-1}) + \ell(f) < \ell(x^{-1}) + \ell(w)$$

□

3.2.2 Remark. Since $X_\infty = \mathrm{Stab}_{\mathrm{PGL}_2(\mathbb{Z})}(\infty)$ and $\mathrm{PGL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, we have a bijection

$$X_\infty \backslash \mathrm{PGL}_2(\mathbb{Z}) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{Q}), \quad [w] \mapsto w^{-1} \bullet \infty$$

By lemma 3.2.1, there exists a complete set $\mathfrak{F} \subseteq \mathrm{PGL}_2(\mathbb{Z})$ for the action of X_∞, and therefore a bijection

$$\mathfrak{F} \xrightarrow{\sim} \mathbb{P}^1(\mathbb{Q}), \quad f \mapsto f^{-1} \bullet \infty$$

The set $\mathfrak{F}$ (indicated in blue) and its labelling are illustrated in figure 6

3.3 The subalgebra $\mathcal{A}_\infty \subseteq \mathcal{H}$

We let

$$X_\infty = \langle s_2, s_3 \rangle = \mathrm{Stab}_W(\infty \in \mathbb{P}^1(\mathbb{Q}))$$

as before.

3.3.1 Definition. The parabolic Hecke subalgebra associated to the special subgroup $(X_\infty, \{s_2, s_3\})$ of W is denoted by $\mathcal{A}_\infty$. In other words,

$$\mathcal{A}_\infty = \bigoplus_{x \in X_\infty} RT_x \subseteq \mathcal{H}$$

3.3.2 Remark. The Coxeter group $(X_\infty, \{s_2, s_3\})$ is nothing but the *infinite dihedral group* D_∞, which is also the same as the *extended affine Weyl group* $W = X^\vee \rtimes W_0$ of the root datum

$$(X, \Phi, X^\vee, \Phi^\vee) = (\mathbb{Z}, \{\pm 2\}, \mathbb{Z}, \{\pm 1\})$$

of the group $\mathbf{SL}_2$. Under the identification $X_\infty = X^\vee \rtimes W_0 = \mathbb{Z} \rtimes S_2$, the subgroup W_0 corresponds to $\langle s_3 \rangle = \{1, s_3\}$ and $X^\vee$ corresponds to $\langle s_2 s_3 \rangle = \{(s_2 s_3)^k : k \in \mathbb{Z}\}$.

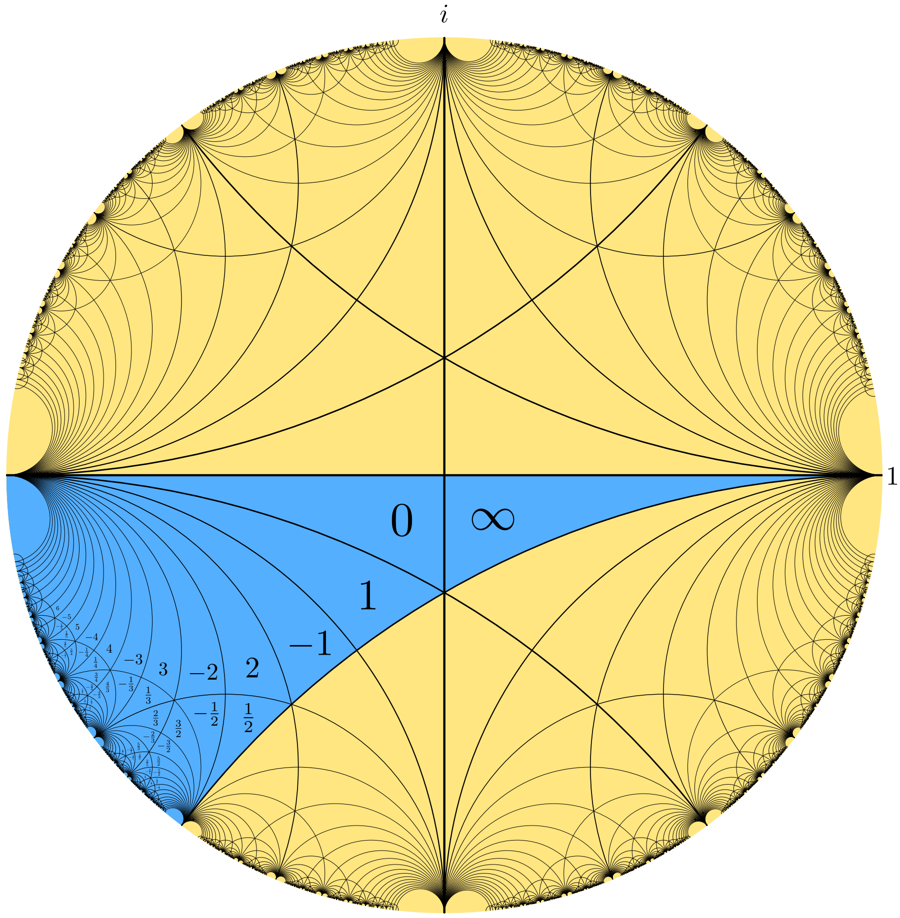


Figure 6: The fundamental domain $\mathfrak{F}$ (coloured in blue) for the left action of X_∞ on $\mathrm{PGL}_2(\mathbb{Z})$ and the labelling of its elements via the bijection $\mathfrak{F} \xrightarrow{\sim} \mathbb{P}^1(\mathbb{Q})$, $f \mapsto f^{-1} \bullet \infty$.

In particular, $\mathcal{A}_\infty$ is the affine Hecke algebra of $\mathbf{SL}_2$ with parameters a_s, b_s, $s \in \{s_2, s_3\}$, and the structure theorem 2.7.1 applies (cf. also section 2.2.1), showing that $\mathcal{A}_\infty$ is noetherian when R is, and that the center of $\mathcal{A}_\infty$ is given by the invariants of each of the two commutative subalgebras ($\varepsilon \in \{\pm\}$)

$$\mathcal{A}_\infty^\varepsilon := \mathcal{A}^{(1)}_{\mathfrak{o}^\varepsilon_\infty}(X^\vee) = \bigoplus_{k\in\mathbb{Z}} R\widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}((s_2s_3)^k) \subseteq \mathcal{A}_\infty$$

under the action of $W_0 = \{1, s_3\}$, where $s_3 \bullet \widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}((s_2s_3)^k) = \widehat{\theta}_{\mathfrak{o}^{\varepsilon\infty}}((s_2s_3)^{-k})$. Note here that the restriction of the orientations $\mathfrak{o}^+_\infty, \mathfrak{o}^-_\infty$ to X_∞ give precisely the two spherical orientations of X_∞. More explicitly, we have

$$\widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}((s_2s_3)^\varepsilon) = T_2T_3, \quad \widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}((s_2s_3)^{-\varepsilon}) = (T_3 - b_3)(T_2 - b_2)$$

and $\widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}((s_2s_3)^{\pm k}) = \widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}((s_2s_3)^{\pm 1})^k$ for $k \geq 0$, and

$$\begin{aligned} R[X,Y]/(XY - a_2a_3) &\xrightarrow{\sim} \mathcal{A}^\varepsilon_\infty \\ X &\longmapsto \widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}(s_2s_3) \\ Y &\longmapsto \widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}((s_2s_3)^{-1}) \end{aligned}$$

Under this isomorphism, $s_3 \bullet \overline{X} = \overline{Y}$ where $\overline{X}, \overline{Y}$ denote the images of X, Y under $R[X,Y] \twoheadrightarrow R[X,Y]/(XY - a_2a_3)$. Moreover, there is an isomorphism of R-algebras

$$\begin{aligned} R[Z,T]/(T^2 - ZT + a_2a_3) &\xrightarrow{\sim} R[X,Y]/(XY - a_2a_3) \\ Z &\longmapsto \overline{X} + \overline{Y} \\ T &\longmapsto \overline{X} \end{aligned}$$

with the induced action of s_3 being trivial on Z and satisfying $s_3 \bullet T = Z - T$. The invariant subalgebra is $R[Z]$, and hence

$$R[Z] \xrightarrow{\sim} Z(\mathcal{A}_\infty), \quad Z \longmapsto T_2T_3 + (T_3 - b_3)(T_2 - b_2)$$

Moreover, $\mathcal{H}$ is free as a left-module over $\mathcal{A}^\varepsilon_\infty$ with basis $\{1, \widehat{\theta}_{\mathfrak{o}^\varepsilon_\infty}(s_2)\}$, and $\widehat{\theta}_{\mathfrak{o}^+_\infty}(s_2) = T_2$. As an $R[Z]$-module, $\mathcal{H}$ is therefore free with basis $\{1, T_2, \underbrace{T_2T_3}_{\overline{T}}, \underbrace{T_2T_3}_{\overline{T}}T_2\}$.

3.3.3 Lemma. *$\mathcal{H}$ is free as a left-$\mathcal{A}_\infty$-module, with basis given by the elements*

$$[f]_\infty := T_f = \widehat{\theta}_{-\mathfrak{o}^+_\infty}(f) = \widehat{\theta}_{-\mathfrak{o}^-_\infty}(f), \quad f \in \mathfrak{F}$$

Moreover, the relation ($\varepsilon \in \{\pm\}$)

$$[f]_\infty \cdot \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty \bullet f}(w) = \mathbb{X}(f,w)\widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty}(x)[f']_\infty, \quad x \in X_\infty, f' \in \mathfrak{F}, \; fw = xf'$$

holds true for all $f \in \mathfrak{F}$ and $w \in W$.

Proof. Let us begin by proving the equalities

$$T_f = \widehat{\theta}_{-\mathfrak{o}^+_\infty}(f) = \widehat{\theta}_{-\mathfrak{o}^-_\infty}(f)$$

By definition

$$\mathfrak{o}^\varepsilon_\infty = \lim_{n\to\infty} \mathfrak{o}_{(s_2s_3)^{\varepsilon n}}$$

Therefore, for $f \in \mathfrak{F}$, $i \in \{1,2,3\}$ and $n > 0$ sufficiently large we have (cf. definition 1.5.7)

$$\mathfrak{o}^\varepsilon_\infty(f, s_i) = \mathfrak{o}_{(s_2s_3)^{\varepsilon n}}(f, s_i) = \begin{cases} +1 & \text{if } \ell((s_2s_3)^{-\varepsilon n} f s_i) < \ell((s_2s_3)^{-\varepsilon n} f) \\ -1 & \text{if } \ell((s_2s_3)^{-\varepsilon n} f s_i) > \ell((s_2s_3)^{-\varepsilon n} f) \end{cases} \tag{3.3.1}$$

If moreover $fs_i \in \mathfrak{F}$, then

$$\ell((s_2s_3)^{\varepsilon n} f s_i) = \ell((s_2s_3)^{\varepsilon n}) + \ell(fs_i), \quad \ell((s_2s_3)^{\varepsilon n} f) = \ell((s_2s_3)^{\varepsilon n}) + \ell(f)$$

by eq. (3.2.1), hence eq. (3.3.1) simplifies to

$$\mathfrak{o}^\varepsilon_\infty(f, s_i) = \begin{cases} +1 & \text{if } \ell(fs_i) < \ell(f) \\ -1 & \text{if } \ell(fs_i) > \ell(f) \end{cases} \tag{3.3.2}$$

In particular, $\mathfrak{o}^+_\infty(f, s_i) = \mathfrak{o}^-_\infty(f, s_i)$ in that case, and

$$-\mathfrak{o}^\varepsilon_\infty(f, s_i) = 1 \quad \text{if} \quad f s_i \in \mathfrak{F},\ \ell(f s_i) > \ell(f)$$

Moreover, $\mathfrak{F}$ is *convex*, i.e. given a reduced expression $f = s_{i_1} \dots s_{i_r}$ we have

$$s_{i_1} \dots s_{i_j} \in \mathfrak{F} \quad \forall j = 1, \dots, r$$

Therefore, we can apply the above formula to conclude that $\mathfrak{o}^\varepsilon(s_{i_1} \dots s_{i_{j-1}}, s_{i_j}) = -1$ for all j, and it follows that (cf. theorem 1.10.1)

$$\widehat{\theta}_{\mathfrak{o}^+_\infty}(f) = \widehat{\theta}_{\mathfrak{o}^-_\infty}(f) = T_{s_{i_1}} \dots T_{s_{i_r}} = T_f$$

Using this, we compute

$$\begin{aligned}
[f]_\infty \cdot \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty \bullet f}(w) &= \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty}(f) \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty \bullet f}(w) \\
&= \mathbb{X}(f, w) \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty}(fw) \\
&= \mathbb{X}(f, w) \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty}(xf') \\
&= \mathbb{X}(f, w) \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty}(x) \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty \bullet x}(f') \\
&= \mathbb{X}(f, w) \widehat{\theta}_{-\mathfrak{o}^\varepsilon_\infty}(x) [f']_\infty
\end{aligned}$$

Here we have used that $\ell(xf') = \ell(x) + \ell(f')$ in the second-to-last step, and the fact that X_∞ leaves $\{\mathfrak{o}^+_\infty, \mathfrak{o}^-_\infty\}$ invariant in the last step.

That the $[f]_\infty$, $f \in \mathfrak{F}$ constitute a basis of $\mathcal{H}$ as left module over $\mathcal{A}_\infty$ follows from the fact that $\mathcal{A}_\infty$ is free as an R-module with basis T_x, $x \in X_\infty$, and the fact that the elements

$$T_x T_f = T_{xf}, \quad x \in X_\infty,\ f \in \mathfrak{F}$$

are an R-basis of $\mathcal{H}$. □

3.3.4 Remark. Recall from remark 3.2.2 that we have a bijection

$$\mathfrak{F} \longrightarrow \mathbb{P}^1(\mathbb{Q}), \quad f \mapsto f^{-1} \bullet \infty$$

Let $x \mapsto f_x$ denote the inverse of this bijection, and let us write

$$[x]_\infty := [f_x]_\infty \in \mathcal{H}$$

for $x \in \mathbb{P}^1(\mathbb{Q})$. Then the induction functor $M \mapsto M \otimes_{\mathcal{A}_\infty} \mathcal{H}$ can also be written as

$$M \otimes_{\mathcal{A}_\infty} \mathcal{H} \simeq \bigoplus_{x \in \mathbb{P}^1(\mathbb{Q})} M[x]_\infty$$

Let us now complete explicitly the action of the generators $T_i := T_{s_i}$, $i = 1, 2, 3$ of $\mathcal{H}$ on the basis elements $[x]_\infty$.

3.3.5 Lemma. *For all $x \in \mathbb{P}^1(\mathbb{Q})$ we have*

$$[x]_\infty \cdot T_1 = \begin{cases} T_2[x]_\infty & \text{if } x \in \{-1, 1\} \\ \mathbb{X}(f_x, s_1)[\frac{1}{x}]_\infty & \text{if } x \notin \{-1, 1\},\ \ell(f_x s_1) > \ell(f_x) \\ \mathbb{X}(f_x, s_1)[\frac{1}{x}]_\infty + b_1[x]_\infty & \text{if } x \notin \{-1, 1\},\ \ell(f_x s_1) < \ell(f_x) \end{cases}$$

$$[x]_\infty \cdot T_2 = \begin{cases} T_2[x]_\infty & \text{if } x \in \{\frac{1}{2}, \infty\} \\ \mathbb{X}(f_x, s_2)[-x+1]_\infty & \text{if } x \notin \{\frac{1}{2}, \infty\},\ \ell(f_x s_2) > \ell(f_x) \\ \mathbb{X}(f_x, s_2)[-x+1]_\infty + b_2[x]_\infty & \text{if } x \notin \{\frac{1}{2}, \infty\},\ \ell(f_x s_2) < \ell(f_x) \end{cases}$$

$$[x]_\infty \cdot T_3 = \begin{cases} T_3[x]_\infty & \text{if } x \in \{0, \infty\} \\ \mathbb{X}(f_x, s_3)[-x]_\infty & \text{if } x \notin \{0, \infty\},\ \ell(f_x s_3) > \ell(f_x) \\ \mathbb{X}(f_x, s_3)[-x]_\infty + b_3[x]_\infty & \text{if } x \notin \{0, \infty\},\ \ell(f_x s_3) < \ell(f_x) \end{cases}$$

Proof. By lemma 3.3.3, we have

$$[f_x]_\infty \widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon \bullet f_x}(s_i) = \mathbb{X}(f_x, s_i)\widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon}(y)[f']_\infty$$

where $y \in X_\infty$ and $f \in \mathfrak{F}$ are such that

$$f_x s_i = y f'$$

In particular

$$[f_x]_\infty \widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon \bullet f_x}(s_i) = \mathbb{X}(f_x, s_i)[f_x s_i]_\infty$$

if $f_x s_i \in \mathfrak{F}$. Moreover in that case we can apply eq. (3.3.2) to see that

$$(-\mathfrak{o}_\infty^\varepsilon \bullet f_x)(1, s_i) = -\mathfrak{o}_\infty^\varepsilon(f_x, s_i) = \begin{cases} +1 & \text{if } \ell(f_x s_i) > \ell(f_x) \\ -1 & \text{if } \ell(f_x s_i) < \ell(f_x) \end{cases}$$

Accordingly

$$\widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon \bullet f_x}(s_i) = \begin{cases} T_i & \text{if } \ell(f_x s_i) > \ell(f_x) \\ T_i - b_i & \text{if } \ell(f_x s_i) < \ell(f_x) \end{cases}$$

which proves the claimed formulas in the case where $f_x s_i \in \mathfrak{F}$ because $(f_x s_i)^{-1} \bullet \infty = s_i \bullet x$ and therefore

$$f_x s_i = f_{s_i \bullet x}$$

Now, $f_x \in \mathfrak{F}$, $f_x s_i \notin \mathfrak{F}$ happens if and only if the hyperplane $H = f_x s_i f_x^{-1}$ separating f_x and $f_x s_i$ is one of the hyperplanes s_2, s_3 defining the boundary of $\mathfrak{F}$, i.e.

$$f_x s_i \notin \mathfrak{F} \quad \Leftrightarrow \quad f_x s_i f_x^{-1} \in \{s_2, s_3\}$$

and in that case we have

$$[f_x]_\infty \widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon \bullet f_x}(s_i) = \mathbb{X}(f_x, s_i)\widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon}(s_j)[f_x]_\infty = \widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon}(s_j)[f_x]_\infty$$

if $j \in \{2, 3\}$ is such that $f_x s_i = s_j f_x$, as the convexity of $\mathfrak{F}$ and $f_x \in \mathfrak{F}$, $f_x s_i \notin \mathfrak{F}$ implies that $\ell(f_x s_i) > \ell(f_x)$ and therefore $\mathbb{X}(f_x, s_i) = 1$. But, as figure figure 6 shows, there are only finitely many $f \in \mathfrak{F}$ which have a facet lying in the boundary hyperplanes s_2 and s_3, namely f_x for $x \in \{\infty, 0, 1, -1, \frac{1}{2}\}$. More precisely, inspecting figure figure 6 again one sees that $f_x s_i \notin \mathfrak{F}$ happens precisely when $s_i \bullet x = x$, and that

$$\begin{aligned} f_1 s_1 &= s_2 f_1, & f_{-1} s_1 &= s_2 f_{-1} \\ f_{\frac{1}{2}} s_2 &= s_2 f_{\frac{1}{2}}, & f_\infty s_2 &= s_2 f_\infty \\ f_\infty s_3 &= s_3 f_\infty, & f_0 s_3 &= s_3 f_0 \end{aligned}$$

Using eq. (3.3.1) and eq. (3.2.1) it follows that

$$\begin{aligned} -\mathfrak{o}_\infty^\varepsilon(1, s_i) &= \begin{cases} +1 & \text{if } \ell((s_2 s_3)^{-\varepsilon n} s_i) > \ell((s_2 s_3)^{-\varepsilon n}) \\ -1 & \text{if } \ell((s_2 s_3)^{-\varepsilon n} s_i) < \ell((s_2 s_3)^{-\varepsilon n}) \end{cases} \\ &= -\mathfrak{o}_\infty^\varepsilon(1, s_j) \end{aligned}$$

Moreover, elementary arguments show that

$$-\mathfrak{o}_\infty^\varepsilon(1, s_i) = \begin{cases} +1 & \text{if } i = 1 \vee (i = 2 \wedge \varepsilon = -1) \vee (i = 3 \wedge \varepsilon = +1) \\ -1 & \text{if } (i = 2 \wedge \varepsilon = +1) \vee (i = 3 \wedge \varepsilon = -1) \end{cases}$$

and hence ε can always be chosen so that

$$(-\mathfrak{o}_\infty^\varepsilon \bullet f_x)(1, s_i) = -\mathfrak{o}_\infty^\varepsilon(1, s_j) = 1$$

which gives

$$\widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon \bullet f_x}(s_i) = T_i, \quad \widehat{\theta}_{-\mathfrak{o}_\infty^\varepsilon}(s_j) = T_j$$

and therefore implies the claimed formulas in the case where $f_x s_i \notin \mathfrak{F}$. ∎

3.4 Intertwiners

3.4.1 Lemma.

$$\forall w \in \mathrm{PGL}_2(\mathbb{Z}) \quad \ell(ws_2) > \ell(w) \vee \ell(ws_3) > \ell(w)$$

Proof. Assume that $\ell(ws_2) < \ell(w)$. Then there exists a reduced expression

$$w = s_{i_1} \dots s_{i_r}$$

of w ending in s_2 (i.e. $i_r = 2$). If we also had $\ell(ws_3) < \ell(w)$, then there would exist another reduced expression ending in s_3. Since two reduced expressions of the same element of a Coxeter group are connected by a finite sequence of braid moves, this would imply in particular that $i_j = 3$ for some $j < r$, and so

$$\max\{j : i_j = 3\} < \max\{j : i_j = 2\}$$

But this condition is preserved under braid moves (since $\mathrm{m}(s_2, s_3) = \infty$), which leads to a contradiction. Hence we must have $\ell(ws_3) > \ell(w)$. □

3.4.2 Theorem. *Let $\chi : \mathcal{A}_\infty \longrightarrow R$ be a character. If $a_i \in R^\times$ for all i, then*

$$\mathrm{End}(\chi \otimes_{\mathcal{A}_\infty} \mathcal{H}) = R$$

i.e. the induced $\mathcal{H}$-module $\chi \otimes_{\mathcal{A}_\infty} \mathcal{H} \simeq \bigoplus_{x \in \mathbb{P}^1(\mathbb{Q})} \chi[x]_\infty$ is Schur-simple.

Proof. Frobenius reciprocity gives

$$\begin{aligned} \mathrm{End}_{\mathcal{H}}(\chi \otimes_{\mathcal{A}_\infty} \mathcal{H}) &\simeq \mathrm{Hom}_{\mathcal{A}_\infty}(\chi, \chi \otimes_{\mathcal{A}_\infty} \mathcal{H}) \\ &\simeq \{\phi \in \chi \otimes_{\mathcal{A}_\infty} \mathcal{H} \ : \ \phi \cdot T_i = \chi(T_i)\phi \ \text{ for } i = 2, 3\} \end{aligned}$$

Let $\phi = \sum_{x \in \mathbb{P}^1(\mathbb{Q})} c_x [x]_\infty \neq 0$ be a nonvanishing element of the set in the second line above. We must show that $c_x = 0$ for all $x \neq \infty$. First, let us show that

$$\{x \in \mathbb{P}^1(\mathbb{Q}) \ : \ c_x \neq 0\} \subseteq \{0, \frac{1}{2}, \infty\} \tag{3.4.1}$$

To this end, let f be an element of

$$\{f \in \mathcal{F} \ : \ c_{f^{-1} \bullet \infty} \neq 0\} - \{f_x : x \in \{0, \frac{1}{2}, \infty\}\} \subseteq W$$

of maximal length. By lemma 3.4.1, we have $\ell(fs_2) > \ell(f)$ or $\ell(fs_3) > \ell(f)$. Without loss of generality, we may assume that $\ell(fs_2) > \ell(f)$ (the other case is treated similar). Because $f \notin \{f_x : \{0, \frac{1}{2}, \infty\}\}$ we have $fs_2 \in \mathfrak{F}$, and by lemma 3.3.5 it follows therefore that ($x := f^{-1} \bullet \infty$)

$$[x]_\infty T_2 = a_2 [s_2 \bullet x]_\infty$$

Since f was assumed to be of maximum length and $\ell(fs_2) > \ell(f)$, it follows[24] that the coefficient of $[s_2 \bullet x]_\infty$ in $\chi(T_2)\phi$ vanishes; for the same reasons, the coefficient of $[s_2 \bullet x]_\infty$ in $\phi \cdot T_2$ equals $a_2 c_x$. Since $\phi \cdot T_2 = \chi(T_2)\phi$ by assumption, it follows that

$$a_2 c_x = 0$$

Since $a_2 \in R^\times$, it follows that $c_x = 0$ in contradiction to our assumptions. This proves eq. (3.4.1), and we can therefore write

$$\phi = c_0[0]_\infty + c_{\frac{1}{2}}[\frac{1}{2}]_\infty + c_\infty[\infty]_\infty$$

Using lemma 3.3.5, we compute

$$\phi \cdot T_2 = a_2 c_0 [1]_\infty + \chi(T_2) c_{\frac{1}{2}} [\frac{1}{2}]_\infty + \chi(T_2) c_\infty [\infty]_\infty$$

$$\chi(T_2)\phi = \chi(T_2) c_0 [0]_\infty + \chi(T_2) c_{\frac{1}{2}} [\frac{1}{2}]_\infty + \chi(T_2) c_\infty [\infty]_\infty$$

Thus $c_0 = 0$. Since

$$\phi \cdot T_3 = a_3 c_{\frac{1}{2}} [-\frac{1}{2}]_\infty + \chi(T_3) c_\infty [\infty]_\infty$$

$$\chi(T_3)\phi = \chi(T_3) c_{\frac{1}{2}} [\frac{1}{2}]_\infty + \chi(T_3) c_\infty [\infty]_\infty$$

it follows that $c_{\frac{1}{2}} = 0$. □

[24] None of the element f_x, $x \in \{0, \frac{1}{2}, \infty\}$ is of the form $f_x = fs_i$ with $\ell(f_x) > \ell(f)$, $i \in \{2, 3\}$; in particular, $fs_2 \notin \{f_x : x \in \{0, \frac{1}{2}, \infty\}$

4 Normalizers of Tori

Let $\mathbf{G}$ be a split reductive group over a field k with maximal split torus $\mathbf{T} \leq \mathbf{G}$ and let $\mathbf{N} \leq \mathbf{G}$ be its normalizer. Then there is a canonical short exact sequence

$$1 \longrightarrow T \longrightarrow N \longrightarrow W_0 \longrightarrow 1 \tag{4.0.1}$$

where $N = \mathbf{N}(k)$, $T = \mathbf{T}(k)$ denotes the groups of k-rational points and W_0 denotes the Weyl group $W_0 = N/T$ of the pair $(\mathbf{G}, \mathbf{T})$. Our goal is to determine when this sequence splits in the case when $\mathbf{G}$ is *almost simple*, i.e. when $\mathbf{G}$ has a finite center $\mathbf{Z}$ and $\mathbf{G}/\mathbf{Z}$ is simple, or equivalently, when the root datum of $\mathbf{G}$ is *semisimple* and its Dynkin diagram connected.

We point out, that this question has already been resolved. In fact, various versions of it have been resolved by several different people independently, starting with Tits himself who announced their existence in [Tit66] but never published them (see also [Pop75]). For compact simple real Lie groups, this questions has been settled by Curtis, Wiederhold and Williams in [CWW74], for the case of split simple groups over an arbitrary field in [AH17, Theorem 4.16]. Over fields of positive characteristics, these results had also been obtained previously by Galt in several articles [Gal15], [Gal14], [Gal17a], [Gal17b]. Here, we will provide an answer to this question for *all rings*. Note that (a posteriori) the answer for rings is easily deduced from the one for fields (cf. remark 4.2.2). However, as our method of proof is different and gives new information (the cohomology groups $H^k(W_0, X^\vee)$ and $H^k(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ for small k), we think it is still of interest.

By Schreier theory (see [BS06, 1.5]), the splitness of eq. (4.0.1) is determined by the vanishing of a cohomology class $[\phi] \in H^2(W_0, T)$ corresponding to this extension. In section 1.8 we explicitly computed a representing 2-cocycle for the canonical extension

$$1 \longrightarrow T \longrightarrow W^{(1)} \longrightarrow W \longrightarrow 1$$

of a pro-p Coxeter group $W^{(1)}$. In order to make use of this result, we will see in the next section how N can be endowed with the structure of a pro-p Coxeter group.

4.1 The normalizer as a pro-p Coxeter group and its description in terms of root data

Let $(\mathbf{G}, \mathbf{T})$ be as before. Our goal in this section is to describe the extension eq. (4.0.1) entirely in terms of the root datum

$$\mathcal{R} := (X, \Phi, X^\vee, \Phi^\vee) := (X^*(\mathbf{T}), \Phi, X_*(\mathbf{T}), \Phi^\vee)$$

of $(\mathbf{G}, \mathbf{T})$.

First of all, the Weyl group $W_0 = N/T$ is equal to the Weyl group of the root datum $\mathcal{R}$ as a subgroup of

$$\mathrm{Aut}(\mathbf{T}) \simeq \mathrm{GL}_{\mathbb{Z}}(X^*(\mathbf{T}))^{\mathrm{op}}$$

and the two may therefore be identified. Moreover, the group $T = \mathbf{T}(k)$ of k-rational points of $\mathbf{T}$ naturally identifies with

$$T \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{T}), k^\times) \simeq X_*(\mathbf{T}) \otimes k^\times$$

and this identification respects the action of W_0.

By Schreier theory, the extension eq. (4.0.1) is determined (up to isomorphism) by its class in $H^2(W_0, T)$. To compute a representing 2-cocycle $\phi \in Z^2(W_0, T)$, we will use the results of section 1.8 by giving N the structure of a pro-p Coxeter group, i.e. by exhibiting lifts $n_s \in N$ of the $s \in S$ that satisfy the braid relations.

Recall from section 2.2.3 that for every element $u \in U_\alpha$, $u \neq 1$, the intersection $U_{-\alpha} u U_{-\alpha} \cap N$ consists of a single element $m(u)$, where $U_\alpha = \mathbf{U}_\alpha(k)$ denotes the group of k-rational points of the root subgroup $\mathbf{U}_\alpha \leq G$ corresponding to a root $\alpha \in \Phi$.

For every root $\alpha \in \Phi$, fix an element $u_\alpha \in U_\alpha, u_\alpha \neq 1$ and let

$$n_\alpha := m(u_\alpha) \in U_{-\alpha} u_\alpha U_{-\alpha} \cap N$$

4.1.1 Lemma. *With the above notation, it holds that*

(i) $n_\alpha^2 = \alpha^\vee(-1)$, *where* $\alpha^\vee$ *denotes coroot* $\alpha^\vee \in X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ *dual to* α

(ii) $n_\alpha n_\beta n_\alpha \cdots = n_\beta n_\alpha n_\beta \ldots$ *if* $\alpha, \beta \in \Phi$ *are part of a root basis* Δ *of* Φ, *where the number of factors on both sides equals the order* m *of* $s_\alpha s_\beta \in W_0$

Proof. Formula (ii) follows from [BT72, Prop. 6.1.8]. Formula (i) is proven in [Spr98, Lemma 8.1.4] (for a special choice of the u_α; but since $m(u_\alpha)$, $m(u'_\alpha)$ for $u, u' \in U_\alpha - \{1\}$ differ only by $\alpha^\vee(t)$ for some $t \in k^\times$, they square to the same element). $\square$

To make N into a pro-p Coxeter group, fix a root basis $\Delta \subseteq \Phi$ and consequently a set $S \subseteq W_0$, $S = \{s_\alpha : \alpha \in \Delta\}$, making W_0 into a Coxeter group, and put

$$n_{s_\alpha} := n_\alpha = m(u_\alpha) \quad \text{for } \alpha \in \Delta$$

By section 1.8, a 2-cocycle representing eq. (4.0.1) is explicitly given as follows. First, recall that the lifts n_{s_α} induce a canonical set-theoretic section

$$n : W_0 \longrightarrow N$$

of $N \twoheadrightarrow W_0$, determined by

$$n(s) = n_s, \quad n(ww') = n(w)n(w') \text{ if } \ell(ww') = \ell(w) + \ell(w')$$

The 2-cocycle

$$\phi(w, w') = n(w)n(w')n(ww')^{-1} \tag{4.1.1}$$

given by this section is then given in terms of the universal 2-cocycle $\mathbb{X}$ by

$$\phi = h \circ \mathbb{X} \tag{4.1.2}$$

where

$$h : \mathbb{Z}[\mathfrak{H}] \longrightarrow T$$

denotes the $\mathbb{Z}[W_0]$-module homomorphism from the free abelian group on the set $\mathfrak{H} := \{wsw^{-1} : w \in W_0,\ s \in S\} = \{s_\alpha : \alpha \in \Phi\}$ of reflections that is determined by

$$h(s_\alpha) = n_{s_\alpha}^2 = \alpha^\vee(-1) \quad \forall \alpha \in \Delta$$

4.1.2 Remark. The homomorphism $h : \mathbb{Z}[\mathfrak{H}] \longrightarrow T = X_*(\mathbf{T})$ can be factorized into the composition

$$\begin{array}{ccccc} \mathbb{Z}[\mathfrak{H}] & \xrightarrow{\widetilde{h}} & X_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \xrightarrow{\iota} & X_*(\mathbf{T}) \otimes_{\mathbb{Z}} k^\times \\ s_\alpha & \longmapsto & \alpha^\vee \otimes 1 & & \\ & & \chi \otimes x & \longmapsto & \chi \otimes (-1)^x \end{array}$$

of W_0-equivariant homomorphisms. In particular, the 2-cocycle $\phi \in Z^2(W_0, T)$ defined in eq. (4.1.1) is the push-forward of

$$\phi_u := \widetilde{h} \circ \mathbb{X} \in Z^2(W_0, X_*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{F}_2) \tag{4.1.3}$$

along ι.

4.2 Characterizing splitting in terms of $H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$

In the previous section we have explicitly computed a representative ϕ of the class in $H^2(W_0, T)$ corresponding to the extension eq. (4.0.1), and we have seen that it only depends on the root datum

$$(X, \Phi, X^\vee, \Phi^\vee) := (X^*(\mathbf{T}), \Phi, X_*(\mathbf{T}), \Phi^\vee)$$

of $(\mathbf{G}, \mathbf{T})$ and the ground field k. Moreover, we have seen in remark 4.1.2 that ϕ is the pushforward $\iota_*(\phi_u) = \iota \circ \phi_u$ of the 2-cocycle $\phi_u = \widetilde{h}_*(\mathbb{X}) \in Z^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ along the map $\iota : X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2 \longrightarrow X^\vee \otimes_{\mathbb{Z}} k^\times$ induced by $\mathbb{F}_2 \to k^\times, x \mapsto (-1)^x$.

Since these definitions make sense for any root datum and any (commutative) ring k, we will in the following assume that $\mathcal{R} = (X, \Phi, X^\vee, \Phi^\vee)$ is an arbitrary root datum (not necessarily semisimple), that $W_0 = W_0(\mathcal{R})$ is the Weyl group of $\mathcal{R}$, that k is a commutative ring, and that

$$\phi_u := \widetilde{h} \circ \mathbb{X} \in Z^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2) \tag{4.2.1}$$

and

$$\phi := \iota \circ \phi_u \in Z^2(W_0, X^\vee \otimes_{\mathbb{Z}} k^\times) \tag{4.2.2}$$

with $\widetilde{h}$ and ι given as before.

Our goal in this section will be to characterize the vanishing of $[\phi] \in H^2(W_0, X^\vee \otimes_{\mathbb{Z}} k^\times)$ in terms of the cohomology of $X^\vee$, $X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2$ and the element $[\phi_u] \in H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$. Note that if $-1 = 1$ in k, then ι and hence ϕ vanish. We will therefore assume in the following that $-1 \neq 1$ in k.

The idea is to isolate the dependence on k of the question whether $[\phi] \in H^2(W_0, X^\vee \otimes_{\mathbb{Z}} k^\times)$ vanishes using the following corollary of the Künneth theorem (see theorem 4.4.15).

4.2.1 Corollary (Universal coefficient theorem for group cohomology)**.** *Let G be a finite group, M a $\mathbb{Z}[G]$-module that is flat as a $\mathbb{Z}$-module, and N be any $\mathbb{Z}$-module. Then for any $n \in \mathbb{Z}$, there is a split short exact sequence*

$$0 \longrightarrow H^n(G, M) \otimes_{\mathbb{Z}} N \longrightarrow H^n(G, M \otimes_{\mathbb{Z}} N) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G, M), N) \longrightarrow 0 \tag{4.2.3}$$

natural in M and N, where $M \otimes_{\mathbb{Z}} N$ is viewed as a $\mathbb{Z}[G]$-module by action on the left factor only.

Proof. The group cohomology $H^n(G, M)$ can be computed as the cohomology of the cochain complex

$$(\mathrm{Hom}_{\mathbb{Z}[G]}(C_k, M))_{k \in \mathbb{Z}}$$

where $(C_k)_{k\in\mathbb{Z}}$ is any resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules of finite dimension (for example, the bar resolution). If S_k denotes a basis of the $\mathbb{Z}[G]$-module C_k, then it follows that we have a natural identification

$$\mathrm{Hom}_{\mathbb{Z}[G]}(C_k, M) \simeq \bigoplus_{s \in S_k} M \tag{4.2.4}$$

of $\mathbb{Z}$-modules. This implies that the natural map

$$\mathrm{Hom}_{\mathbb{Z}[G]}(C_k, M) \otimes_{\mathbb{Z}} N \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(C_k, M \otimes_{\mathbb{Z}} \mathbb{N})$$

is an isomorphism, and hence the cochain complex $(\mathrm{Hom}_{\mathbb{Z}[G]}(C_k, M) \otimes_{\mathbb{Z}} N)_{k\in\mathbb{Z}}$ computes $(H^k(G, M \otimes_{\mathbb{Z}} \mathbb{N}))_{k\in\mathbb{Z}}$. Moreover it follows from eq. (4.2.4) that the cochain complex $(\mathrm{Hom}_{\mathbb{Z}[G]}(C_k, M))_{k\in\mathbb{Z}}$ is degreewise flat, and we can apply theorem 4.4.15 with $\mathbf{C} := (\mathrm{Hom}_{\mathbb{Z}[G]}(C_k, M))_{k\in\mathbb{Z}}$ and $\mathbf{D} = N$ considered as a complex concentrated in degree zero. □

We now apply corollary 4.2.1 with $M = X^\vee$ and $j : N = \mathbb{F}_2 \hookrightarrow k^\times = N'$ to obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^2(W_0, X^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_2 \arrow[r, "\eta"] \arrow[d, "\psi = \mathrm{id} \otimes j"] & H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2) \arrow[r] \arrow[d] & \mathrm{Tor}_1^{\mathbb{Z}}(H^3(W_0, X^\vee), \mathbb{F}_2) \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & H^2(W_0, X^\vee) \otimes_{\mathbb{Z}} k^\times \arrow[r] & H^2(W_0, X^\vee \otimes_{\mathbb{Z}} k^\times) \arrow[r] & \mathrm{Tor}_1^{\mathbb{Z}}(H^3(W_0, X^\vee), k^\times) \arrow[r] & 0 \end{tikzcd} \tag{4.2.5}$$

of split exact sequences, where the leftmost horizontal maps are induced by the canonical maps

$$Z^2(W_0, M) \otimes_{\mathbb{Z}} N \longrightarrow Z^2(W_0, M \otimes_{\mathbb{Z}} N), \quad \phi \otimes n \mapsto ((w, w') \mapsto \phi(w, w') \otimes n)$$

on the level of cocycles, and the injectivity of the rightmost vertical map follows from the injectivity of $\mathbb{F}_2 \hookrightarrow k^\times$ and [HS96, Ch. III, Cor. 8.4]. The class $[\phi] \in H^2(W_0, X^\vee \otimes_{\mathbb{Z}} k^\times)$ is the image of $[\phi_u] \in H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ under the middle vertical map. A diagram chase therefore shows that

$$[\phi] = 1 \Leftrightarrow [\phi_u] \in \mathrm{im}(\eta) \wedge \eta^{-1}([\phi_u]) \in \ker(\psi) \tag{4.2.6}$$

In particular, to decide the vanishing of $[\phi]$ it suffices to compute the cohomology groups $H^2(W_0, X^\vee)$ and $H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ together with the map η. This computation will be carried out for all semisimple root data of rank ≤ 8 whose underlying root system is *irreducible* (i.e. such that the Dynkin diagram is connected) in section appendix A.

4.2.2 Remark. The following are equivalent:

(i) The class $[\phi_u] \in H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ vanishes.

(ii) The extension eq. (4.0.1) splits for all rings k.

(iii) The extension eq. (4.0.1) split for $k = \mathbb{F}_3$.

Proof. As the 2-cocycle classifying the extension eq. (4.0.1) over a ring k is the pushforward of ϕ_u, it's clear that (i) implies (ii). It's also clear that (ii) implies (iii). Finally, $\mathbb{F}_3^\times = \{\pm 1\} \simeq (\mathbb{F}_2, +)$, and hence the cocycle classifying the extension eq. (4.0.1) over $k = \mathbb{F}_3$ is precisely ϕ_u, hence (iii) implies (i). ☐

4.3 Direct products of root data

We preserve the notation and assumptions of the previous section. In particular,

$$\mathcal{R} = (X, \Phi, X^\vee, \Phi^\vee)$$

is an arbitrary root datum, *not* assumed to be semisimple. Instead, we assume in this section that $\mathcal{R}$ is a direct product [DG70, Exposé XXI, 6.4]

$$\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$$

of root data

$$\mathcal{R}_i = (X_i, \Phi_i, X_i^\vee, \Phi_i^\vee), \quad i = 1, 2$$

i.e.

$$X = X_1 \oplus X_2, \quad X^\vee = X_1^\vee \oplus X_2^\vee$$

and the duality pairing $\langle \cdot, \cdot \rangle : X^\vee \otimes X \longrightarrow \mathbb{Z}$ is given in terms of the duality pairings $\langle \cdot, \cdot \rangle_i$ of $\mathcal{R}_i$ by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2$$

and the sets Φ, $\Phi^\vee$ of roots and coroots are given by

$$\Phi = \Phi_1 \times \{0\} \cup \{0\} \times \Phi_2, \quad \Phi^\vee = \Phi_1^\vee \times \{0\} \cup \{0\} \times \Phi_2^\vee$$

Finally, the bijection $\Phi \leftrightarrow \Phi^\vee$ between roots and coroots is given by

$$\begin{array}{rl} \Phi \leftrightarrow & \Phi^\vee \\ (\alpha_1, 0) \leftrightarrow & (\alpha_1^\vee, 0) \\ (0, \alpha_2) \leftrightarrow & (0, \alpha_2^\vee) \end{array}$$

It follows that the Weyl group W_0 of $\mathcal{R}$ decomposes into a direct product

$$W_0 = W_0(\mathcal{R}_1) \times W_0(\mathcal{R}_2)$$

of (commuting) subgroups canonically isomorphic to the Weyl groups of $\mathcal{R}_1$ and $\mathcal{R}_2$ ([DG70, Exposé XXI, Prop. 6.4.2]). More precisely, we have embeddings

$$\begin{aligned} W_0(\mathcal{R}_1) &\hookrightarrow W_0, \\ s_{\alpha_1} &\longmapsto s_{(\alpha_1, 0)}, \end{aligned} \qquad \begin{aligned} W_0(\mathcal{R}_2) &\hookrightarrow W_0 \\ s_{\alpha_2} &\longmapsto s_{(0, \alpha_2)} \end{aligned}$$

and the decomposition $X = X_1 \oplus X_2$ respects the decomposition $W_0 = W_0(\mathcal{R}_1) \times W_0(\mathcal{R}_2)$, that is

$$(w_1, w_2) \bullet (x_1, x_2) = (w_1 \bullet x_1, w_2 \bullet x_2)$$

4.3.1 Remark. Our goal is now to relate the vanishing of the class $[\phi] \in H^2(W_0, X^\vee \otimes_{\mathbb{Z}} k^\times)$ associated to $\mathcal{R}$ to the vanishing of the classes $[\phi_1] \in H^2(W_0(\mathcal{R}_1), X_1^\vee \otimes_{\mathbb{Z}} k^\times)$, $[\phi_2] \in H^2(W_0(\mathcal{R}_2), X_2^\vee \otimes_{\mathbb{Z}} k^\times)$ associated to $\mathcal{R}_1$ and $\mathcal{R}_2$. Note that it's a priori clear that $[\phi]$ vanishes if and only if both $[\phi_1]$ and $[\phi_2]$ vanish because the extension eq. (4.0.1) associated to $\mathcal{R}$ splits into the direct product

$$1 \longrightarrow X_1^\vee \otimes_{\mathbb{Z}} k^\times \times X_2^\vee \otimes_{\mathbb{Z}} k^\times \longrightarrow N_1 \times N_2 \longrightarrow W_0(\mathcal{R}_1) \times W_0(\mathcal{R}_2) \longrightarrow 1$$

of the extension associated to $\mathcal{R}_1$ and $\mathcal{R}_2$, and a direct product of extensions clearly splits if and only if each factor does. However, since we are also interested in the computation of the cohomology groups $H^k(W_0(\mathcal{R}), X^\vee)$ in itself, it's more useful to take the more complicated route using the Künneth theorem.

4.3.2 Lemma. *Abbreviate*

$$W_1 = W_0(\mathcal{R}_1), \quad W_2 = W_0(\mathcal{R}_2), \quad W = W_0(\mathcal{R}) = W_1 \times W_2$$

and

$$\overline{X}_1^\vee = X_1^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2, \quad \overline{X}_2^\vee = X_2^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2, \quad \overline{X}^\vee = X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2 = \overline{X}_1^\vee \oplus \overline{X}_2^\vee$$

The standard 2-cocycle $\phi_u : Z^2(W, \overline{X}^\vee)$ associated to the root datum $\mathcal{R}$ by eq. (4.2.1) *is reducible to the standard 2-cocycles $\phi_{1,u} \in Z^2(W_1, \overline{X}_1^\vee)$ and $\phi_{2,u} \in Z^2(W_2, \overline{X}_2^\vee)$ associated to the root data $\mathcal{R}_1$ and $\mathcal{R}_2$ in the sense of corollary 4.4.17, and therefore*

$$[\phi_u] = \Phi([\phi_{1,u}] \otimes 1) + \Phi(1 \otimes [\phi_{2,u}])$$

where

$$\Phi : \bigoplus_{p=0}^{2} \Big(H^p(W_1, \overline{X}_1^\vee) \otimes H^{2-p}(W_2, \mathbb{F}_2) \ \oplus \ H^p(W_1, \mathbb{F}_2) \otimes H^{2-p}(W_2, \overline{X}_2^\vee) \Big) \hookrightarrow H^2(W_1 \times W_2, \overline{X}_1^\vee \oplus \overline{X}_2^\vee)$$

denote the injection defined in corollary 4.4.17.

In particular for any ring k, the induced 2-cocycle $\phi \in Z^2(W, X^\vee \otimes_{\mathbb{Z}} k^\times)$ (see eq. (4.2.2)*) is reducible to the respective induced cocycles $\phi_1 \in Z^2(W_1, X_1^\vee \otimes_{\mathbb{Z}} k^\times)$ and $\phi_2 \in Z^2(W_2, X_2^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$, and*

$$[\phi] = \Phi([\phi_1] \otimes 1) + \Phi(1 \otimes [\phi_2])$$

with Φ the injection defined by corollary 4.4.17 in this case.

Proof. By remark 1.7.3 the 2-cocycle $\mathbb{X}$ is normalized and therefore ϕ_u is normalized, too. Moreover, from lemma 1.7.10 it follows that $\mathbb{X}(w_1, w_2) = \mathbb{X}(w_2, w_1)$ whenever w_1 and w_2 lie in special subgroups that commute with each other, as

$$\mathbb{X}(w_1, w_2)^2 = \mathbb{L}(w_1)\ w_1(\mathbb{L}(w_2))\mathbb{L}(w_1 w_2)^{-1} \overset{(!)}{=} \mathbb{L}(w_1)\mathbb{L}(w_2)\mathbb{L}(w_1 w_2)^{-1} = \mathbb{X}(w_2, w_1)^2$$

Therefore ϕ_u has the same property, and it follows that condition eq. (4.4.11) is satisfied. The validity of eq. (4.4.12) follows immediately from the definition of the homomorphism $\widetilde{h}$ (see remark 4.1.2) appearing in $\phi_u := \widetilde{h} \circ \mathbb{X}$ and the definition of $\mathbb{X}$ (see definition 1.7.1).

Finally, if ϕ_u is reducible then obviously also its pushforward ϕ. ☐

4.4 Some results from homological algebra

In this section, we will recall some standard (and not so standard) results from homological algebra, in particular the theory of group cohomology. In the following unless stated otherwise, R will denote an arbitrary commutative ring and G will denote an arbitrary group.

First, let's recall some basic definitions (cf. [Bro82, Ch. 0]).

4.4.1 Definition. The category $\mathrm{Ch}(\mathcal{A})$ of chain complexes over an additive category $\mathcal{A}$ has as objects the **chain complexes** over $\mathcal{A}$, i.e. sequences $(C_k)_{k\in\mathbb{Z}}$ of objects $C_k \in \mathcal{A}$ together with a sequence of morphisms (**differentials**)

$$\partial_k : C_k \longrightarrow C_{k-1}$$

in $\mathcal{A}$ satisfying the chain relation

$$\partial_{k-1} \circ \partial_k = 0 \quad \forall k \in \mathbb{Z}$$

A chain complex $(C_k)_{k\in\mathbb{Z}}$ will often be denoted by $C_\bullet$, or simply by C, and if $x \in C_k$, we will write $\deg(x) := k$.

A morphism $f_\bullet : C_\bullet \longrightarrow D_\bullet$ of chain complexes (chain map) is a sequence $f_k : C_k \longrightarrow D_k$ of morphisms in $\mathcal{A}$ satisfying

$$\partial_k \circ f_k = f_{k-1} \circ \partial_k \quad \forall k \in \mathbb{Z}$$

More generally, a **chain map $f_\bullet : C_\bullet \longrightarrow D_\bullet$ of degree** $r \in \mathbb{Z}$ (a regular chain map being of degree *zero*) is a sequence of morphisms $f_k : C_k \longrightarrow D_{k+r}$ satisfying

$$\partial_{k+r} \circ f_k = f_{k-1} \circ \partial_k \quad \forall k \in \mathbb{Z}$$

A **chain homotopy** $h_\bullet : f_\bullet \longrightarrow g_\bullet$ between parallel chain maps $f, g : C \to D$ is a sequence $(h_k)_{k\in\mathbb{Z}}$ of morphisms

$$h_k : C_k \longrightarrow D_{k+1}$$

satisfying the relation

$$f_k - g_k = \partial_{k+1} \circ h_k + h_{k-1} \circ \partial_k$$

A chain complex C is said to be **contractible**, with **contracting chain homotopy** h, if there exists a chain homotopy $h : \mathrm{id}_C \to 0_C$ between the identity and the zero map on C, i.e. if

$$\mathrm{id}_{C_k} = \partial_{k+1} \circ h_k + h_{k-1} \circ \partial_k \tag{4.4.1}$$

for all $k \in \mathbb{Z}$.

If the category $\mathcal{A}$ is moreover endowed with a biadditive bifunctor

$$- \otimes - : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

and $\mathcal{A}$ has countable coproducts, then for each pair C, D of chain complexes, one defines their **tensor product** $C \otimes D$ degreewise as

$$(C \otimes D)_n := \bigoplus_{p+q=n} C_p \otimes D_q$$

with differential determined by

$$\partial_k \circ ((C \otimes D)_n \leftarrow C_p \otimes D_q) = \partial^C_p \otimes \mathrm{id}_{D_q} + (-1)^p \,\mathrm{id}_{C_p} \otimes \partial^D_q$$

Given chain maps $f : C \to D$, $g : C' \to D'$ between chain complexes, their tensor product

$$f \otimes g : C \otimes D \longrightarrow C' \otimes D'$$

is defined by

$$(f \otimes g)_n := f_n \otimes g_n$$

More generally, if f and g are chain maps of degrees $\deg(f)$ and $\deg(g)$, then the tensor product $f \otimes g$ is the chain map $C \otimes D \longrightarrow C' \otimes D'$ of degree $\deg(f) + \deg(g)$ determined by

$$(f \otimes g)_n(x \otimes y) = (-1)^{\deg(g)\deg(x)} f(x) \otimes g(y)$$

Given any two chain complexes C, D and assuming that $\mathcal{A}$ has countable products, there is the (outer[25]) **Hom complex** $\mathbf{Hom}(C, D)$, which is a chain complex of abelian groups (or whatever the hom groups in $\mathcal{A}$ are enriched over in addition), defined by

$$\mathbf{Hom}(C, D)_n := \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(C_k, D_{k+n})$$

with differential

$$(\partial_n f)_k := \partial_{n+k} f_k - (-1)^n f_{k-1} \partial_k$$

If the category $\mathcal{A}$ is not only additive but also *abelian*, then for each chain complex $(C_\bullet, \partial_\bullet)$ over $\mathcal{A}$ and each $k \in \mathbb{Z}$ one defines the **k-th homology (group)** by

$$H_k(C) := \frac{\ker(\partial_k)}{\mathrm{im}(\partial_{k+1})}$$

4.4.2 Remark. There exists a notion dual to chain complexes: a **cochain complex** C over an additive category $\mathcal{A}$ is a sequence $(C_k)_{k \in \mathbb{Z}}$ of objects $C_k \in \mathcal{A}$ together with maps $d_k : C_k \to C_{k+1}$ satisfying $d_{k+1} \circ d_k = 0$. Correspondingly, when $\mathcal{A}$ is abelian, one defines the k-th **cohomology (group)** by

$$H^k(C) := \frac{\ker(d_k)}{\mathrm{im}(d_{k-1})}$$

The distinction between chain complexes and cochain complexes is somewhat formal (and confusing), as there is an isomorphism between the categories of chain complexes and cochain complexes over $\mathcal{A}$, given by sending a chain complex $(C_k, \partial_k)_{k \in \mathbb{Z}}$ to the cochain complex C' with $C'_k := C_{-k}$ and $d_k := \partial_{-k}$. Under this identification,

$$H^k(C') = H_{-k}(C)$$

[25] when $\mathcal{A}$ is a *closed* $\otimes$-category, there is an ‘inner‘ version of the hom complex that is again a complex over $\mathcal{A}$, defined using the inner hom object, which recovers the outer one as the set of maps from the unit object to the inner one.

Moreover, let us recall the *standard* (*bar*) resolution for a group G over a ring R.

4.4.3 Definition. Given a group R (not necessarily finite) and a (commutative) ring R, the **standard resolution** over R for R is the chain complex $P_\bullet$ of $R[G]$-modules given by

$$P_k := R[G][G^{\times k}] := \bigoplus_{[g_1,\ldots,g_k]\in G^{\times k}} R[G]\cdot[g_1,\ldots,g_k]$$

, where for a set X we denote by $R[G][X]$ the free $R[G]$-module with basis X and the elements of the cartesian power $G^{\times k}$ are denote using square brackets, with differential given on basis elements by

$$\partial_k[g_1,\ldots,g_k] := g_1[g_2,\ldots,g_k] + \sum_{1\le j\le k-1} (-1)^j[g_1,\ldots,g_{j-1},g_jg_{j+1},g_{j+2},\ldots g_k] + (-1)^k[g_1,\ldots,g_{k-1}]$$

for $k \geq 1$, and $\partial_k := 0$ for $k \leq 0$. This complex is contractible as a chain complex of R-modules[26], as witnessed by the contracting chain homotopy h given on its canonical R-basis by

$$h_k(g_0[g_1,\ldots,g_k]) := [g_0,\ldots g_k] \tag{4.4.2}$$

4.4.4 Remark. Given an $R[G]$-module M, the cohomology groups $H^k(G,M)$ can be computed as the cohomology groups of the cochain complex $\mathrm{Hom}_{R[G]}(P_\bullet,M)$ induced by the standard resolution $P_\bullet$ for G. It is customary and convenient to identify degree n piece $\mathrm{Hom}_{R[G]}(P_\bullet,M)_n = \mathrm{Hom}_{R[G]}(P_n,M)$ with the set of all functions $G^{\times n} \longrightarrow M$, and hence to denote the value of an element $\phi \in \mathrm{Hom}_{R[G]}(P_\bullet,M)_n$ at a basis element $[g_1,\ldots,g_n]$ by

$$\phi(g_1,\ldots,g_n) := \phi([g_1,\ldots,g_n])$$

4.4.5 Remark. The cohomology groups $H^k(G,M)$ can be viewed as functors in *two* variables, as follows (cf. [Bro82, III.8]). Consider the category $\mathcal{D}$ whose objects are pairs (G,M) consisting of a group G and a $\mathbb{Z}[G]$-module M. A morphism

$$(G,M) \longrightarrow (G',M')$$

between two such pairs shall be a pair (φ_0,φ_1), where $\varphi_0 : G \to G'$ is a homomorphism of groups and where $\varphi_1 : M' \to M$ is a homomorphism of $\mathbb{Z}$-modules which is equivariant in the sense that

$$\varphi_1(\varphi_0(g)\bullet m') = g\bullet\varphi_1(m') \quad \forall g\in G,\ m'\in M' \tag{4.4.3}$$

Such a pair (φ_0,φ_1) then induces a map

$$(\varphi_0,\varphi_1)^* : H^k(G',M') \longrightarrow H^k(G,M)$$

in a functorial way, given as follows. Let $F_\bullet$ and $F'_\bullet$ be projective resolutions of G and G' respectively, with augmentations maps $\varepsilon : F_\bullet \to \mathbb{Z}$ and $\varepsilon' : F'_\bullet \to \mathbb{Z}$. Let $\varphi_0^*(F')$ denote F' considered as a chain complex of $\mathbb{Z}[G]$-modules using the map φ_0. Then $\varphi_0^*(F')$ is still acyclic (even though it may not be degreewise projective anymore), and therefore we can apply the 'fundamental lemma of homological algebra' ([Bro82, I.7.4]) to the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & F_2 \arrow[r] \arrow[d, dashed, "f_2"] & F_1 \arrow[r] \arrow[d, dashed, "f_1"] & F_0 \arrow[r, "\varepsilon_0"] \arrow[d, dashed] & \mathbb{Z} \arrow[r] \arrow[d, "f_{-1}:=\mathrm{id}"] & 0 \\
\ldots \arrow[r] & F'_2 \arrow[r] & F'_1 \arrow[r] & F'_0 \arrow[r, "\varepsilon'_0"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

yielding a map $f : F \to \varphi_0^*(F')$ of chain complexes of $\mathbb{Z}[G]$-modules that makes the diagram

$$\begin{tikzcd}
F \arrow[dr, "\varepsilon"] \arrow[dd, "f"'] & \\
& \mathbb{Z} \\
\varphi_0^*(F') \arrow[ur, "\varepsilon'"] &
\end{tikzcd}$$

[26]but not as a complex of $R[G]$-modules

commutative, and f is uniquely determined up to chain homotopy by this property. Since the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(--, M)$ from chain complexes of $\mathbb{Z}[G]$-modules to cochain complexes of $\mathbb{Z}$-modules preserves the relation of chain homotopy, and chain homotopic maps induce the same maps on (co-)homology, it follows that the induced map

$$H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(f, M) : H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(\varphi_0^*(F'), M) \longrightarrow H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(F, M)) = H^k(G, M)$$

is independent of the choice of f. Precomposing this map with

$$H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(\varphi_0^*(F'), \varphi_1)) : H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(\varphi_0^*(F'), \varphi_0^*(M'))) \longrightarrow H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(\varphi_0^*(F'), M))$$

and the map

$$H^k(G', M') = H^k(\mathrm{Hom}_{\mathbb{Z}[G']}(F', M')) \longrightarrow H^k(\mathrm{Hom}_{\mathbb{Z}[G]}(\varphi_0^*(F'), \varphi_0^*(M'))$$

induced by the forgetful map $\mathrm{Hom}_{\mathbb{Z}[G']}(F', M') \to \mathrm{Hom}_{\mathbb{Z}[G]}(\varphi_0^*(F'), \varphi_0^*(M'))$ then yields the desired

$$H^k(G', M') \longrightarrow H^k(G, M)$$

4.4.6 Remark. The functoriality described in remark 4.4.5 has a simple description when using the standard resolution (see definition 4.4.3, remark 4.4.4) to compute the cohomology groups. Namely, there exists a *natural* map

$$f : P \longrightarrow \varphi_0^*(P')$$

between the standard resolutions P and P' of G and G' that is compatible with the augmentations. This map is given in degree k on basis elements by

$$f_k([g_1, \dots, g_k]) = [\varphi_0(g_1), \dots, \varphi_0(g_k)]$$

One can check that the map

$$\mathrm{Hom}_{\mathbb{Z}[G']}(P', M') \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P, M)$$

of cochain complexes of $\mathbb{Z}$-modules that induces the maps $H^k(G', M') \longrightarrow H^k(G, M)$ by definition (in remark 4.4.5) is given in terms of standard cochains by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}}((G')^{\times k}, M') &\longrightarrow \mathrm{Hom}_{\mathrm{Set}}(G, M) \\ \phi &\longmapsto \varphi_1 \circ \phi \circ \varphi_0^{\times k} \end{aligned}$$

4.4.7 Remark. From the identification of $\mathrm{Hom}_{R[G]}(P_\bullet, M)_n \simeq \mathrm{Hom}_{\mathrm{Set}}(G^{\times n}, M)$ given in remark 4.4.4, it follows that the cohomology groups $H^k(G, M)$ of an $R[G]$-module M and the cohomology groups of the underlying $\mathbb{Z}[G]$-module are canonically identified. In particular, there is no need to reference the underlying coefficient ring R explicitly in the notation. However, it is still useful to consider a coefficient ring e.g. in statements of theorems as an $R[G]$-module can be flat over R without being flat over $\mathbb{Z}$.

We now specialize our situation, and assume that the group G decomposes into a direct product $G = G_1 \times G_2$ of commuting normal subgroups.

4.4.8 Lemma. *Let $P_{1,\bullet}$ and $P_{2,\bullet}$ be free (resp. projective) resolutions of the trivial $R[G_1]$-module R and the trivial $R[G_2]$-module R, respectively, and let $\varepsilon_1 : P_{1,\bullet} \longrightarrow R$ and $\varepsilon_2 : P_{2,\bullet} \longrightarrow R$ be the associated augmentation maps.*

Then, the tensor product complex $P_1 \otimes P_2$, taken in the category of R-chain complexes and endowed with the natural $R[G_1] \otimes_R R[G_2] \simeq R[G_1 \times G_2] = R[G]$ action, is a free (resp. projective) resolution of R as a trivial $R[G]$-module, with augmentation map given by $\varepsilon_1 \otimes \varepsilon_2$ (tensor product of chain maps), given in degree zero by

$$(\varepsilon_1 \otimes \varepsilon_2)(x \otimes y) = \varepsilon_1(x)\varepsilon_2(y) \in R, \quad x \in P_{1,0},\ y \in P_{2,0}$$

Proof. Cf. [Bro82, Ch. V]. Because P_1 and P_2 are resolutions of R, it follows that ε_1 and ε_2 are chain homotopy equivalences (see [Bro82, Ch. 0, Cor. 7.6]) of complexes of R-modules. Since the tensor product of maps of chain complexes preserves the relation of chain homotopy and therefore preserves chain homotopy equivalences (see [Bro82, Ch. 0, Sec. I, Ex. 7c]), it follows that the tensor product

$$\varepsilon_1 \otimes \varepsilon_2 : P_1 \otimes P_2 \longrightarrow R \otimes R = R$$

of R-chain complexes is again a chain homotopy equivalence (here R considered as a chain complex concentrated in degree zero), and therefore $P_1 \otimes P_2$ is a resolution of R as a trivial $R[G]$-module. Moreover, the tensor product $C_1 \otimes C_2$ of a complex C_1 of $R[G_1]$-modules with a complex C_2 of $R[G_2]$-modules is again degreewise free (resp.

projective) if C_1 and C_2 are degreewise free (resp. projective) as $R[G_1]$- and $R[G_2]$-modules, respectively. This follows immediately from the fact that the tensor product $M_1 \otimes_R M_2$ of a free (resp. projective) $R[G_1]$-module M_1 with a free (resp. projective) $R[G_2]$-module M_2 is a free (resp. projective) $R[G]$-module. Indeed, for free modules this follows immediately from the special case $M_1 = R[G_1]$, $M_2 = R[G_2]$ and the identification

$$R[G_1] \otimes_R R[G_2] \simeq R[G_1 \times G_2] = R[G]$$

Because projective modules are characterized as the direct summands of free modules, the statement for projective modules follows from the one for free modules. □

We now have two ways to compute the cohomology groups $H^k(G, M)$ of a G-module M, using the bar resolution $P_\bullet$ for G as the cohomology of the cochain complex $\mathrm{Hom}_{R[G]}(P_\bullet, M)$, or using the product resolution $P_{1,\bullet} \otimes P_{2,\bullet}$ as the cohomology of the cochain complex $\mathrm{Hom}_{R[G]}(P_{1,\bullet} \otimes P_{2,\bullet}, M)$. It is a basic result of homological algebra that these groups are canonically isomorphic; however, in order to make this isomorphism explicit, we need a constructive version of this basic result, which we will recall now.

4.4.9 Lemma (Fundamental Lemma of Homological Algebra)**.** *Let R any commutative ring, let A be an R-algebra, and let $C = (C_\bullet, \partial_\bullet)$ and $D = (D_\bullet, \partial_\bullet)$ be chain complexes of left-A-modules. Assume that, in each degree k, C_k is a free left-A-module with basis $\mathfrak{B}_k$, and moreover assume that D is chain contractible*[27] *as a chain complex of R-modules, with explicit contracting chain homotopy $h_\bullet$.*

Then, given any integer r and any family $f_k : C_k \to D_k$ $(k \le r)$ of A-linear maps that satisfy the chain map relation

$$\partial_k \circ f_k = f_{k-1} \circ \partial_k \tag{4.4.4}$$

for all $k \le r$, there exist maps $f_k : C_k \to D_k$ $(k > r)$ extending this family to all integers, to a map $f_\bullet : C_\bullet \to D_\bullet$ of chain complexes of left-A-modules. Any two such extensions are chain homotopic via a chain homotopy $h'_\bullet : C_\bullet \to D_{\bullet+1}$ vanishing $h'_k = 0$ in all degree $k \le r$.

Moreover, in this situation there is a canonical extension, determined recursively by the conditions

$$f_{k+1}|_{\mathfrak{B}_{k+1}} = h_k \circ f_k \circ \partial_{k+1}|_{\mathfrak{B}_{k+1}} \quad \forall k \ge r \tag{4.4.5}$$

Proof. We refer to [Bro82, Ch. 0, Lemma 7.4] for details; here, we'll only give a sketch and explain why the formula eq. (4.4.5) gives rise to such an extension.

To prove the existence of such an extension, one argues by induction of course. So assume that such an extension satisfying eq. (4.4.4) has been constructed up to degree k, and consider the mapping problem

$$\begin{tikzcd} C_{k+1} \arrow[r, "\partial_{k+1}"] \arrow[d, dashed, "f_{k+1}"] & C_k \arrow[r, "\partial_k"] \arrow[d, "f_k"] & C_{k-1} \arrow[d, "f_{k-1}"] \\ D_{k+1} \arrow[r, "\partial_{k+1}"] & D_k \arrow[r, "\partial_k"] & D_{k-1} \end{tikzcd}$$

Now, a map f_{k+1} completing the diagram amounts to a lift $C_{k+1} \longrightarrow D_{k+1}$ of $f_k \circ \partial_{k+1} : C_{k+1} \longrightarrow D_k$ along $\partial_{k+1} : D_{k+1} \longrightarrow D_k$. However, because

$$\partial_k \circ f_k \circ \partial_{k+1} = f_{k-1} \circ \partial_k \circ \partial_{k+1} = 0$$

$\varphi := f_k \circ \partial_{k+1}$ restricts to a map $C_{k+1} \longrightarrow \ker(\partial_k^D)$, and f_{k+1} must in fact be a lift of φ along $D_{k+1} \twoheadrightarrow \ker(\partial_k^D)$, where the surjectivity of the map $D_{k+1} \to \ker(\partial_k^D)$ follows abstractly from the acyclicity of D, and the existence of f_{k+1} follows abstractly from the projectivity of C_{k+1}. However, in our situation we are given a basis $\mathfrak{B}_{k+1}$ of C_{k+1}, providing an explicit witness of the projectivity of C_{k+1}, and we are given a contracting chain homotopy $h_\bullet$, providing an explicit witness of the acyclicity of D, and it follows immediately that formula eq. (4.4.5) defines such an extension f_{k+1}. □

4.4.10 Remark. The previous lemma illustrates the difference between *abstract* and *constructive* mathematics (where 'abstract' is not to be equated with 'conceptual'), the former of which most mathematicians are more accustomed to, since it is easier. That is, it is easier to prove and use *abstract* existence statements because one can essentially 'forget' their proof. In contrast, in constructive mathematics the proof of a property, in this example the proof of projectivity by exhibiting a basis or the proof of acyclicity by providing a contracting chain

[27] and therefore in particular acyclic

homotopy, becomes a necessary ‘input‘ into any proof using this property, much like an input into a computer program. In fact, exactly like an input into a computer program, as the emerging marriage of *constructive* and *formal* mathematics via type theory and the ‘Curry-Howard isomorphism‘ shows.

It is the conviction of the author that the formalization of mathematics, long thought to be a hopeless endeavour, is inevitable, and that therefore the constructive nature of proofs of classic results, which have been mostly used abstractly, needs to be rediscovered and appreciated.

We now return to our previous situation of a group $G = G_1 \times G_2$, and will compute the ‘comparison map‘

$$f_\bullet : P_{1,\bullet} \otimes P_{2,\bullet} \longrightarrow P_\bullet$$

between the product $P_{1,\bullet} \otimes P_{2,\bullet}$ of the standard resolutions for G_1 and G_2, and the standard resolution $P_\bullet$ for G, up to degree 2.

4.4.11 Lemma. *The map*

$$f_\bullet : P_{1,\bullet} \otimes P_{2,\bullet} \longrightarrow P_\bullet \tag{4.4.6}$$

of chain complexes of $R[G]$-modules, defined by eq. (4.4.5) *of lemma 4.4.9 (with $A = R[G]$), is explicitly given on the canonical $R[G]$-basis elements (see definition 4.4.3 for notation) by*

$$\begin{aligned}
f_0([] \otimes []) &= [] \\
f_1([g] \otimes []) &= [g] - [1] && g \in G_1 \\
f_1([] \otimes [g]) &= [g] - [1] && g \in G_2 \\
f_2([g_1, g_2] \otimes []) &= [g_1, g_2] - [g_1, 1] - [1, g_1 g_2] + [1, g_1] && g_1, g_2 \in G_1 \\
f_2([g_1] \otimes [g_2]) &= [g_1, g_2] - [g_1, 1] - [1, g_2] - [g_2, g_1] + [g_2, 1] + [1, g_1] && g_1 \in G_1,\ g_2 \in G_2 \\
f_2([] \otimes [g_1, g_2]) &= [g_1, g_2] - [g_1, 1] - [1, g_1 g_2] + [1, g_1] && g_1, g_2 \in G_2
\end{aligned}$$

Proof. Left to the reader. □

4.4.12 Corollary. *Given an $R[G]$-module M, the map*

$$\mathrm{Hom}_{R[G]}(P_\bullet, M) \xrightarrow{f^*} \mathrm{Hom}_{R[G]}(P_{1,\bullet} \otimes P_{2,\bullet}, M)$$

of cochain complexes induced by eq. (4.4.6) *is given in degree two by*

$$f^*(\phi)(x) = \begin{cases} \phi(g_1, g_2) - \phi(g_1, 1) + \phi(1, g_1) - \phi(1, g_1 g_2) & \text{if } x = [g_1, g_2] \otimes [] \\ \phi(g_1, g_2) - \phi(g_2, g_1) + \phi(1, g_1) - \phi(g_1, 1) + \phi(g_2, 1) - \phi(1, g_2) & \text{if } x = [g_1] \otimes [g_2] \\ \phi(g_1, g_2) - \phi(g_1, 1) + \phi(1, g_1) - \phi(1, g_1 g_2) & \text{if } x = [] \otimes [g_1, g_2] \end{cases}$$

(we keep the notation of lemma 4.4.11; see also remark 4.4.4).

The reason for considering the resolution $P_{1,\bullet} \otimes P_{2,\bullet}$ is that it allows the comparison between the cohomology groups of G_1 and G_2, and the product $G = G_1 \times G_2$ via the following map:

4.4.13 Definition. Given an $R[G_1]$-module M_1 and an $R[G_2]$-module M_2, the **cochain cross product** (cf. [Bro82, Ch. V]) is the map

$$\mathrm{Hom}_{R[G_1]}(P_{1,\bullet}, M_1) \otimes \mathrm{Hom}_{R[G_2]}(P_{2,\bullet}, M_2) \longrightarrow \mathrm{Hom}_{R[G]}(P_{1,\bullet} \otimes P_{2,\bullet}, M_1 \otimes_R M) \tag{4.4.7}$$

of cochain complexes given in degree n by $\phi \otimes \psi \mapsto \phi \times \psi$, where

$$(\phi \times \psi)([g_1, \dots, g_p] \otimes [h_1, \dots, h_q]) := \begin{cases} (-1)^{pq} \phi(g_1, \dots, g_p) \otimes \psi(h_1, \dots, h_q) & \text{if } p = \deg(\phi),\ q = \deg(\psi) \\ 0 & \text{otherwise} \end{cases}$$

4.4.14 Lemma. *If G_1 and G_2 are finite, then eq.* (4.4.7) *is an isomorphism.*

Proof. (Cf. [Bro82, Ch. V, Sec. 3, Exercise 2]). First of all, since the chain complexes $P_{1,\bullet}$ and $P_{2,\bullet}$ vanish in negative degree, the direct sum

$$(P_{1,\bullet} \otimes P_{2,\bullet})_n = \bigoplus_{p+q=n} P_{1,p} \otimes_R P_{2,q}$$

is finite for all $n \in \mathbb{Z}$. Therefore, the direct product in

$$\begin{aligned}\operatorname{Hom}_{R[G]}(P_{1,\bullet} \otimes P_{2,\bullet}, M_1 \otimes_R M_2)_n &= \operatorname{Hom}_{R[G]}((P_{1,\bullet} \otimes P_{2,\bullet})_n, M_1 \otimes_R M_2) \\ &\simeq \prod_{p+q=n} \operatorname{Hom}_{R[G]}(P_{1,p} \otimes_R P_{2,q}, M_1 \otimes_R M_2)\end{aligned}$$

is actually a direct sum and hence it suffices (we don't need to bother with the sign $(-1)^{pq}$) to show that for all $p, q \in \mathbb{N}$ the map

$$\begin{aligned}\operatorname{Hom}_{R[G_1]}(P_{1,p}, M_1) \otimes_R \operatorname{Hom}_{R[G_2]}(P_{2,q}, M_2) &\longrightarrow \operatorname{Hom}_{R[G]}(P_{1,p} \otimes_R P_{2,q}, M_1 \otimes_R M_2) \\ \phi \otimes \psi &\longmapsto \phi \otimes_R \psi\end{aligned}$$

is an isomorphism, where $\phi \otimes_R \psi$ denotes the map $\phi \otimes_R \psi : P_{1,p} \otimes_R P_{2,q} \to M_1 \otimes_R M_2$ induced by ϕ, ψ and the functoriality of $\otimes_R$. Now, since $P_{1,p}$, $P_{2,q}$ and $P_{1,p} \otimes_R P_{2,q}$ are free modules over $R[G_1]$, $R[G_2]$ and $R[G]$ with bases $G_1^{\times p}$, $G_2^{\times q}$, $G_1^{\times p} \times G_2^{\times q}$ respectively, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{R[G_1]}(P_{1,p}, M_1) \otimes_R \operatorname{Hom}_{R[G_2]}(P_{2,q}, M_2) \arrow[r] \arrow[d, "\sim"] & \operatorname{Hom}_{R[G]}(P_{1,p} \otimes_R P_{2,q}, M_1 \otimes_R M_2) \arrow[d, "\sim"] \\
\left(\prod_{[g_1,\ldots,g_p] \in G_1^{\times p}} M_1\right) \otimes_R \left(\prod_{[h_1,\ldots,h_q] \in G_2^{\times q}} M_2\right) \arrow[r] & \prod_{([g_1,\ldots,g_p],[h_1,\ldots,h_q]) \in G_1^{\times p} \times G_2^{\times q}} M_1 \otimes_R M_2
\end{tikzcd}$$

Since G_1 and G_2 are finite by assumptions, the bases are finite and the direct products above are actually direct sums, and it follows that the bottom homomorphism is an isomorphism since the bifunctor $- \otimes_R -$ commutes with direct sums. ☐

Finally, to use the cochain cross product isomorphism eq. (4.4.7) to relate the group cohomology of finite groups G_1, G_2 to that of their product $G = G_1 \times G_2$, we apply the Künneth theorem which we will recall now.

4.4.15 Theorem (Künneth)**.** *Let $\mathbf{C}, \mathbf{D}$ be cochain complexes over a principal ideal domain R, and suppose that one of $\mathbf{C}$, $\mathbf{D}$ is degreewise flat. Then for any $n \in \mathbb{Z}$, there is a short exact sequence*

(4.4.8)
$$0 \longrightarrow \bigoplus_{p+q=n} H^p(\mathbf{C}) \otimes_R H^q(\mathbf{D}) \xrightarrow{\eta} H^n(\mathbf{C} \otimes_R \mathbf{D}) \longrightarrow \bigoplus_{p+q=n+1} \operatorname{Tor}_1^R(H^p(\mathbf{C}), H^q(\mathbf{D})) \longrightarrow 0$$

natural in $\mathbf{C}$ and $\mathbf{D}$. This sequence splits (but not naturally), and the map η is induced by the inclusion maps

$$Z^p(\mathbf{C}) \otimes_R Z^q(\mathbf{D}) \hookrightarrow Z^{p+q}(\mathbf{C} \otimes \mathbf{D}), \quad x \otimes y \mapsto x \otimes y$$

of cocycles.

Proof. For a proof of this statement for chain complexes instead of cochain complexes, see [HS96, Ch. V, Thm. 2.1]. The statement for cochain complexes follows by viewing a cochain complex $(\mathbf{C}, d)$ as a chain complex $(\mathbf{C}', \partial)$ with $\mathbf{C}'_n = \mathbf{C}_{-n}$ and $\partial_n = d_{-n}$. ☐

4.4.16 Corollary. *Let R be a principal ideal domain, G_1, G_2 finite groups and $G = G_1 \times G_2$ their product. Then given an $R[G_1]$-module M_1 and an $R[G_2]$-module M_2 such that at least one of them is flat as an R-module, we have for all $n \in \mathbb{Z}$ a split exact sequence*

(4.4.9)
$$0 \longrightarrow \bigoplus_{p+q=n} H^p(G_1, M_1) \otimes_R H^q(G_2, M_2) \xrightarrow{\varphi} H^n(G, M_1 \otimes_R M_2) \longrightarrow \ldots$$

$$\ldots \longrightarrow \bigoplus_{p+q=n+1} \operatorname{Tor}_1^R(H^p(G_1, M_1), H^q(G_2, M_2)) \longrightarrow 0$$

natural in M_1 and M_2, where, if $H^n(G, M_1 \otimes_R M_2)$ is computed as the cohomology group of the cochain complex $\operatorname{Hom}_{R[G]}(P_{1,\bullet} \otimes P_{2,\bullet}, M_1 \otimes_R M_2)$ and $H^p(G_1, M_1)$ and $H^q(G_2, M_2)$ are computed as the cohomology groups of *the cochain complexes* $\operatorname{Hom}_{R[G_1]}(P_{1,\bullet}, M_1)$ *and* $\operatorname{Hom}_{R[G_2]}(P_{2,\bullet}, M_2)$*, the map* φ *is given in terms of the cochain cross product eq.* (4.4.7) *by*

$$\varphi([\phi] \otimes [\psi]) = [\phi \times \psi]$$

for representing cocycles $\phi \in Z^p(G_1, M_1)$*,* $\psi \in Z^q(G_2, M_2)$*.*

Moreover, if $R = k$ *is a field, this exact sequence simplifies to an isomorphism*

$$\varphi : \bigoplus_{p+q=n} H^p(G_1, M_1) \otimes_k H^q(G_2, M_2) \xrightarrow{\sim} H^n(G, M_1 \otimes_k M_2)$$

Proof. This follows immediately by applying theorem 4.4.15 with

$$\mathbf{C} := \operatorname{Hom}_{R[G_1]}(P_{1,\bullet}, M_1), \quad \text{and} \quad \mathbf{D} := \operatorname{Hom}_{R[G_2]}(P_{2,\bullet}, M_2)$$

and using the isomorphism between $C \otimes D$ and $\operatorname{Hom}_{R[G]}(P_{1,\bullet} \otimes P_{2,\bullet}, M_1 \otimes_R M_2)$ of lemma 4.4.14, observing that $\mathbf{C}$ and $\mathbf{D}$ are degreewise finite direct sums of copies of M_1 and M_2, respectively. The final remark follows from standard properties of the Tor functor and the fact that every module over a field is flat. □

$R[G]$-modules of the form $M_1 \oplus M_2$, where $G = G_1 \times G_2$ acts on each summand through the respective projection $p_i : G_1 \times G_2 \to G_i$, can also be viewed as the direct sum

$$M_1 \oplus M_2 \simeq M_1 \otimes_R R \ \oplus \ R \otimes_R M_2$$

as tensor product $R[G]$-modules, where R is considered as a trivial $R[G_2]$-module or a trivial $R[G_1]$-module respectively. Since cohomology functors are additive, we can therefore derive the following version of corollary 4.4.16 for $R[G]$-modules of that form.

4.4.17 Corollary. *Given an* $R[G_1]$*-module* M_1 *and an* $R[G_2]$*-module* M_2*, for a principal ideal domain* R *and finite groups* G_1*,* G_2*, the cohomology groups of the* $R[G]$*-module* $M_1 \oplus M_2$ *over the product* $G = G_1 \times G_2$ *sit in a natural split exact sequence*

$$\begin{aligned} &\bigoplus_{p+q=n} (H^p(G_1, M_1) \otimes_R H^q(G_2, R)) \oplus (H^p(G_1, R) \otimes_R H^q(G_2, M_2)) \overset{\Phi}{\hookrightarrow} H^n(G, M_1 \oplus M_2) \\ &\twoheadrightarrow \bigoplus_{p+q=n+1} \operatorname{Tor}_1^R(H^p(G_1, M_1), H^q(G_2, R)) \oplus \operatorname{Tor}_1^R(H^p(G_1, R), H^q(G_2, M_2)) \end{aligned} \tag{4.4.10}$$

where, if $H^n(G, M_1 \oplus M_2)$ *is computed using the resolution* $P_{1,\bullet} \otimes P_{2,\bullet}$*, the injection* Φ *is explicitly described in terms of representing standard cocycles and the cochain cross product eq.* (4.4.7) *by*

$$\begin{aligned} \Phi([\phi] \otimes [\psi]) &= [\iota_1 \circ (\phi \times \psi)] \qquad && \phi : G_1^{\times p} \to M_1,\ \psi : G_2^{\times q} \to R \\ \Phi([\phi] \otimes [\psi]) &= [\iota_2 \circ (\phi \times \psi)] && \phi : G_1^{\times p} \to R,\ \psi : G_2^{\times q} \to M_2 \end{aligned}$$

where $\iota_1 : M_1 \otimes_R R \hookrightarrow M_1 \oplus M_2$*,* $\iota_2 : R \otimes_R M_2 \hookrightarrow M_1 \oplus M_2$ *are the canonical inclusions. If* $R = k$ *is a field, this exact sequence simplifies to an isomorphism*

$$\bigoplus_{p+q=n} (H^p(G_1, M_1) \otimes_R H^q(G_2, k)) \oplus (H^p(G_1, k) \otimes_k H^q(G_2, M_2)) \xrightarrow{\sim} H^n(G, M_1 \oplus M_2)$$

Moreover, if $[\phi] \in H^2(G, M_1 \oplus M_2)$ *is a class represented by a standard* 2*-cocycle* $\phi : G^{\times 2} \longrightarrow M_1 \oplus M_2$ *that is* ***reducible to*** *standard* 2*-cocycles* $\phi_1 : G_1^{\times 2} \to M_1$*,* $\phi_2 : G_2^{\times 2} \to M_2$*, in the sense that*

$$\phi(g_1, g_2) = \phi(g_2, g_1) \quad \forall g_1 \in G_1,\ g_2 \in G_2 \qquad\qquad \phi(1,1) = 0 \quad (\phi \text{ normalized}) \tag{4.4.11}$$

and

$$\phi(g_1, g_1') = j_1(\phi_1(g_1, g_1')) \quad \forall g_1, g_1' \in G_1 \qquad\qquad \phi(g_2, g_2') = j_2(\phi_2(g_2, g_2')) \quad \forall g_2, g_2' \in G_2 \tag{4.4.12}$$

(where $j_i : M_i \hookrightarrow M_1 \oplus M_2$ *denote the inclusions) then* $[\phi]$ *is the sum of the images of* $[\phi_1] \in H^2(G_1, M_1)$ *and* $[\phi_2] \in H^2(G_2, M_2)$ *under*

$$H^2(G_1, M_1) \simeq H^2(G_1, M_1) \otimes_R \overbrace{H^0(G_2, R)}^{R} \overset{\Phi}{\hookrightarrow} H^2(G, M_1 \oplus M_2)$$

and

$$H^2(G_2,M_2) \simeq \overbrace{H^0(G_1,R)}^{R} \otimes_R H^2(G_2,M_2) \overset{\Phi}{\hookrightarrow} H^2(G,M_1\oplus M_2)]$$

In particular,

$$[\phi]=0 \quad\Leftrightarrow\quad [\phi_1]=0 \textit{ and } [\phi_2]=0 \tag{4.4.13}$$

Proof. The statement about the existence of the exact sequence follows by applying corollary 4.4.16 to the pairs M_1 and R, and R and M_2, and taking the direct sum of those two split exact sequences, noting that since R is flat, the hypotheses of the previous corollary are satisfied. The statement about reducible 2-cocycles follows from the description of the map φ of corollary 4.4.16 (signs!) and the combination of corollary 4.4.12 with the properties eq. (4.4.11) and eq. (4.4.12) (using that for a normalized cocycle $\phi(1,g)=\phi(g,1)=0$ for all g).

Finally, eq. (4.4.13) follows from the injectivity of Φ. □

4.5 Maps between root data

In this preparatory section, we consider conditions under which there exist Weyl-equivariant maps (in a sense to be defined)

$$\mathcal{R}_1=(X_1,\Phi_1^\vee,X_2^\vee,\Phi_2^\vee)\longrightarrow(X_2,\Phi_2,X_2^\vee,\Phi_2^\vee)=\mathcal{R}_2$$

between root data $\mathcal{R}_1$ and $\mathcal{R}_2$. In particular, such a map should yield a group homomorphism

$$\varphi_0: W_0(\mathcal{R}_1)\longrightarrow W_0(\mathcal{R}_2)$$

and $\mathbb{Z}$-linear maps

$$\varphi_1: X_1\longrightarrow X_2 \quad\text{and}\quad \varphi_1^\vee: X_2^\vee\longrightarrow X_1^\vee$$

which are equivariant in the sense that

$$\varphi_1(w\bullet x)=\varphi_0(w)\bullet\varphi_1(x) \quad\text{and}\quad \varphi_1^\vee(\varphi_0(w)\bullet y)=w\bullet\varphi_1^\vee(y)$$

for all $w\in W_0(\mathcal{R}_1)$, $x\in X_1$ and $y\in X_2^\vee$.

There does exist a notion of *morphisms between root data* ([DG70, Exposé XXI, 6.1]), but it is very restrictive. In particular, a morphism between $\mathcal{R}_1$ and $\mathcal{R}_2$ in the sense of [DG70] always yields an isomorphism between $W_0(\mathcal{R}_1)$ and $W_0(\mathcal{R}_2)$.

The following definition is more useful for our purposes:

4.5.1 Definition. A **frugal morphism** $\varphi:\mathcal{R}_1\to\mathcal{R}_2$ **between root data** $\mathcal{R}_i=(X_i,\Phi_i,X_i^\vee,\Phi_i^\vee)$ $(i=1,2)$ is a $\mathbb{Z}$-linear map

$$\varphi: X_1\longrightarrow X_2$$

satisfying

$$\varphi(\Phi_1)\subseteq\Phi_2 \tag{4.5.1}$$

and

$$\forall\alpha\in\Phi_1\ \ \varphi^\vee(\varphi(\alpha)^\vee)=\alpha^\vee \tag{4.5.2}$$

, where $\varphi^\vee: X_2^\vee\to X_1^\vee$ denotes the adjoint of φ determined by

$$\forall x\in X_1,\ y\in X_2^\vee \quad \langle x,\varphi^\vee(y)\rangle=\langle\varphi(x),y\rangle$$

4.5.2 Lemma. *(i) If $\varphi:\mathcal{R}_1\to\mathcal{R}_2$ and $\psi:\mathcal{R}_2\to\mathcal{R}_3$ are frugal morphisms, then $\psi\circ\varphi: X_1\to X_3$ defines a frugal morphism $\mathcal{R}_1\to\mathcal{R}_3$, and $(\psi\circ\varphi)^\vee=\varphi^\vee\circ\psi^\vee$ (as $\mathbb{Z}$-linear maps). In particular, root data form the objects of a category whose morphisms are frugal morphisms.*

(ii) For every frugal morphism $\varphi:\mathcal{R}_1\longrightarrow\mathcal{R}_2$, restriction defines an injection

$$\varphi|:\Phi_1\hookrightarrow\Phi_2$$

(iii) *Given a frugal morphism $\varphi : \mathcal{R}_1 \to \mathcal{R}_2$ of root data, there exists a morphism*

$$\varphi_0 : W_0(\mathcal{R}_1) \longrightarrow W_0(\mathcal{R}_2)$$

of groups determined by

$$\varphi_0(s_\alpha) = s_{\varphi(\alpha)} \quad \forall \alpha \in \Phi_1 \tag{4.5.3}$$

Moreover, the maps φ and $\varphi^\vee$ are equivariant with respect to φ_0 in the sense that

$$\varphi(w \bullet x) = \varphi_0(w) \bullet \varphi(x) \quad \textit{and} \quad \varphi^\vee(\varphi_0(w) \bullet y) = w \bullet \varphi^\vee(y)$$

for all $x \in X_1$ and $y \in X_2^\vee$.

Proof. ad (i): It follows from the definitions that $(\psi \circ \varphi)^\vee = \varphi^\vee \circ \psi^\vee$. Moreover, clearly $\psi(\varphi(\Phi_1)) \subseteq \Phi_3$ and

$$\varphi^\vee(\psi^\vee(\psi(\varphi(\alpha))^\vee))) = \varphi^\vee(\varphi(\alpha)^\vee) = \alpha^\vee \quad \forall \alpha \in \Phi_1$$

Since the identity $\mathrm{id}_X : X \to X$ is obviously a frugal morphism, it follows that this defines the structure of a category.

ad (ii): Follows immediately from eq. (4.5.2) and the fact that the map $\alpha \mapsto \alpha^\vee$ is a bijection between roots and coroots.

ad (iii): First of all, we have

$$\langle \varphi(\alpha), \varphi(\beta)^\vee \rangle = \langle \alpha, \varphi^\vee(\varphi(\beta)^\vee) \rangle = \langle \alpha, \beta^\vee \rangle$$

and therefore also

$$\mathrm{ord}(s_{\varphi(\alpha)} s_{\varphi(\beta)}) = \mathrm{ord}(s_\alpha s_\beta) \tag{4.5.4}$$

for all $\alpha, \beta \in \Phi_1$. Chosing a root basis $\Delta_1 \subseteq \Phi_1$, the pair $(W_0(\mathcal{R}_1), \{s_\alpha : \alpha \in \Delta_1\})$ is a Coxeter group, and by the characterization of Coxeter groups via generators and relations it follows from eq. (4.5.4) that there exists a unique morphism

$$\varphi_0 : W_0(\mathcal{R}_1) \longrightarrow W_0(\mathcal{R}_2)$$

satisfying eq. (4.5.3) for all $\alpha \in \Delta_1$. But, since

$$w s_\alpha w^{-1} = s_{w(\alpha)} \quad \forall w \in W_0(\mathcal{R}_1),\ \alpha \in \Phi_1$$

and every $\alpha \in \Phi_1$ is $W_0(\mathcal{R}_1)$-conjugate to some $\alpha \in \Delta_1$ up to a rational multiple, and parallel roots define the same element of the Weyl group, it follows that eq. (4.5.3) holds for all $\alpha \in \Phi_1$.

Finally, the equivariance of $\varphi^\vee$ follows formally from the equivariance of φ, and it suffices to check the equivariance of φ on the generators $w = s_\alpha$, for which they follow from explicit computation

$$\begin{aligned} \varphi(s_\alpha \bullet x) &= \varphi(x - \langle x, \alpha^\vee \rangle \alpha) = \varphi(x) - \langle x, \varphi^\vee(\varphi(\alpha)^\vee) \rangle \varphi(\alpha) \\ &= \varphi(x) - \langle \varphi(x), \varphi(\alpha)^\vee \rangle \varphi(\alpha) \\ &= s_{\varphi(\alpha)}(\varphi(x)) = \varphi_0(s_\alpha) \bullet \varphi(x) \end{aligned}$$

□

The fundamental example of a frugal morphism is the following:

4.5.3 Lemma. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be root data with bases $\Delta_1 \subseteq \Phi_1$ and $\Delta_2 \subseteq \Phi_2$, and assume that $\mathcal{R}_1$ is reduced and satisfies $X_1 = \mathbb{Z}\langle \Phi_1 \rangle$. Then, every map*

$$\varphi : \Delta_1 \longrightarrow \Phi_2$$

satisfying

$$\forall \alpha, \beta \in \Delta_1 \quad \langle \varphi(\alpha), \varphi(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \tag{4.5.5}$$

extends uniquely to a frugal morphism

$$\varphi : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$$

Proof. (Cf. [Bou07, Ch. VI, §1.5, Cor. of Prop. 15]) From eq. (4.5.5) it follows that

$$\operatorname{ord}(s_{\varphi(\alpha)}s_{\varphi(\beta)}) = \operatorname{ord}(s_\alpha s_\beta) \quad \forall \alpha, \beta \in \Delta_1$$

Therefore, by the characterization of Coxeter groups in terms of generators and relations it follows that there exists a unique group homomorphism

$$\varphi_0 : W_0(\mathcal{R}_1) \longrightarrow W_0(\mathcal{R}_2)$$

satisfying $\varphi_0(s_\alpha) = s_{\varphi(\alpha)}$ for all $\alpha \in \Delta_1$. Moreover, since the $\mathbb{Z}$-module is spanned by Φ_1, it follows that Δ_1 is a basis of it. Hence, the map φ extends uniquely to a $\mathbb{Z}$-module homomorphism

$$\varphi : X_1 \longrightarrow X_2$$

, and by assumption this morphism satisfies $\varphi(\alpha) \in \Phi_2$ for all $\alpha \in \Delta_1$. Now, since $\mathcal{R}_1$ is *reduced*, every $\beta \in \Phi_1$ can be written in the form $\beta = w \bullet \alpha$ with $\alpha \in \Delta_1$. But,

$$\begin{aligned}\varphi(s_\alpha \bullet x) &= \varphi(x - \langle x, \alpha^\vee\rangle\, \alpha) = \varphi(x) - \langle x, \alpha^\vee\rangle\, \varphi(\alpha)\\ &= \varphi(x) - \langle \varphi(x), \varphi(alpha)^\vee\rangle\, \varphi(\alpha)\\ &= s_{\varphi(\alpha)} \bullet \varphi(x)\end{aligned}$$

for all $x \in X_1$ and $\alpha \in \Delta_1$, which shows that $\varphi(\Phi_1) \subseteq \Phi_2$ and that $\varphi : X_1 \to X_2$ is equivariant with respect to φ_0. Here we have used that

$$\langle \varphi(x), \varphi(\alpha)^\vee\rangle = \langle x, \alpha^\vee\rangle \tag{4.5.6}$$

for all $x \in X_1$ and $\alpha \in \Delta_1$, which follows by linearity from the assumption that this equation holds for all $x \in \Delta_1$.

Moreover, eq. (4.5.6) also holds for $x \in X_1$ and all $\alpha \in \Phi_1$. To see this, note first that for every root datum $\mathcal{R}$ the bijection $\alpha \mapsto \alpha^\vee$ between roots and coroots respects the action of the Weyl group, i.e.

$$w(\alpha)^\vee = (w^\vee)^{-1}(\alpha^\vee) \quad \forall \alpha \in \Phi,\ w \in W_0(\mathcal{R}) \tag{4.5.7}$$

(see [DG70, Exposé XXI, Proposition 1.2.9] for a proof of this). Now, given any $\alpha \in \Phi_1$, use the reducedness of $\mathcal{R}_1$ to write $\alpha = w(\beta)$ with $\beta \in \Delta_1$. It then follows that

$$\begin{aligned}\langle \varphi(x), \varphi(\alpha)^\vee\rangle &= \langle \varphi(x), \varphi(w(\beta))^\vee\rangle = \langle \varphi(x), (\varphi_0(w)(\varphi(\beta)))^\vee\rangle\\ &= \left\langle \varphi(x), (\varphi_0(w)^\vee)^{-1}(\varphi(\beta)^\vee)\right\rangle\\ &= \langle \varphi_0(w)(\varphi(x)), \varphi(\beta)^\vee\rangle\\ &= \langle \varphi(w(x)), \varphi(\beta)^\vee\rangle\\ &= \langle w(x), \beta^\vee\rangle\\ &= \left\langle x, (w^\vee)^{-1}(\beta^\vee)\right\rangle\\ &= \langle x, w(\beta)^\vee\rangle\\ &= \langle x, \alpha^\vee\rangle\end{aligned}$$

for all $x \in X_1$. But since we can rewrite eq. (4.5.6) equivalently as

$$\langle \varphi(x), \varphi(\alpha)^\vee\rangle = \langle x, \alpha^\vee\rangle \quad \Leftrightarrow \quad \langle x, \varphi^\vee(\varphi(\alpha)^\vee)\rangle = \langle x, \alpha^\vee\rangle$$

, the validity of this equation for all $x \in X_1$ implies that

$$\varphi^\vee(\varphi(\alpha)^\vee) = \alpha^\vee$$

for all $\alpha \in \Phi_1$, showing that eq. (4.5.2) holds. We have therefore shown that the linear extension $\varphi : X_1 \to X_2$ of $\varphi : \Delta_1 \to \Phi_2$ is a frugal morphism of root data. Uniqueness is clear. □

Until now, the fact that the Weyl group $W_0(\mathcal{R})$ is a Coxeter group didn't enter, except in proofs. But of course, this aspect is very important, and so we need to make up a new definition:

4.5.4 Definition. A root datum together with a choice of a root basis $\Delta \subseteq \Phi$ is called a **based root datum**. Given based root data $\mathcal{R}_i = (X_i, \Phi_i, X_i^\vee, \Phi_i^\vee, \Delta_i)$ $(i = 1, 2)$, a frugal morphism

$$\varphi : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$$

is called **basic** if it preserves the chosen root bases, i.e. if

$$\varphi(\Delta_1) \subseteq \Delta_2$$

4.5.5 Lemma. *Given a basic frugal morphism*

$$\varphi : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$$

between based root data $\mathcal{R}_i = (X_i, \Phi_i, X_i^\vee, \Phi_i^\vee, \Delta_i)$*, the morphism*

$$\varphi_0 : W_0(\mathcal{R}_1) \longrightarrow W_0(\mathcal{R}_2)$$

defined in lemma 4.5.2 is a morphism of Coxeter groups for the canonical structures of Coxeter groups on $W_0(\mathcal{R}_i)$*, i.e.*

$$\varphi_0(S(\Delta_1)) \subseteq S(\Delta_2)$$

where

$$S(\Delta_i) = \{s_\alpha : \alpha \in \Delta_i\}$$

Moreover, φ_0 *is isometric, i.e. it preserves the length functions*

$$\forall w \in W_0(\mathcal{R}_1) \quad \ell(\varphi_0(w)) = \ell(w)$$

and (cf. lemma 4.7.4) restricts to an isomorphism

$$\varphi_0 : (W_0(\mathcal{R}_1), S(\Delta_1)) \xrightarrow{\sim} (\langle \varphi_0(S(\Delta_1)) \rangle, \varphi_0(S(\Delta_1)))$$

of Coxeter groups.

Proof. Omitted. □

4.5.6 Lemma. *Given based root data* $\mathcal{R}_i = (X_i, \Phi_i, X_i^\vee, \Phi_i^\vee, \Delta_i)$ $(i = 1, 2)$*, such that* $X_1 = \mathbb{Z}\langle \Phi_1 \rangle$*, every map*

$$\varphi : \Delta_1 \longrightarrow \Delta_2$$

satisfying eq. (4.5.5) *extends uniquely to a basic frugal morphism*

$$\varphi : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$$

Proof. Follows immediately from lemma 4.5.3. □

4.6 The theory of $\mathbf{FI}_W$-modules

The goal of this section is to review the theory of $\mathbf{FI}_W$-modules, and to construct examples associated to the classical families A_ℓ, B_ℓ, C_ℓ, D_ℓ of root systems.

The theory of **FI**-modules (without $_W$) goes back to Church, Ellenberg and Farb [CEF15], and relates to the earlier theory of *representation stability* of Church and Farb [CF13]. Both theories are aimed at providing a framework in which to establish stability properties for families $(G_n)_{n\in\mathbb{N}}$ of groups that naturally embed $G_n \hookrightarrow G_{n+1}$ into each other, the most notable example being the family $G_n = S_n$ of symmetric groups.

A representative example of the kind of stability phenomena studied is *homological stability.* A family $(G_n)_{n\in\mathbb{N}}$ as above is called *(co-)homologically stable* with respect to a coefficient group A, if for every fixed $k \in \mathbb{N}$ and all sufficiently large n, the natural map

$$H_k(G_n, A) \longrightarrow H^k(G_{n+1}, A) \quad \big(\text{resp. } H^k(G_{n+1}, A) \longrightarrow H^k(G_n, A)\big)$$

induced by $G_n \hookrightarrow G_{n+1}$ is an isomorphism. For example, the family of symmetric groups is homologically and cohomologically stable for all finite abelian groups A by a theorem of Nakaoka [Nak60].

The newer theory of **FI**-modules algebraizes these stability phenomena, at least for the family $(S_n)_{n\in\mathbb{N}}$, by turning them into finiteness properties of the name-bearing **FI**-modules, more precisely the property of being *finitely generated.* Later, the theory of **FI**-modules was generalized by Wilson [Wil14] into the theory of $\mathbf{FI}_W$-modules, which extends the former by allowing the symmetric groups S_n to be replaced by the Weyl groups of *any* of the classical families A_ℓ, B_ℓ, C_ℓ, and D_ℓ of root systems.

4.6.1 Definition ([Wil14, 1.1])**.** The category[28] $\mathbf{FI}_{BC}$ is the (nonfull) subcategory of the category of finite sets with objects

$$\mathbf{n} := \{k \in \mathbb{Z} : 1 \leq |k| \leq n\}$$

[28]We will write $\mathbf{FI}_B = \mathbf{FI}_C = \mathbf{FI}_{BC}$ for convenience.

for every natural number $n \in \{0,1,2,\dots\}$, and with morphisms

$$\mathrm{Hom}_{\mathbf{FI}_{BC}}(\mathbf{n},\mathbf{m}) = \{f \in \mathrm{Hom}_{\mathrm{Set}}(\mathbf{n},\mathbf{m}) \ : \ f \text{ injective},\ f(-a) = -f(a)\ \forall a \in \mathbf{n}\}$$

The category $\mathbf{FI}_D$ is the subcategory of $\mathbf{FI}_{BC}$ with the same objects and

$$\mathrm{Hom}_{\mathbf{FI}_D}(\mathbf{n},\mathbf{m}) = \{f \in \mathrm{Hom}_{\mathbf{FI}_{BC}}(\mathbf{n},\mathbf{m}) \ : \ \#\{i : i > 0,\ f(i) < 0\} \text{ even}\}$$

The category $\mathbf{FI}_A$ is the subcategory of $\mathbf{FI}_D$ with the same objects and

$$\mathrm{Hom}_{\mathbf{FI}_A}(\mathbf{n},\mathbf{m}) = \{f \in \mathrm{Hom}_{\mathbf{FI}_D}(\mathbf{n},\mathbf{m}) \ : \ \forall i\ i > 0 \Rightarrow f(i) > 0\}$$

The categories $\mathbf{FI}_X$ $(X \in \{A,B,C,D\})$ relate to the corresponding families of root systems X_ℓ via the identifications

$$W_0(X_\ell) \simeq \begin{cases} \mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{\ell+1}) & \text{if } X = A \\ \mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{\ell}) & \text{if } X \in \{B,C,D\} \end{cases}$$

These identifications are canonical (even natural), once we have numbered the simple roots of the families X_ℓ coherently. So let us do this.

4.6.2 Remark. In the following, we will describe the root system X_ℓ for $X \in \{A,B,C,D\}$ and ℓ a natural number (≥ 0 for type A and ≥ 1 for the other ones). In particular, this means (see [Bou07, Ch. VI, §1.1]) giving a $\mathbb{Q}$-vector space V_ℓ and a subset $\Phi_\ell \subseteq V_\ell$. Moreover, we also want to describe a set Δ_ℓ of simple roots together with a (coherent) numbering. Let us therefore agree that, if we write

$$\alpha_1 = \dots, \alpha_2 = \dots, \dots, \alpha_\ell = \dots$$

, that this means that $\Delta_\ell = \{\alpha_1,\dots,\alpha_\ell\}$ with the numbering implied in the notation. Also, $e_1, e_2, \dots$ denote the standard basis vectors of the standard vector space relevant to the given context. For convenience, we will also describe the Weyl group $W_\ell = W_0(X_\ell)$ as a subgroup of $\mathrm{GL}_{\mathbb{Q}}(V_\ell)$ as well as the Cartan matrix

$$C = (C_{i,j})_{1\leq i,j\leq \ell} \text{ with } C_{i,j} = \left\langle \alpha_i, \alpha_j^\vee \right\rangle$$

each case.

(i) Type A:

$$\begin{aligned}
V_\ell &= \{x \in \mathbb{Q}^{\ell+1} : \sum_i x_i = 0\} \\
\Phi_\ell &= \{e_i - e_j : i \neq j,\ 1 \leq i,j \leq \ell+1\} \\
\alpha_i &= e_{i+1} - e_i \\
W_\ell &= S_{\ell+1} \text{ (permuting the basis vectors)} \\
C &= \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}
\end{aligned}$$

(ii) Type B:

$$\begin{aligned}
V_\ell &= \mathbb{Q}^\ell \\
\Phi_\ell &= \{\pm e_i : 1 \leq i \leq \ell\} \cup \{\pm e_i \pm e_j : i \neq j,\ 1 \leq i < j \leq \ell\} \\
\alpha_1 &= e_1 \text{ and } \alpha_i = e_i - e_{i-1} \text{ for } 2 \leq i \leq \ell \\
W_\ell &= \{\pm 1\}^\ell \rtimes S_\ell \text{ with } \{\pm 1\}^\ell \text{ acting by multiplication} \\
C &= \begin{pmatrix} 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ 2 & 2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2 & -1 & 0 & 0 \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}
\end{aligned}$$

(iii) Type C:

$$\begin{aligned}
&V_\ell = \mathbb{Q}^\ell \\
&\Phi_\ell = \{\pm 2e_i : 1 \le i \le \ell\} \cup \{\pm e_i \pm e_j : i \neq j,\ 1 \le i < j \le \ell\} \\
&\alpha_1 = 2e_1 \text{ and } \alpha_i = e_i - e_{i-1} \text{ for } 2 \le i \le \ell \\
&W_\ell = \{\pm 1\}^\ell \rtimes S_\ell \text{ with } \{\pm 1\}^\ell \text{ acting by multiplication}
\end{aligned}$$

$$C = \begin{pmatrix} 2 & 2 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2 & -1 & 0 & 0 \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}$$

(iv) Type D:

$$\begin{aligned}
&V_\ell = \mathbb{Q}^\ell \\
&\Phi_\ell = \{\pm e_i \pm e_j : i \neq j,\ 1 \le i < j \le \ell\} \\
&\alpha_1 = e_1 + e_2 \text{ and } \alpha_i = e_i - e_{i-1} \text{ for } 2 \le i \le \ell \\
&W_\ell = \{(\varepsilon_i)_i \in \{\pm 1\}^\ell : \prod_i \varepsilon_i = 1\} \rtimes S_\ell
\end{aligned}$$

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

Via the identification

$$\boldsymbol{\ell} \simeq \{\pm e_i : 1 \le i \le \ell\}$$

we therefore have an equality

$$\operatorname{End}_{\mathbf{FI}_X}(\boldsymbol{\ell}) = \begin{cases} W_{\ell-1} & \text{if } X = A \\ W_\ell & \text{if } X \in \{B, C, D\} \end{cases} \tag{4.6.1}$$

as subgroups of the permutation group of $\boldsymbol{\ell}$.

4.6.3 Definition ([Wil14, Definition 1.2])**.** For $X \in \{A, B, C, D\}$, an $\mathbf{FI}_X$-module with coefficients in a ring k is a functor

$$\mathbf{FI}_X \longrightarrow \operatorname{Mod}(k)$$

Because of eq. (4.6.1), every $\mathbf{FI}_X$-module M gives rise to a sequence $M_n := M(\mathbf{n})$ of representations of W_{n-1} (type A) resp. of W_n (type B, C, D). Moreover, for every $n \ge 0$, the inclusion

$$\mathbf{n} \subseteq \mathbf{n+1}$$

of sets defines a canonical element $I_n \in \operatorname{Hom}_{\mathbf{FI}_A}(\mathbf{n}, \mathbf{n+1})$, and more generally, for $n \ge m \ge 0$ the inclusion $\mathbf{m} \subseteq \mathbf{n}$ defines a canonical element $I_{m,n} \in \operatorname{Hom}_{\mathbf{FI}_A}(\mathbf{m}, \mathbf{n})$, which is also the composition

$$I_{m,n} = I_{n-1} \circ I_{n-2} \circ \dots \circ I_{m+1} \circ \dots I_m$$

In particular, the sequence M_n of representations comes equipped with maps

$$\phi_n = M(I_n) : M_n \longrightarrow M_{n+1}$$

and these maps are equivariant (in the obvious way) with respect to the canonical inclusion

$$\operatorname{End}_{\mathbf{FI}_X}(\mathbf{n}) \hookrightarrow \operatorname{End}_{\mathbf{FI}_X}(\mathbf{n+1})$$

A sequence (M_n, ϕ_n) with these properties is also called a **consistent sequence**. Not every such sequence (M_n, ϕ_n) comes from an $\mathbf{FI}_X$-module, however there is the following lemma:

4.6.4 Lemma ([Wil14, Lemma 3.4])**.** *A consistent sequence (M_n, ϕ_n) of representations of* $\mathrm{End}_{\mathbf{FI}_X}(\mathbf{n})$ *is obtained from an $\mathbf{FI}_X$-module if and only if for all $n \geq m \geq 0$ the stabilizer*

$$H_{m,n} := \{\sigma \in \mathrm{End}_{\mathbf{FI}_X}(\mathbf{n}) : \sigma \circ I_{m,n} = I_{m,n}\} \leq \mathrm{End}_{\mathbf{FI}_X}(\mathbf{n})$$

acts trivial on the image $\phi_{m,n}(M_m) \subseteq M_n$ of the map

$$\phi_{m,n} = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_{m+1} \circ \phi_m$$

Let us now show that, for a fixed type X and varying ℓ, the root lattices

$$Q_\ell = \mathbb{Z}\Phi_\ell \subseteq V_\ell$$

and coroot lattices

$$Q_\ell^\vee = \mathbb{Z}\Phi_\ell^\vee \subseteq V_\ell^\vee$$

of the root system X_ℓ form an $\mathbf{FI}_X$-module. Here, $V_\ell^\vee = \mathrm{Hom}_{\mathbb{Q}}(V_\ell, \mathbb{Q})$ denotes the dual vector space and $\Phi_\ell^\vee \subseteq V_\ell^\vee$ denotes the set of dual roots (which is determined by the roots; see [Bou07, Ch. VI, §1.1]).

4.6.5 Lemma. *Let $X \in \{A, B, C, D\}$. For $\ell' \geq \ell$, the map of sets*

$$\begin{aligned} \varphi_{\ell,\ell'} : \Delta_\ell &\hookrightarrow \Delta_{\ell'} \\ \alpha_i &\longmapsto \alpha_i \end{aligned}$$

is isometric, i.e. it respects the Cartan matrices in the sense that

$$\langle \varphi_{\ell,\ell'}(\alpha), \varphi_{\ell,\ell'}(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle \quad \forall \alpha, \beta \in \Delta_\ell$$

Moreover, its linear extension to a map of $\mathbb{Z}$-modules

$$\varphi_{\ell,\ell'} : Q_\ell \longrightarrow Q_{\ell'}$$

is equivariant with respect to $W_\ell \hookrightarrow W_{\ell'}$, and

$$\varphi_{\ell,\ell'}(Q_\ell) \subseteq Q_{\ell'}$$

is invariant under the action of $I_{\ell+1,\ell'+1}$ (type A) resp. $I_{\ell,\ell'}$ (types B, C, D). Therefore, the consistent sequence $(Q_{\ell-1}, \varphi_{\ell-1,\ell})$ (type A) resp. $(Q_\ell, \varphi_{\ell,\ell+1})$ (types B, C, D) defines an $\mathbf{FI}_X$-module Q over $\mathbb{Z}$.

Proof. The isometry follows immediately from the description of the Cartan matrices (to be honest, it's probably more immediate from the Dynkin graphs; see the diagrams provided in appendix A). The equivariance of $\varphi_{\ell,\ell'}$ follows from lemma 4.5.2 since the linear extension $\varphi_{\ell,\ell'} : Q_\ell \longrightarrow Q_{\ell'}$ is a basic frugal morphism (for the 'adjoint' root data of types X_ℓ and $X_{\ell'}$) by lemma 4.5.6. Now, the invariance of

$$\varphi_{\ell,\ell'}(Q_\ell) = \mathbb{Z}\{\alpha_i : 1 \leq i \leq \ell\} \subseteq Q_{\ell'}$$

under the action of $I_{\ell+1,\ell'+1}$ in type A resp. $I_{\ell,\ell'}$ in the other types is clear, since this group fixed the vectors e_i, $1 \leq i \leq \ell+1$ of the ambient vector space $\mathbb{Q}^{\ell'+1}$ for type A resp. the vectors e_i, $1 \leq i \leq \ell$ of the ambient vector space $\mathbb{Q}^{\ell'}$ for types B, C, D, and the roots α_i, $1 \leq i \leq \ell$ in both cases lie in the subspace spanned by these vectors. ☐

4.6.6 Remark. To avoid confusion, we denote by $Q(n)$ the degree n piece of the $\mathbf{FI}_X$-module Q, and by Q_ℓ the root lattice in X_ℓ. Note that $Q(\ell) = Q_{\ell-1}$ in type A, but $Q(\ell) = Q_\ell$ for the other types.

4.6.7 Corollary. *For fixed type $X \in \{A, B, C, D\}$ and varying ℓ, the coroot lattices $Q_\ell^\vee$ together with the transition maps*

$$\begin{aligned} Q_\ell^\vee &\hookrightarrow Q_{\ell+1}^\vee \\ \alpha_i^\vee &\longmapsto \alpha_i^\vee \end{aligned}$$

form an $\mathbf{FI}_X$-module.

Proof. The same proof as for lemma 4.6.5 works; however, this also follows from the fact that the coroot lattice is the root lattice in the *dual root system* (B and C are dual, whereas A and D are self-dual). ☐

The main ‘slogan’ of the theory of $\mathbf{FI}_X$-modules is that periodicity should be equivalent to the property of being finitely generated. Let us therefore recall this important notion.

4.6.8 Definition ([Wil14, Definition 3.13])**.** Given an $\mathbf{FI}_X$-module M and a subset

$$S \subseteq \coprod_{n\geq 0} M_n$$

the **span of** S—denoted by $\mathrm{span}_M(S)$—is the minimal $\mathbf{FI}_X$-submodule[29] of M containing S. This module is also called **the $\mathbf{FI}_X$-submodule generated by** S.

An $\mathbf{FI}_X$-module M is called **finitely generated**, if there exists a finite subset $S \subseteq \coprod_{n\geq 0} M_n$ such that

$$M = \mathrm{span}_M(S)$$

More specifically, M is said to be **finitely generated in degree** $\leq m$ if S can be chosen to lie in $\coprod_{n\leq m} M_n$.

The above ‘abstract’ notion of being finitely generated is equivalent to another, more concrete one, which we are going to explain now.

4.6.9 Definition ([Wil14, Definition 3.7])**.** For a given $m \geq 0$, there exists an $\mathbf{FI}_X$-module $M(m)$, where

$$M(m)_n := k[\mathrm{Hom}_{\mathbf{FI}_X}(\boldsymbol{m}, \boldsymbol{n})] \quad \text{(free module over } k)$$

with the action of $\mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{n})$ being given by post-composition, and the transition map induced by the natural map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{FI}_X}(\boldsymbol{m}, \boldsymbol{n}) &\longrightarrow \mathrm{Hom}_{\mathbf{FI}_X}(\boldsymbol{m}, \boldsymbol{n+1}) \\ f &\longmapsto I_n \circ f \end{aligned}$$

The fundamental property of the modules $M(m)$ is the following:

4.6.10 Proposition ([Wil14, Proposition 3.11])**.** *For every $m \geq 0$, the ‘forgetful’ functor*

$$\begin{aligned} \mathrm{Fun}(\mathbf{FI}_X, \mathrm{Mod}(k)) &\longrightarrow \mathrm{Rep}(\mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{m})) \\ M &\longrightarrow M_m \end{aligned}$$

admits a left adjoint, given by $V \longmapsto M(m) \otimes_{k[\mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{m})]} V$, where

$$(M(m) \otimes_{k[\mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{m})]} V)_n := M(m)_n \otimes_{k[\mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{m})]} V$$

with $M(m)$ being a right-$k[\mathrm{End}_{\mathbf{FI}_X}(\boldsymbol{m})]$-module in the obvious way. In particular, for every $\mathbf{FI}_X$-module M we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{FI}_X, \mathrm{Mod}(k))}(M(m), M) \simeq M_m$$

Using this lemma, one deduces:

4.6.11 Proposition ([Wil14, Proposition 3.15])**.** *An $\mathbf{FI}_X$-module M is finitely generated in degree $\leq m$ if and only if there exists a (degree-wise) surjection*

$$\bigoplus_{n=0}^{m} M(n)^{\oplus k_n} \twoheadrightarrow M$$

for some integers $k_n \geq 0$.

This characterization of being finitely generated is more useful, since allows one to define the notion of being finitely presented in a straight-forward way:

4.6.12 Definition ([Wil14, Definition 3.18])**.** An $\mathbf{FI}_X$-module M is called **finitely presented** with generators in degree $\leq g$ and relations in degree $\leq r$, if there exists a right exact sequence

$$\textstyle\bigoplus_{n=0}^{r} M(n)^{\oplus k_n} \longrightarrow \bigoplus_{n=0}^{g} M(n)^{\oplus k'_n} \longrightarrow M \longrightarrow 0$$

for some integers $k_n, k'_n \geq 0$.

[29]The notion of $\mathbf{FI}_X$-submodule is the obvious one.

There is also a notion of *degree*, at least when k is a field (and in any case for $X = A$):

4.6.13 Definition (Cf. [NS18, 1.2])**.** Let k be a field. An $\mathbf{FI}_X$-module is said to be **of degree** m, if the function

$$n \longmapsto \dim_k M_n$$

agrees with a (necessarily unique) polynomial of degree m, for sufficiently large n.

We come now to the main goal of this section, which is to prepare the ground for applying [NS18, Theorem 1.6] to the $\mathbf{FI}_A$-module $Q^\vee$.

4.6.14 Proposition. *The $\mathbf{FI}_X$-modules Q and $Q^\vee$ (over $\mathbb{Z}$) are finitely generated with generators in degree ≤ 2 and their base change $Q \otimes_{\mathbb{Z}} k$, $Q^\vee \otimes_{\mathbb{Z}} k$ is of degree 1 for every field k. Further, for type A the $\mathbf{FI}_A$-modules Q and $Q^\vee$ is finitely presented with generators in degree ≤ 2 and relations in degree ≤ 2.*

Proof. By duality, it suffices to prove everything for Q. Now,

$$\mathrm{rk}_{\mathbb{Z}}\, Q_\ell = \ell = \mathrm{rk}_{\mathbb{Z}}\, Q_\ell^\vee$$

hence the claim regarding the degree is clear. Moreover for $\ell \geq 2$, Q_ℓ is generated as a $\mathbb{Z}[W_\ell]$-module by $\alpha_1 \in Q_1 = Q(2)$ for type A, and by $\alpha_2 \in Q_2 = Q(\ell)$ for the other types. Hence, the map of $\mathbf{FI}_X$-modules

$$M(2) \twoheadrightarrow Q \tag{4.6.2}$$

sending the canonical element $\mathrm{id}_{\mathbf{2}} \in \mathrm{Hom}_{\mathbf{FI}_X}(\mathbf{2},\mathbf{2}) \subseteq M(2)_2$ to the mentioned generator is surjective for all degree $\ell \geq 2$, and hence it follows that Q is finitely generated in degree ≤ 2. Assume now that $X = A$. We can identify $\mathrm{Hom}_{\mathbf{FI}_A}(\mathbf{2}, n)$ with the set of pairs (i,j) of distinct integers $1 \leq i,j \leq n$, $i \neq j$. An arbitrary element of $M(2)_n$ is therefore of the form

$$\sum_{1 \leq i,j \leq n,\ i \neq j} \lambda_{i,j}(i,j) \quad \text{with } \lambda_{i,j} \in \mathbb{Z}$$

It lies in the kernel of eq. (4.6.2) if and only if

$$0 = \sum_{i,j} \lambda_{i,j}(e_i - e_j) = \sum_{i=1}^{n} (\sum_{j \neq i} \lambda_{i,j} - \lambda_{j,i}) e_i$$

which is equivalent to

$$\forall i \ \sum_{j \neq i} \lambda_{i,j} = \sum_{j \neq i} \lambda_{j,i} \tag{4.6.3}$$

It follows that the kernel is generated as a $\mathbb{Z}$-module by expressions of the form

$$\sum_{j \neq i} \lambda_{i,j}(i,j) + \lambda_{j,i}(j,i)$$

with $\lambda_{i,j}, \lambda_{j,i} \in \mathbb{Z}$, $j \in \{1,\ldots,n\} - \{i\}$ satisfying eq. (4.6.3). But, by applying eq. (4.6.3) with j instead of i for some $j \in \{1,\ldots,n\} - \{i\}$, it follows that we must have

$$\lambda_{j,i} = \sum_{k \neq j} \lambda_{j,k} = \sum_{k \neq j} \lambda_{k,j} = \lambda_{i,j}$$

Hence the kernel is generated as a $\mathbb{Z}$-module by expressions of the form

$$(i,j) + (j,i)$$

hence as a $\mathbb{Z}[S_n]$-module by the $I_{2,n}$-invariant element

$$(1,2) + (2,1) \in M(2)_n^{I_{2,n}}$$

Since $M(2)_n \simeq \mathrm{Ind}_{I_{2,n}}^{S_n} k$ (see the remark following [Wil14, Definition 3.7]), it follows that for $n \geq 2$, the $\mathbb{Z}[S_n]$-module homomorphism

$$M(2)_n \longrightarrow M(2)_n$$

sending the canonical generator of $M(2)_n \simeq \mathrm{Ind}_{I_{2,n}}^{S_n}$ to $(1,2) + (2,1)$ maps surjectively onto the kernel of eq. (4.6.2) in degree n, hence the morphism

$$M(2) \longrightarrow M(2)$$

of $\mathbf{FI}_A$-modules sending the generator $\mathrm{id}_{\mathbf{2}} \in \mathrm{Hom}_{\mathbf{FI}_A}(\mathbf{2},\mathbf{2})$ to $(1,2) + (2,1)$ maps surjectively onto the kernel of eq. (4.6.2) in degrees ≥ 2. $\square$

4.7 The DeConcini-Salvetti resolution

Throughout this section (W, S) denotes a Coxeter group with S countable[30]. Moreover, we assume that a total ordering $S = \{s_1, s_2, \dots\}$ has been chosen.

The goal of this section is to recall a free resolution $\mathcal{CS}_\bullet$ of the trivial $\mathbb{Z}[W]$-module $\mathbb{Z}$ found by DeConcini and Salvetti. Its usefulness for concrete computations rests on the fact that the rank (for $\#S < \infty$) grows only polynomially in $\#S$

$$\operatorname{rk}_{\mathbb{Z}[W]} \mathcal{CS}_k = \binom{\#S + k - 1}{k} = O((\#S)^k)$$

whereas the rank of the standard resolution grows polynomially in $\#W$

$$\operatorname{rk}_{\mathbb{Z}[W]} P_k = (\#W)^k$$

For example, in the case $W = S_n$ of symmetric groups the growth rates would be $(n-1)^k$ and $(n!)^k$ respectively.

This resolution was originally [CS00] only explicitly stated for finite Coxeter groups, even though it was remarked [CS00, p. 215] that the construction would carry over with minor modifications to the finitely generated case ($\#S < \infty$), without providing details however. The definition in the general finitely generated case was given later in [MSV12, 2.5] (see also [Sal02]).

We will now state the resolution in the countably generated case, noting that it is easily reduced to the finitely generated by considering W as the union $W = \bigcup_{n=1}^{\infty} \langle s_1, \dots, s_n \rangle$.

4.7.1 Definition. (Cf. [MSV12, 2.5].) For $k \in \mathbb{Z}$, let $\mathcal{CS}_k$ be the free $\mathbb{Z}[W]$-module over the set

$$\mathcal{F}_k := \{\Gamma = (\Gamma_i)_{i \in \mathbb{N}, i \geq 1} : S \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots, \quad \#\Gamma = k, \quad \#\langle \Gamma_1 \rangle < \infty\} \tag{4.7.1}$$

of descending flags of cardinality k of subsets Γ_i of S generating finite subgroups $\langle \Gamma_i \rangle$ of W, where the cardinality of a flag Γ is defined by

$$\#\Gamma := \sum_{i \geq 1} \#\Gamma_i = k$$

In particular, $\#\Gamma < \infty$ implies that $\Gamma_i = \emptyset$ for $i \gg 0$. Note also that $\mathcal{F}_k = \emptyset$ for $k < 0$.

For $k \in \mathbb{Z}$, define the differential

$$\partial_k : \mathcal{CS}_k \longrightarrow \mathcal{CS}_{k-1}$$

on basis elements by

$$\partial_k([\Gamma]) := \sum_{\substack{i \geq 1 \\ \#\Gamma_i > \#\Gamma_{i+1}}} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in W_{\Gamma_i}^{\Gamma_i - \{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta \subseteq \Gamma_i - \{\tau\}}} (-1)^{\alpha(\Gamma, i, \tau, \beta)} \beta [\Gamma^{i,\tau,\beta}] \tag{4.7.2}$$

where $\Gamma^{i,\tau,\beta} \in \mathcal{F}_{k-1}$ is defined by

$$\Gamma_j^{i,\tau,\beta} := \begin{cases} \Gamma_j & \text{if } j < i \\ \Gamma_i - \{\tau\} & \text{if } j = i \\ \beta^{-1}\Gamma_j\beta & \text{if } j > i \end{cases} \tag{4.7.3}$$

where

$$\alpha(\Gamma, i, \tau, \beta) := i \cdot \ell(\beta) + \sum_{j<i} \#\Gamma_j + \mu(\Gamma_i, \tau) + \sum_{j>i} \sigma(\beta, \Gamma_j) \tag{4.7.4}$$

with

$$\mu(\Gamma_i, \tau) := \#\{s \in \Gamma_i : s \leq \tau\}$$

the number of elements of Γ_i smaller or equal than τ in the fixed ordering of S and

$$\sigma(\beta, \Gamma_j) := \#\{(x, y) \in \Gamma_j \times \Gamma_j : x < y \text{ and } \beta^{-1}x\beta > \beta^{-1}y\beta\}$$

the number of inversions of the map $\Gamma_j \longrightarrow \Gamma_i - \{\tau\}$ given by $x \mapsto \beta^{-1}x\beta$. Finally,

$$W_{\Gamma_i}^{\Gamma_i - \{\tau\}} := \{g \in \langle \Gamma_i \rangle : \forall h \in \langle \Gamma_i - \{\tau\} \rangle \quad \ell(gh) \geq \ell(g)\}$$

denotes the set of *minimal coset representatives*[31] of the group $\langle \Gamma_i \rangle$ with respect to the special subgroup $\langle \Gamma_i - \{\tau\} \rangle$.

[30] The reader may assume that W is finite, since we only need the resolution in that case. Stating it in greater generality is useful however, since this allows one to treat e.g. the infinite symmetric group $S_\infty := \bigcup_{n \geq 1} S_n$ directly.

[31] By the theory of Coxeter groups [Bou07, Ch. IV, Exercises §1, Ex. 3], every such coset contains a *unique* element of minimal length.

4.7.2 Notation. A flag $\Gamma \in \mathcal{F}_k$ with $\Gamma_i = \emptyset$ for $i > n$ will also be denoted by the expression

$$\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_n$$

and the basis element of $\mathcal{CS}_k$ corresponding to Γ will be denoted alternatively by

$$[\Gamma] \quad \text{or} \quad [\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_n]$$

For example,

$$[S] = [S \supseteq \emptyset] = [S \supseteq \emptyset \supseteq \emptyset]$$

all denote the same basis element in $\mathcal{CS}_2$, corresponding to the flag Γ with $\Gamma_1 = S$ and $\Gamma_i = \emptyset$ for $i > 0$. Also,

$$[] = [\emptyset]$$

both denote the (unique) canonical basis element of $\mathcal{CS}_0 \simeq \mathbb{Z}[W]$.

4.7.3 Lemma. *The chain complex $\mathcal{CS}$ of definition 4.7.1 together with the augmentation map*

$$\varepsilon : \mathcal{CS} \longrightarrow \mathbb{Z}$$

given in degree zero by

$$\varepsilon_0(g[]) = \varepsilon(g), \quad g \in \mathbb{Z}[W]$$

, where $\varepsilon : \mathbb{Z}[W] \to \mathbb{Z}$ denotes the augmentation map of the group algebra, is a (free) resolution of the complex $\mathbb{Z}$ (concentrated in degree zero).

Proof. This is proven in [CS00, Theorem 3.1.7] in the finite and in [MSV12, Theorem 8] in the finitely generated case; the infinite case is easily reduced to this. For example, to prove that it is exact, given an element $x \in \mathcal{CS}_k$ with $\partial_k(x) = 0$, there exists a finite subsets $S' \subseteq S$ such that the formula for $\partial_k(x)$ only involves flags Γ with $S' \supseteq \Gamma_1$. □

In 4.4.6 we described how the map

$$H^k(G', M') \longrightarrow H^k(G, M)$$

induced by a morphism $(\varphi_0, \varphi_1) : (G, M) \to (G', M')$ of the category $\mathcal{D}$ (see remark 4.4.5) has a simple description when using the standard resolution to compute the cohomology groups. A similar situation holds for the DeConcini-Salvetti resolution, at least when one restricts to morphisms of Coxeter groups:

4.7.4 Lemma. *Let (W, S) and (W', S') be Coxeter groups with S and S' countable, together with total orderings on S and S'. Moreover, let*

$$\varphi_0 : W \longrightarrow W'$$

be a monotonous isometric morphism of Coxeter groups, i.e. a homomorphism of groups such that $\varphi_0(S) \subseteq S'$ and the induced map

$$\varphi_0|_S : S \hookrightarrow S'$$

is strictly monotonous (i.e. $s < t \Leftrightarrow \varphi_0(s) < \varphi_0(t)$), and such that the lengths are preserved ('isometry'), i.e.

$$\forall w \in W \quad \ell(\varphi_0(w)) = \ell(w)$$

Then there is a morphism

$$f : \mathcal{CS} \longrightarrow \varphi_0^*(\mathcal{CS}')$$

of chain complexes of $\mathbb{Z}[W]$-modules, where $\mathcal{CS}$ and $\mathcal{CS}'$ denote the DeConcini-Salvetti complexes of (W, S) and (W', S') (and the chosen total orderings) respectively, compatible with the augmentations defined in lemma 4.7.3, given in degree k on basis elements by

$$f_k([\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots]) = [\varphi_0(\Gamma_1) \supseteq \varphi_1(\Gamma_2) \supseteq \ldots]$$

Proof. Omitted. □

4.7.5 Corollary. *If $(\varphi_0, \varphi_1) : (W, M) \longrightarrow (W', M')$ is a morphism of the category $\mathcal{D}$ of remark 4.4.5 such that $\varphi_0 : W \to W'$ is as in lemma 4.7.4, then the functoriality map*

$$(\varphi_0, \varphi_1)^* : H^k(W', M') \longrightarrow H^k(W, M)$$

is induced by the map

$$\mathrm{Hom}_{\mathbb{Z}[W']}(\mathcal{CS}'_k, M') \longrightarrow \mathrm{Hom}_{\mathbb{Z}[W]}(\mathcal{CS}_k, M)$$

on cochains sending a cochain α to the cochain α' given by

$$\alpha'([\Gamma_1 \supseteq \Gamma_2 \supseteq \ldots]) = \varphi_1(\alpha([\varphi_0(\Gamma_1) \supseteq \varphi_0(\Gamma_2) \supseteq \ldots]))$$

4.7.6 Remark. Given a morphism $\varphi : W \to W'$ between Coxeter groups (W, S) and (W', S') satisfying $\varphi(S) \subseteq S'$, the condition

$$\forall w \in W \quad \ell(\varphi(w)) = \ell(w)$$

is equivalent to

$$\forall s, t \in S \quad \mathrm{ord}(\varphi(st)) = \mathrm{ord}(st)$$

Moreover, if these conditions are satisfied, then φ is necessarily *injective* and induces an isomorphism of W with the parabolic subgroups $\langle \varphi(S) \rangle \subseteq W'$ of W'.

We can use the resolution $\mathcal{CS}$ to compute the cohomology group $H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ for the Weyl group W_0 of a root system $(X, \Phi, X^\vee, \Phi^\vee)$ as an abstract group. But in order to determine when eq. (4.0.1) splits, we also need to know $[\phi_u]$ as an element of this group. Hence, we need to explicitly compute a comparison map between the standard resolution and $\mathcal{CS}$, which we will do now.

4.7.7 Lemma. *The map*

$$f_\bullet : \mathcal{CS}_\bullet \longrightarrow P_\bullet \tag{4.7.5}$$

of chain complexes of $\mathbb{Z}[W]$-modules between the DeConcini-Salvetti complex $\mathcal{CS}$ (definition 4.7.1) and the standard resolution P (definition 4.4.3) for W that is given by the recursion eq. (4.4.5) *of lemma 4.4.9, the contracting chain homotopy h of definition 4.4.3, and by requiring that*

$$f_0([]) = []$$

is given up to degree 2 by (for notations see remark 4.4.4 and notation 4.7.2

$$\begin{aligned}
f_1([\{s\}]) &= [s] - [1] && s \in S \\
f_2([\{s\} \supseteq \{s\}] &= [s, s] - [s, 1] + [1, s] - [1, 1] && s \in S \\
f_2([\{s, t\}]) &= \sum_{k=0}^{m(s,t)-1} (-1)^k \left([\mathrm{prod}(t, s; k), s] - [\mathrm{prod}(t, s; k), 1]\right. \\
&\qquad \left. -[\mathrm{prod}(s, t; k), t] + [\mathrm{prod}(s, t; k), 1]\right) && s, t \in S, \ s < t
\end{aligned}$$

Proof. First, we explicitly compute the differential $\partial_\bullet^{\mathcal{CS}}$ on basis elements using eq. (4.7.2). In degree one, a basis element is of the form $[\Gamma]$ with $\Gamma \in \mathcal{F}_1$ of the form $\Gamma_1 = \{s\}$ and $\Gamma_i = \emptyset$ for $i > 0$. It follows that

$$\partial_1^{\mathcal{CS}}[\Gamma] = \sum_{\beta \in \{1,s\}} (-1)^{\alpha(\Gamma,1,s,\beta)} \beta[\Gamma^{1,s,\beta}] = s[] - [] \tag{4.7.6}$$

in this case (here $\tau = s$, $\Gamma_i = \{s\}$ and $W_{\Gamma_i}^{\Gamma_i - \{\tau\}} = W_{\{s\}}^{\emptyset} = \{1, s\}$). In degree two there are two kinds of basis elements, the 'degenerate' ones of the form $[\Gamma]$ with $\Gamma_1 = \{s\} = \Gamma_2$ and $\Gamma_i = \emptyset$ for $i > 1$, for which

$$\partial_2^{\mathcal{CS}}[\Gamma] = \sum_{\beta \in \{1,s\}} (-1)^{\alpha(\Gamma,2,s,\beta)} \beta[\Gamma^{2,s,\beta}] = [\{s\}] + s[\{s\}] \tag{4.7.7}$$

and the 'non-degenerate' ones of the form $[\Gamma]$ with $\Gamma_1 = \{s, t\}$, $\Gamma_i = \emptyset$ for $i > 0$ and $s, t \in S$ with $s < t$ (recall that we fix a total ordering of S), for which

$$\begin{aligned}
\partial_2^{\mathcal{CS}}[\Gamma] &= \sum_{\tau \in \{s,t\}} \sum_{\beta \in W_{\{s,t\}}^{\{s,t\}-\{\tau\}}} (-1)^{\alpha(\Gamma,1,\tau,\beta)} \beta[\Gamma^{1,\tau,\beta}] \\
&= \sum_{\beta \in W_{\{s,t\}}^{\{t\}}} (-1)^{\ell(\beta)+1} \beta[\{t\}] + \sum_{\beta \in W_{\{s,t\}}^{\{s\}}} (-1)^{\ell(\beta)+2} \beta[\{s\}] \\
&= \sum_{k=0}^{m(s,t)-1} (-1)^{k+1} \mathrm{prod}(s, t; k)[\{t\}] + \sum_{k=0}^{m(s,t)-1} (-1)^k \mathrm{prod}(t, s; k)[\{s\}]
\end{aligned} \tag{4.7.8}$$

where we have used that the minimal coset representatives are given by

$$W^{\{t\}}_{\{s,t\}} = \{\mathrm{prod}(s,t;k) : 0 \le k \le m(s,t)-1\} \quad \text{and} \quad W^{\{s\}}_{\{s,t\}} = \{\mathrm{prod}(t,s;k) : 0 \le k \le m(s,t)-1\}$$

with

$$\mathrm{prod}(s,t;k) := \underbrace{\ldots sts}_{k \text{ factors}}$$

and $m(s,t) = \mathrm{ord}(st) < \infty$ equal to the order of st.

Using the recursion eq. (4.4.5) and the definition eq. (4.7.2) of the differential of $\mathcal{CS}$, it follows from eq. (4.7.6) that

$$f_1([\{s\}]) = h_0(f_0(\partial_1^{\mathcal{CS}}[\{s\}])) = h_0(f_0(s[] - [])) = h_0(s[] - []) = [s] - [1]$$

From eq. (4.7.7) and eq. (4.7.8) it follows that

$$\begin{aligned} f_2([\{s\} \supseteq \{s\}]) &= h_1(f_1(\partial_2^{\mathcal{CS}}[\{s\} \supseteq \{s\}])) = h_1(f_1(s[\{s\}] + [\{s\}])) \\ &= h_1(s([s]-[1]) + [s] - [1]) = [s,s] - [s,1] + [1,s] - [1,1] \end{aligned}$$

and

$$\begin{aligned} f_2([\{s,t\}]) = h_1(f_1(\partial_2^{\mathcal{CS}}[\{s,t\}])) &= h_1\left(f_1\left(\sum_{k=0}^{m(s,t)-1} (-1)^k \left(\mathrm{prod}(t,s;k)[\{s\}] - \mathrm{prod}(s,t;k)[\{t\}]\right)\right)\right) \\ &= h_1\left(\sum_{k=0}^{m(s,t)-1} (-1)^k \left(\mathrm{prod}(t,s;k)([s]-[1]) - \mathrm{prod}(s,t;k)([t]-[1])\right)\right) \\ &= \sum_{k=0}^{m(s,t)-1} (-1)^k \left([\mathrm{prod}(t,s;k),s] - [\mathrm{prod}(t,s;k),1] - [\mathrm{prod}(s,t;k),t] + [\mathrm{prod}(s,t;k),1]\right) \end{aligned}$$

□

4.7.8 Corollary. *Given a $\mathbb{Z}[W]$-module M, the induced map*

$$f^* : \mathrm{Hom}_{\mathbb{Z}[W]}(P_\bullet, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[W]}(\mathcal{CS}_\bullet, M)$$

on cochains is given in degree two by

$$\begin{aligned} f_2^*(\phi)([\{s\} \supseteq \{s\}]) &= \phi(s,s) - \phi(s,1) + \phi(1,s) - \phi(1,1) && s \in S \\ f_2^*(\phi)([\{s,t\}]) &= \sum_{k=0}^{m(s,t)-1} (-1)^k \left(\phi(\mathrm{prod}(t,s;k),s) - \phi(\mathrm{prod}(t,s;k),1)\right. \\ &\qquad \left. -\phi(\mathrm{prod}(s,t;k),t) + \phi(\mathrm{prod}(s,t;k),1)\right) && s,t \in S,\ s<t \end{aligned} \tag{4.7.9}$$

In particular, if the 2-cocycle ϕ satisfies

$$\forall w,w' \in W \ \ \ell(ww') = \ell(w) + \ell(w') \quad \Rightarrow \phi(w,w') = 1 \tag{4.7.10}$$

then $f^*(\phi)$ is given by

$$\begin{aligned} f_2^*(\phi)([\{s\} \supseteq \{s\}]) &= \phi(s,s) \\ f_2^*(\phi)([\{s,t\}]) &= 0 \qquad s,t \in S,\ s<t \end{aligned} \tag{4.7.11}$$

Proof. Equation (4.7.9) follows directly from lemma 4.7.7 and the definitions. To see that eq. (4.7.11) holds, note first that ϕ is normalized, i.e. $\phi(1,w) = 0 = \phi(w,1)$ for all $w \in W$; this follows immediately from eq. (4.7.10). Moreover,

$$\ell(\mathrm{prod}(s,t;k)\cdot t) = k+1 = \ell(\mathrm{prod}(t,s;k)\cdot s) \quad \forall\, 0 \le k \le m(s,t)-1$$

and therefore

$$\phi(\mathrm{prod}(s,t;k),t) = 0 = \phi(\mathrm{prod}(t,s;k),s) \quad \forall\, 0 \le k \le m(s,t)-1$$

Equation (4.7.9) then follows. □

4.7.9 Corollary. *Let* $(X, \Phi, X^\vee, \Phi^\vee)$ *be a root datum and* W_0 *its Weyl group. Fix a root basis* $\Delta \subseteq \Phi$ *and a total ordering* $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ *of the simple roots, and let* $(W_0, S) = (W_0, \{s_{\alpha_1}, \dots, s_{\alpha_\ell}\})$ *be the corresponding Coxeter group and* $\mathcal{CS}$ *the DeConcini-Salvetti complex associated to* (W_0, S) *and the total ordering of* S.

The 2*-cocycle* $f_2^*(\phi_u) \in Z^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2) \subseteq \mathrm{Hom}_{\mathbb{Z}[W_0]}(\mathcal{CS}_2, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ *induced by the standard* 2*-cocycle* ϕ_u *defined in eq.* (4.1.3) *is given by*

$$\begin{aligned} f_2^*(\phi_u)([\{s_\alpha\} \supseteq \{s_\alpha\}]) &= \alpha^\vee \otimes 1 \qquad && \alpha \in \Delta \\ f_2^*(\phi_u)([\{s_\alpha, s_\beta\}]) &= 0 && \alpha, \beta \in \Delta,\ \alpha < \beta \end{aligned} \tag{4.7.12}$$

Proof. Follows immediately from corollary 4.7.8—noting that eq. (4.7.10) is satisfied by remark 1.7.2—by remark 4.1.2 and the fact that

$$\mathbb{X}(s_\alpha, s_\alpha) = s_\alpha \quad \forall \alpha \in \Delta$$

by definition of $\mathbb{X}$ (see definition 1.7.1). □

4.7.10 Corollary. *Given a basic frugal morphism (see definition 4.5.4)*

$$\varphi : (X_1, \Phi_1, X_1^\vee, \Phi_1^\vee, \Delta_1) \longrightarrow (X_2, \Phi_2, X_2^\vee, \Phi_2^\vee, \Delta_2)$$

between based root data, the map

$$(\varphi_0, \varphi^\vee)^* : H^2(W_0(R_2), X_2^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2) \longrightarrow H^2(W_0(R_1), X_1^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2) \tag{4.7.13}$$

induced by the pair $(\varphi_0, \varphi^\vee)$ *(belonging to the category* $\mathcal{D}$ *defined in remark 4.4.5) preserves the canonical classes* $[\phi_{u,i}] \in H^2(W_0(\mathcal{R}_i), X_i^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$*. Here,*

$$\varphi_0 : W_0(\mathcal{R}_1) \longrightarrow W_0(\mathcal{R}_2)$$

denotes the morphism defined in lemma 4.5.2 and $\varphi^\vee : X_2^\vee \to X_1^\vee$ *denotes the adjoint of* $\varphi : X_1 \to X_2$.

Proof. First, note that by lemma 4.5.2 the pair $(\varphi_0, \varphi^\vee)$ really belongs to the category $\mathcal{D}$, i.e. $\varphi^\vee$ is equivariant with respect to φ_0. Moreover, by lemma 4.5.5 the morphism φ_0 is *isometric*, i.e. preserves the length functions of the Coxeter groups. Also, the restriction

$$\varphi| : \Phi_1 \longrightarrow \Phi_2$$

is injective by lemma 4.5.2, hence we may chose a total ordering of Δ_2 and endow Δ_1 with the ordering induced by identifying it with the subset $\varphi(\Delta_1) \subseteq \Delta_2$. With these ordering, the maps φ_0 is hence a monotonous isometric morphism of Coxeter groups, and by corollary 4.7.5 the map eq. (4.7.13) is explicitly described in terms of cochains of the DeConcini-Salvetti-complexes. Moreover, a (canonical) DeConcini-Salvetti-cochain representing the class $[\phi_u]$ is explicitly described in corollary 4.7.9. It therefore follows that the image of $[\phi_{u,2}]$ under eq. (4.7.13) is represented by the cocycle f determined by (where $\alpha, \beta \in \Delta_1$)

$$\begin{aligned} f([\{s_\alpha\} \supseteq \{s_\alpha\}]) &= (\varphi^\vee \otimes \mathrm{id})(\phi_{u,2}([\{\varphi_0(s_\alpha)\} \supseteq \{\varphi_0(s_\alpha)\}])) \\ &= (\varphi^\vee \otimes \mathrm{id})(\phi_{u,2}([\{s_{\varphi(\alpha)}\} \supseteq \{s_{\varphi(\alpha)}\}]) \\ &= (\varphi^\vee \otimes \mathrm{id})(\varphi(\alpha)^\vee \otimes 1) \\ &= (\varphi^\vee(\varphi(\alpha)^\vee) \otimes 1 \\ &= \alpha^\vee \otimes 1 = \phi_{u,1}([\{s_\alpha\} \supseteq \{s_\alpha\}]) \\ f([\{s_\alpha, s_\beta\}]) &= (\varphi^\vee \otimes \mathrm{id})(\phi_{u,2}([\{\varphi_0(s_\alpha), \varphi_0(s_\beta)\}])) \\ &= (\varphi^\vee \otimes \mathrm{id})(\phi_{u,2}([\{s_{\varphi(\alpha)}, s_{\varphi(\beta)}\}]) \\ &= (\varphi^\vee \otimes \mathrm{id})(0) \\ &= 0 = \phi_{u,1}([\{s_\alpha, s_\beta\}]) \end{aligned}$$

where we have used the equality $\varphi^\vee(\varphi(\alpha)^\vee) = \alpha^\vee$ which holds since φ is frugal (see eq. (4.5.2)). Therefore it follows that $f = \phi_{u,1}$, in particular $[f] = [\phi_{u,1}]$, and hence the claim follows. □

4.8 Discussion of the computational results

We have computed the cohomology groups $H^k(W_0, X^\vee)$ and $H^k(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2) = H^k(W_0, \overline{X^\vee})$ for $k = 0, 1, 2, 3$ and all sublattices $Q^\vee \subseteq X^\vee \subseteq P^\vee$ of all irreducible reduced root systems of rank $\ell \leq 8$. We have also determined, for each such $X^\vee$, whether the class $[\phi_u] \in H^2(W_0, \overline{X^\vee})$ vanishes, and—if it doesn't—whether or not this class lies in the image of the comparison map

$$\mathrm{comp}_2 : H^2(W_0, X^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_2 \hookrightarrow H^2(W_0, \overline{X^\vee})$$

coming from the Künneth theorem (see eq. (4.2.5)). The detailed results of these computations are given in the appendix (appendix A).

Let us discuss some conclusions to draw from these computations (in particular, what these imply for the question of the splitness of eq. (4.0.1)). First of all, it turned out that in all examples computed, the class $[\phi_u]$ never lies in the image of comp_2 unless it is already zero. In particular:

4.8.1 Observation. For the lattices $X^\vee$ considered in appendix A, the splitting of eq. (4.0.1) is independent of the ring k. That is, this sequence either splits for all rings, or it never splits except in the trivial case where $2 = 0$ in k.

Proof. This follows from eq. (4.2.6). □

That the splitness of eq. (4.0.1) turns out to be independent of the ground ring k may not be surprising *morally*, given the rigid behaviour of semisimple groups in general. But, this is far from being obvious! Indeed, before the final resolution of this question for all fields by Adams and He in [AH17], this question had been decided separately in the case of fields of positive characteristic by Galt in several articles ([Gal15], [Gal14], [Gal17a], [Gal17b]) using delicate computations.

4.8.2 Observation. We have

$$H^1(W_0, P^\vee) = 0 \tag{4.8.1}$$

for all irreducible reduced root systems of rank $\ell \le 8$, except for A_1 and the ones of type B (for which always $H^1(W_0, P^\vee) \approx \mathbb{Z}/2\mathbb{Z})$).

In particular for the former root systems, eq. (4.8.1) implies that for every sublattice $Q^\vee \subseteq X^\vee \subseteq P^\vee$, the long exact sequence in cohomology induced by the short exact sequence

$$0 \longrightarrow X^\vee \longrightarrow P^\vee \longrightarrow P^\vee/X^\vee \longrightarrow 0$$

gives a canonical isomorphism

$$P^\vee/X^\vee \xrightarrow{\sim} H^1(W_0, X^\vee)$$

Regarding the question of the splitness of eq. (4.0.1), i.e. the vanishing of the class $[\phi_u]$ we find in accordance with [AH17] that

4.8.3 Observation. For simple root systems up to rank 8, the class $[\phi_u] \in H^2(W_0, \overline{X^\vee})$ vanishes precisely in the following cases (where $\mathbf{G}$ denotes the corresponding split almost-simple semisimple group):

(i) For type A_ℓ ($\ell \ge 1$), when the order of $P^\vee/X^\vee$ (isomorphic to the center of $\mathbf{G}$) is odd or if $\ell = 3$ and $X^\vee = Q^\vee + \Omega$ for the unique subgroup $\Omega \le P^\vee/Q^\vee \approx \mathbb{Z}/4\mathbb{Z}$ of order 2 (corresponding to $\mathbf{G} = \mathbf{SL}_4/\{\pm\}$).

(ii) For type B_ℓ ($\ell \ge 2$), when $X^\vee = P^\vee$ (corresponding to $\mathbf{G} = \mathbf{SO}(2\ell + 1)$).

(iii) For type C_ℓ ($\ell \ge 2$), when $\ell = 2$ and $X^\vee = P^\vee$ (corresponding to $\mathbf{G} = \mathbf{PSp}_4$).

(iv) For type D_ℓ ($\ell \ge 3$), when ℓ is odd and $X^\vee = Q^\vee + \Omega$ for a nonzero subgroup

$$\Omega \le P^\vee/Q^\vee = \left\langle \overline{\Lambda_\ell^\vee} \right\rangle \simeq \mathbb{Z}/4\mathbb{Z}$$

(corresponding to $\mathbf{G} = \mathbf{SO}(2\ell)$ and $\mathbf{G} = \mathbf{PSO}(2\ell)$). Or, when ℓ is even, and $X^\vee = Q^\vee + \Omega$ for a subgroup

$$\Omega \le P^\vee/Q^\vee = \left\langle \overline{\Lambda_{\ell-1}^\vee}, \overline{\Lambda_\ell^\vee} \right\rangle \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

of the form $\Omega = \left\langle \overline{\Lambda_{\ell-1}^\vee} \right\rangle$ or $\Omega = \left\langle \overline{\Lambda_\ell^\vee} \right\rangle$ (both corresponding to $\mathbf{G} = \mathbf{SO}(2\ell)$) or of the form $\Omega = P^\vee/Q^\vee$ (corresponding to $\mathbf{G} = \mathbf{PSO}(2\ell)$).

Moreover, when $\ell = 4$ also when $X^\vee = Q^\vee + \Omega$ with $\Omega = \left\langle \overline{\Lambda_3^\vee + \Lambda_4^\vee} \right\rangle$ (corresponding to $\mathbf{G} = \mathbf{Semispin}(8)$), because in this case all proper sublattices $Q^\vee \subsetneq X^\vee \subsetneq P^\vee$ are conjugate under the action of the automorphism group of the Dykin diagram (corresponding to the isomorphism $\mathbf{SO}(8) \simeq \mathbf{Semispin}(8)$ provided by *triality*).

(v) For type G_2.

Unfortunately, computations can only ever a finite number of cases, but there are infinitely many irreducible root systems. The question therefore becomes, can we infer any information about an infinite number of cases from finitely many?

To answer this question, let's follow the time-honored tradition of putting things you don't understand into tables, and let's group together all the first cohomology groups $H^1(W_0, Q^\vee)$ of the coroot lattices of the root systems X_ℓ, $X \in \{A, B, C, D\}$, $1 \le \ell \le 8$:

ℓ	**A**	**B**	**C**	**D**
1	2	–	–	–
2	3	2	2	–
3	4	2	2	4
4	5	2	2	2, 2
5	6	2	2	4
6	7	2	2	2, 2
7	8	2	2	4
8	9	2	2	2, 2

Table 2: The invariants of $H^1(W_0, Q^\vee)$ for the classical root systems.

Here we list in each row the invariants of the finite abelian groups $H^1(W_0, Q^\vee)$, i.e. the entry 2, 2 stands for the group $\mathbb{Z}2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Clearly, all the columns except the first one exhibit *periodicity.* Let's now look at the first cohomology groups of the reductions $\overline{Q^\vee} = Q^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2$, describing now each group via its dimension as an $\mathbb{F}_2$-vector space:

ℓ	**A**	**B**	**C**	**D**
1	1	–	–	–
2	0	2	2	–
3	2	3	3	2
4	0	2	3	4
5	2	3	3	2
6	0	2	3	4
7	2	3	3	2
8	0	2	3	4

Table 3: The $\mathbb{F}_2$-dimensions of $H^1(W_0, \overline{Q^\vee})$ for the classical root systems.

This time, we see that *all* columns exhibit periodicity (with period at most two) *eventually.* Moreover, the same periodicity can also be observed for the higher cohomology groups:

ℓ	**A**	**B**	**C**	**D**
1	1	–	–	–
2	0	3	3	–
3	2	5	6	2

ℓ	**A**	**B**	**C**	**D**
4	0	4	7	8
5	3	7	8	4
6	0	4	8	8
7	3	7	8	4
8	0	4	8	8

Table 4: The $\mathbb{F}_2$-dimensions of $H^2(W_0, \overline{Q^\vee})$ for the classical root systems.

ℓ	**A**	**B**	**C**	**D**
1	1	–	–	–
2	0	4	4	–
3	3	8	10	3
4	0	9	14	17
5	5	15	18	8
6	0	10	19	19
7	6	17	20	10
8	0	10	20	19

Table 5: The $\mathbb{F}_2$-dimensions of $H^3(W_0, \overline{Q^\vee})$ for the classical root systems.

At least for the family A_ℓ, this periodicity phenomenon is explained by the theory of **FI**-modules and the following theorem of Nagpal and Snowden:

4.8.4 Theorem. *[Let M be a finitely generated* **FI***-module over a field k of characteristic p.] Suppose that M is generated in degrees $\leq g$ with relations in degrees $\leq r$ and has degree δ. Let q be the smallest power of p such that $\delta < p$. Then*

$$\dim_k H^t(S_n, M_n) = \dim_k H^t(S_{n+q}, M_{n+q})$$

holds for all $n \geq \max(g + r, 2t + \delta)$.

([NS18, Theorem 1.6])

In proposition 4.6.14, we have shown that the root and coroot lattices of the families A_ℓ, B_ℓ, C_ℓ, D_ℓ naturally form finitely generated $\mathbf{FI}_W$-modules. Moreover, we have shown that the coroot lattices of A_ℓ form a finitely presented **FI**-module generated in degrees ≤ 2 with relations in degrees ≤ 2 having degree $\delta = 1$. Therefore, we have shown that

4.8.5 Theorem. *The dimension*

$$d_k(\ell) := \dim_{\mathbb{F}_2} H^k(S_{\ell+1}, Q_\ell^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$$

of the first cohomology group of the mod 2 reduction of the coroot lattice $Q_\ell^\vee$ of the root system A_ℓ is given in degrees $k = 1$ by

$$d_1(\ell) = \begin{cases} 1 & \textit{if } \ell = 1 \\ 0 & \textit{if } \ell \geq 2, \textit{ and } \ell \textit{ even} \\ 2 & \textit{if } \ell \geq 2, \textit{ and } \ell \textit{ odd} \end{cases}$$

in degree $k = 2$ by

$$d_2(\ell) = \begin{cases} 1 & \textit{if } \ell = 1 \\ 0 & \textit{if } \ell = 2 \\ 2 & \textit{if } \ell = 3 \\ 0 & \textit{if } \ell \geq 4, \textit{ and } \ell \textit{ even} \\ 3 & \textit{if } \ell \geq 4, \textit{ and } \ell \textit{ odd} \end{cases}$$

and in degree $k = 3$ by

$$d_3(\ell) = \begin{cases} 1 & \textit{if } \ell = 1 \\ 0 & \textit{if } \ell = 2 \\ 3 & \textit{if } \ell = 3 \\ 0 & \textit{if } \ell = 4 \\ 5 & \textit{if } \ell = 5 \\ 0 & \textit{if } \ell \geq 6, \textit{ and } \ell \textit{ even} \\ 6 & \textit{if } \ell \geq 6, \textit{ and } \ell \textit{ odd} \end{cases}$$

Proof. The bound of theorem 4.8.4 in this case is ($t = k$ in our notation)

$$\max(g + r, 2k + \delta) = \max(2 + 2, 2k + 1) = 2k + 1$$

and $1 = \delta < p = 2$. Therefore $d_k(\ell)$ satisfies

$$d_k(\ell) = d_k(\ell + 2) \quad \text{for } \ell + 1 \geq 2k + 1$$

The claim then follows from table 3, table 4, and table 5. Note, that in the cases $k = 2, 3$ the bound provided by the theorem of Nagpal and Snowden is *optimal.* □

From the theorem, the following corollary follows formally.

4.8.6 Corollary. *For $k = 1, 2, 3$ and ℓ even, the group*

$$H^k(S_{\ell+1}, Q_\ell^\vee)$$

has no 2-torsion.

Proof. By the Künneth theorem, for every prime p we have a natural identification

$$H^k(S_{\ell+1}, Q_\ell^\vee)_{(p)} \simeq H^k(S_{\ell+1}, Q_{\ell,(p)}^\vee)$$

Moreover, again by the Künneth theorem, we have a natural injection

$$H^k(S_{\ell+1}, Q_{\ell,(p)}^\vee) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}/p\mathbb{Z} \simeq H^k(S_{\ell+1}, Q_\ell^\vee \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})$$

Since the $\mathbb{Z}_{(p)}$-module $H^k(S_{\ell+1}, Q_{\ell,(p)}^\vee)$ is finitely generated (as $S_{\ell+1}$ is finite and $Q_\ell^\vee$ is a finitely generated $\mathbb{Z}$-module), it follows from Nakayama's lemma that

$$H^k(S_{\ell+1}, Q_\ell^\vee \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}) = 0 \quad \Rightarrow \quad H^k(S_{\ell+1}, Q_{\ell,(p)}^\vee) = 0$$

Since we know from theorem 4.8.5 that the left hand side vanishes for $p = 2$ and ℓ even, the claim follows. □

What about the other families? Unfortunately, there is (yet) no analogue of theorem 4.8.4 for $\mathbf{FI}_W$-modules of the other types, even though table 3, table 4, and table 5 highly suggest that it should exist.

Also, what about the other lattices $X^\vee$, especially the coweight lattice $P^\vee$? We know from lemma 4.6.5 that the coweight lattices of the classical families—being the duals of the *root* lattices—form $\mathbf{FI}_W^{\mathrm{op}}$-modules. Furthermore, the first cohomology group of the coweight lattices clearly exhibits the same periodicity:

ℓ	**A**	**B**	**C**	**D**
1	1	–	–	–
2	0	2	2	–

ℓ	**A**	**B**	**C**	**D**
3	1	3	1	1
4	0	3	1	2
5	0	3	0	0
6	0	3	0	0
7	0	3	0	0
8	0	3	0	0

Table 6: The $\mathbb{F}_2$-dimensions of $H^1(W_0, \overline{P^\vee})$ for the classical root systems.

In the second and third cohomology groups, the periodicity is still discernible even though less clearly.

ℓ	**A**	**B**	**C**	**D**
1	1	–	–	–
2	0	3	3	–
3	2	6	3	2
4	0	7	4	7
5	1	8	2	2
6	0	8	2	3
7	0	8	1	1
8	0	8	1	1

Table 7: The $\mathbb{F}_2$-dimensions of $H^2(W_0, \overline{P^\vee})$ for the classical root systems.

ℓ	**A**	**B**	**C**	**D**
1	1	–	–	–
2	0	4	4	–
3	2	10	5	2
4	0	14	7	11
5	2	18	6	4
6	0	19	6	8
7	1	20	4	3
8	0	20	4	4

Table 8: The $\mathbb{F}_2$-dimensions of $H^3(W_0, \overline{P^\vee})$ for the classical root systems.

Unfortunately, there is also no analogue of theorem 4.8.4 for $\mathbf{FI}^{\mathrm{op}}$-modules, let alone $\mathbf{FI}_W^{\mathrm{op}}$-modules. Moreover—now coming back to our original question of the splitness of eq. (4.0.1)—even theorem 4.8.4 is clearly insufficient insofar as it only show that certain cohomology groups are isomorphic without providing an actual isomorphism. In particular, we can't hope to use theorem 4.8.4 to establish the vanishing/non-vanishing of the class $[\phi_u]$, despite the fact that not only the cohomology groups $H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ clearly become periodic but also the coefficients of the class $[\phi_u]$ expressed as a linear combination of the generating cocycles given in appendix A.

Given the naturality of $[\phi_u]$ under basic frugal morphisms (corollary 4.7.10), we therefore conjecture the following:

4.8.7 Conjecture. *Given a type $X \in \{A, B, C, D\}$, let $\varphi_\ell : Q_\ell \hookrightarrow Q_{\ell+1}$ denote the canonical embedding between the root lattices of the root systems X_ℓ and $X_{\ell+1}$. Then, for types A, B, and D, the restriction map*

$$H^2(W_0, Q^\vee_{\ell+2} \otimes_{\mathbb{Z}} \mathbb{F}_2) \longrightarrow H^2(W_0, Q^\vee_\ell \otimes_{\mathbb{Z}} \mathbb{F}_2)$$

(induced by the composition $\varphi^\vee_\ell \circ \varphi^\vee_{\ell+1} : Q^\vee_{\ell+2} \to Q^\vee_\ell$) is an isomorphism for ℓ sufficiently large.

Furthermore, for type C the restriction map

$$H^2(W_0, Q^\vee_{\ell+1} \otimes_{\mathbb{Z}} \mathbb{F}_2) \longrightarrow H^2(W_0, Q^\vee_\ell \otimes_{\mathbb{Z}} \mathbb{F}_2)$$

(induced by $\varphi^\vee_\ell : Q^\vee_{\ell+1} \to Q^\vee_\ell$) is an isomorphism for ℓ sufficiently large.

This conjecture gains further plausibility by the fact that the *theorem of Nakaoka* ([Nak60]), which is cited in the beginning of [NS18], states that the restriction map

$$H^k(S_{\ell+1}, A) \longrightarrow H^k(S_\ell, A)$$

(for A a finite abelian group with trivial action) is an isomorphism of sufficiently large ℓ. Moreover, despite the fact that the periodicity result of Nagpal and Snowden provides no explicit isomorphism, their proofs do involve a specific *connection* ∇ defined in terms of the restriction map (cf. [NS18, Proposition 4.7]). This question is also acknowledged by Nagpal and Snowden in their introduction [NS18, 1.6] (despite their claim that their main theorem [NS18, Theorem 1.2] generalizes Nakaoka's theorem).

A Computational Results

A.1 User's guide

Before listing the computational results, let us explain how they can be reproduced as well as the form in which they are presented.

A.1.1 Reproducing the results

For as long as entropy allows, the most convenient way to reproduce the computational results is by downloading the software from the author's git repository hosted on GitHub.com, by running

```
git clone https://github.com/mr-infty/crd.git
cd crd
```

If that fails, I'm afraid you have to type in the listings in appendix B by hand. In any case, once you have all necessary files, you should edit the file `Makefile` and in the line

```
SAGE = /Applications/SageMath/sage
```

replace `/Applications/SageMath/sage` with the path to your `Sage` executable (which you should have already installed); for example, if `Sage` is globally available from your command line, then

```
SAGE = sage
```

should work. Once you have edited `Makefile` correctly, you should run

```
make compute
```

in the command line, wait one day, and then have the results as LaTeX files in your working directory (they follow the naming scheme `cohomology_of_<X>_<l>.tex`). You can include these files in another LaTeX files using the command `\include{<name-of-the-file>}`, but make sure that you have imported the packages `tabu`, `longtable`, `bm`, `float`, `tikz`, and `booktabs`. Also, you should have defined LaTeX macros `\Z` and `\F` producing the symbols $\mathbb{Z}$ and $\mathbb{F}$.

If the computation fails, then you probably have a different version of Sage installed (I tested it with versions 8.1 and 8.2). In this case, please feel free to open a new issue on my GitHub page or write me an email at `zero@fromzerotoinfinity.xyz`.

A.1.2 Description of root systems

For a simple root systems X_ℓ, where X is any of the letters A, B, C, D, E, F, G and $1 \leq \ell \leq 8$ denotes the rank (= the number of simple roots) of the root system, there is a separate section named 'Root system X_ℓ'. The beginning of this section describes the Dynkin diagram of this root system—together with a labelling of its nodes by positive integers—and its fundamental group $P^\vee/Q^\vee$, which is the quotient of the coweight lattice $P^\vee$ by the root lattice $Q^\vee$.[32]

The nodes of the Dynkin diagram are in bijection with a set Δ of simple roots of the root system, and the labelling of the Dynkin diagram therefore gives a numbering $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the simple roots. The dual basis $\Lambda_1^\vee, \ldots, \Lambda_\ell^\vee$ of the simple roots (determined by $\Lambda_i^\vee(\alpha_j) = \delta_{i,j}$) gives a $\mathbb{Z}$-basis of $P^\vee$, and a set of generators of $P^\vee/Q^\vee$ is described in terms of these 'fundamental coweights' $\Lambda_i^\vee$. More precisely, if it is written that

$$\begin{aligned} P^\vee/Q^\vee &\simeq \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \mathbb{Z}/d_m\mathbb{Z} \\ &\text{generated by } x_1, \ldots, x_m \in P^\vee \mod Q^\vee \end{aligned}$$

this should be interpreted as saying that the map

$$\bigoplus_{i=1}^m \mathbb{Z}e_i \longrightarrow P^\vee, \quad e_i \mapsto x_i$$

induces an isomorphism

$$(\bigoplus_{i=1}^m \mathbb{Z}e_i)/(\bigoplus_{i=1}^m d_i\mathbb{Z}e_i) \xrightarrow{\sim} P^\vee/Q^\vee$$

A.1.3 Description of the cohomology groups of sublattices $X^\vee$

After the description of the Dynkin diagram and the fundamental group of this root system follow the computations of the cohomology groups for all the sublattices $Q^\vee \subseteq X^\vee \subseteq P^\vee$ as well as for the trivial coefficients $\mathbb{Z}$ and $\mathbb{F}_2$, given in separate subsections.

The coroot lattice $X^\vee = Q^\vee$ and the coweight lattice $X^\vee = P^\vee$ are referred to by name, the other sublattices $X^\vee$ are denoted by the subgroups Ω of the fundamental groups to which they correspond under the bijection

$$\begin{aligned} \{Q^\vee \subseteq X^\vee \subseteq \text{ sublattice}\} &\xrightarrow{\sim} \{\text{subgroups } \Omega \leq P^\vee/Q^\vee\} \\ X^\vee &\longmapsto \Omega = X^\vee/Q^\vee \end{aligned}$$

These subgroups are denoted by the coefficients of their generators in the given generators $x_1, \ldots, x_m$ of $P^\vee/Q^\vee$, i.e.

$$\Omega = \langle(\mu_{1,1}, \ldots, \mu_{1,m}),\ \ldots,\ (\mu_{n,1}, \ldots, \mu_{n,m})\rangle$$

denotes the subgroup generated by the elements $\sum_j \mu_{i,j}x_j$ $(i = 1, \ldots, n)$.

For every sublattice $X^\vee$, the corresponding subsection describes the cocycle ϕ_u (defined in remark 4.1.2), the cohomology groups $H^k(W_0, X^\vee)$ and $H^k(W_0, \overline{X^\vee})$ for $k = 0, 1, 2, 3$, and the comparison map

$$\mathrm{comp}_k : H^k(W_0, X^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_2 \hookrightarrow H^k(W_0, \overline{X^\vee})$$

coming from the Künneth sequence (see eq. (4.2.5) for the case $k = 2$). Here (and later) we use the abbreviation

$$\overline{X^\vee} := X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2$$

More precisely, the integral cohomology groups $H^k(W_0, X^\vee)$ are described by tables with three columns, where the first column contains the integer k, the second column contains an abelian group isomorphic to $H^k(W_0, X^\vee)$, written in the form $\mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \mathbb{Z}/d_n\mathbb{Z}$, and the third column contains a list of cocycles whose classes generate $H^k(W_0, X^\vee)$ and which corresponds with the second column in the sense that, if $\phi_1, \ldots, \phi_n$ is the list of these cocycles (from top to bottom), then the map

$$\begin{aligned} \bigoplus_{i=1}^n (\mathbb{Z}/d_i\mathbb{Z})e_i &\longrightarrow H^k(W_0, X^\vee) \\ e_i &\longmapsto [\phi_i] \end{aligned}$$

[32] see [Bou07, Ch. VI, §1.9], where they are denoted $P(R^\vee)$ and $Q(R^\vee)$, for the definitions of $P^\vee$ and $Q^\vee$

is a well-defined isomorphism of abelian groups. More precisely, the cocycles given in the third column are cocycles (of degree k) of the cochain complex

$$\mathrm{Hom}_{\mathbb{Z}[W_0]}(\mathcal{CS}_\bullet, X^\vee)$$

induced by the DeConcini-Salvetti resolution $\mathcal{CS}_\bullet$ (see definition 4.7.1) of the Weyl group W_0 of X_ℓ for the total ordering[33] of the simple reflections

$$S = \{s_{\alpha_1}, \dots, s_{\alpha_\ell}\} \subseteq W_0$$

induced by the numbering of the simple roots given by the Dynkin diagram (i.e. $s_{\alpha_i} < s_{\alpha_j}$ iff $i < j$). Recall that $\mathcal{CS}_k$ is the free $\mathbb{Z}[W_0]$-module over the set $\mathcal{F}_k$ of flags $\Gamma = (\Gamma_i)_{i\geq 1}$ in S of cardinality k, i.e.

$$S \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots, \quad \text{and} \quad \sum_{i\geq 1} \#\Gamma_i = k$$

Consequently, it follows that

$$\mathrm{Hom}_{\mathbb{Z}[W_0]}(\mathcal{CS}_\bullet, X^\vee)_k = \mathrm{Hom}_{\mathbb{Z}[W_0]}(\mathcal{CS}_k, X^\vee) \simeq \mathrm{Hom}_{\mathrm{Set}}(\mathcal{F}_k, X^\vee)$$

i.e. degree k cochains identify with maps of sets $\mathcal{F}_k \longrightarrow X^\vee$. Since $\mathcal{F}_k$ is finite, we can also identify degree k cochains also with formal sums

$$\sum_{\Gamma\in\mathcal{F}_k} a_\Gamma[\Gamma], \quad a_\Gamma \in X^\vee$$

the corresponding map of sets being $\Gamma \mapsto a_\Gamma$. We use this identification to denote the generating cochains $\phi_i \in \mathrm{Hom}_{\mathbb{Z}[W_0]}(\mathcal{CS}_\bullet, X^\vee)_k$ by such formal sum, where we use an abbreviated notation for the flags Γ. For example,

$$4\Lambda_3^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + (-4\Lambda_3^\vee)\,[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$$

would denote the cochain of degree 3 given by the formal sum

$$x[\Gamma] + x'[\Gamma']$$

where $x, x' \in X^\vee$ are the elements given in terms of the basis $\Lambda_1^\vee, \dots, \Lambda_\ell^\vee$ of $P^\vee \supseteq X^\vee$ by

$$x = 4\Lambda_3^\vee, \quad x' = -4\Lambda_3^\vee$$

and where $\Gamma, \Gamma' \in \mathcal{F}_3$ are the flags

$$\begin{aligned} \Gamma: \quad & \Gamma_1 = \{s_{\alpha_1}, s_{\alpha_2}\} \ \supseteq\ \Gamma_2 = \{s_{\alpha_1}\} \ \supseteq\ \Gamma_3 = \emptyset \ \supseteq \dots \\ \Gamma': \quad & \Gamma_1' = \{s_{\alpha_1}, s_{\alpha_2}\} \ \supseteq\ \Gamma_2' = \{s_{\alpha_2}\} \ \supseteq\ \Gamma_3' = \emptyset \ \supseteq \dots \end{aligned}$$

The cohomology groups $H^k(W_0, \overline{X}^\vee)$ of the reduction $\overline{X^\vee} = X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2$ are described in a similar vein, using a table of three columns, but there are some small differences. First, since $H^k(W_0, \overline{X}^\vee)$ is an $\mathbb{F}_2$-vector space, it suffices to denote its dimension (in the second column), which we abbreviate as

$$h^k(\overline{X^\vee}) := \dim_{\mathbb{F}_2} H^k(W_0, \overline{X^\vee})$$

In the third column, we again give a list of generating cocycles denoted using the abbreviated formal sum notation $x_1[\Gamma_1] + x_2[\Gamma_2] + \dots$. However, the expressions for the 'coefficients' x_i strictly speaking don't denote elements of $\overline{X^\vee}$ but rather elements of $X^\vee$: this is of course to be understood as defining an element of $\overline{X^\vee}$ by specifying a lift under the projection $\mathrm{pr}: X^\vee \twoheadrightarrow \overline{X^\vee}$. For example, the expression

$$(2\Lambda_1^\vee + \Lambda_2^\vee)[1, 3, 5 \supseteq 3] + \Lambda_3^\vee[2, 3 \supseteq 2, 3]$$

would denote the degree 4 cochain given by $x[\Gamma] + x'[\Gamma']$ with

$$x = \mathrm{pr}(2\Lambda_1^\vee + \Lambda_2^\vee), \quad x' = \mathrm{pr}(\Lambda_3^\vee)$$

and

$$\begin{aligned} \Gamma: \quad & \Gamma_1 = \{s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_5}\} \ \supseteq\ \Gamma_2 = \{s_{\alpha_3}\} \ \supseteq\ \Gamma_3 = \emptyset \ \supseteq \dots \\ \Gamma': \quad & \Gamma_1' = \{s_{\alpha_2}, s_{\alpha_3}\} \ \supseteq\ \Gamma_2' = \{s_{\alpha_2}, s_{\alpha_3}\} \ \supseteq\ \Gamma_3' = \emptyset \ \supseteq \dots \end{aligned}$$

Beware here that although pr is the reduction modulo two, we cannot conclude that

$$\mathrm{pr}(2\Lambda_1^\vee + \Lambda_2^\vee) = \mathrm{pr}(\Lambda_2^\vee)$$

because the summands in the expression $2\Lambda_1^\vee + \Lambda_2^\vee$ don't have to lie in $X^\vee$.

[33]Recall from section 4.7 that the DeConcini-Salvetti resolution depends on a Coxeter group (W, S) as well as a choice of a total ordering on S.

A.1.4 Description of the comparison maps $\mathrm{comp}_k : H^k(W_0, X^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_2 \hookrightarrow H^k(W_0, \overline{X^\vee})$

This concludes the description of the form in which the cohomology groups $H^k(W_0, X^\vee)$ and $H^k(W_0, \overline{X^\vee})$ are presented in each subsection. The comparison map

$$\mathrm{comp}_k : H^k(W_0, X^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_2 \hookrightarrow H^k(W_0, \overline{X^\vee})$$

is described at the end of each subsection, by exhibiting the matrix of comp_k relative to the lists of generating cocycles of $H^k(W_0, X^\vee)$ and $H^k(W_0, \overline{X^\vee})$ that are given.

More precisely, if $\phi_1, \ldots, \phi_n$ is the list of generators of $H^k(W_0, X^\vee)$ and $\psi_1, \ldots, \psi_m$ is the list of generators of $H^k(W_0, \overline{X^\vee})$, then comp_k is described by the matrix A with m rows and n columns determined by

$$\mathrm{comp}_k(\phi_i) = \sum_{j=1}^{m} A_{j,i} \psi_j$$

A.1.5 Description of the cocycle $\phi_u \in Z^2(W_0, \overline{X^\vee})$

The cocycle ϕ_u is described in the beginning of each subsection, in the following way. If the cocycle ϕ_u happens to vanish (this can happen), this is indicated. If $\phi_u \neq 0$ but its class $[\phi_u] = 0$ vanishes, then a cochain τ of degree 1 exhibiting ϕ_u as its coboundary

$$\phi_u = \partial \tau$$

is described (using the conventions for describing cochains detailed above).

Finally, if $[\phi_u] \neq 0$, then the class $[\phi_u] \in H^2(W_0, \overline{X^\vee})$ is described in terms of its basis expansion relative to the generating cocycles $\psi_1, \ldots, \psi_m$ of $H^2(W_0, \overline{X^\vee})$ described earlier. For example,

$$[\phi_u] = (1, 1, 1, 0, 1)$$

would mean that

$$[\phi_u] = \psi_1 + \psi_2 + \psi_3 + \psi_5$$

Moreover whenever $[\phi_u] \neq 0$, is is explicitly stated whether this class lies in the image of the comparison map comp_2 or not, and if it does[34] a preimage in $H^2(W_0, X^\vee)$ is described.

A.1.6 Description of the cohomology of W_0 with trivial coefficients

Lastly, after the subsections describing the cohomology groups (and ϕ_u and comp_k) for all sublattices $X^\vee$, follows a subsection that describes the cohomology groups $H^k(W_0, \mathbb{Z})$ and $H^k(W_0, \mathbb{F}_2)$, using the same notations conventions as for the groups $H^k(W_0, X^\vee)$.

A.1.7 Redundancy in the computational results

The computational results presented below contain some redundance that we have decided to keep for convenience' sake.

First, since the computation of the zeroth cohomology group is elementary (for $H^k(W_0, \overline{X^\vee})$) or even trivial (for the others, as $H^0(W_0, X^\vee) = 0$, $H^0(W_0, \mathbb{Z}) = \mathbb{Z}$ and $H^0(W_0, \mathbb{F}_2) = \mathbb{F}_2$), it would not be necessary to list them. Second, by the duality theory for Tate cohomology [Bro82, VI.7], it follows that the groups $H^3(W_0, \mathbb{Z})$ are dual to the second homology groups $H_2(W_0, \mathbb{Z})$, which in turn are dual to the *Schur multipliers* $H^2(W_0, \mathbb{C}^\times)$, and these have been computed for all finite reflection groups by Ihara and Yokonuma [IY65].

Third, the cohomology groups $H^k(W_0, \mathbb{F}_2)$ with trivial modular coefficients are easily computed from cohomology with trivial integer coefficients using the Künneth theorem. Fourth, for the root system G_2 the Weyl group W_0 is a dihedral group of order 12, and the cohomology groups $H^k(D_m, \mathbb{Z})$ of the dihedral groups D_m with integer coefficients are known [Han93, Proof of Theorem 5.2].

Fourth and final, because of the presence of the automorphism group of the Dynkin diagram, it can happen that some sublattices $Q^\vee \subseteq X_1^\vee, X_2^\vee \subseteq P^\vee$ are isomorphic as W_0-modules, in the sense that there is a group automorphism $\varphi_0 : W_0 \xrightarrow{\sim} W_0$ and a $\mathbb{Z}$-linear isomorphism $\varphi_1 : X_1^\vee \xrightarrow{\sim} X_2^\vee$ such that

$$\varphi_1(\varphi_0(w) \bullet x) = w \bullet \varphi_1(x) \quad \forall w \in W_0 \ \forall x \in X_1^\vee$$

In other words, the pair (φ_0, φ_1) is an object of the category $\mathcal{D}$ defined in remark 4.4.5, and therefore induces an isomorphism

$$H^k(W_0, X_1^\vee) \xrightarrow{\sim} H^k(W_0, X_2^\vee)$$

[34] This case actually never happens.

, which by remark 4.4.6 is given in terms of standard cochains (in degree k) by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}}(W_0^{\times k}, X_1^\vee) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(W_0^{\times k}, X_2^\vee) \\ \phi &\longmapsto \varphi_1 \circ \phi \circ \phi_0^{\times k} \end{aligned} \tag{A.1.1}$$

Such pairs (φ_0, φ_1) of isomorphisms between sublattices are provided by the automorphism group of the Dykin diagram, as follows.

First, the automorphism group of the Dynkin diagram can be identified with the group

$$\widetilde{\Omega} = \{u \in \mathrm{Aut}_{\mathrm{Set}}(\Delta) \ : \ \forall \alpha, \beta \in \Delta \quad \langle u(\alpha), u(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle\}$$

i.e. the subgroup of the group permutations of the set Δ of simple roots that preserve the Cartan matrix. By linear extension, every $u \in \widetilde{\Omega}$ gives rise to a $\mathbb{Z}$-linear automorphism of the root lattice Q, by permutation of its basis Δ. This automorphism is compatible with the action of W_0 in the sense that (cf. [DG70, Exposé XXI, Lemma 6.7.1])

$$u \circ s_\alpha u^{-1} = s_{u(\alpha)} \quad \forall \alpha \in \Phi \tag{A.1.2}$$

By duality, the group $\widetilde{\Omega}$ also acts on the coweight lattice $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$, and this action preserves the coroots lattice $Q^\vee \subseteq P^\vee$. Concretely, if $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\Lambda_1^\vee, \ldots, \Lambda_\ell^\vee$ denotes the corresponding dual basis of $P^\vee$ (i.e. $\langle \alpha_i, \Lambda_j^\vee \rangle = \delta_{i,j}$), then we can identify $\widetilde{\Omega}$ with a subgroup $\mathrm{Aut}_{\mathrm{Set}}(\{1, \ldots, \ell\})$ (i.e. $u(\alpha_i) = \alpha_{u(i)}$), and the action of $\widetilde{\Omega}$ on $P^\vee$ is determined by

$$u(\Lambda_i^\vee) = \Lambda_{u(i)}^\vee$$

For every sublattice $Q^\vee \subseteq X^\vee \subseteq P^\vee$ and every $u \in \widetilde{\Omega}$, we then have a pair (φ_0, φ_1) as above, providing an isomorphism between $X^\vee$ and the sublattice $u(X^\vee)$, given by

$$\varphi_0(w) = u^{-1} \circ w \circ u, \quad \varphi_1(x) := u(x)$$

Moreover, from eq. (A.1.2) it follows that φ_0 is an automorphism of Coxeter groups (i.e. it preserves S), and from corollary 4.7.10 induced isomorphism

$$H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2) \xrightarrow{\sim} H^2(W_0, u(X^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_2)$$

preserves the canonical classes $[\phi_u]$. In particular, the canonical class in $H^2(W_0, X^\vee \otimes_{\mathbb{Z}} \mathbb{F}_2)$ vanishes iff the canonical class in $H^2(W_0, u(X^\vee) \otimes_{\mathbb{Z}} \mathbb{F}_2)$ vanishes.

As an example, consider the root system $X_\ell = D_4$. Then $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, and $\widetilde{\Omega}$ identifies with $\mathrm{Aut}_{\mathrm{Set}}(\{1, 3, 4\})$. It is then not hard to see that $\widetilde{\Omega}$ acts transitively on the three proper sublattices $Q^\vee \subsetneq X^\vee \subseteq P^\vee$ (corresponding to the exceptional isomorphism $\mathrm{SO}(8) \simeq \mathrm{Semispin}(8)$ given by *triality*).

A.2 Root system A_1

Dynkin diagram	$\bigcirc$ 1
Fundamental group	$P^\vee / Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_1^\vee \in P^\vee \mod Q^\vee$

A.2.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\boxed{\begin{aligned} [\phi_u] &= (1) \\ &\textbf{does not lie in the image of } \mathbf{comp}_2 \end{aligned}}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
1	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_1^\vee[\mathbf{1}]$
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$2\Lambda_1^\vee[]$
1	1	$2\Lambda_1^\vee[\mathbf{1}]$
2	1	$2\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1}]$
3	1	$2\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	(1)	()	(1)

A.2.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad 0}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_1^\vee[\mathbf{1}]$
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$\Lambda_1^\vee[]$
1	1	$\Lambda_1^\vee[\mathbf{1}]$
2	1	$\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1}]$
3	1	$\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$

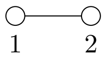
$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	(1)	()	(1)

A.2.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1} \supseteq \mathbf{1}]$
3	0	

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	1	$[\mathbf{1}]$
2	1	$[\mathbf{1} \supseteq \mathbf{1}]$
3	1	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$

A.3 Root system A_2

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/3\mathbb{Z}$ generated by $\Lambda_2^\vee \in P^\vee \mod Q^\vee$

A.3.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_2^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)\,[\mathbf{2}]}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/3\mathbb{Z}$	$\left(\Lambda_1^\vee - 2\Lambda_2^\vee\right)[\mathbf{2}]$
2	0	
3	0	

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
2	0	
3	0	

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

A.3.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + \Lambda_2^\vee[\mathbf{2}]}$$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/3\mathbb{Z}$	$\Lambda_1^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	0	
1	0	
2	0	
3	0	

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

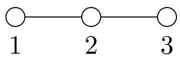
A.3.3 Cohomology with trivial coefficients

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}]$

$\mathbf{k}$	$\mathbf{H^k(W_0,\mathbb{Z})}$	generating cocycles
3	0	

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	1	$[\mathbf{1}]+[\mathbf{2}]$
2	1	$[\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}]$
3	1	$[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]$

A.4 Root system A_3

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee\simeq\mathbb{Z}/4\mathbb{Z}$ generated by $\Lambda_3^\vee\in P^\vee \mod Q^\vee$

A.4.1 Cohomology of coroot lattice $X^\vee=Q^\vee$

$$\boxed{\begin{array}{rcl}[\phi_u] & = & (1,0)\\ & & \textbf{does not lie in the image of comp}_2\end{array}}$$

$\mathbf{k}$	$\mathbf{H^k(W_0,X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/4\mathbb{Z}$	$\left(\Lambda_2^\vee-2\Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$4\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}]+\left(-4\Lambda_3^\vee\right)[\mathbf{2}\supseteq\mathbf{2}]+4\Lambda_3^\vee[\mathbf{1},\mathbf{2}]+\left(-2\Lambda_2^\vee\right)[\mathbf{3}\supseteq\mathbf{3}]+\left(-2\Lambda_2^\vee+4\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}]+\left(-2\Lambda_1^\vee+6\Lambda_2^\vee-6\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$4\Lambda_3^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+2\Lambda_2^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(-2\Lambda_1^\vee+2\Lambda_2^\vee-2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$4\Lambda_3^\vee[]$
1	2	$\left(\Lambda_2^\vee+2\Lambda_3^\vee\right)[\mathbf{1}]$
		$4\Lambda_3^\vee[\mathbf{1}]+4\Lambda_3^\vee[\mathbf{2}]+4\Lambda_3^\vee[\mathbf{3}]$
2	2	$\left(\Lambda_2^\vee+2\Lambda_3^\vee\right)[\mathbf{1}\supseteq\mathbf{1}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+3\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}]+\left(\Lambda_2^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}]$
		$4\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}]+4\Lambda_3^\vee[\mathbf{2}\supseteq\mathbf{2}]+4\Lambda_3^\vee[\mathbf{3}\supseteq\mathbf{3}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
3	3	$\left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 6\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + 4\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$4\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 4\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 4\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\left(\Lambda_2^\vee + 6\Lambda_3^\vee\right)[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 4\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$

A.4.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(2)\rangle$

$$\boxed{\phi_u \quad = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_3^\vee)[\mathbf{1}] + (\Lambda_2^\vee + 2\Lambda_3^\vee)[\mathbf{2}] + (\Lambda_1^\vee + \Lambda_3^\vee)[\mathbf{3}]}$$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{2}] + \left(-\Lambda_2^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{3}] + \left(-\Lambda_1^\vee + 3\Lambda_2^\vee - 3\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(2\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	1	$2\Lambda_3^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3}]$
2	3	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{3}]$
3	4	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\\1&0\\0&1\end{pmatrix}$

A.4.3 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)[\mathbf{2}] + \Lambda_3^\vee[\mathbf{3}]}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{2}\supseteq\mathbf{2}] + \Lambda_3^\vee[\mathbf{1},\mathbf{2}] + \left(-\Lambda_1^\vee\right)[\mathbf{3}\supseteq\mathbf{3}] + \left(\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}] + \left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
3	$\mathbb{Z}/4\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	1	$\Lambda_3^\vee[\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}] + \Lambda_1^\vee[\mathbf{3}]$
2	2	$\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}\supseteq\mathbf{2}] + \Lambda_1^\vee[\mathbf{3}\supseteq\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$ $\Lambda_3^\vee[\mathbf{1},\mathbf{2}]$
3	2	$\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}] + \Lambda_1^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}] + \Lambda_1^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}] + \Lambda_3^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}] + \Lambda_2^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}] + \Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]$ $\Lambda_3^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}] + \Lambda_1^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$

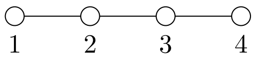
A.4.4 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1}\supseteq\mathbf{1}] + (-1)[\mathbf{2}\supseteq\mathbf{2}] + [\mathbf{1},\mathbf{2}] + (-1)[\mathbf{3}\supseteq\mathbf{3}]$

k	$\mathbf{H^k(W_0,\mathbb{Z})}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+(-1)\,[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	1	$[\mathbf{1}]+[\mathbf{2}]+[\mathbf{3}]$
2	2	$[\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{3}\supseteq\mathbf{3}]$
		$[\mathbf{1},\mathbf{3}]$
3	3	$[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]$
		$[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]$
		$[\mathbf{1},\mathbf{2},\mathbf{3}]$

A.5 Root system A_4

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/5\mathbb{Z}$ generated by $\Lambda_4^\vee \in P^\vee \mod Q^\vee$

A.5.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\phi_u = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + 6\Lambda_4^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 8\Lambda_4^\vee)\,[\mathbf{2}] + (\Lambda_3^\vee + 3\Lambda_4^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + 3\Lambda_4^\vee)\,[\mathbf{4}]$$

k	$\mathbf{H^k(W_0,X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/5\mathbb{Z}$	$\left(\Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{4}]$
2	0	
3	0	

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	0	
1	0	
2	0	

$\mathbf{k}$	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
3	0	

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

A.5.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee)\,[\mathbf{3}] + \Lambda_4^\vee[\mathbf{4}]}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	0	
2	0	
3	0	

$\mathbf{k}$	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
0	0	
1	0	
2	0	
3	0	

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

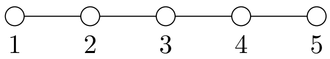
A.5.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)\,[\mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{4} \supseteq \mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{1}, \mathbf{3}, \mathbf{4}] + [\mathbf{2}, \mathbf{3}, \mathbf{4}]$

k	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
0	1	$[]$
1	1	$[\mathbf{1}]+[\mathbf{2}]+[\mathbf{3}]+[\mathbf{4}]$
2	2	$[\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{3}\supseteq\mathbf{3}]+[\mathbf{4}\supseteq\mathbf{4}]$
		$[\mathbf{1},\mathbf{3}]+[\mathbf{1},\mathbf{4}]+[\mathbf{2},\mathbf{4}]$
3	3	$[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]$
		$[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]$
		$[\mathbf{1},\mathbf{2},\mathbf{3}]+[\mathbf{2},\mathbf{3},\mathbf{4}]$

A.6 Root system A_5

Dynkin diagram



Fundamental group

$P^\vee/Q^\vee \simeq \mathbb{Z}/6\mathbb{Z}$
generated by $\Lambda_5^\vee \in P^\vee \mod Q^\vee$

A.6.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] = (1,0,0)$
does not lie in the image of comp$_2$

k	$\mathbf{H^k}(\mathbf{W_0},\mathbf{X}^\vee)$	generating cocycles
0	0	
1	$\mathbb{Z}/6\mathbb{Z}$	$\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$6\Lambda_5^\vee[\mathbf{1}\supseteq\mathbf{1}]+\left(-6\Lambda_5^\vee\right)[\mathbf{2}\supseteq\mathbf{2}]+6\Lambda_5^\vee[\mathbf{1},\mathbf{2}]+\left(-6\Lambda_5^\vee\right)[\mathbf{3}\supseteq\mathbf{3}]+\left(-6\Lambda_5^\vee\right)[\mathbf{4}\supseteq\mathbf{4}]+\left(-3\Lambda_4^\vee\right)[\mathbf{5}\supseteq\mathbf{5}]+\left(-3\Lambda_4^\vee+6\Lambda_5^\vee\right)[\mathbf{1},\mathbf{5}]+\left(3\Lambda_4^\vee-6\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}]+\left(3\Lambda_4^\vee-6\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}]+\left(-3\Lambda_3^\vee+9\Lambda_4^\vee-9\Lambda_5^\vee\right)[\mathbf{4},\mathbf{5}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+\left(-\Lambda_4^\vee-10\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\left(-6\Lambda_4^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+\left(-\Lambda_4^\vee-4\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{5}]+\left(-\Lambda_4^\vee-10\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\left(-\Lambda_2^\vee+2\Lambda_3^\vee-7\Lambda_4^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\left(6\Lambda_4^\vee-6\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}]+\left(\Lambda_2^\vee+4\Lambda_3^\vee-4\Lambda_4^\vee+4\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$6\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+6\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(-6\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+\left(-6\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+\left(-6\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+\left(-6\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+\left(-6\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+6\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{4}]+6\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+\left(-6\Lambda_5^\vee\right)[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+\left(-3\Lambda_4^\vee\right)[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+\left(-6\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\left(-3\Lambda_4^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+\left(-6\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\left(-3\Lambda_4^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\left(-3\Lambda_4^\vee+6\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}]+\left(-3\Lambda_3^\vee+9\Lambda_4^\vee-9\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}]+\left(-3\Lambda_3^\vee+9\Lambda_4^\vee-9\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}]+\left(3\Lambda_3^\vee-3\Lambda_4^\vee+3\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	1	$6\Lambda_5^\vee[]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
1	2	$\left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1}]$
		$6\Lambda_5^\vee[\mathbf{1}] + 6\Lambda_5^\vee[\mathbf{2}] + 6\Lambda_5^\vee[\mathbf{3}] + 6\Lambda_5^\vee[\mathbf{4}] + 6\Lambda_5^\vee[\mathbf{5}]$
2	3	$\left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1} \supseteq \mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}] + \left(\Lambda_4^\vee + 10\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}] + \left(\Lambda_2^\vee + 8\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}]$
		$6\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1}] + 6\Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2}] + 6\Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3}] + 6\Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4}] + 6\Lambda_5^\vee[\mathbf{5} \supseteq \mathbf{5}]$
		$6\Lambda_5^\vee[\mathbf{1},\mathbf{3}] + 6\Lambda_5^\vee[\mathbf{1},\mathbf{4}] + 6\Lambda_5^\vee[\mathbf{2},\mathbf{4}] + 6\Lambda_5^\vee[\mathbf{1},\mathbf{5}] + 6\Lambda_5^\vee[\mathbf{2},\mathbf{5}] + 6\Lambda_5^\vee[\mathbf{3},\mathbf{5}]$
3	5	$\left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + 5\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(\Lambda_4^\vee + 10\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \left(\Lambda_4^\vee + 10\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + \left(\Lambda_2^\vee + 8\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}] + \left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 7\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$6\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 6\Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 6\Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 6\Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 6\Lambda_5^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$\left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 6\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}] + \left(\Lambda_4^\vee + 4\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}]$
		$6\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + 6\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + 6\Lambda_5^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + 6\Lambda_5^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + 6\Lambda_5^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + 6\Lambda_5^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + 6\Lambda_5^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + 6\Lambda_5^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + 6\Lambda_5^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + 6\Lambda_5^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + 6\Lambda_5^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + 6\Lambda_5^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$6\Lambda_5^\vee[\mathbf{1},\mathbf{2},\mathbf{3}] + 6\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}] + 6\Lambda_5^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\\0&1\\0&1\\0&1\end{pmatrix}$

A.6.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(3)\rangle$

$$\boxed{\phi_u = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + 4\Lambda_5^\vee)[\mathbf{1}] + (\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee)[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee)[\mathbf{3}] + (\Lambda_4^\vee + \Lambda_5^\vee)[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee)[\mathbf{5}]}$$

k	$\mathbf{H^k}(\mathbf{W_0},\mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/3\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{5}]$
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$3\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + 3\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + \left(-3\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(-3\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \left(-3\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \left(-3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + 3\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{4}] + 3\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}] + \left(-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_1^\vee - 4\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee - 2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_1^\vee + \Lambda_3^\vee - 2\Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}] + \left(-\Lambda_2^\vee + \Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}] + \left(\Lambda_1^\vee + 2\Lambda_3^\vee - 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
0	0	

k	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
1	0	
2	1	$\left(\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{1,3}]+3\Lambda_5^\vee[\mathbf{1,4}]+3\Lambda_5^\vee[\mathbf{2,4}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{3,4}]+\Lambda_3^\vee[\mathbf{1,5}]+\Lambda_3^\vee[\mathbf{2,5}]+\left(\Lambda_1^\vee+4\Lambda_5^\vee\right)[\mathbf{3,5}]+\Lambda_3^\vee[\mathbf{4,5}]$
3	2	$\left(\Lambda_2^\vee+2\Lambda_5^\vee\right)[\mathbf{1,3\supseteq 1}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+3\Lambda_5^\vee\right)[\mathbf{1,2,3}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1,3,4}]+3\Lambda_5^\vee[\mathbf{2,3,4}]+\left(\Lambda_1^\vee+4\Lambda_5^\vee\right)[\mathbf{2,3,5}]+\left(\Lambda_1^\vee+4\Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$\left(\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{1,3\supseteq 1}]+3\Lambda_5^\vee[\mathbf{1,3\supseteq 3}]+\Lambda_3^\vee[\mathbf{2,3\supseteq 2}]+\left(\Lambda_1^\vee+4\Lambda_5^\vee\right)[\mathbf{2,3\supseteq 3}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1,2,3}]+3\Lambda_5^\vee[\mathbf{1,4\supseteq 1}]+3\Lambda_5^\vee[\mathbf{1,4\supseteq 4}]+3\Lambda_5^\vee[\mathbf{2,4\supseteq 2}]+3\Lambda_5^\vee[\mathbf{2,4\supseteq 4}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1,3,4}]+\Lambda_3^\vee[\mathbf{2,3,4}]+\Lambda_3^\vee[\mathbf{1,5\supseteq 1}]+\Lambda_3^\vee[\mathbf{1,5\supseteq 5}]+\Lambda_3^\vee[\mathbf{2,5\supseteq 2}]+\Lambda_3^\vee[\mathbf{2,5\supseteq 5}]+\left(\Lambda_1^\vee+4\Lambda_5^\vee\right)[\mathbf{3,5\supseteq 3}]+\left(\Lambda_1^\vee+4\Lambda_5^\vee\right)[\mathbf{3,5\supseteq 5}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1,3,5}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_5^\vee\right)[\mathbf{2,3,5}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{1,4,5}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{2,4,5}]+\left(\Lambda_1^\vee+4\Lambda_5^\vee\right)[\mathbf{3,4,5}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$

A.6.3 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(2)\rangle$

$$[\phi_u] = (1,1,0)$$
does not lie in the image of $\mathbf{comp}_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_5^\vee[\mathbf{1\supseteq 1}]+\left(-2\Lambda_5^\vee\right)[\mathbf{2\supseteq 2}]+2\Lambda_5^\vee[\mathbf{1,2}]+\left(-2\Lambda_5^\vee\right)[\mathbf{3\supseteq 3}]+\left(-2\Lambda_5^\vee\right)[\mathbf{4\supseteq 4}]+\left(-\Lambda_4^\vee\right)[\mathbf{5\supseteq 5}]+\left(-\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{1,5}]+\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{2,5}]+\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{3,5}]+\left(-\Lambda_3^\vee+3\Lambda_4^\vee-3\Lambda_5^\vee\right)[\mathbf{4,5}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{1,5\supseteq 1}]+\left(-\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{2,5\supseteq 2}]+\left(-2\Lambda_4^\vee\right)[\mathbf{2,5\supseteq 5}]+\left(-\Lambda_4^\vee\right)[\mathbf{1,2,5}]+\left(-\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{3,5\supseteq 3}]+\left(-\Lambda_2^\vee+2\Lambda_3^\vee-3\Lambda_4^\vee\right)[\mathbf{3,5\supseteq 5}]+\left(2\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{2,4,5}]+\Lambda_2^\vee[\mathbf{3,4,5}]$
		$2\Lambda_5^\vee[\mathbf{1,3\supseteq 1}]+2\Lambda_5^\vee[\mathbf{1,3\supseteq 3}]+\left(-2\Lambda_5^\vee\right)[\mathbf{1,2,3}]+\left(-2\Lambda_5^\vee\right)[\mathbf{1,4\supseteq 1}]+\left(-2\Lambda_5^\vee\right)[\mathbf{1,4\supseteq 4}]+\left(-2\Lambda_5^\vee\right)[\mathbf{2,4\supseteq 2}]+\left(-2\Lambda_5^\vee\right)[\mathbf{2,4\supseteq 4}]+2\Lambda_5^\vee[\mathbf{1,3,4}]+2\Lambda_5^\vee[\mathbf{2,3,4}]+\left(-2\Lambda_5^\vee\right)[\mathbf{1,5\supseteq 1}]+\left(-\Lambda_4^\vee\right)[\mathbf{1,5\supseteq 5}]+\left(-2\Lambda_5^\vee\right)[\mathbf{2,5\supseteq 2}]+\left(-\Lambda_4^\vee\right)[\mathbf{2,5\supseteq 5}]+\left(-2\Lambda_5^\vee\right)[\mathbf{3,5\supseteq 3}]+\left(-\Lambda_4^\vee\right)[\mathbf{3,5\supseteq 5}]+\left(-\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{1,3,5}]+\left(-\Lambda_3^\vee+3\Lambda_4^\vee-3\Lambda_5^\vee\right)[\mathbf{1,4,5}]+\left(-\Lambda_3^\vee+3\Lambda_4^\vee-3\Lambda_5^\vee\right)[\mathbf{2,4,5}]+\left(\Lambda_3^\vee-\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{3,4,5}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
0	1	$2\Lambda_5^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}]+2\Lambda_5^\vee[\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}]+2\Lambda_5^\vee[\mathbf{5}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
2	3	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}]+\Lambda_2^\vee[\mathbf{3},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{5}\supseteq\mathbf{5}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{4}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{5}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{5}]+\Lambda_4^\vee[\mathbf{3},\mathbf{5}]$
3	5	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_2^\vee[\mathbf{1},\mathbf{3},\mathbf{5}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]$
		$\left(\Lambda_1^\vee+\Lambda_3^\vee+4\Lambda_5^\vee\right)[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_4^\vee[\mathbf{1},\mathbf{3},\mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]+\left(\Lambda_3^\vee+3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_2^\vee+\Lambda_4^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\1&1\\0&1\\1&0\end{pmatrix}$

A.6.4 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee+\Lambda_2^\vee)[\mathbf{2}] + (\Lambda_2^\vee+\Lambda_3^\vee)[\mathbf{3}] + (\Lambda_3^\vee+\Lambda_4^\vee)[\mathbf{4}] + \Lambda_5^\vee[\mathbf{5}]}$$

k	$\mathbf{H^k}(\mathbf{W_0},\mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(-\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+\left(-\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+\left(-\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+\left(-\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+\left(-\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{4}]+\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+\left(-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+\left(-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+\left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+\left(-\Lambda_1^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\left(-\Lambda_1^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\left(-\Lambda_1^\vee+\Lambda_3^\vee-\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}]+\left(-\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{5}]+\left(\Lambda_4^\vee-\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}]+\left(\Lambda_4^\vee-\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}]+\Lambda_1^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	$h^k(\overline{X}^\vee)$	generating cocycles
0	0	
1	0	
2	1	$\Lambda_5^\vee[1,3] + \Lambda_5^\vee[1,4] + \Lambda_5^\vee[2,4] + \Lambda_3^\vee[1,5] + \Lambda_3^\vee[2,5] + \Lambda_1^\vee[3,5]$
3	2	$\Lambda_5^\vee[1,3 \supseteq 1] + \Lambda_5^\vee[1,3 \supseteq 3] + \Lambda_5^\vee[1,4 \supseteq 1] + \Lambda_5^\vee[1,4 \supseteq 4] + \Lambda_5^\vee[2,4 \supseteq 2] + \Lambda_5^\vee[2,4 \supseteq 4] + \Lambda_3^\vee[1,5 \supseteq 1] + \Lambda_3^\vee[1,5 \supseteq 5] + \Lambda_3^\vee[2,5 \supseteq 2] + \Lambda_3^\vee[2,5 \supseteq 5] + \Lambda_1^\vee[3,5 \supseteq 3] + \Lambda_1^\vee[3,5 \supseteq 5] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)[1,3,5] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[2,3,5] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[1,4,5] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[2,4,5]$
		$\Lambda_5^\vee[1,2,3] + \Lambda_5^\vee[2,3,4] + \Lambda_1^\vee[2,3,5] + \Lambda_3^\vee[1,4,5] + \Lambda_3^\vee[2,4,5] + \Lambda_1^\vee[3,4,5]$

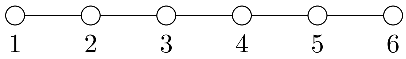
k	0	1	2	3
$\mathbf{comp}_k$	()	()	()	$\begin{pmatrix}1\\1\end{pmatrix}$

A.6.5 Cohomology with trivial coefficients

k	$H^k(W_0, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[1 \supseteq 1] + (-1)[2 \supseteq 2] + [1,2] + (-1)[3 \supseteq 3] + (-1)[4 \supseteq 4] + (-1)[5 \supseteq 5]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[1,3 \supseteq 1] + [1,3 \supseteq 3] + (-1)[1,2,3] + (-1)[1,4 \supseteq 1] + (-1)[1,4 \supseteq 4] + (-1)[2,4 \supseteq 2] + (-1)[2,4 \supseteq 4] + [1,3,4] + [2,3,4] + (-1)[1,5 \supseteq 1] + (-1)[1,5 \supseteq 5] + (-1)[2,5 \supseteq 2] + (-1)[2,5 \supseteq 5] + (-1)[3,5 \supseteq 3] + (-1)[3,5 \supseteq 5] + [3,4,5]$

k	$h^k(\mathbb{F}_2)$	generating cocycles
0	1	[]
1	1	$[1] + [2] + [3] + [4] + [5]$
2	2	$[1 \supseteq 1] + [2 \supseteq 2] + [3 \supseteq 3] + [4 \supseteq 4] + [5 \supseteq 5]$
		$[1,3] + [1,4] + [2,4] + [1,5] + [2,5] + [3,5]$
3	4	$[1 \supseteq 1 \supseteq 1] + [2 \supseteq 2 \supseteq 2] + [3 \supseteq 3 \supseteq 3] + [4 \supseteq 4 \supseteq 4] + [5 \supseteq 5 \supseteq 5]$
		$[1,3 \supseteq 1] + [1,3 \supseteq 3] + [1,4 \supseteq 1] + [1,4 \supseteq 4] + [2,4 \supseteq 2] + [2,4 \supseteq 4] + [1,5 \supseteq 1] + [1,5 \supseteq 5] + [2,5 \supseteq 2] + [2,5 \supseteq 5] + [3,5 \supseteq 3] + [3,5 \supseteq 5]$
		$[1,2,3] + [2,3,4] + [3,4,5]$
		$[1,3,5]$

A.7 Root system A_6

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/7\mathbb{Z}$ generated by $\Lambda_6^\vee \in P^\vee \mod Q^\vee$

A.7.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\phi_u = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + 8\Lambda_6^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 10\Lambda_6^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + 12\Lambda_6^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee + 14\Lambda_6^\vee)\,[\mathbf{4}] + (\Lambda_5^\vee + 5\Lambda_6^\vee)\,[\mathbf{5}] + (\Lambda_5^\vee + 5\Lambda_6^\vee)\,[\mathbf{6}]$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/7\mathbb{Z}$	$\left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{6}]$
2	0	
3	0	

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	0	
1	0	
2	0	
3	0	

k	0	1	2	3
comp$_k$	()	()	()	()

A.7.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\phi_u = \partial\tau \text{ with } \tau = \Lambda_1^\vee\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee)\,[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee)\,[\mathbf{5}] + \Lambda_6^\vee\,[\mathbf{6}]$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	0	

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	0	

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	0	
3	0	

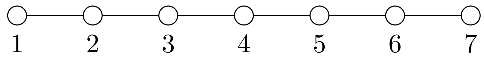
k	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

A.7.3 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1 \supseteq 1}] + (-1)\,[\mathbf{2 \supseteq 2}] + [\mathbf{1, 2}] + (-1)\,[\mathbf{3 \supseteq 3}] + (-1)\,[\mathbf{4 \supseteq 4}] + (-1)\,[\mathbf{5 \supseteq 5}] + (-1)\,[\mathbf{6 \supseteq 6}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1, 3 \supseteq 1}] + [\mathbf{1, 3 \supseteq 3}] + (-1)\,[\mathbf{1, 2, 3}] + (-1)\,[\mathbf{1, 4 \supseteq 1}] + (-1)\,[\mathbf{1, 4 \supseteq 4}] +$ $(-1)\,[\mathbf{2, 4 \supseteq 2}] + (-1)\,[\mathbf{2, 4 \supseteq 4}] + [\mathbf{1, 3, 4}] + [\mathbf{2, 3, 4}] + (-1)\,[\mathbf{1, 5 \supseteq 1}] + (-1)\,[\mathbf{1, 5 \supseteq 5}] +$ $(-1)\,[\mathbf{2, 5 \supseteq 2}] + (-1)\,[\mathbf{2, 5 \supseteq 5}] + (-1)\,[\mathbf{3, 5 \supseteq 3}] + (-1)\,[\mathbf{3, 5 \supseteq 5}] + [\mathbf{3, 4, 5}] +$ $(-1)\,[\mathbf{1, 6 \supseteq 1}] + (-1)\,[\mathbf{1, 6 \supseteq 6}] + (-1)\,[\mathbf{2, 6 \supseteq 2}] + (-1)\,[\mathbf{2, 6 \supseteq 6}] + (-1)\,[\mathbf{3, 6 \supseteq 3}] +$ $(-1)\,[\mathbf{3, 6 \supseteq 6}] + (-1)\,[\mathbf{4, 6 \supseteq 4}] + (-1)\,[\mathbf{4, 6 \supseteq 6}] + [\mathbf{4, 5, 6}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}]$
2	2	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}] + [\mathbf{6 \supseteq 6}]$
		$[\mathbf{1, 3}] + [\mathbf{1, 4}] + [\mathbf{2, 4}] + [\mathbf{1, 5}] + [\mathbf{2, 5}] + [\mathbf{3, 5}] + [\mathbf{1, 6}] + [\mathbf{2, 6}] + [\mathbf{3, 6}] + [\mathbf{4, 6}]$
3	4	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}] + [\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}] + [\mathbf{5 \supseteq 5 \supseteq 5}] + [\mathbf{6 \supseteq 6 \supseteq 6}]$
		$[\mathbf{1, 3 \supseteq 1}] + [\mathbf{1, 3 \supseteq 3}] + [\mathbf{1, 4 \supseteq 1}] + [\mathbf{1, 4 \supseteq 4}] + [\mathbf{2, 4 \supseteq 2}] + [\mathbf{2, 4 \supseteq 4}] + [\mathbf{1, 5 \supseteq 1}] + [\mathbf{1, 5 \supseteq 5}] +$ $[\mathbf{2, 5 \supseteq 2}] + [\mathbf{2, 5 \supseteq 5}] + [\mathbf{3, 5 \supseteq 3}] + [\mathbf{3, 5 \supseteq 5}] + [\mathbf{1, 6 \supseteq 1}] + [\mathbf{1, 6 \supseteq 6}] + [\mathbf{2, 6 \supseteq 2}] + [\mathbf{2, 6 \supseteq 6}] +$ $[\mathbf{3, 6 \supseteq 3}] + [\mathbf{3, 6 \supseteq 6}] + [\mathbf{4, 6 \supseteq 4}] + [\mathbf{4, 6 \supseteq 6}]$
		$[\mathbf{1, 2, 3}] + [\mathbf{2, 3, 4}] + [\mathbf{3, 4, 5}] + [\mathbf{4, 5, 6}]$
		$[\mathbf{1, 3, 5}] + [\mathbf{1, 3, 6}] + [\mathbf{1, 4, 6}] + [\mathbf{2, 4, 6}]$

A.8 Root system A_7

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/8\mathbb{Z}$ generated by $\Lambda_7^\vee \in P^\vee \mod Q^\vee$

A.8.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (1,0,0)$$
$$\textbf{does not lie in the image of comp}_2$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/8\mathbb{Z}$	$(\Lambda_6^\vee - 2\Lambda_7^\vee)\,\mathbf{[7]}$
2	$\mathbb{Z}/2\mathbb{Z}$	$8\Lambda_7^\vee\,\mathbf{[1 \supseteq 1]} + (-8\Lambda_7^\vee)\,\mathbf{[2 \supseteq 2]} + 8\Lambda_7^\vee\,\mathbf{[1,2]} + (-8\Lambda_7^\vee)\,\mathbf{[3 \supseteq 3]} + (-8\Lambda_7^\vee)\,\mathbf{[4 \supseteq 4]} + (-8\Lambda_7^\vee)\,\mathbf{[5 \supseteq 5]} + (-8\Lambda_7^\vee)\,\mathbf{[6 \supseteq 6]} + (-4\Lambda_6^\vee)\,\mathbf{[7 \supseteq 7]} + (-4\Lambda_6^\vee + 8\Lambda_7^\vee)\,\mathbf{[1,7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[2,7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[3,7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[4,7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[5,7]} + (-4\Lambda_5^\vee + 12\Lambda_6^\vee - 12\Lambda_7^\vee)\,\mathbf{[6,7]}$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$(\Lambda_6^\vee - 2\Lambda_7^\vee)\,\mathbf{[1,7 \supseteq 1]} + (-\Lambda_6^\vee - 14\Lambda_7^\vee)\,\mathbf{[2,7 \supseteq 2]} + (-8\Lambda_6^\vee)\,\mathbf{[2,7 \supseteq 7]} + (-\Lambda_6^\vee - 6\Lambda_7^\vee)\,\mathbf{[1,2,7]} + (-\Lambda_6^\vee - 14\Lambda_7^\vee)\,\mathbf{[3,7 \supseteq 3]} + (-8\Lambda_6^\vee)\,\mathbf{[3,7 \supseteq 7]} + (-\Lambda_6^\vee - 14\Lambda_7^\vee)\,\mathbf{[4,7 \supseteq 4]} + (-8\Lambda_6^\vee)\,\mathbf{[4,7 \supseteq 7]} + (-\Lambda_6^\vee - 14\Lambda_7^\vee)\,\mathbf{[5,7 \supseteq 5]} + (-\Lambda_4^\vee + 2\Lambda_5^\vee - 9\Lambda_6^\vee)\,\mathbf{[5,7 \supseteq 7]} + (8\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[2,6,7]} + (8\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[3,6,7]} + (8\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[4,6,7]} + (\Lambda_4^\vee + 6\Lambda_5^\vee - 6\Lambda_6^\vee + 6\Lambda_7^\vee)\,\mathbf{[5,6,7]}$
		$8\Lambda_7^\vee\,\mathbf{[1,3 \supseteq 1]} + 8\Lambda_7^\vee\,\mathbf{[1,3 \supseteq 3]} + (-8\Lambda_7^\vee)\,\mathbf{[1,2,3]} + (-8\Lambda_7^\vee)\,\mathbf{[1,4 \supseteq 1]} + (-8\Lambda_7^\vee)\,\mathbf{[1,4 \supseteq 4]} + (-8\Lambda_7^\vee)\,\mathbf{[2,4 \supseteq 2]} + (-8\Lambda_7^\vee)\,\mathbf{[2,4 \supseteq 4]} + 8\Lambda_7^\vee\,\mathbf{[1,3,4]} + 8\Lambda_7^\vee\,\mathbf{[2,3,4]} + (-8\Lambda_7^\vee)\,\mathbf{[1,5 \supseteq 1]} + (-8\Lambda_7^\vee)\,\mathbf{[1,5 \supseteq 5]} + (-8\Lambda_7^\vee)\,\mathbf{[2,5 \supseteq 2]} + (-8\Lambda_7^\vee)\,\mathbf{[2,5 \supseteq 5]} + (-8\Lambda_7^\vee)\,\mathbf{[3,5 \supseteq 3]} + (-8\Lambda_7^\vee)\,\mathbf{[3,5 \supseteq 5]} + 8\Lambda_7^\vee\,\mathbf{[3,4,5]} + (-8\Lambda_7^\vee)\,\mathbf{[1,6 \supseteq 1]} + (-8\Lambda_7^\vee)\,\mathbf{[1,6 \supseteq 6]} + (-8\Lambda_7^\vee)\,\mathbf{[2,6 \supseteq 2]} + (-8\Lambda_7^\vee)\,\mathbf{[2,6 \supseteq 6]} + (-8\Lambda_7^\vee)\,\mathbf{[3,6 \supseteq 3]} + (-8\Lambda_7^\vee)\,\mathbf{[3,6 \supseteq 6]} + (-8\Lambda_7^\vee)\,\mathbf{[4,6 \supseteq 4]} + (-8\Lambda_7^\vee)\,\mathbf{[4,6 \supseteq 6]} + 8\Lambda_7^\vee\,\mathbf{[4,5,6]} + (-8\Lambda_7^\vee)\,\mathbf{[1,7 \supseteq 1]} + (-4\Lambda_6^\vee)\,\mathbf{[1,7 \supseteq 7]} + (-8\Lambda_7^\vee)\,\mathbf{[2,7 \supseteq 2]} + (-4\Lambda_6^\vee)\,\mathbf{[2,7 \supseteq 7]} + (-8\Lambda_7^\vee)\,\mathbf{[3,7 \supseteq 3]} + (-4\Lambda_6^\vee)\,\mathbf{[3,7 \supseteq 7]} + (-4\Lambda_6^\vee + 8\Lambda_7^\vee)\,\mathbf{[1,3,7]} + (-8\Lambda_7^\vee)\,\mathbf{[4,7 \supseteq 4]} + (-4\Lambda_6^\vee)\,\mathbf{[4,7 \supseteq 7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[1,4,7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[2,4,7]} + (-8\Lambda_7^\vee)\,\mathbf{[5,7 \supseteq 5]} + (-4\Lambda_6^\vee)\,\mathbf{[5,7 \supseteq 7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[1,5,7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[2,5,7]} + (4\Lambda_6^\vee - 8\Lambda_7^\vee)\,\mathbf{[3,5,7]} + (-4\Lambda_5^\vee + 12\Lambda_6^\vee - 12\Lambda_7^\vee)\,\mathbf{[1,6,7]} + (-4\Lambda_5^\vee + 12\Lambda_6^\vee - 12\Lambda_7^\vee)\,\mathbf{[2,6,7]} + (-4\Lambda_5^\vee + 12\Lambda_6^\vee - 12\Lambda_7^\vee)\,\mathbf{[3,6,7]} + (-4\Lambda_5^\vee + 12\Lambda_6^\vee - 12\Lambda_7^\vee)\,\mathbf{[4,6,7]} + (4\Lambda_5^\vee - 4\Lambda_6^\vee + 4\Lambda_7^\vee)\,\mathbf{[5,6,7]}$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$8\Lambda_7^\vee\,[]$
1	2	$(\Lambda_2^\vee + 2\Lambda_7^\vee)\,\mathbf{[1]}$
		$8\Lambda_7^\vee\,\mathbf{[1]} + 8\Lambda_7^\vee\,\mathbf{[2]} + 8\Lambda_7^\vee\,\mathbf{[3]} + 8\Lambda_7^\vee\,\mathbf{[4]} + 8\Lambda_7^\vee\,\mathbf{[5]} + 8\Lambda_7^\vee\,\mathbf{[6]} + 8\Lambda_7^\vee\,\mathbf{[7]}$
2	3	$(\Lambda_2^\vee + 2\Lambda_7^\vee)\,\mathbf{[1 \supseteq 1]} + (\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_7^\vee)\,\mathbf{[1,2]} + (\Lambda_6^\vee + 14\Lambda_7^\vee)\,\mathbf{[2,7]} + (\Lambda_6^\vee + 14\Lambda_7^\vee)\,\mathbf{[3,7]} + (\Lambda_6^\vee + 14\Lambda_7^\vee)\,\mathbf{[4,7]} + (\Lambda_4^\vee + 12\Lambda_7^\vee)\,\mathbf{[5,7]}$
		$8\Lambda_7^\vee\,\mathbf{[1 \supseteq 1]} + 8\Lambda_7^\vee\,\mathbf{[2 \supseteq 2]} + 8\Lambda_7^\vee\,\mathbf{[3 \supseteq 3]} + 8\Lambda_7^\vee\,\mathbf{[4 \supseteq 4]} + 8\Lambda_7^\vee\,\mathbf{[5 \supseteq 5]} + 8\Lambda_7^\vee\,\mathbf{[6 \supseteq 6]} + 8\Lambda_7^\vee\,\mathbf{[7 \supseteq 7]}$
		$8\Lambda_7^\vee\,\mathbf{[1,3]} + 8\Lambda_7^\vee\,\mathbf{[1,4]} + 8\Lambda_7^\vee\,\mathbf{[2,4]} + 8\Lambda_7^\vee\,\mathbf{[1,5]} + 8\Lambda_7^\vee\,\mathbf{[2,5]} + 8\Lambda_7^\vee\,\mathbf{[3,5]} + 8\Lambda_7^\vee\,\mathbf{[1,6]} + 8\Lambda_7^\vee\,\mathbf{[2,6]} + 8\Lambda_7^\vee\,\mathbf{[3,6]} + 8\Lambda_7^\vee\,\mathbf{[4,6]} + 8\Lambda_7^\vee\,\mathbf{[1,7]} + 8\Lambda_7^\vee\,\mathbf{[2,7]} + 8\Lambda_7^\vee\,\mathbf{[3,7]} + 8\Lambda_7^\vee\,\mathbf{[4,7]} + 8\Lambda_7^\vee\,\mathbf{[5,7]}$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	generating cocycles
3	6	$\left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + 5\Lambda_7^\vee\right)[\mathbf{1,2} \supseteq \mathbf{2}] + \left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_6^\vee + 14\Lambda_7^\vee\right)[\mathbf{2,7} \supseteq \mathbf{2}] + \left(\Lambda_6^\vee + 14\Lambda_7^\vee\right)[\mathbf{3,7} \supseteq \mathbf{3}] + \left(\Lambda_6^\vee + 14\Lambda_7^\vee\right)[\mathbf{4,7} \supseteq \mathbf{4}] + \left(\Lambda_6^\vee + 14\Lambda_7^\vee\right)[\mathbf{5,7} \supseteq \mathbf{5}] + \left(\Lambda_4^\vee + 12\Lambda_7^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 6\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + 11\Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$8\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 8\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 8\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 8\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 8\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}] + 8\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}] + 8\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7} \supseteq \mathbf{7}]$
		$\left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3} \supseteq \mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 6\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,3,4}] + \left(\Lambda_6^\vee + 14\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee + 14\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(\Lambda_4^\vee + 12\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_4^\vee + 12\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_2^\vee + 10\Lambda_7^\vee\right)[\mathbf{3,5,7}]$
		$8\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 8\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + 8\Lambda_7^\vee[\mathbf{1,4} \supseteq \mathbf{1}] + 8\Lambda_7^\vee[\mathbf{1,4} \supseteq \mathbf{4}] + 8\Lambda_7^\vee[\mathbf{2,4} \supseteq \mathbf{2}] + 8\Lambda_7^\vee[\mathbf{2,4} \supseteq \mathbf{4}] + 8\Lambda_7^\vee[\mathbf{1,5} \supseteq \mathbf{1}] + 8\Lambda_7^\vee[\mathbf{1,5} \supseteq \mathbf{5}] + 8\Lambda_7^\vee[\mathbf{2,5} \supseteq \mathbf{2}] + 8\Lambda_7^\vee[\mathbf{2,5} \supseteq \mathbf{5}] + 8\Lambda_7^\vee[\mathbf{3,5} \supseteq \mathbf{3}] + 8\Lambda_7^\vee[\mathbf{3,5} \supseteq \mathbf{5}] + 8\Lambda_7^\vee[\mathbf{1,6} \supseteq \mathbf{1}] + 8\Lambda_7^\vee[\mathbf{1,6} \supseteq \mathbf{6}] + 8\Lambda_7^\vee[\mathbf{2,6} \supseteq \mathbf{2}] + 8\Lambda_7^\vee[\mathbf{2,6} \supseteq \mathbf{6}] + 8\Lambda_7^\vee[\mathbf{3,6} \supseteq \mathbf{3}] + 8\Lambda_7^\vee[\mathbf{3,6} \supseteq \mathbf{6}] + 8\Lambda_7^\vee[\mathbf{4,6} \supseteq \mathbf{4}] + 8\Lambda_7^\vee[\mathbf{4,6} \supseteq \mathbf{6}] + 8\Lambda_7^\vee[\mathbf{1,7} \supseteq \mathbf{1}] + 8\Lambda_7^\vee[\mathbf{1,7} \supseteq \mathbf{7}] + 8\Lambda_7^\vee[\mathbf{2,7} \supseteq \mathbf{2}] + 8\Lambda_7^\vee[\mathbf{2,7} \supseteq \mathbf{7}] + 8\Lambda_7^\vee[\mathbf{3,7} \supseteq \mathbf{3}] + 8\Lambda_7^\vee[\mathbf{3,7} \supseteq \mathbf{7}] + 8\Lambda_7^\vee[\mathbf{4,7} \supseteq \mathbf{4}] + 8\Lambda_7^\vee[\mathbf{4,7} \supseteq \mathbf{7}] + 8\Lambda_7^\vee[\mathbf{5,7} \supseteq \mathbf{5}] + 8\Lambda_7^\vee[\mathbf{5,7} \supseteq \mathbf{7}]$
		$8\Lambda_7^\vee[\mathbf{1,2,3}] + 8\Lambda_7^\vee[\mathbf{2,3,4}] + 8\Lambda_7^\vee[\mathbf{3,4,5}] + 8\Lambda_7^\vee[\mathbf{4,5,6}] + 8\Lambda_7^\vee[\mathbf{5,6,7}]$
		$8\Lambda_7^\vee[\mathbf{1,3,5}] + 8\Lambda_7^\vee[\mathbf{1,3,6}] + 8\Lambda_7^\vee[\mathbf{1,4,6}] + 8\Lambda_7^\vee[\mathbf{2,4,6}] + 8\Lambda_7^\vee[\mathbf{1,3,7}] + 8\Lambda_7^\vee[\mathbf{1,4,7}] + 8\Lambda_7^\vee[\mathbf{2,4,7}] + 8\Lambda_7^\vee[\mathbf{1,5,7}] + 8\Lambda_7^\vee[\mathbf{2,5,7}] + 8\Lambda_7^\vee[\mathbf{3,5,7}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\\0&0\\0&1\\0&1\\0&0\end{pmatrix}$

A.8.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(4)\rangle$

$$\begin{aligned}[\phi_u] &= (1,0,0)\\ &\textbf{does not lie in the image of } \mathbf{comp}_2\end{aligned}$$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	$\mathbb{Z}/4\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$4\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 4\Lambda_7^\vee[\mathbf{1,2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(-2\Lambda_6^\vee + 4\Lambda_7^\vee\right)[\mathbf{1,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{4,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{5,7}] + \left(-2\Lambda_5^\vee + 6\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{6,7}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-4\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-4\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-4\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 5\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $4\Lambda_7^\vee[\mathbf{1,3 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 4}] + 4\Lambda_7^\vee[\mathbf{1,3,4}] + 4\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 5}] + 4\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 6}] + 4\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,7 \supseteq 7}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-2\Lambda_6^\vee + 4\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(-4\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(-2\Lambda_5^\vee + 6\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-2\Lambda_5^\vee + 6\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-2\Lambda_5^\vee + 6\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-2\Lambda_5^\vee + 6\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$4\Lambda_7^\vee[]$
1	2	$\left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1}]$ $4\Lambda_7^\vee[\mathbf{1}] + 4\Lambda_7^\vee[\mathbf{2}] + 4\Lambda_7^\vee[\mathbf{3}] + 4\Lambda_7^\vee[\mathbf{4}] + 4\Lambda_7^\vee[\mathbf{5}] + 4\Lambda_7^\vee[\mathbf{6}] + 4\Lambda_7^\vee[\mathbf{7}]$
2	3	$\left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1 \supseteq 1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_7^\vee\right)[\mathbf{1,2}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{2,7}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{3,7}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{4,7}] + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)[\mathbf{5,7}]$ $4\Lambda_7^\vee[\mathbf{1 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{2 \supseteq 2}] + 4\Lambda_7^\vee[\mathbf{3 \supseteq 3}] + 4\Lambda_7^\vee[\mathbf{4 \supseteq 4}] + 4\Lambda_7^\vee[\mathbf{5 \supseteq 5}] + 4\Lambda_7^\vee[\mathbf{6 \supseteq 6}] + 4\Lambda_7^\vee[\mathbf{7 \supseteq 7}]$ $\Lambda_4^\vee[\mathbf{1,3}] + 4\Lambda_7^\vee[\mathbf{1,4}] + 4\Lambda_7^\vee[\mathbf{2,4}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{3,4}] + 4\Lambda_7^\vee[\mathbf{1,5}] + 4\Lambda_7^\vee[\mathbf{2,5}] + 4\Lambda_7^\vee[\mathbf{3,5}] + 4\Lambda_7^\vee[\mathbf{1,6}] + 4\Lambda_7^\vee[\mathbf{2,6}] + 4\Lambda_7^\vee[\mathbf{3,6}] + 4\Lambda_7^\vee[\mathbf{4,6}] + 4\Lambda_7^\vee[\mathbf{1,7}] + 4\Lambda_7^\vee[\mathbf{2,7}] + 4\Lambda_7^\vee[\mathbf{3,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,7}] + \Lambda_4^\vee[\mathbf{5,7}]$
3	6	$\left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1 \supseteq 1 \supseteq 1}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,2 \supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + 5\Lambda_7^\vee\right)[\mathbf{1,2 \supseteq 2}] + \left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 6\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $4\Lambda_7^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + 4\Lambda_7^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + 4\Lambda_7^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 4\Lambda_7^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + 4\Lambda_7^\vee[\mathbf{6 \supseteq 6 \supseteq 6}] + 4\Lambda_7^\vee[\mathbf{7 \supseteq 7 \supseteq 7}]$ $\left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3 \supseteq 1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 6\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,3,4}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,5,7}]$ $\Lambda_4^\vee[\mathbf{1,3 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{2,3 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{2,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + 4\Lambda_7^\vee\right)[\mathbf{1,2,3}] + 4\Lambda_7^\vee[\mathbf{1,4 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{1,4 \supseteq 4}] + 4\Lambda_7^\vee[\mathbf{2,4 \supseteq 2}] + 4\Lambda_7^\vee[\mathbf{2,4 \supseteq 4}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,3,4}] + \left(\Lambda_3^\vee + \Lambda_5^\vee + 8\Lambda_7^\vee\right)[\mathbf{2,3,4}] + 4\Lambda_7^\vee[\mathbf{1,5 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{1,5 \supseteq 5}] + 4\Lambda_7^\vee[\mathbf{2,5 \supseteq 2}] + 4\Lambda_7^\vee[\mathbf{2,5 \supseteq 5}] + 4\Lambda_7^\vee[\mathbf{3,5 \supseteq 3}] + 4\Lambda_7^\vee[\mathbf{3,5 \supseteq 5}] + 4\Lambda_7^\vee[\mathbf{1,6 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{1,6 \supseteq 6}] + 4\Lambda_7^\vee[\mathbf{2,6 \supseteq 2}] + 4\Lambda_7^\vee[\mathbf{2,6 \supseteq 6}] + 4\Lambda_7^\vee[\mathbf{3,6 \supseteq 3}] + 4\Lambda_7^\vee[\mathbf{3,6 \supseteq 6}] + 4\Lambda_7^\vee[\mathbf{4,6 \supseteq 4}] + 4\Lambda_7^\vee[\mathbf{4,6 \supseteq 6}] + 4\Lambda_7^\vee[\mathbf{1,7 \supseteq 1}] + 4\Lambda_7^\vee[\mathbf{1,7 \supseteq 7}] + 4\Lambda_7^\vee[\mathbf{2,7 \supseteq 2}] + 4\Lambda_7^\vee[\mathbf{2,7 \supseteq 7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + 4\Lambda_7^\vee[\mathbf{3,7 \supseteq 7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + 4\Lambda_7^\vee[\mathbf{4,7 \supseteq 7}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee + 6\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \Lambda_4^\vee[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $\Lambda_4^\vee[\mathbf{1,2,3}] + 4\Lambda_7^\vee[\mathbf{2,3,4}] + 4\Lambda_7^\vee[\mathbf{3,4,5}] + 4\Lambda_7^\vee[\mathbf{4,5,6}] + 4\Lambda_7^\vee[\mathbf{5,6,7}]$ $\Lambda_4^\vee[\mathbf{1,3,5}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{1,4,5}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{2,4,5}] + 4\Lambda_7^\vee[\mathbf{1,3,6}] + 4\Lambda_7^\vee[\mathbf{1,4,6}] + 4\Lambda_7^\vee[\mathbf{2,4,6}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{1,5,6}] + \left(\Lambda_3^\vee + 3\Lambda_7^\vee\right)[\mathbf{2,5,6}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{3,5,6}] + 4\Lambda_7^\vee[\mathbf{1,3,7}] + 4\Lambda_7^\vee[\mathbf{1,4,7}] + 4\Lambda_7^\vee[\mathbf{2,4,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,5,7}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1&1\\0&0\\0&1\\0&1\\0&1\\0&0\end{pmatrix}$

A.8.3 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(2)\rangle$

$$[\phi_u] = (1,1,0)$$
$$\textbf{does not lie in the image of } \mathbf{comp}_2$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda^\vee_7[\mathbf{1 \supseteq 1}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2 \supseteq 2}] + 2\Lambda^\vee_7[\mathbf{1, 2}] + \left(-2\Lambda^\vee_7\right)[\mathbf{3 \supseteq 3}] + \left(-2\Lambda^\vee_7\right)[\mathbf{4 \supseteq 4}] + \left(-2\Lambda^\vee_7\right)[\mathbf{5 \supseteq 5}] + \left(-2\Lambda^\vee_7\right)[\mathbf{6 \supseteq 6}] + \left(-\Lambda^\vee_6\right)[\mathbf{7 \supseteq 7}] + \left(-\Lambda^\vee_6 + 2\Lambda^\vee_7\right)[\mathbf{1, 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{2, 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{3, 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{4, 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{5, 7}] + \left(-\Lambda^\vee_5 + 3\Lambda^\vee_6 - 3\Lambda^\vee_7\right)[\mathbf{6, 7}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{1, 7 \supseteq 1}] + \left(-\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{2, 7 \supseteq 2}] + \left(-2\Lambda^\vee_6\right)[\mathbf{2, 7 \supseteq 7}] + \left(-\Lambda^\vee_6\right)[\mathbf{1, 2, 7}] + \left(-\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{3, 7 \supseteq 3}] + \left(-2\Lambda^\vee_6\right)[\mathbf{3, 7 \supseteq 7}] + \left(-\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{4, 7 \supseteq 4}] + \left(-2\Lambda^\vee_6\right)[\mathbf{4, 7 \supseteq 7}] + \left(-\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{5, 7 \supseteq 5}] + \left(-\Lambda^\vee_4 + 2\Lambda^\vee_5 - 3\Lambda^\vee_6\right)[\mathbf{5, 7 \supseteq 7}] + \left(2\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{2, 6, 7}] + \left(2\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{3, 6, 7}] + \left(2\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{4, 6, 7}] + \Lambda^\vee_4[\mathbf{5, 6, 7}]$ $2\Lambda^\vee_7[\mathbf{1, 3 \supseteq 1}] + 2\Lambda^\vee_7[\mathbf{1, 3 \supseteq 3}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 2, 3}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 4 \supseteq 1}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 4 \supseteq 4}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2, 4 \supseteq 2}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2, 4 \supseteq 4}] + 2\Lambda^\vee_7[\mathbf{1, 3, 4}] + 2\Lambda^\vee_7[\mathbf{2, 3, 4}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 5 \supseteq 1}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 5 \supseteq 5}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2, 5 \supseteq 2}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2, 5 \supseteq 5}] + \left(-2\Lambda^\vee_7\right)[\mathbf{3, 5 \supseteq 3}] + \left(-2\Lambda^\vee_7\right)[\mathbf{3, 5 \supseteq 5}] + 2\Lambda^\vee_7[\mathbf{3, 4, 5}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 6 \supseteq 1}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 6 \supseteq 6}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2, 6 \supseteq 2}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2, 6 \supseteq 6}] + \left(-2\Lambda^\vee_7\right)[\mathbf{3, 6 \supseteq 3}] + \left(-2\Lambda^\vee_7\right)[\mathbf{3, 6 \supseteq 6}] + \left(-2\Lambda^\vee_7\right)[\mathbf{4, 6 \supseteq 4}] + \left(-2\Lambda^\vee_7\right)[\mathbf{4, 6 \supseteq 6}] + 2\Lambda^\vee_7[\mathbf{4, 5, 6}] + \left(-2\Lambda^\vee_7\right)[\mathbf{1, 7 \supseteq 1}] + \left(-\Lambda^\vee_6\right)[\mathbf{1, 7 \supseteq 7}] + \left(-2\Lambda^\vee_7\right)[\mathbf{2, 7 \supseteq 2}] + \left(-\Lambda^\vee_6\right)[\mathbf{2, 7 \supseteq 7}] + \left(-2\Lambda^\vee_7\right)[\mathbf{3, 7 \supseteq 3}] + \left(-\Lambda^\vee_6\right)[\mathbf{3, 7 \supseteq 7}] + \left(-\Lambda^\vee_6 + 2\Lambda^\vee_7\right)[\mathbf{1, 3, 7}] + \left(-2\Lambda^\vee_7\right)[\mathbf{4, 7 \supseteq 4}] + \left(-\Lambda^\vee_6\right)[\mathbf{4, 7 \supseteq 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{1, 4, 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{2, 4, 7}] + \left(-2\Lambda^\vee_7\right)[\mathbf{5, 7 \supseteq 5}] + \left(-\Lambda^\vee_6\right)[\mathbf{5, 7 \supseteq 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{1, 5, 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{2, 5, 7}] + \left(\Lambda^\vee_6 - 2\Lambda^\vee_7\right)[\mathbf{3, 5, 7}] + \left(-\Lambda^\vee_5 + 3\Lambda^\vee_6 - 3\Lambda^\vee_7\right)[\mathbf{1, 6, 7}] + \left(-\Lambda^\vee_5 + 3\Lambda^\vee_6 - 3\Lambda^\vee_7\right)[\mathbf{2, 6, 7}] + \left(-\Lambda^\vee_5 + 3\Lambda^\vee_6 - 3\Lambda^\vee_7\right)[\mathbf{3, 6, 7}] + \left(-\Lambda^\vee_5 + 3\Lambda^\vee_6 - 3\Lambda^\vee_7\right)[\mathbf{4, 6, 7}] + \left(\Lambda^\vee_5 - \Lambda^\vee_6 + \Lambda^\vee_7\right)[\mathbf{5, 6, 7}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda^\vee_7[]$
1	2	$\Lambda^\vee_2[\mathbf{1}] + 2\Lambda^\vee_7[\mathbf{2}] + 2\Lambda^\vee_7[\mathbf{3}] + 2\Lambda^\vee_7[\mathbf{4}] + 2\Lambda^\vee_7[\mathbf{5}] + 2\Lambda^\vee_7[\mathbf{6}] + 2\Lambda^\vee_7[\mathbf{7}]$ $2\Lambda^\vee_7[\mathbf{1}] + 2\Lambda^\vee_7[\mathbf{2}] + 2\Lambda^\vee_7[\mathbf{3}] + 2\Lambda^\vee_7[\mathbf{4}] + 2\Lambda^\vee_7[\mathbf{5}] + 2\Lambda^\vee_7[\mathbf{6}] + 2\Lambda^\vee_7[\mathbf{7}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
2	3	$\Lambda_2^\vee[\mathbf{1\supseteq 1}]+2\Lambda_7^\vee[\mathbf{2\supseteq 2}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{1,2}]+2\Lambda_7^\vee[\mathbf{3\supseteq 3}]+2\Lambda_7^\vee[\mathbf{4\supseteq 4}]+2\Lambda_7^\vee[\mathbf{5\supseteq 5}]+2\Lambda_7^\vee[\mathbf{6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{7\supseteq 7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3,7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{4,7}]+\Lambda_4^\vee[\mathbf{5,7}]$ $2\Lambda_7^\vee[\mathbf{1\supseteq 1}]+2\Lambda_7^\vee[\mathbf{2\supseteq 2}]+2\Lambda_7^\vee[\mathbf{3\supseteq 3}]+2\Lambda_7^\vee[\mathbf{4\supseteq 4}]+2\Lambda_7^\vee[\mathbf{5\supseteq 5}]+2\Lambda_7^\vee[\mathbf{6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{7\supseteq 7}]$ $\Lambda_2^\vee[\mathbf{1,3}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{2,3}]+2\Lambda_7^\vee[\mathbf{1,4}]+2\Lambda_7^\vee[\mathbf{2,4}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{3,4}]+2\Lambda_7^\vee[\mathbf{1,5}]+2\Lambda_7^\vee[\mathbf{2,5}]+2\Lambda_7^\vee[\mathbf{3,5}]+2\Lambda_7^\vee[\mathbf{1,6}]+2\Lambda_7^\vee[\mathbf{2,6}]+2\Lambda_7^\vee[\mathbf{3,6}]+2\Lambda_7^\vee[\mathbf{4,6}]+2\Lambda_7^\vee[\mathbf{1,7}]+2\Lambda_7^\vee[\mathbf{2,7}]+\Lambda_6^\vee[\mathbf{3,7}]+2\Lambda_7^\vee[\mathbf{4,7}]+2\Lambda_7^\vee[\mathbf{5,7}]$
3	6	$\Lambda_2^\vee[\mathbf{1\supseteq 1\supseteq 1}]+2\Lambda_7^\vee[\mathbf{2\supseteq 2\supseteq 2}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 2}]+2\Lambda_7^\vee[\mathbf{3\supseteq 3\supseteq 3}]+\Lambda_2^\vee[\mathbf{1,2,3}]+2\Lambda_7^\vee[\mathbf{4\supseteq 4\supseteq 4}]+2\Lambda_7^\vee[\mathbf{2,3,4}]+2\Lambda_7^\vee[\mathbf{5\supseteq 5\supseteq 5}]+2\Lambda_7^\vee[\mathbf{3,4,5}]+2\Lambda_7^\vee[\mathbf{6\supseteq 6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{4,5,6}]+2\Lambda_7^\vee[\mathbf{7\supseteq 7\supseteq 7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,7\supseteq 2}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3,7\supseteq 3}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{4,7\supseteq 4}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{5,7\supseteq 5}]+\Lambda_4^\vee[\mathbf{5,7\supseteq 7}]+\left(\Lambda_2^\vee+\Lambda_4^\vee+2\Lambda_7^\vee\right)[\mathbf{3,5,7}]+\left(\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{4,5,7}]+\left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $2\Lambda_7^\vee[\mathbf{1\supseteq 1\supseteq 1}]+2\Lambda_7^\vee[\mathbf{2\supseteq 2\supseteq 2}]+2\Lambda_7^\vee[\mathbf{3\supseteq 3\supseteq 3}]+2\Lambda_7^\vee[\mathbf{4\supseteq 4\supseteq 4}]+2\Lambda_7^\vee[\mathbf{5\supseteq 5\supseteq 5}]+2\Lambda_7^\vee[\mathbf{6\supseteq 6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{7\supseteq 7\supseteq 7}]$ $\left(\Lambda_1^\vee+\Lambda_3^\vee+4\Lambda_7^\vee\right)[\mathbf{2\supseteq 2\supseteq 2}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 2}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 2}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 3}]+\Lambda_2^\vee[\mathbf{1,2,3}]+2\Lambda_7^\vee[\mathbf{2,3,4}]+2\Lambda_7^\vee[\mathbf{3,4,5}]+2\Lambda_7^\vee[\mathbf{4,5,6}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,7\supseteq 2}]+2\Lambda_7^\vee[\mathbf{5,6,7}]$ $\Lambda_2^\vee[\mathbf{1,3\supseteq 1}]+2\Lambda_7^\vee[\mathbf{1,3\supseteq 3}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_7^\vee\right)[\mathbf{1,2,3}]+2\Lambda_7^\vee[\mathbf{1,4\supseteq 1}]+2\Lambda_7^\vee[\mathbf{1,4\supseteq 4}]+2\Lambda_7^\vee[\mathbf{2,4\supseteq 2}]+2\Lambda_7^\vee[\mathbf{2,4\supseteq 4}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,3,4}]+2\Lambda_7^\vee[\mathbf{1,5\supseteq 1}]+2\Lambda_7^\vee[\mathbf{1,5\supseteq 5}]+2\Lambda_7^\vee[\mathbf{2,5\supseteq 2}]+2\Lambda_7^\vee[\mathbf{2,5\supseteq 5}]+2\Lambda_7^\vee[\mathbf{3,5\supseteq 3}]+2\Lambda_7^\vee[\mathbf{3,5\supseteq 5}]+2\Lambda_7^\vee[\mathbf{1,6\supseteq 1}]+2\Lambda_7^\vee[\mathbf{1,6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{2,6\supseteq 2}]+2\Lambda_7^\vee[\mathbf{2,6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{3,6\supseteq 3}]+2\Lambda_7^\vee[\mathbf{3,6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{4,6\supseteq 4}]+2\Lambda_7^\vee[\mathbf{4,6\supseteq 6}]+2\Lambda_7^\vee[\mathbf{1,7\supseteq 1}]+2\Lambda_7^\vee[\mathbf{1,7\supseteq 7}]+2\Lambda_7^\vee[\mathbf{2,7\supseteq 2}]+2\Lambda_7^\vee[\mathbf{2,7\supseteq 7}]+2\Lambda_7^\vee[\mathbf{3,7\supseteq 3}]+2\Lambda_7^\vee[\mathbf{3,7\supseteq 7}]+2\Lambda_7^\vee[\mathbf{4,7\supseteq 4}]+2\Lambda_7^\vee[\mathbf{4,7\supseteq 7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{1,4,7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,4,7}]+2\Lambda_7^\vee[\mathbf{5,7\supseteq 5}]+2\Lambda_7^\vee[\mathbf{5,7\supseteq 7}]+\Lambda_4^\vee[\mathbf{1,5,7}]+\Lambda_4^\vee[\mathbf{2,5,7}]+\left(\Lambda_2^\vee+2\Lambda_7^\vee\right)[\mathbf{3,5,7}]$ $\Lambda_4^\vee[\mathbf{1,3\supseteq 1}]+\left(\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 2}]+\left(\Lambda_1^\vee+3\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 3}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_7^\vee\right)[\mathbf{1,2,3}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,3,4}]+\left(\Lambda_3^\vee+\Lambda_5^\vee+2\Lambda_7^\vee\right)[\mathbf{2,3,4}]+2\Lambda_7^\vee[\mathbf{3,4,5}]+2\Lambda_7^\vee[\mathbf{4,5,6}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3,7\supseteq 3}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{4,7\supseteq 4}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{1,4,7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,4,7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{5,7\supseteq 5}]+\Lambda_4^\vee[\mathbf{5,7\supseteq 7}]+\Lambda_4^\vee[\mathbf{1,5,7}]+\Lambda_4^\vee[\mathbf{2,5,7}]+\Lambda_4^\vee[\mathbf{3,5,7}]+\left(\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{4,5,7}]+\left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $\Lambda_2^\vee[\mathbf{1,3,5}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{2,3,5}]+2\Lambda_7^\vee[\mathbf{1,3,6}]+2\Lambda_7^\vee[\mathbf{1,4,6}]+2\Lambda_7^\vee[\mathbf{2,4,6}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{3,5,6}]+2\Lambda_7^\vee[\mathbf{1,3,7}]+2\Lambda_7^\vee[\mathbf{1,4,7}]+2\Lambda_7^\vee[\mathbf{2,4,7}]+2\Lambda_7^\vee[\mathbf{1,5,7}]+2\Lambda_7^\vee[\mathbf{2,5,7}]+\Lambda_6^\vee[\mathbf{3,5,7}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1&0\\1&0\\1&1\\0&1\\0&0\\0&0\end{pmatrix}$

A.8.4 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{aligned}\phi_u \;=\; &\partial\tau \text{ with } \tau=\Lambda_1^\vee[\mathbf{1}]+(\Lambda_1^\vee+\Lambda_2^\vee)[\mathbf{2}]+(\Lambda_2^\vee+\Lambda_3^\vee)[\mathbf{3}]+(\Lambda_3^\vee+\Lambda_4^\vee)[\mathbf{4}]+\\ &(\Lambda_4^\vee+\Lambda_5^\vee)[\mathbf{5}]+(\Lambda_5^\vee+\Lambda_6^\vee)[\mathbf{6}]+\Lambda_7^\vee[\mathbf{7}]\end{aligned}}$$

k	$\mathbf{H^k}(\mathbf{W_0},\mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	0	

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
2	0	
3	0	

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	0	
3	1	$\Lambda_7^\vee[\mathbf{1,3,5}] + \Lambda_7^\vee[\mathbf{1,3,6}] + \Lambda_7^\vee[\mathbf{1,4,6}] + \Lambda_7^\vee[\mathbf{2,4,6}] + \Lambda_5^\vee[\mathbf{1,3,7}] + \Lambda_5^\vee[\mathbf{1,4,7}] + \Lambda_5^\vee[\mathbf{2,4,7}] + \Lambda_3^\vee[\mathbf{1,5,7}] + \Lambda_3^\vee[\mathbf{2,5,7}] + \Lambda_1^\vee[\mathbf{3,5,7}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

A.8.5 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1 \supseteq 1}] + (-1)\,[\mathbf{2 \supseteq 2}] + [\mathbf{1,2}] + (-1)\,[\mathbf{3 \supseteq 3}] + (-1)\,[\mathbf{4 \supseteq 4}] + (-1)\,[\mathbf{5 \supseteq 5}] + (-1)\,[\mathbf{6 \supseteq 6}] + (-1)\,[\mathbf{7 \supseteq 7}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)\,[\mathbf{1,2,3}] + (-1)\,[\mathbf{1,4 \supseteq 1}] + (-1)\,[\mathbf{1,4 \supseteq 4}] + (-1)\,[\mathbf{2,4 \supseteq 2}] + (-1)\,[\mathbf{2,4 \supseteq 4}] + [\mathbf{1,3,4}] + [\mathbf{2,3,4}] + (-1)\,[\mathbf{1,5 \supseteq 1}] + (-1)\,[\mathbf{1,5 \supseteq 5}] + (-1)\,[\mathbf{2,5 \supseteq 2}] + (-1)\,[\mathbf{2,5 \supseteq 5}] + (-1)\,[\mathbf{3,5 \supseteq 3}] + (-1)\,[\mathbf{3,5 \supseteq 5}] + [\mathbf{3,4,5}] + (-1)\,[\mathbf{1,6 \supseteq 1}] + (-1)\,[\mathbf{1,6 \supseteq 6}] + (-1)\,[\mathbf{2,6 \supseteq 2}] + (-1)\,[\mathbf{2,6 \supseteq 6}] + (-1)\,[\mathbf{3,6 \supseteq 3}] + (-1)\,[\mathbf{3,6 \supseteq 6}] + (-1)\,[\mathbf{4,6 \supseteq 4}] + (-1)\,[\mathbf{4,6 \supseteq 6}] + [\mathbf{4,5,6}] + (-1)\,[\mathbf{1,7 \supseteq 1}] + (-1)\,[\mathbf{1,7 \supseteq 7}] + (-1)\,[\mathbf{2,7 \supseteq 2}] + (-1)\,[\mathbf{2,7 \supseteq 7}] + (-1)\,[\mathbf{3,7 \supseteq 3}] + (-1)\,[\mathbf{3,7 \supseteq 7}] + (-1)\,[\mathbf{4,7 \supseteq 4}] + (-1)\,[\mathbf{4,7 \supseteq 7}] + (-1)\,[\mathbf{5,7 \supseteq 5}] + (-1)\,[\mathbf{5,7 \supseteq 7}] + [\mathbf{5,6,7}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}] + [\mathbf{7}]$
2	2	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}] + [\mathbf{6 \supseteq 6}] + [\mathbf{7 \supseteq 7}]$
		$[\mathbf{1,3}]+[\mathbf{1,4}]+[\mathbf{2,4}]+[\mathbf{1,5}]+[\mathbf{2,5}]+[\mathbf{3,5}]+[\mathbf{1,6}]+[\mathbf{2,6}]+[\mathbf{3,6}]+[\mathbf{4,6}]+[\mathbf{1,7}]+[\mathbf{2,7}]+[\mathbf{3,7}]+[\mathbf{4,7}]+[\mathbf{5,7}]$

k	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
3	4	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}] + [\mathbf{7} \supseteq \mathbf{7} \supseteq \mathbf{7}]$
		$[\mathbf{1,3} \supseteq \mathbf{1}] + [\mathbf{1,3} \supseteq \mathbf{3}] + [\mathbf{1,4} \supseteq \mathbf{1}] + [\mathbf{1,4} \supseteq \mathbf{4}] + [\mathbf{2,4} \supseteq \mathbf{2}] + [\mathbf{2,4} \supseteq \mathbf{4}] + [\mathbf{1,5} \supseteq \mathbf{1}] + [\mathbf{1,5} \supseteq \mathbf{5}] + [\mathbf{2,5} \supseteq \mathbf{2}] + [\mathbf{2,5} \supseteq \mathbf{5}] + [\mathbf{3,5} \supseteq \mathbf{3}] + [\mathbf{3,5} \supseteq \mathbf{5}] + [\mathbf{1,6} \supseteq \mathbf{1}] + [\mathbf{1,6} \supseteq \mathbf{6}] + [\mathbf{2,6} \supseteq \mathbf{2}] + [\mathbf{2,6} \supseteq \mathbf{6}] + [\mathbf{3,6} \supseteq \mathbf{3}] + [\mathbf{3,6} \supseteq \mathbf{6}] + [\mathbf{4,6} \supseteq \mathbf{4}] + [\mathbf{4,6} \supseteq \mathbf{6}] + [\mathbf{1,7} \supseteq \mathbf{1}] + [\mathbf{1,7} \supseteq \mathbf{7}] + [\mathbf{2,7} \supseteq \mathbf{2}] + [\mathbf{2,7} \supseteq \mathbf{7}] + [\mathbf{3,7} \supseteq \mathbf{3}] + [\mathbf{3,7} \supseteq \mathbf{7}] + [\mathbf{4,7} \supseteq \mathbf{4}] + [\mathbf{4,7} \supseteq \mathbf{7}] + [\mathbf{5,7} \supseteq \mathbf{5}] + [\mathbf{5,7} \supseteq \mathbf{7}]$
		$[\mathbf{1,2,3}] + [\mathbf{2,3,4}] + [\mathbf{3,4,5}] + [\mathbf{4,5,6}] + [\mathbf{5,6,7}]$
		$[\mathbf{1,3,5}] + [\mathbf{1,3,6}] + [\mathbf{1,4,6}] + [\mathbf{2,4,6}] + [\mathbf{1,3,7}] + [\mathbf{1,4,7}] + [\mathbf{2,4,7}] + [\mathbf{1,5,7}] + [\mathbf{2,5,7}] + [\mathbf{3,5,7}]$

A.9 Root system A_8

Dynkin diagram	$\underset{1}{\circ}-\underset{2}{\circ}-\underset{3}{\circ}-\underset{4}{\circ}-\underset{5}{\circ}-\underset{6}{\circ}-\underset{7}{\circ}-\underset{8}{\circ}$
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/9\mathbb{Z}$ generated by $\Lambda_8^\vee \in P^\vee \mod Q^\vee$

A.9.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\begin{aligned}\phi_u \;=\; & \partial\tau \text{ with } \tau = (\Lambda_1^\vee + 10\Lambda_8^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 12\Lambda_8^\vee)\,[\mathbf{2}] + \\ & (\Lambda_2^\vee + \Lambda_3^\vee + 14\Lambda_8^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee + 16\Lambda_8^\vee)\,[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee + 18\Lambda_8^\vee)\,[\mathbf{5}] + \\ & (\Lambda_5^\vee + \Lambda_6^\vee + 20\Lambda_8^\vee)\,[\mathbf{6}] + (\Lambda_7^\vee + 7\Lambda_8^\vee)\,[\mathbf{7}] + (\Lambda_7^\vee + 7\Lambda_8^\vee)\,[\mathbf{8}]\end{aligned}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/9\mathbb{Z}$	$\left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{8}]$
2	0	
3	0	

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
0	0	
1	0	
2	0	
3	0	

k	0	1	2	3
comp$_k$	()	()	()	()

A.9.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(3)\rangle$

$\phi_u = \partial\tau$ with $\tau = (\Lambda_1^\vee + 4\Lambda_8^\vee)\,[\mathbf{1}] + (\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_8^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_8^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee + 4\Lambda_8^\vee)\,[\mathbf{4}] + (\Lambda_5^\vee + \Lambda_6^\vee + 2\Lambda_8^\vee)\,[\mathbf{5}] + (\Lambda_5^\vee + \Lambda_6^\vee + 2\Lambda_8^\vee)\,[\mathbf{6}] + (\Lambda_7^\vee + \Lambda_8^\vee)\,[\mathbf{7}] + (\Lambda_7^\vee + \Lambda_8^\vee)\,[\mathbf{8}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/3\mathbb{Z}$	$(\Lambda_7^\vee - 2\Lambda_8^\vee)\,[\mathbf{8}]$
2	0	
3	0	

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
0	0	
1	0	
2	0	
3	0	

k	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

A.9.3 Cohomology of coweight lattice $X^\vee = P^\vee$

$\phi_u = \partial\tau$ with $\tau = \Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee)\,[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee)\,[\mathbf{5}] + (\Lambda_5^\vee + \Lambda_6^\vee)\,[\mathbf{6}] + (\Lambda_6^\vee + \Lambda_7^\vee)\,[\mathbf{7}] + \Lambda_8^\vee[\mathbf{8}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	0	
3	0	

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
0	0	

$\mathbf{k}$	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
1	0	
2	0	
3	0	

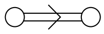
$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

A.9.4 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0,\mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1}\supseteq\mathbf{1}]+(-1)\,[\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{1,2}]+(-1)\,[\mathbf{3}\supseteq\mathbf{3}]+(-1)\,[\mathbf{4}\supseteq\mathbf{4}]+(-1)\,[\mathbf{5}\supseteq\mathbf{5}]+(-1)\,[\mathbf{6}\supseteq\mathbf{6}]+(-1)\,[\mathbf{7}\supseteq\mathbf{7}]+(-1)\,[\mathbf{8}\supseteq\mathbf{8}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1,3}\supseteq\mathbf{1}]+[\mathbf{1,3}\supseteq\mathbf{3}]+(-1)\,[\mathbf{1,2,3}]+(-1)\,[\mathbf{1,4}\supseteq\mathbf{1}]+(-1)\,[\mathbf{1,4}\supseteq\mathbf{4}]+(-1)\,[\mathbf{2,4}\supseteq\mathbf{2}]+(-1)\,[\mathbf{2,4}\supseteq\mathbf{4}]+[\mathbf{1,3,4}]+[\mathbf{2,3,4}]+(-1)\,[\mathbf{1,5}\supseteq\mathbf{1}]+(-1)\,[\mathbf{1,5}\supseteq\mathbf{5}]+(-1)\,[\mathbf{2,5}\supseteq\mathbf{2}]+(-1)\,[\mathbf{2,5}\supseteq\mathbf{5}]+(-1)\,[\mathbf{3,5}\supseteq\mathbf{3}]+(-1)\,[\mathbf{3,5}\supseteq\mathbf{5}]+[\mathbf{3,4,5}]+(-1)\,[\mathbf{1,6}\supseteq\mathbf{1}]+(-1)\,[\mathbf{1,6}\supseteq\mathbf{6}]+(-1)\,[\mathbf{2,6}\supseteq\mathbf{2}]+(-1)\,[\mathbf{2,6}\supseteq\mathbf{6}]+(-1)\,[\mathbf{3,6}\supseteq\mathbf{3}]+(-1)\,[\mathbf{3,6}\supseteq\mathbf{6}]+(-1)\,[\mathbf{4,6}\supseteq\mathbf{4}]+(-1)\,[\mathbf{4,6}\supseteq\mathbf{6}]+[\mathbf{4,5,6}]+(-1)\,[\mathbf{1,7}\supseteq\mathbf{1}]+(-1)\,[\mathbf{1,7}\supseteq\mathbf{7}]+(-1)\,[\mathbf{2,7}\supseteq\mathbf{2}]+(-1)\,[\mathbf{2,7}\supseteq\mathbf{7}]+(-1)\,[\mathbf{3,7}\supseteq\mathbf{3}]+(-1)\,[\mathbf{3,7}\supseteq\mathbf{7}]+(-1)\,[\mathbf{4,7}\supseteq\mathbf{4}]+(-1)\,[\mathbf{4,7}\supseteq\mathbf{7}]+(-1)\,[\mathbf{5,7}\supseteq\mathbf{5}]+(-1)\,[\mathbf{5,7}\supseteq\mathbf{7}]+[\mathbf{5,6,7}]+(-1)\,[\mathbf{1,8}\supseteq\mathbf{1}]+(-1)\,[\mathbf{1,8}\supseteq\mathbf{8}]+(-1)\,[\mathbf{2,8}\supseteq\mathbf{2}]+(-1)\,[\mathbf{2,8}\supseteq\mathbf{8}]+(-1)\,[\mathbf{3,8}\supseteq\mathbf{3}]+(-1)\,[\mathbf{3,8}\supseteq\mathbf{8}]+(-1)\,[\mathbf{4,8}\supseteq\mathbf{4}]+(-1)\,[\mathbf{4,8}\supseteq\mathbf{8}]+(-1)\,[\mathbf{5,8}\supseteq\mathbf{5}]+(-1)\,[\mathbf{5,8}\supseteq\mathbf{8}]+(-1)\,[\mathbf{6,8}\supseteq\mathbf{6}]+(-1)\,[\mathbf{6,8}\supseteq\mathbf{8}]+[\mathbf{6,7,8}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	1	$[\mathbf{1}]+[\mathbf{2}]+[\mathbf{3}]+[\mathbf{4}]+[\mathbf{5}]+[\mathbf{6}]+[\mathbf{7}]+[\mathbf{8}]$
2	2	$[\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{3}\supseteq\mathbf{3}]+[\mathbf{4}\supseteq\mathbf{4}]+[\mathbf{5}\supseteq\mathbf{5}]+[\mathbf{6}\supseteq\mathbf{6}]+[\mathbf{7}\supseteq\mathbf{7}]+[\mathbf{8}\supseteq\mathbf{8}]$
		$[\mathbf{1,3}]+[\mathbf{1,4}]+[\mathbf{2,4}]+[\mathbf{1,5}]+[\mathbf{2,5}]+[\mathbf{3,5}]+[\mathbf{1,6}]+[\mathbf{2,6}]+[\mathbf{3,6}]+[\mathbf{4,6}]+[\mathbf{1,7}]+[\mathbf{2,7}]+[\mathbf{3,7}]+[\mathbf{4,7}]+[\mathbf{5,7}]+[\mathbf{1,8}]+[\mathbf{2,8}]+[\mathbf{3,8}]+[\mathbf{4,8}]+[\mathbf{5,8}]+[\mathbf{6,8}]$
3	4	$[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]+[\mathbf{6}\supseteq\mathbf{6}\supseteq\mathbf{6}]+[\mathbf{7}\supseteq\mathbf{7}\supseteq\mathbf{7}]+[\mathbf{8}\supseteq\mathbf{8}\supseteq\mathbf{8}]$
		$[\mathbf{1,3}\supseteq\mathbf{1}]+[\mathbf{1,3}\supseteq\mathbf{3}]+[\mathbf{1,4}\supseteq\mathbf{1}]+[\mathbf{1,4}\supseteq\mathbf{4}]+[\mathbf{2,4}\supseteq\mathbf{2}]+[\mathbf{2,4}\supseteq\mathbf{4}]+[\mathbf{1,5}\supseteq\mathbf{1}]+[\mathbf{1,5}\supseteq\mathbf{5}]+[\mathbf{2,5}\supseteq\mathbf{2}]+[\mathbf{2,5}\supseteq\mathbf{5}]+[\mathbf{3,5}\supseteq\mathbf{3}]+[\mathbf{3,5}\supseteq\mathbf{5}]+[\mathbf{1,6}\supseteq\mathbf{1}]+[\mathbf{1,6}\supseteq\mathbf{6}]+[\mathbf{2,6}\supseteq\mathbf{2}]+[\mathbf{2,6}\supseteq\mathbf{6}]+[\mathbf{3,6}\supseteq\mathbf{3}]+[\mathbf{3,6}\supseteq\mathbf{6}]+[\mathbf{4,6}\supseteq\mathbf{4}]+[\mathbf{4,6}\supseteq\mathbf{6}]+[\mathbf{1,7}\supseteq\mathbf{1}]+[\mathbf{1,7}\supseteq\mathbf{7}]+[\mathbf{2,7}\supseteq\mathbf{2}]+[\mathbf{2,7}\supseteq\mathbf{7}]+[\mathbf{3,7}\supseteq\mathbf{3}]+[\mathbf{3,7}\supseteq\mathbf{7}]+[\mathbf{4,7}\supseteq\mathbf{4}]+[\mathbf{4,7}\supseteq\mathbf{7}]+[\mathbf{5,7}\supseteq\mathbf{5}]+[\mathbf{5,7}\supseteq\mathbf{7}]+[\mathbf{1,8}\supseteq\mathbf{1}]+[\mathbf{1,8}\supseteq\mathbf{8}]+[\mathbf{2,8}\supseteq\mathbf{2}]+[\mathbf{2,8}\supseteq\mathbf{8}]+[\mathbf{3,8}\supseteq\mathbf{3}]+[\mathbf{3,8}\supseteq\mathbf{8}]+[\mathbf{4,8}\supseteq\mathbf{4}]+[\mathbf{4,8}\supseteq\mathbf{8}]+[\mathbf{5,8}\supseteq\mathbf{5}]+[\mathbf{5,8}\supseteq\mathbf{8}]+[\mathbf{6,8}\supseteq\mathbf{6}]+[\mathbf{6,8}\supseteq\mathbf{8}]$
		$[\mathbf{1,2,3}]+[\mathbf{2,3,4}]+[\mathbf{3,4,5}]+[\mathbf{4,5,6}]+[\mathbf{5,6,7}]+[\mathbf{6,7,8}]$
		$[\mathbf{1,3,5}]+[\mathbf{1,3,6}]+[\mathbf{1,4,6}]+[\mathbf{2,4,6}]+[\mathbf{1,3,7}]+[\mathbf{1,4,7}]+[\mathbf{2,4,7}]+[\mathbf{1,5,7}]+[\mathbf{2,5,7}]+[\mathbf{3,5,7}]+[\mathbf{1,3,8}]+[\mathbf{1,4,8}]+[\mathbf{2,4,8}]+[\mathbf{1,5,8}]+[\mathbf{2,5,8}]+[\mathbf{3,5,8}]+[\mathbf{1,6,8}]+[\mathbf{2,6,8}]+[\mathbf{3,6,8}]+[\mathbf{4,6,8}]$

A.10 Root system B_2

Dynkin diagram	 1 2
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_1^\vee \in P^\vee \mod Q^\vee$

A.10.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] = (1,1,0)$
does not lie in the image of comp$_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(2\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_1^\vee + 2\Lambda_2^\vee\right)[\mathbf{1},\mathbf{2}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}]$
		$\left(2\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_1^\vee[]$
1	2	$2\Lambda_1^\vee[\mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1}]$
2	3	$2\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}]$
		$2\Lambda_1^\vee[\mathbf{1},\mathbf{2}]$
3	4	$2\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$
		$2\Lambda_1^\vee[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}]$

k	0	1	2	3
comp$_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\\0&0\\1&0\end{pmatrix}$

A.10.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}]}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{2}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_1^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(-2\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1}, \mathbf{2}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_1^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$
		$\left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$\Lambda_2^\vee[]$
1	2	$\Lambda_1^\vee[\mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{2}]$
2	3	$\Lambda_1^\vee[\mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{2}]$
3	4	$\Lambda_1^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_1^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\\1&0\\0&0\end{pmatrix}$

A.10.3 Cohomology with trivial coefficients

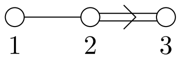
k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{1} \supseteq \mathbf{1}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1,2 \supseteq 1}] + [\mathbf{1,2 \supseteq 2}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}]$
		$[\mathbf{2}]$
2	3	$[\mathbf{1 \supseteq 1}]$
		$[\mathbf{2 \supseteq 2}]$
		$[\mathbf{1,2}]$
3	4	$[\mathbf{1 \supseteq 1 \supseteq 1}]$
		$[\mathbf{2 \supseteq 2 \supseteq 2}]$
		$[\mathbf{1,2 \supseteq 1}]$
		$[\mathbf{1,2 \supseteq 2}]$

A.11 Root system B_3

Dynkin diagram



Fundamental group

$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$
generated by $\Lambda_3^\vee \in P^\vee \mod Q^\vee$

A.11.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (1,1,1,0,1)$$
does not lie in the image of $\mathbf{comp}_2$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(2\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{3 \supseteq 3}] + \left(\Lambda_1^\vee - 2\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2,3}]$
		$2\Lambda_3^\vee[\mathbf{1 \supseteq 1}] + \left(-2\Lambda_3^\vee\right)[\mathbf{2 \supseteq 2}] + 2\Lambda_3^\vee[\mathbf{1,2}] + \left(-2\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1,3}] + \left(4\Lambda_2^\vee - 4\Lambda_3^\vee\right)[\mathbf{2,3}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_3^\vee[\mathbf{2,3 \supseteq 2}] + 2\Lambda_2^\vee[\mathbf{2,3 \supseteq 3}] + \left(4\Lambda_1^\vee - 4\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1,2,3}]$
		$\left(2\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1,3 \supseteq 3}]$
		$2\Lambda_3^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_2^\vee[\mathbf{1,3 \supseteq 3}] + \left(4\Lambda_1^\vee - 4\Lambda_2^\vee\right)[\mathbf{1,2,3}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
0	1	$2\Lambda_3^\vee[]$
1	3	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}]$
		$2\Lambda_3^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3}]$
2	5	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}]$
3	8	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\0&0\\1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\1&0&0\\1&0&1\\0&0&0\\0&1&0\\0&0&0\\0&0&0\end{pmatrix}$

A.11.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_3^\vee)[\mathbf{1}] + \Lambda_2^\vee[\mathbf{2}]}$$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee - 2\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_3^\vee[\mathbf{1}, \mathbf{2}] + \left(-\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{3}] + \left(2\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}] + \left(2\Lambda_1^\vee - 2\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(2\Lambda_1^\vee - 2\Lambda_2^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$\left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$\Lambda_3^\vee[]$
1	3	$\Lambda_3^\vee[\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{3}]$
2	6	$\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{1}, \mathbf{3}]$
		$\Lambda_1^\vee[\mathbf{2}, \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{2}, \mathbf{3}]$
3	10	$\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}]$
		$\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_1^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}]$
		$\Lambda_1^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

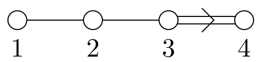
k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

A.11.3 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + [\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	2	$[\mathbf{1}] + [\mathbf{2}]$
		$[\mathbf{3}]$
2	4	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{3}]$
		$[\mathbf{2}, \mathbf{3}]$
3	7	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}]$
		$[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

A.12 Root system B_4

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_3^\vee \in P^\vee \mod Q^\vee$

A.12.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (1,1,1,0)$$
does not lie in the image of comp$_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{2}] + \left(-\Lambda_2^\vee - \Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{3}] + \left(-\Lambda_1^\vee + 3\Lambda_2^\vee - 3\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2}, \mathbf{3}] + \left(2\Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{3}, \mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - \Lambda_4^\vee\right)[\mathbf{3}, \mathbf{4} \supseteq \mathbf{3}]$
		$\Lambda_4^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_4^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}] + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1}, \mathbf{3}, \mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}] + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1}, \mathbf{3}, \mathbf{4}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
0	1	$2\Lambda_3^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3}]$
2	4	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$\left(\Lambda_2^\vee + 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3}]$
3	9	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \Lambda_4^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$\left(\Lambda_2^\vee + 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \Lambda_4^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + 3\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$
		$\left(\Lambda_2^\vee + 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{3}, \mathbf{4} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\1&1&0\\1&0&0\\0&0&1\\0&0&0\\1&1&0\\0&0&0\\0&0&0\end{pmatrix}$

A.12.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_4^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_4^\vee)\,[\mathbf{2}] + \Lambda_3^\vee[\mathbf{3}]}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{4}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_4^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_4^\vee[\mathbf{1},\mathbf{2}] + \left(-\Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{4}] + \left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{2},\mathbf{4}] + \left(2\Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{3}] + \Lambda_3^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee - 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_4^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}] + \left(-2\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}] + \left(-2\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
0	1	$\Lambda_4^\vee[]$
1	3	$\Lambda_4^\vee[\mathbf{1}] + \Lambda_4^\vee[\mathbf{2}] + \Lambda_4^\vee[\mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{4}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	generating cocycles
2	7	$\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_4^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{4}] + \Lambda_4^\vee[\mathbf{2},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{3},\mathbf{4}]$
3	14	$\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \Lambda_4^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \Lambda_4^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{3},\mathbf{4}]$
		$\Lambda_1^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$

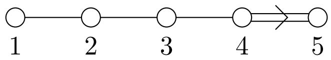
k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&0&1&1&0\\0&0&0&0&0\\0&1&0&0&0\\1&0&0&0&0\\1&1&0&0&0\\0&0&0&0&0\\0&0&1&1&0\\0&1&0&0&0\\0&1&0&0&0\end{pmatrix}$

A.12.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)\,[\mathbf{3} \supseteq \mathbf{3}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{3}, \mathbf{4} \supseteq \mathbf{3}] + [\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}]$
		$[\mathbf{4}]$
2	5	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{3}]$
		$[\mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{4}] + [\mathbf{2}, \mathbf{4}]$
		$[\mathbf{3}, \mathbf{4}]$
3	10	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}]$
		$[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{3}, \mathbf{4} \supseteq \mathbf{3}]$
		$[\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3}, \mathbf{4}]$
		$[\mathbf{2}, \mathbf{3}, \mathbf{4}]$

A.13 Root system B_5

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_5^\vee \in P^\vee \mod Q^\vee$

A.13.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\boxed{\begin{array}{rcl}[\phi_u] & = & (1,1,0,1,0,1,0) \\ & & \textbf{does not lie in the image of comp}_2\end{array}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_4^\vee[\mathbf{5} \supseteq \mathbf{5}] + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{4},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{5}] + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}] + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}] + \left(4\Lambda_4^\vee - 4\Lambda_5^\vee\right)[\mathbf{4},\mathbf{5}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_5^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{4}] + 2\Lambda_4^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_2^\vee + 4\Lambda_3^\vee - 4\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$\left(\Lambda_2^\vee - 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + \left(-2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}] + \left(2\Lambda_3^\vee - 4\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}] + \left(-2\Lambda_3^\vee + 4\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}] + \left(-4\Lambda_3^\vee + 4\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{5}] + \left(-\Lambda_4^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + \left(-\Lambda_4^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}] + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}] + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}] + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}] + \left(-2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}] + \left(4\Lambda_4^\vee - 4\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}] + \left(4\Lambda_4^\vee - 4\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_5^\vee[]$
1	3	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}]$
		$2\Lambda_5^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{5}]$
2	7	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{5}] + 2\Lambda_5^\vee[\mathbf{3},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{3},\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{3},\mathbf{5}]$
		$\left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{5} \supseteq \mathbf{5}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{4},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5}] + \Lambda_4^\vee[\mathbf{2},\mathbf{5}] + \Lambda_4^\vee[\mathbf{3},\mathbf{5}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
3	15	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]$
		$2\Lambda_5^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]$
		$\left(\Lambda_1^\vee+\Lambda_3^\vee+4\Lambda_5^\vee\right)[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\Lambda_4^\vee[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+\Lambda_4^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{3}]+\left(\Lambda_2^\vee+2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]+\left(\Lambda_3^\vee+3\Lambda_5^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]$
		$\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4}\supseteq\mathbf{3}]+\Lambda_4^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}]$
		$\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+\Lambda_4^\vee[\mathbf{1},\mathbf{2},\mathbf{4}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{4},\mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{4},\mathbf{5}\supseteq\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{5}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{3},\mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+\Lambda_4^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\Lambda_4^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_4^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_4^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{4},\mathbf{5}\supseteq\mathbf{5}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\0&0\\0&0\\1&0\\0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&0\\0&0&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$

A.13.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \ \partial\tau \text{ with } \tau = (\Lambda_1^\vee+\Lambda_5^\vee)\,[\mathbf{1}]+(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_5^\vee)\,[\mathbf{2}]+(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee)\,[\mathbf{3}]+\Lambda_4^\vee[\mathbf{4}]}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[5]}$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_4^\vee\mathbf{[5 \supseteq 5]} + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[4, 5]}$
		$\Lambda_5^\vee\mathbf{[1 \supseteq 1]} + \left(-\Lambda_5^\vee\right)\mathbf{[2 \supseteq 2]} + \Lambda_5^\vee\mathbf{[1, 2]} + \left(-\Lambda_5^\vee\right)\mathbf{[3 \supseteq 3]} + \left(-\Lambda_5^\vee\right)\mathbf{[4 \supseteq 4]} + \left(-\Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[1, 5]} + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[2, 5]} + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[3, 5]} + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)\mathbf{[4, 5]}$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee - \Lambda_4^\vee\right)\mathbf{[4, 5 \supseteq 5]}$
		$\Lambda_5^\vee\mathbf{[4, 5 \supseteq 4]} + \Lambda_4^\vee\mathbf{[4, 5 \supseteq 5]} + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 2\Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[3, 4, 5]}$
		$\Lambda_5^\vee\mathbf{[1, 5 \supseteq 1]} + \Lambda_4^\vee\mathbf{[1, 5 \supseteq 5]} + \left(-\Lambda_5^\vee\right)\mathbf{[2, 5 \supseteq 2]} + \left(-\Lambda_4^\vee\right)\mathbf{[2, 5 \supseteq 5]} + \left(-\Lambda_5^\vee\right)\mathbf{[1, 2, 5]} + \left(-\Lambda_5^\vee\right)\mathbf{[3, 5 \supseteq 3]} + \left(-\Lambda_4^\vee\right)\mathbf{[3, 5 \supseteq 5]} + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[1, 4, 5]} + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[2, 4, 5]} + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee\right)\mathbf{[3, 4, 5]}$
		$\Lambda_4^\vee\mathbf{[1, 5 \supseteq 1]} + \Lambda_4^\vee\mathbf{[1, 5 \supseteq 5]} + \left(-\Lambda_4^\vee\right)\mathbf{[2, 5 \supseteq 2]} + \left(-\Lambda_4^\vee\right)\mathbf{[2, 5 \supseteq 5]} + \left(-\Lambda_4^\vee\right)\mathbf{[1, 2, 5]} + \left(-\Lambda_4^\vee\right)\mathbf{[3, 5 \supseteq 3]} + \left(-\Lambda_4^\vee\right)\mathbf{[3, 5 \supseteq 5]} + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[1, 4, 5]} + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[2, 4, 5]} + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee\right)\mathbf{[3, 4, 5]}$
		$\left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[5 \supseteq 5 \supseteq 5]}$
		$\Lambda_5^\vee\mathbf{[1, 3 \supseteq 1]} + \Lambda_5^\vee\mathbf{[1, 3 \supseteq 3]} + \left(-\Lambda_5^\vee\right)\mathbf{[1, 2, 3]} + \left(-\Lambda_5^\vee\right)\mathbf{[1, 4 \supseteq 1]} + \left(-\Lambda_5^\vee\right)\mathbf{[1, 4 \supseteq 4]} + \left(-\Lambda_5^\vee\right)\mathbf{[2, 4 \supseteq 2]} + \left(-\Lambda_5^\vee\right)\mathbf{[2, 4 \supseteq 4]} + \Lambda_5^\vee\mathbf{[1, 3, 4]} + \Lambda_5^\vee\mathbf{[2, 3, 4]} + \left(-\Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[1, 3, 5]} + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)\mathbf{[1, 4, 5]} + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)\mathbf{[2, 4, 5]}$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$\Lambda_5^\vee\mathbf{[]}$
1	3	$\Lambda_5^\vee\mathbf{[1]} + \Lambda_5^\vee\mathbf{[2]} + \Lambda_5^\vee\mathbf{[3]} + \Lambda_5^\vee\mathbf{[4]}$
		$\Lambda_4^\vee\mathbf{[5]}$
		$\Lambda_5^\vee\mathbf{[5]}$
2	8	$\Lambda_5^\vee\mathbf{[1 \supseteq 1]} + \Lambda_5^\vee\mathbf{[2 \supseteq 2]} + \Lambda_5^\vee\mathbf{[3 \supseteq 3]} + \Lambda_5^\vee\mathbf{[4 \supseteq 4]}$
		$\Lambda_5^\vee\mathbf{[1, 3]} + \Lambda_5^\vee\mathbf{[1, 4]} + \Lambda_5^\vee\mathbf{[2, 4]}$
		$\Lambda_4^\vee\mathbf{[5 \supseteq 5]}$
		$\Lambda_5^\vee\mathbf{[5 \supseteq 5]}$
		$\Lambda_4^\vee\mathbf{[1, 5]} + \Lambda_4^\vee\mathbf{[2, 5]} + \Lambda_4^\vee\mathbf{[3, 5]}$
		$\Lambda_5^\vee\mathbf{[1, 5]} + \Lambda_5^\vee\mathbf{[2, 5]} + \Lambda_5^\vee\mathbf{[3, 5]}$
		$\Lambda_3^\vee\mathbf{[4, 5]}$
		$\Lambda_5^\vee\mathbf{[4, 5]}$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
3	18	$\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + \Lambda_5^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \Lambda_5^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \Lambda_5^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{2},\mathbf{3}] + \Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \Lambda_4^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \Lambda_5^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \Lambda_4^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \Lambda_4^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \Lambda_5^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \Lambda_5^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{5}]$
		$\Lambda_3^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4},\mathbf{5}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4},\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{4},\mathbf{5}] + \Lambda_5^\vee[\mathbf{2},\mathbf{4},\mathbf{5}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	0	1	2	3
$\mathbf{comp}_k$	$\begin{pmatrix}\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

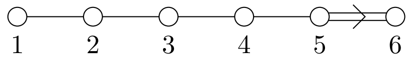
A.13.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)\,[\mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{4} \supseteq \mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{4}, \mathbf{5} \supseteq \mathbf{4}] + [\mathbf{4}, \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{5} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + (-1)\,[\mathbf{2}, \mathbf{5} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{5} \supseteq \mathbf{5}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{5}] + (-1)\,[\mathbf{3}, \mathbf{5} \supseteq \mathbf{3}] + (-1)\,[\mathbf{3}, \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{1}, \mathbf{3}, \mathbf{4}] + [\mathbf{2}, \mathbf{3}, \mathbf{4}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_\mathbf{2})$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}]$
		$[\mathbf{5}]$
2	5	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3}] + [\mathbf{1}, \mathbf{4}] + [\mathbf{2}, \mathbf{4}]$
		$[\mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{5}] + [\mathbf{2}, \mathbf{5}] + [\mathbf{3}, \mathbf{5}]$
		$[\mathbf{4}, \mathbf{5}]$
3	11	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{3}] + [\mathbf{2}, \mathbf{3}, \mathbf{4}]$
		$[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{5} \supseteq \mathbf{1}] + [\mathbf{2}, \mathbf{5} \supseteq \mathbf{2}] + [\mathbf{3}, \mathbf{5} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{2}, \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{3}, \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{3}, \mathbf{5}]$
		$[\mathbf{4}, \mathbf{5} \supseteq \mathbf{4}]$
		$[\mathbf{4}, \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{4}, \mathbf{5}] + [\mathbf{2}, \mathbf{4}, \mathbf{5}]$
		$[\mathbf{3}, \mathbf{4}, \mathbf{5}]$

A.14 Root system B_6

Dynkin diagram



Fundamental group

$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$
generated by $\Lambda_5^\vee \in P^\vee \mod Q^\vee$

A.14.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] \; = \; (1,1,0,1)$
does not lie in the image of comp$_2$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_4^\vee - \Lambda_6^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{1},\mathbf{5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2},\mathbf{5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{4},\mathbf{5}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{5},\mathbf{6}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - \Lambda_6^\vee\right)[\mathbf{5},\mathbf{6} \supseteq \mathbf{5}]$ $\left(\Lambda_4^\vee - 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee - 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \left(-2\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{5}] + \left(-\Lambda_4^\vee - 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 3\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}] + \Lambda_6^\vee[\mathbf{1},\mathbf{4},\mathbf{5}] + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}] + \left(\Lambda_2^\vee + 3\Lambda_6^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}] + \left(4\Lambda_5^\vee - 4\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5},\mathbf{6}] + \left(4\Lambda_5^\vee - 4\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5},\mathbf{6}] + \left(4\Lambda_5^\vee - 4\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5},\mathbf{6}]$ $2\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + \left(-\Lambda_4^\vee - \Lambda_6^\vee\right)[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \left(-\Lambda_4^\vee - \Lambda_6^\vee\right)[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + \left(-\Lambda_4^\vee - \Lambda_6^\vee\right)[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}] + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{5}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5},\mathbf{6}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5},\mathbf{6}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5},\mathbf{6}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_5^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5}]$ $2\Lambda_5^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5}]$
2	4	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_5^\vee\right)[\mathbf{1},\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5} \supseteq \mathbf{5}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2},\mathbf{5}] + \left(\Lambda_2^\vee + \Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}]$ $2\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5} \supseteq \mathbf{5}]$ $\Lambda_2^\vee[\mathbf{1},\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2},\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{3},\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{5}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{5}] + \left(\Lambda_4^\vee + \Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}]$ $\left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{4},\mathbf{5}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
3	10	$\Lambda_2^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1, 2, 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{2, 3, 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2, 5 \supseteq 2}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3, 5 \supseteq 3}] + \left(\Lambda_2^\vee + \Lambda_6^\vee\right)[\mathbf{3, 5 \supseteq 5}] + \Lambda_2^\vee[\mathbf{1, 3, 5}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3, 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3, 4, 5}] + 2\Lambda_5^\vee[\mathbf{3, 5, 6}]$
		$2\Lambda_5^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5 \supseteq 5}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + 4\Lambda_5^\vee\right)[\mathbf{2 \supseteq 2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 2}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 2}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1, 2, 3}] + 2\Lambda_5^\vee[\mathbf{2, 3, 4}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2, 5 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3, 4, 5}]$
		$\Lambda_2^\vee[\mathbf{1, 3 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1, 3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 2, 3}] + 2\Lambda_5^\vee[\mathbf{1, 4 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1, 4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{2, 4 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{2, 4 \supseteq 4}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 3, 4}] + 2\Lambda_5^\vee[\mathbf{1, 5 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1, 5 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{2, 5 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{2, 5 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{3, 5 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{3, 5 \supseteq 5}] + \Lambda_4^\vee[\mathbf{1, 3, 5}]$
		$\Lambda_4^\vee[\mathbf{1, 3 \supseteq 1}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 2}] + \left(\Lambda_1^\vee + 3\Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 2, 3}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 3, 4}] + \left(\Lambda_3^\vee + 3\Lambda_5^\vee\right)[\mathbf{2, 3, 4}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3, 5 \supseteq 3}] + \left(\Lambda_2^\vee + \Lambda_6^\vee\right)[\mathbf{3, 5 \supseteq 5}] + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1, 3, 5}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3, 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3, 4, 5}] + 2\Lambda_5^\vee[\mathbf{3, 5, 6}]$
		$\left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{5 \supseteq 5 \supseteq 5}] + \left(\Lambda_2^\vee + \Lambda_6^\vee\right)[\mathbf{3, 5 \supseteq 5}] + \Lambda_2^\vee[\mathbf{1, 3, 5}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3, 5}] + \Lambda_4^\vee[\mathbf{4, 5 \supseteq 4}] + \left(\Lambda_3^\vee + 3\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{4, 5 \supseteq 5}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{3, 4, 5}] + 2\Lambda_5^\vee[\mathbf{3, 5, 6}]$
		$\left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 5 \supseteq 1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 2, 5}] + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1, 3, 5}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 4, 5}] + 2\Lambda_5^\vee[\mathbf{2, 4, 5}]$
		$\left(\Lambda_4^\vee + 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1, 5 \supseteq 5}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2, 5}] + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1, 3, 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1, 4, 5}] + \left(2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2, 4, 5}] + 2\Lambda_5^\vee[\mathbf{2, 5, 6}] + 2\Lambda_5^\vee[\mathbf{3, 5, 6}]$
		$2\Lambda_5^\vee[\mathbf{5, 6 \supseteq 5}]$
		$\Lambda_4^\vee[\mathbf{5, 6 \supseteq 6}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\0&1&1\\0&0&1\\0&0&0\\1&0&0\\1&1&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

A.14.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\phi_u = \partial\tau \text{ with } \tau = \left(\Lambda_1^\vee + \Lambda_6^\vee\right)[\mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_6^\vee\right)[\mathbf{2}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_6^\vee\right)[\mathbf{3}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_6^\vee\right)[\mathbf{4}] + \Lambda_5^\vee[\mathbf{5}]$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{6}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_5^\vee[\mathbf{6 \supseteq 6}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{5, 6}]$
		$\Lambda_6^\vee[\mathbf{1 \supseteq 1}] + \left(-\Lambda_6^\vee\right)[\mathbf{2 \supseteq 2}] + \Lambda_6^\vee[\mathbf{1, 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{3 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{4 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{5 \supseteq 5}] + \left(-\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1, 6}] + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{2, 6}] + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{3, 6}] + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{4, 6}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{5, 6}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{5, 6 \supseteq 6}]$
		$\Lambda_6^\vee[\mathbf{5, 6 \supseteq 5}] + \Lambda_5^\vee[\mathbf{5, 6 \supseteq 6}] + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{4, 5, 6}]$
		$\Lambda_6^\vee[\mathbf{1, 6 \supseteq 1}] + \Lambda_5^\vee[\mathbf{1, 6 \supseteq 6}] + \left(-\Lambda_6^\vee\right)[\mathbf{2, 6 \supseteq 2}] + \left(-\Lambda_5^\vee\right)[\mathbf{2, 6 \supseteq 6}] + \left(-\Lambda_6^\vee\right)[\mathbf{1, 2, 6}] + \left(-\Lambda_6^\vee\right)[\mathbf{3, 6 \supseteq 3}] + \left(-\Lambda_5^\vee\right)[\mathbf{3, 6 \supseteq 6}] + \left(-\Lambda_6^\vee\right)[\mathbf{4, 6 \supseteq 4}] + \left(-\Lambda_5^\vee\right)[\mathbf{4, 6 \supseteq 6}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1, 5, 6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{2, 5, 6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{3, 5, 6}] + \left(-2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{4, 5, 6}]$
		$\Lambda_5^\vee[\mathbf{1, 6 \supseteq 1}] + \Lambda_5^\vee[\mathbf{1, 6 \supseteq 6}] + \left(-\Lambda_5^\vee\right)[\mathbf{2, 6 \supseteq 2}] + \left(-\Lambda_5^\vee\right)[\mathbf{2, 6 \supseteq 6}] + \left(-\Lambda_5^\vee\right)[\mathbf{1, 2, 6}] + \left(-\Lambda_5^\vee\right)[\mathbf{3, 6 \supseteq 3}] + \left(-\Lambda_5^\vee\right)[\mathbf{3, 6 \supseteq 6}] + \left(-\Lambda_5^\vee\right)[\mathbf{4, 6 \supseteq 4}] + \left(-\Lambda_5^\vee\right)[\mathbf{4, 6 \supseteq 6}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1, 5, 6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{2, 5, 6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{3, 5, 6}] + \left(-2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{4, 5, 6}]$
		$\left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{6 \supseteq 6 \supseteq 6}]$
		$\Lambda_6^\vee[\mathbf{1, 3 \supseteq 1}] + \Lambda_6^\vee[\mathbf{1, 3 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{1, 2, 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{1, 4 \supseteq 1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1, 4 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{2, 4 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2, 4 \supseteq 4}] + \Lambda_6^\vee[\mathbf{1, 3, 4}] + \Lambda_6^\vee[\mathbf{2, 3, 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{1, 5 \supseteq 1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1, 5 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{2, 5 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2, 5 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{3, 5 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3, 5 \supseteq 5}] + \Lambda_6^\vee[\mathbf{3, 4, 5}] + \left(-\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1, 3, 6}] + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{1, 4, 6}] + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{2, 4, 6}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{1, 5, 6}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{2, 5, 6}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{3, 5, 6}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$\Lambda_6^\vee[]$
1	3	$\Lambda_6^\vee[\mathbf{1}] + \Lambda_6^\vee[\mathbf{2}] + \Lambda_6^\vee[\mathbf{3}] + \Lambda_6^\vee[\mathbf{4}] + \Lambda_6^\vee[\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{6}]$
2	8	$\Lambda_6^\vee[\mathbf{1 \supseteq 1}] + \Lambda_6^\vee[\mathbf{2 \supseteq 2}] + \Lambda_6^\vee[\mathbf{3 \supseteq 3}] + \Lambda_6^\vee[\mathbf{4 \supseteq 4}] + \Lambda_6^\vee[\mathbf{5 \supseteq 5}]$
		$\Lambda_6^\vee[\mathbf{1, 3}] + \Lambda_6^\vee[\mathbf{1, 4}] + \Lambda_6^\vee[\mathbf{2, 4}] + \Lambda_6^\vee[\mathbf{1, 5}] + \Lambda_6^\vee[\mathbf{2, 5}] + \Lambda_6^\vee[\mathbf{3, 5}]$
		$\Lambda_5^\vee[\mathbf{6 \supseteq 6}]$
		$\Lambda_6^\vee[\mathbf{6 \supseteq 6}]$
		$\Lambda_5^\vee[\mathbf{1, 6}] + \Lambda_5^\vee[\mathbf{2, 6}] + \Lambda_5^\vee[\mathbf{3, 6}] + \Lambda_5^\vee[\mathbf{4, 6}]$
		$\Lambda_6^\vee[\mathbf{1, 6}] + \Lambda_6^\vee[\mathbf{2, 6}] + \Lambda_6^\vee[\mathbf{3, 6}] + \Lambda_6^\vee[\mathbf{4, 6}]$
		$\Lambda_4^\vee[\mathbf{5, 6}]$
		$\Lambda_6^\vee[\mathbf{5, 6}]$

$\mathbf{k}$	$\mathbf{h}^{\mathbf{k}}(\overline{\mathbf{X}}^{\vee})$	**generating cocycles**
3	19	$\Lambda_6^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+\Lambda_6^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\Lambda_6^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+\Lambda_6^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+\Lambda_6^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]$
		$\Lambda_6^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\Lambda_6^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\Lambda_6^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+\Lambda_6^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+\Lambda_6^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+\Lambda_6^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+$ $\Lambda_6^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+\Lambda_6^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_6^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\Lambda_6^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_6^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\Lambda_6^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]$
		$\Lambda_6^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+\Lambda_6^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+\Lambda_6^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{6}\supseteq\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{6}\supseteq\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{6}\supseteq\mathbf{1}]+\Lambda_5^\vee[\mathbf{2},\mathbf{6}\supseteq\mathbf{2}]+\Lambda_5^\vee[\mathbf{3},\mathbf{6}\supseteq\mathbf{3}]+\Lambda_5^\vee[\mathbf{4},\mathbf{6}\supseteq\mathbf{4}]$
		$\Lambda_6^\vee[\mathbf{1},\mathbf{6}\supseteq\mathbf{1}]+\Lambda_6^\vee[\mathbf{2},\mathbf{6}\supseteq\mathbf{2}]+\Lambda_6^\vee[\mathbf{3},\mathbf{6}\supseteq\mathbf{3}]+\Lambda_6^\vee[\mathbf{4},\mathbf{6}\supseteq\mathbf{4}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{6}\supseteq\mathbf{6}]+\Lambda_5^\vee[\mathbf{2},\mathbf{6}\supseteq\mathbf{6}]+\Lambda_5^\vee[\mathbf{3},\mathbf{6}\supseteq\mathbf{6}]+\Lambda_5^\vee[\mathbf{4},\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{1},\mathbf{6}\supseteq\mathbf{6}]+\Lambda_6^\vee[\mathbf{2},\mathbf{6}\supseteq\mathbf{6}]+\Lambda_6^\vee[\mathbf{3},\mathbf{6}\supseteq\mathbf{6}]+\Lambda_6^\vee[\mathbf{4},\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{6}]+\Lambda_5^\vee[\mathbf{1},\mathbf{4},\mathbf{6}]+\Lambda_5^\vee[\mathbf{2},\mathbf{4},\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{1},\mathbf{3},\mathbf{6}]+\Lambda_6^\vee[\mathbf{1},\mathbf{4},\mathbf{6}]+\Lambda_6^\vee[\mathbf{2},\mathbf{4},\mathbf{6}]$
		$\Lambda_4^\vee[\mathbf{5},\mathbf{6}\supseteq\mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{5},\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{5},\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{5},\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5},\mathbf{6}]+\Lambda_4^\vee[\mathbf{2},\mathbf{5},\mathbf{6}]+\Lambda_4^\vee[\mathbf{3},\mathbf{5},\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{1},\mathbf{5},\mathbf{6}]+\Lambda_6^\vee[\mathbf{2},\mathbf{5},\mathbf{6}]+\Lambda_6^\vee[\mathbf{3},\mathbf{5},\mathbf{6}]$
		$\Lambda_3^\vee[\mathbf{4},\mathbf{5},\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{4},\mathbf{5},\mathbf{6}]$

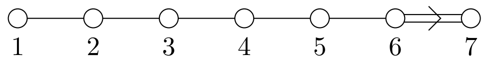
$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

A.14.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{6 \supseteq 6}]$
		$[\mathbf{1 \supseteq 1}] + (-1)\,[\mathbf{2 \supseteq 2}] + [\mathbf{1,2}] + (-1)\,[\mathbf{3 \supseteq 3}] + (-1)\,[\mathbf{4 \supseteq 4}] + (-1)\,[\mathbf{5 \supseteq 5}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{5,6 \supseteq 5}] + [\mathbf{5,6 \supseteq 6}]$
		$[\mathbf{1,6 \supseteq 1}] + [\mathbf{1,6 \supseteq 6}] + (-1)\,[\mathbf{2,6 \supseteq 2}] + (-1)\,[\mathbf{2,6 \supseteq 6}] + (-1)\,[\mathbf{1,2,6}] + (-1)\,[\mathbf{3,6 \supseteq 3}] + (-1)\,[\mathbf{3,6 \supseteq 6}] + (-1)\,[\mathbf{4,6 \supseteq 4}] + (-1)\,[\mathbf{4,6 \supseteq 6}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)\,[\mathbf{1,2,3}] + (-1)\,[\mathbf{1,4 \supseteq 1}] + (-1)\,[\mathbf{1,4 \supseteq 4}] + (-1)\,[\mathbf{2,4 \supseteq 2}] + (-1)\,[\mathbf{2,4 \supseteq 4}] + [\mathbf{1,3,4}] + [\mathbf{2,3,4}] + (-1)\,[\mathbf{1,5 \supseteq 1}] + (-1)\,[\mathbf{1,5 \supseteq 5}] + (-1)\,[\mathbf{2,5 \supseteq 2}] + (-1)\,[\mathbf{2,5 \supseteq 5}] + (-1)\,[\mathbf{3,5 \supseteq 3}] + (-1)\,[\mathbf{3,5 \supseteq 5}] + [\mathbf{3,4,5}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}]$
		$[\mathbf{6}]$
2	5	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}]$
		$[\mathbf{1,3}] + [\mathbf{1,4}] + [\mathbf{2,4}] + [\mathbf{1,5}] + [\mathbf{2,5}] + [\mathbf{3,5}]$
		$[\mathbf{6 \supseteq 6}]$
		$[\mathbf{1,6}] + [\mathbf{2,6}] + [\mathbf{3,6}] + [\mathbf{4,6}]$
		$[\mathbf{5,6}]$
3	12	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}] + [\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}] + [\mathbf{5 \supseteq 5 \supseteq 5}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + [\mathbf{1,4 \supseteq 1}] + [\mathbf{1,4 \supseteq 4}] + [\mathbf{2,4 \supseteq 2}] + [\mathbf{2,4 \supseteq 4}] + [\mathbf{1,5 \supseteq 1}] + [\mathbf{1,5 \supseteq 5}] + [\mathbf{2,5 \supseteq 2}] + [\mathbf{2,5 \supseteq 5}] + [\mathbf{3,5 \supseteq 3}] + [\mathbf{3,5 \supseteq 5}]$
		$[\mathbf{1,2,3}] + [\mathbf{2,3,4}] + [\mathbf{3,4,5}]$
		$[\mathbf{1,3,5}]$
		$[\mathbf{6 \supseteq 6 \supseteq 6}]$
		$[\mathbf{1,6 \supseteq 1}] + [\mathbf{2,6 \supseteq 2}] + [\mathbf{3,6 \supseteq 3}] + [\mathbf{4,6 \supseteq 4}]$
		$[\mathbf{1,6 \supseteq 6}] + [\mathbf{2,6 \supseteq 6}] + [\mathbf{3,6 \supseteq 6}] + [\mathbf{4,6 \supseteq 6}]$
		$[\mathbf{1,3,6}] + [\mathbf{1,4,6}] + [\mathbf{2,4,6}]$
		$[\mathbf{5,6 \supseteq 5}]$
		$[\mathbf{5,6 \supseteq 6}]$
		$[\mathbf{1,5,6}] + [\mathbf{2,5,6}] + [\mathbf{3,5,6}]$
		$[\mathbf{4,5,6}]$

A.15 Root system B_7

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_7^\vee \in P^\vee \mod Q^\vee$

A.15.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] \;=\; (1,1,0,1,0,1,0)$$
does not lie in the image of $\mathbf{comp}_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_6^\vee[\mathbf{7\supseteq 7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{6,7}]$ $2\Lambda_7^\vee[\mathbf{1\supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{1,2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3\supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4\supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5\supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{6\supseteq 6}] + \left(-2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{6,7}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_7^\vee[\mathbf{6,7\supseteq 6}] + 2\Lambda_6^\vee[\mathbf{6,7\supseteq 7}] + \left(-2\Lambda_4^\vee + 4\Lambda_5^\vee - 4\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $\left(\Lambda_4^\vee - 2\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{5,7\supseteq 7}]$ $2\Lambda_7^\vee[\mathbf{1,7\supseteq 1}] + 2\Lambda_6^\vee[\mathbf{1,7\supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,7\supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,7\supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,7\supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,7\supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,7\supseteq 4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,7\supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,7\supseteq 5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{5,7\supseteq 7}] + \left(2\Lambda_5^\vee - 4\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-2\Lambda_5^\vee + 4\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-2\Lambda_5^\vee + 4\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-2\Lambda_5^\vee + 4\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(-4\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{5,6,7}]$ $\Lambda_6^\vee[\mathbf{1,7\supseteq 1}] + \Lambda_6^\vee[\mathbf{1,7\supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7\supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7\supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7\supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7\supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7\supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7\supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7\supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7\supseteq 7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(-2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{5,6,7}]$ $2\Lambda_7^\vee[\mathbf{1,3\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,3\supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4\supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4\supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4\supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5\supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5\supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5\supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5\supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5\supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6\supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6\supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6\supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6\supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6\supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6\supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6\supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{4,6,7}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_7^\vee[]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
1	3	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}]$
		$2\Lambda_7^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{7}]$
2	7	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_7^\vee\right)[\mathbf{1}, \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{2}, \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{3}, \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{4}, \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{5}, \mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{2}, \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{1}, \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{2}, \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{3}, \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{1}, \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{2}, \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3}, \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{1}, \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{2}, \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{3}, \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4}, \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{3}, \mathbf{7}]$
		$\left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5}, \mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{6}, \mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7}]$
		$\Lambda_6^\vee[\mathbf{1}, \mathbf{7}] + \Lambda_6^\vee[\mathbf{2}, \mathbf{7}] + \Lambda_6^\vee[\mathbf{3}, \mathbf{7}] + \Lambda_6^\vee[\mathbf{4}, \mathbf{7}] + \Lambda_6^\vee[\mathbf{5}, \mathbf{7}]$

k	$h^k(\overline{X}^\vee)$	generating cocycles
3	17	$\Lambda_2^\vee[\mathbf{1\supseteq 1\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2\supseteq 2\supseteq 2}] + \left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 1}] + \left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3\supseteq 3\supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{4\supseteq 4\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{5\supseteq 5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{6\supseteq 6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + 2\Lambda_7^\vee[\mathbf{2,7\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3,7\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4,7\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5,7\supseteq 5}]$
		$2\Lambda_7^\vee[\mathbf{1\supseteq 1\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2\supseteq 2\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3\supseteq 3\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4\supseteq 4\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5\supseteq 5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{6\supseteq 6\supseteq 6}]$
		$\left(\Lambda_1^\vee+\Lambda_3^\vee+4\Lambda_7^\vee\right)[\mathbf{2\supseteq 2\supseteq 2}] + \left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 1}] + \left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 2}] + \left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 2}] + \left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + 2\Lambda_7^\vee[\mathbf{2,7\supseteq 2}]$
		$\Lambda_6^\vee[\mathbf{1,2\supseteq 1}] + \Lambda_6^\vee[\mathbf{1,2\supseteq 2}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{2,6\supseteq 2}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{3,6\supseteq 3}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{4,6\supseteq 4}] + \Lambda_6^\vee[\mathbf{5,6\supseteq 5}] + \left(\Lambda_4^\vee+2\Lambda_7^\vee\right)[\mathbf{5,6\supseteq 6}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{4,5,6}]$
		$\Lambda_2^\vee[\mathbf{1,3\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,3\supseteq 3}] + \left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{1,4\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,4\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{2,4\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,4\supseteq 4}] + \left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{1,5\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{2,5\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,5\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{1,6\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{2,6\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{3,6\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,6\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{1,4,7}] + 2\Lambda_7^\vee[\mathbf{2,4,7}] + 2\Lambda_7^\vee[\mathbf{1,5,7}] + 2\Lambda_7^\vee[\mathbf{2,5,7}] + 2\Lambda_7^\vee[\mathbf{3,5,7}]$
		$\Lambda_4^\vee[\mathbf{1,3\supseteq 1}] + \left(\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 2}] + \left(\Lambda_1^\vee+3\Lambda_7^\vee\right)[\mathbf{2,3\supseteq 3}] + \left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1,3,4}] + \left(\Lambda_3^\vee+\Lambda_5^\vee+2\Lambda_7^\vee\right)[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + 2\Lambda_7^\vee[\mathbf{3,7\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4,7\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,4,7}] + 2\Lambda_7^\vee[\mathbf{2,4,7}] + 2\Lambda_7^\vee[\mathbf{5,7\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{1,5,7}] + 2\Lambda_7^\vee[\mathbf{2,5,7}] + 2\Lambda_7^\vee[\mathbf{3,5,7}]$
		$\Lambda_2^\vee[\mathbf{1,3,5}] + \left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{2,3,5}] + 2\Lambda_7^\vee[\mathbf{1,3,6}] + 2\Lambda_7^\vee[\mathbf{1,4,6}] + 2\Lambda_7^\vee[\mathbf{2,4,6}] + \left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{3,5,6}] + 2\Lambda_7^\vee[\mathbf{3,5,7}]$
		$\left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{6\supseteq 6\supseteq 6}] + \left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee\right)[\mathbf{5,6\supseteq 5}] + \Lambda_6^\vee[\mathbf{5,6\supseteq 6}]$
		$\left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{1,6\supseteq 6}] + \Lambda_6^\vee[\mathbf{1,2,6}] + \left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee\right)[\mathbf{1,5,6}] + 2\Lambda_7^\vee[\mathbf{2,6,7}] + 2\Lambda_7^\vee[\mathbf{3,6,7}] + 2\Lambda_7^\vee[\mathbf{4,6,7}]$
		$\Lambda_6^\vee[\mathbf{7\supseteq 7\supseteq 7}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{6,7\supseteq 7}]$
		$2\Lambda_7^\vee[\mathbf{7\supseteq 7\supseteq 7}]$
		$\Lambda_2^\vee[\mathbf{1,7\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2,7\supseteq 2}] + \left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_5^\vee+2\Lambda_7^\vee\right)[\mathbf{1,2,7}] + 2\Lambda_7^\vee[\mathbf{3,7\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4,7\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5,7\supseteq 5}] + \left(\Lambda_4^\vee+\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_2^\vee+\Lambda_6^\vee\right)[\mathbf{3,5,7}]$
		$\Lambda_6^\vee[\mathbf{1,7\supseteq 1}] + \Lambda_6^\vee[\mathbf{2,7\supseteq 2}] + \Lambda_6^\vee[\mathbf{3,7\supseteq 3}] + \Lambda_6^\vee[\mathbf{4,7\supseteq 4}] + \Lambda_6^\vee[\mathbf{5,7\supseteq 5}]$
		$\Lambda_6^\vee[\mathbf{1,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{2,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{3,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{4,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{5,7\supseteq 7}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{4,6,7}]$
		$\Lambda_6^\vee[\mathbf{1,3,7}] + \Lambda_6^\vee[\mathbf{1,4,7}] + \Lambda_6^\vee[\mathbf{2,4,7}] + \Lambda_6^\vee[\mathbf{1,5,7}] + \Lambda_6^\vee[\mathbf{2,5,7}] + \Lambda_6^\vee[\mathbf{3,5,7}]$
		$\Lambda_6^\vee[\mathbf{6,7\supseteq 7}]$
		$\Lambda_4^\vee[\mathbf{5,6,7}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\0&0\\0&0\\1&0\\0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&0\\0&0&1&0&0\\0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$

A.15.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\phi_u = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_7^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_7^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_7^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_7^\vee)\,[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee + \Lambda_7^\vee)\,[\mathbf{5}] + \Lambda_6^\vee\,[\mathbf{6}]$$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_6^\vee\,[\mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{6},\mathbf{7}]$
		$\Lambda_7^\vee\,[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_7^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_7^\vee\,[\mathbf{1},\mathbf{2}] + \left(-\Lambda_7^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_7^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_7^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_7^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1},\mathbf{7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{5},\mathbf{7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{6},\mathbf{7}]$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{6,7 \supseteq 7}]$
		$\Lambda_7^\vee[\mathbf{6,7 \supseteq 6}] + \Lambda_6^\vee[\mathbf{6,7 \supseteq 7}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$\Lambda_7^\vee[\mathbf{1,7 \supseteq 1}] + \Lambda_6^\vee[\mathbf{1,7 \supseteq 7}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(-2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{5,6,7}]$
		$\Lambda_6^\vee[\mathbf{1,7 \supseteq 1}] + \Lambda_6^\vee[\mathbf{1,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(-2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{5,6,7}]$
		$\left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$\Lambda_7^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 4}] + \Lambda_7^\vee[\mathbf{1,3,4}] + \Lambda_7^\vee[\mathbf{2,3,4}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 5}] + \Lambda_7^\vee[\mathbf{3,4,5}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(-\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 6}] + \Lambda_7^\vee[\mathbf{4,5,6}] + \left(-\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
0	1	$\Lambda_7^\vee[]$
1	3	$\Lambda_7^\vee[\mathbf{1}] + \Lambda_7^\vee[\mathbf{2}] + \Lambda_7^\vee[\mathbf{3}] + \Lambda_7^\vee[\mathbf{4}] + \Lambda_7^\vee[\mathbf{5}] + \Lambda_7^\vee[\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{7}]$
		$\Lambda_7^\vee[\mathbf{7}]$
2	8	$\Lambda_7^\vee[\mathbf{1 \supseteq 1}] + \Lambda_7^\vee[\mathbf{2 \supseteq 2}] + \Lambda_7^\vee[\mathbf{3 \supseteq 3}] + \Lambda_7^\vee[\mathbf{4 \supseteq 4}] + \Lambda_7^\vee[\mathbf{5 \supseteq 5}] + \Lambda_7^\vee[\mathbf{6 \supseteq 6}]$
		$\Lambda_7^\vee[\mathbf{1,3}] + \Lambda_7^\vee[\mathbf{1,4}] + \Lambda_7^\vee[\mathbf{2,4}] + \Lambda_7^\vee[\mathbf{1,5}] + \Lambda_7^\vee[\mathbf{2,5}] + \Lambda_7^\vee[\mathbf{3,5}] + \Lambda_7^\vee[\mathbf{1,6}] + \Lambda_7^\vee[\mathbf{2,6}] + \Lambda_7^\vee[\mathbf{3,6}] + \Lambda_7^\vee[\mathbf{4,6}]$
		$\Lambda_6^\vee[\mathbf{7 \supseteq 7}]$
		$\Lambda_7^\vee[\mathbf{7 \supseteq 7}]$
		$\Lambda_6^\vee[\mathbf{1,7}] + \Lambda_6^\vee[\mathbf{2,7}] + \Lambda_6^\vee[\mathbf{3,7}] + \Lambda_6^\vee[\mathbf{4,7}] + \Lambda_6^\vee[\mathbf{5,7}]$
		$\Lambda_7^\vee[\mathbf{1,7}] + \Lambda_7^\vee[\mathbf{2,7}] + \Lambda_7^\vee[\mathbf{3,7}] + \Lambda_7^\vee[\mathbf{4,7}] + \Lambda_7^\vee[\mathbf{5,7}]$
		$\Lambda_5^\vee[\mathbf{6,7}]$
		$\Lambda_7^\vee[\mathbf{6,7}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	generating cocycles
3	20	$\Lambda_7^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + \Lambda_7^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \Lambda_7^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_7^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + \Lambda_7^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + \Lambda_7^\vee[\mathbf{6 \supseteq 6 \supseteq 6}]$
		$\Lambda_7^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + \Lambda_7^\vee[\mathbf{1,4 \supseteq 1}] + \Lambda_7^\vee[\mathbf{1,4 \supseteq 4}] + \Lambda_7^\vee[\mathbf{2,4 \supseteq 2}] + \Lambda_7^\vee[\mathbf{2,4 \supseteq 4}] + \Lambda_7^\vee[\mathbf{1,5 \supseteq 1}] + \Lambda_7^\vee[\mathbf{1,5 \supseteq 5}] + \Lambda_7^\vee[\mathbf{2,5 \supseteq 2}] + \Lambda_7^\vee[\mathbf{2,5 \supseteq 5}] + \Lambda_7^\vee[\mathbf{3,5 \supseteq 3}] + \Lambda_7^\vee[\mathbf{3,5 \supseteq 5}] + \Lambda_7^\vee[\mathbf{1,6 \supseteq 1}] + \Lambda_7^\vee[\mathbf{1,6 \supseteq 6}] + \Lambda_7^\vee[\mathbf{2,6 \supseteq 2}] + \Lambda_7^\vee[\mathbf{2,6 \supseteq 6}] + \Lambda_7^\vee[\mathbf{3,6 \supseteq 3}] + \Lambda_7^\vee[\mathbf{3,6 \supseteq 6}] + \Lambda_7^\vee[\mathbf{4,6 \supseteq 4}] + \Lambda_7^\vee[\mathbf{4,6 \supseteq 6}]$
		$\Lambda_7^\vee[\mathbf{1,2,3}] + \Lambda_7^\vee[\mathbf{2,3,4}] + \Lambda_7^\vee[\mathbf{3,4,5}] + \Lambda_7^\vee[\mathbf{4,5,6}]$
		$\Lambda_7^\vee[\mathbf{1,3,5}] + \Lambda_7^\vee[\mathbf{1,3,6}] + \Lambda_7^\vee[\mathbf{1,4,6}] + \Lambda_7^\vee[\mathbf{2,4,6}]$
		$\Lambda_6^\vee[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$\Lambda_7^\vee[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$\Lambda_6^\vee[\mathbf{1,7 \supseteq 1}] + \Lambda_6^\vee[\mathbf{2,7 \supseteq 2}] + \Lambda_6^\vee[\mathbf{3,7 \supseteq 3}] + \Lambda_6^\vee[\mathbf{4,7 \supseteq 4}] + \Lambda_6^\vee[\mathbf{5,7 \supseteq 5}]$
		$\Lambda_7^\vee[\mathbf{1,7 \supseteq 1}] + \Lambda_7^\vee[\mathbf{2,7 \supseteq 2}] + \Lambda_7^\vee[\mathbf{3,7 \supseteq 3}] + \Lambda_7^\vee[\mathbf{4,7 \supseteq 4}] + \Lambda_7^\vee[\mathbf{5,7 \supseteq 5}]$
		$\Lambda_6^\vee[\mathbf{1,7 \supseteq 7}] + \Lambda_6^\vee[\mathbf{2,7 \supseteq 7}] + \Lambda_6^\vee[\mathbf{3,7 \supseteq 7}] + \Lambda_6^\vee[\mathbf{4,7 \supseteq 7}] + \Lambda_6^\vee[\mathbf{5,7 \supseteq 7}]$
		$\Lambda_7^\vee[\mathbf{1,7 \supseteq 7}] + \Lambda_7^\vee[\mathbf{2,7 \supseteq 7}] + \Lambda_7^\vee[\mathbf{3,7 \supseteq 7}] + \Lambda_7^\vee[\mathbf{4,7 \supseteq 7}] + \Lambda_7^\vee[\mathbf{5,7 \supseteq 7}]$
		$\Lambda_6^\vee[\mathbf{1,3,7}] + \Lambda_6^\vee[\mathbf{1,4,7}] + \Lambda_6^\vee[\mathbf{2,4,7}] + \Lambda_6^\vee[\mathbf{1,5,7}] + \Lambda_6^\vee[\mathbf{2,5,7}] + \Lambda_6^\vee[\mathbf{3,5,7}]$
		$\Lambda_7^\vee[\mathbf{1,3,7}] + \Lambda_7^\vee[\mathbf{1,4,7}] + \Lambda_7^\vee[\mathbf{2,4,7}] + \Lambda_7^\vee[\mathbf{1,5,7}] + \Lambda_7^\vee[\mathbf{2,5,7}] + \Lambda_7^\vee[\mathbf{3,5,7}]$
		$\Lambda_5^\vee[\mathbf{6,7 \supseteq 6}]$
		$\Lambda_5^\vee[\mathbf{6,7 \supseteq 7}]$
		$\Lambda_6^\vee[\mathbf{6,7 \supseteq 7}]$
		$\Lambda_7^\vee[\mathbf{6,7 \supseteq 7}]$
		$\Lambda_5^\vee[\mathbf{1,6,7}] + \Lambda_5^\vee[\mathbf{2,6,7}] + \Lambda_5^\vee[\mathbf{3,6,7}] + \Lambda_5^\vee[\mathbf{4,6,7}]$
		$\Lambda_7^\vee[\mathbf{1,6,7}] + \Lambda_7^\vee[\mathbf{2,6,7}] + \Lambda_7^\vee[\mathbf{3,6,7}] + \Lambda_7^\vee[\mathbf{4,6,7}]$
		$\Lambda_4^\vee[\mathbf{5,6,7}]$
		$\Lambda_7^\vee[\mathbf{5,6,7}]$

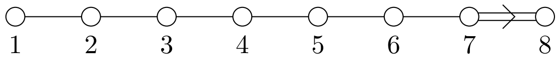
k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

A.15.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{7 \supseteq 7}]$
		$[\mathbf{1 \supseteq 1}] + (-1)\,[\mathbf{2 \supseteq 2}] + [\mathbf{1,2}] + (-1)\,[\mathbf{3 \supseteq 3}] + (-1)\,[\mathbf{4 \supseteq 4}] + (-1)\,[\mathbf{5 \supseteq 5}] + (-1)\,[\mathbf{6 \supseteq 6}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{6,7 \supseteq 6}] + [\mathbf{6,7 \supseteq 7}]$
		$[\mathbf{1,7 \supseteq 1}] + [\mathbf{1,7 \supseteq 7}] + (-1)\,[\mathbf{2,7 \supseteq 2}] + (-1)\,[\mathbf{2,7 \supseteq 7}] + (-1)\,[\mathbf{1,2,7}] + (-1)\,[\mathbf{3,7 \supseteq 3}] + (-1)\,[\mathbf{3,7 \supseteq 7}] + (-1)\,[\mathbf{4,7 \supseteq 4}] + (-1)\,[\mathbf{4,7 \supseteq 7}] + (-1)\,[\mathbf{5,7 \supseteq 5}] + (-1)\,[\mathbf{5,7 \supseteq 7}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)\,[\mathbf{1,2,3}] + (-1)\,[\mathbf{1,4 \supseteq 1}] + (-1)\,[\mathbf{1,4 \supseteq 4}] + (-1)\,[\mathbf{2,4 \supseteq 2}] + (-1)\,[\mathbf{2,4 \supseteq 4}] + [\mathbf{1,3,4}] + [\mathbf{2,3,4}] + (-1)\,[\mathbf{1,5 \supseteq 1}] + (-1)\,[\mathbf{1,5 \supseteq 5}] + (-1)\,[\mathbf{2,5 \supseteq 2}] + (-1)\,[\mathbf{2,5 \supseteq 5}] + (-1)\,[\mathbf{3,5 \supseteq 3}] + (-1)\,[\mathbf{3,5 \supseteq 5}] + [\mathbf{3,4,5}] + (-1)\,[\mathbf{1,6 \supseteq 1}] + (-1)\,[\mathbf{1,6 \supseteq 6}] + (-1)\,[\mathbf{2,6 \supseteq 2}] + (-1)\,[\mathbf{2,6 \supseteq 6}] + (-1)\,[\mathbf{3,6 \supseteq 3}] + (-1)\,[\mathbf{3,6 \supseteq 6}] + (-1)\,[\mathbf{4,6 \supseteq 4}] + (-1)\,[\mathbf{4,6 \supseteq 6}] + [\mathbf{4,5,6}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_\mathbf{2})$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}]$
		$[\mathbf{7}]$
2	5	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}] + [\mathbf{6 \supseteq 6}]$
		$[\mathbf{1,3}] + [\mathbf{1,4}] + [\mathbf{2,4}] + [\mathbf{1,5}] + [\mathbf{2,5}] + [\mathbf{3,5}] + [\mathbf{1,6}] + [\mathbf{2,6}] + [\mathbf{3,6}] + [\mathbf{4,6}]$
		$[\mathbf{7 \supseteq 7}]$
		$[\mathbf{1,7}] + [\mathbf{2,7}] + [\mathbf{3,7}] + [\mathbf{4,7}] + [\mathbf{5,7}]$
		$[\mathbf{6,7}]$
3	12	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}] + [\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}] + [\mathbf{5 \supseteq 5 \supseteq 5}] + [\mathbf{6 \supseteq 6 \supseteq 6}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + [\mathbf{1,4 \supseteq 1}] + [\mathbf{1,4 \supseteq 4}] + [\mathbf{2,4 \supseteq 2}] + [\mathbf{2,4 \supseteq 4}] + [\mathbf{1,5 \supseteq 1}] + [\mathbf{1,5 \supseteq 5}] + [\mathbf{2,5 \supseteq 2}] + [\mathbf{2,5 \supseteq 5}] + [\mathbf{3,5 \supseteq 3}] + [\mathbf{3,5 \supseteq 5}] + [\mathbf{1,6 \supseteq 1}] + [\mathbf{1,6 \supseteq 6}] + [\mathbf{2,6 \supseteq 2}] + [\mathbf{2,6 \supseteq 6}] + [\mathbf{3,6 \supseteq 3}] + [\mathbf{3,6 \supseteq 6}] + [\mathbf{4,6 \supseteq 4}] + [\mathbf{4,6 \supseteq 6}]$
		$[\mathbf{1,2,3}] + [\mathbf{2,3,4}] + [\mathbf{3,4,5}] + [\mathbf{4,5,6}]$
		$[\mathbf{1,3,5}] + [\mathbf{1,3,6}] + [\mathbf{1,4,6}] + [\mathbf{2,4,6}]$
		$[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$[\mathbf{1,7 \supseteq 1}] + [\mathbf{2,7 \supseteq 2}] + [\mathbf{3,7 \supseteq 3}] + [\mathbf{4,7 \supseteq 4}] + [\mathbf{5,7 \supseteq 5}]$
		$[\mathbf{1,7 \supseteq 7}] + [\mathbf{2,7 \supseteq 7}] + [\mathbf{3,7 \supseteq 7}] + [\mathbf{4,7 \supseteq 7}] + [\mathbf{5,7 \supseteq 7}]$
		$[\mathbf{1,3,7}] + [\mathbf{1,4,7}] + [\mathbf{2,4,7}] + [\mathbf{1,5,7}] + [\mathbf{2,5,7}] + [\mathbf{3,5,7}]$
		$[\mathbf{6,7 \supseteq 6}]$
		$[\mathbf{6,7 \supseteq 7}]$
		$[\mathbf{1,6,7}] + [\mathbf{2,6,7}] + [\mathbf{3,6,7}] + [\mathbf{4,6,7}]$
		$[\mathbf{5,6,7}]$

A.16 Root system B_8

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_7^\vee \in P^\vee \mod Q^\vee$

A.16.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (1,1,0,1)$$
does not lie in the image of comp$_2$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{1,2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_6^\vee - \Lambda_8^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{1,7}] +$ $\left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,7}] +$ $\left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7}] + \left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{7,8}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - \Lambda_8^\vee\right)[\mathbf{7,8} \supseteq \mathbf{7}]$ $\left(\Lambda_6^\vee - 2\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{1,7} \supseteq \mathbf{1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{2,7} \supseteq \mathbf{2}] +$ $\left(-2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{3,7} \supseteq \mathbf{3}] +$ $\left(-2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{4,7} \supseteq \mathbf{4}] + \left(-2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,7} \supseteq \mathbf{7}] +$ $\left(-\Lambda_6^\vee - 2\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 3\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] + \Lambda_8^\vee[\mathbf{1,6,7}] +$ $\left(2\Lambda_6^\vee - 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{4,6,7}] +$ $\left(\Lambda_4^\vee + 3\Lambda_8^\vee\right)[\mathbf{5,6,7}] + \left(4\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(4\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{2,7,8}] +$ $\left(4\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(4\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(4\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{5,7,8}]$ $2\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4} \supseteq \mathbf{4}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{2,4} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5} \supseteq \mathbf{1}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{1,5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5} \supseteq \mathbf{3}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{3,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6} \supseteq \mathbf{6}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{2,6} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6} \supseteq \mathbf{6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6} \supseteq \mathbf{6}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{4,6} \supseteq \mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,7} \supseteq \mathbf{1}] +$ $\left(-\Lambda_6^\vee - \Lambda_8^\vee\right)[\mathbf{1,7} \supseteq \mathbf{7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,7} \supseteq \mathbf{2}] + \left(-\Lambda_6^\vee - \Lambda_8^\vee\right)[\mathbf{2,7} \supseteq \mathbf{7}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{3,7} \supseteq \mathbf{3}] + \left(-\Lambda_6^\vee - \Lambda_8^\vee\right)[\mathbf{3,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{1,3,7}] +$ $\left(-2\Lambda_7^\vee\right)[\mathbf{4,7} \supseteq \mathbf{4}] + \left(-\Lambda_6^\vee - \Lambda_8^\vee\right)[\mathbf{4,7} \supseteq \mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,4,7}] +$ $\left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,4,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,7} \supseteq \mathbf{5}] + \left(-\Lambda_6^\vee - \Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] +$ $\left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,5,7}] +$ $\left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,6,7}] +$ $\left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,6,7}] +$ $\left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + \left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,7,8}] +$ $\left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,7,8}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$2\Lambda_7^\vee[]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}]$
2	4	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_7^\vee\right)[\mathbf{1,2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,7}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{5,7}]$
		$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7}]$
		$\Lambda_2^\vee[\mathbf{1,3}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{2,3}] + 2\Lambda_7^\vee[\mathbf{1,4}] + 2\Lambda_7^\vee[\mathbf{2,4}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{3,4}] + 2\Lambda_7^\vee[\mathbf{1,5}] + 2\Lambda_7^\vee[\mathbf{2,5}] + 2\Lambda_7^\vee[\mathbf{3,5}] + 2\Lambda_7^\vee[\mathbf{1,6}] + 2\Lambda_7^\vee[\mathbf{2,6}] + 2\Lambda_7^\vee[\mathbf{3,6}] + 2\Lambda_7^\vee[\mathbf{4,6}] + 2\Lambda_7^\vee[\mathbf{1,7}] + 2\Lambda_7^\vee[\mathbf{2,7}] + \left(\Lambda_6^\vee + \Lambda_8^\vee\right)[\mathbf{3,7}] + 2\Lambda_7^\vee[\mathbf{4,7}] + 2\Lambda_7^\vee[\mathbf{5,7}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{6,7}]$
3	10	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{1,2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + 2\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,7} \supseteq \mathbf{2}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,7} \supseteq \mathbf{3}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,7} \supseteq \mathbf{4}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{5}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + 2\Lambda_7^\vee[\mathbf{5,7,8}]$
		$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7} \supseteq \mathbf{7}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + 4\Lambda_7^\vee\right)[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{1,2} \supseteq \mathbf{2}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{2,3} \supseteq \mathbf{2}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_7^\vee\right)[\mathbf{2,3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,7} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{5,6,7}]$
		$\Lambda_2^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{1,4} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{2,4} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{2,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{1,5} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{2,5} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{2,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3,5} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{3,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{1,6} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{2,6} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{2,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{3,6} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{3,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4,6} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{4,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{1,7} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{2,7} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{2,7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{3,7} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{3,7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{4,7} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{4,7} \supseteq \mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,4,7}] + 2\Lambda_7^\vee[\mathbf{5,7} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{5,7} \supseteq \mathbf{7}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_2^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,5,7}]$
		$\Lambda_4^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + \left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{2,3} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + 3\Lambda_7^\vee\right)[\mathbf{2,3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{1,3,4}] + \left(\Lambda_3^\vee + \Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,7} \supseteq \mathbf{3}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,7} \supseteq \mathbf{4}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,4,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{5}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + 2\Lambda_7^\vee[\mathbf{5,7,8}]$
		$\Lambda_2^\vee[\mathbf{1,3,5}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{2,3,5}] + 2\Lambda_7^\vee[\mathbf{1,3,6}] + 2\Lambda_7^\vee[\mathbf{1,4,6}] + 2\Lambda_7^\vee[\mathbf{2,4,6}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{3,5,6}] + 2\Lambda_7^\vee[\mathbf{1,3,7}] + 2\Lambda_7^\vee[\mathbf{1,4,7}] + 2\Lambda_7^\vee[\mathbf{2,4,7}] + 2\Lambda_7^\vee[\mathbf{1,5,7}] + 2\Lambda_7^\vee[\mathbf{2,5,7}] + \left(\Lambda_6^\vee + \Lambda_8^\vee\right)[\mathbf{3,5,7}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{7} \supseteq \mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_4^\vee + \Lambda_8^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{4,5,7}] + \Lambda_6^\vee[\mathbf{6,7} \supseteq \mathbf{6}] + \left(\Lambda_5^\vee + 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7} \supseteq \mathbf{7}] + \left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{5,6,7}] + 2\Lambda_7^\vee[\mathbf{5,7,8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,7} \supseteq \mathbf{7}] + \left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{1,2,7}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,6,7}] + 2\Lambda_7^\vee[\mathbf{2,7,8}] + 2\Lambda_7^\vee[\mathbf{3,7,8}] + 2\Lambda_7^\vee[\mathbf{4,7,8}] + 2\Lambda_7^\vee[\mathbf{5,7,8}]$
		$2\Lambda_7^\vee[\mathbf{7,8} \supseteq \mathbf{7}]$
		$\Lambda_6^\vee[\mathbf{7,8} \supseteq \mathbf{8}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\\1&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

A.16.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{aligned}\phi_u \quad &= \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_8^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_8^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_8^\vee)\,[\mathbf{3}] + \\ &(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_8^\vee)\,[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee + \Lambda_8^\vee)\,[\mathbf{5}] + (\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_8^\vee)\,[\mathbf{6}] + \Lambda_7^\vee\,[\mathbf{7}]\end{aligned}}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{8}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_7^\vee[\mathbf{8} \supseteq \mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{7,8}]$
		$\Lambda_8^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_8^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_8^\vee[\mathbf{1,2}] + \left(-\Lambda_8^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_8^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_8^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_8^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_8^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(-\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,8}] + \left(\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{2,8}] + \left(\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{3,8}] + \left(\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{4,8}] + \left(\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{5,8}] + \left(\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{6,8}] + \left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{7,8}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus$ $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee-\Lambda_7^\vee\right)[\mathbf{7,8\supseteq 8}]$ $\Lambda_8^\vee[\mathbf{7,8\supseteq 7}]+\Lambda_7^\vee[\mathbf{7,8\supseteq 8}]+\left(-\Lambda_5^\vee+2\Lambda_6^\vee-2\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7,8}]$ $\Lambda_8^\vee[\mathbf{1,8\supseteq 1}]+\Lambda_7^\vee[\mathbf{1,8\supseteq 8}]+\left(-\Lambda_8^\vee\right)[\mathbf{2,8\supseteq 2}]+\left(-\Lambda_7^\vee\right)[\mathbf{2,8\supseteq 8}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,2,8}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{3,8\supseteq 3}]+\left(-\Lambda_7^\vee\right)[\mathbf{3,8\supseteq 8}]+\left(-\Lambda_8^\vee\right)[\mathbf{4,8\supseteq 4}]+\left(-\Lambda_7^\vee\right)[\mathbf{4,8\supseteq 8}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{5,8\supseteq 5}]+\left(-\Lambda_7^\vee\right)[\mathbf{5,8\supseteq 8}]+\left(-\Lambda_8^\vee\right)[\mathbf{6,8\supseteq 6}]+\left(-\Lambda_7^\vee\right)[\mathbf{6,8\supseteq 8}]+$ $\left(\Lambda_6^\vee-2\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{1,7,8}]+\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{2,7,8}]+\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{3,7,8}]+$ $\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{4,7,8}]+\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{5,7,8}]+\left(-2\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{6,7,8}]$ $\Lambda_7^\vee[\mathbf{1,8\supseteq 1}]+\Lambda_7^\vee[\mathbf{1,8\supseteq 8}]+\left(-\Lambda_7^\vee\right)[\mathbf{2,8\supseteq 2}]+\left(-\Lambda_7^\vee\right)[\mathbf{2,8\supseteq 8}]+\left(-\Lambda_7^\vee\right)[\mathbf{1,2,8}]+$ $\left(-\Lambda_7^\vee\right)[\mathbf{3,8\supseteq 3}]+\left(-\Lambda_7^\vee\right)[\mathbf{3,8\supseteq 8}]+\left(-\Lambda_7^\vee\right)[\mathbf{4,8\supseteq 4}]+\left(-\Lambda_7^\vee\right)[\mathbf{4,8\supseteq 8}]+$ $\left(-\Lambda_7^\vee\right)[\mathbf{5,8\supseteq 5}]+\left(-\Lambda_7^\vee\right)[\mathbf{5,8\supseteq 8}]+\left(-\Lambda_7^\vee\right)[\mathbf{6,8\supseteq 6}]+\left(-\Lambda_7^\vee\right)[\mathbf{6,8\supseteq 8}]+$ $\left(\Lambda_6^\vee-2\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{1,7,8}]+\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{2,7,8}]+\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{3,7,8}]+$ $\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{4,7,8}]+\left(-\Lambda_6^\vee+2\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{5,7,8}]+\left(-2\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{6,7,8}]$ $\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{8\supseteq 8\supseteq 8}]$ $\Lambda_8^\vee[\mathbf{1,3\supseteq 1}]+\Lambda_8^\vee[\mathbf{1,3\supseteq 3}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,2,3}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,4\supseteq 1}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,4\supseteq 4}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{2,4\supseteq 2}]+\left(-\Lambda_8^\vee\right)[\mathbf{2,4\supseteq 4}]+\Lambda_8^\vee[\mathbf{1,3,4}]+\Lambda_8^\vee[\mathbf{2,3,4}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,5\supseteq 1}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{1,5\supseteq 5}]+\left(-\Lambda_8^\vee\right)[\mathbf{2,5\supseteq 2}]+\left(-\Lambda_8^\vee\right)[\mathbf{2,5\supseteq 5}]+\left(-\Lambda_8^\vee\right)[\mathbf{3,5\supseteq 3}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{3,5\supseteq 5}]+\Lambda_8^\vee[\mathbf{3,4,5}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,6\supseteq 1}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,6\supseteq 6}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{2,6\supseteq 2}]+\left(-\Lambda_8^\vee\right)[\mathbf{2,6\supseteq 6}]+\left(-\Lambda_8^\vee\right)[\mathbf{3,6\supseteq 3}]+\left(-\Lambda_8^\vee\right)[\mathbf{3,6\supseteq 6}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{4,6\supseteq 4}]+\left(-\Lambda_8^\vee\right)[\mathbf{4,6\supseteq 6}]+\Lambda_8^\vee[\mathbf{4,5,6}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 1}]+\left(-\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 7}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{2,7\supseteq 2}]+\left(-\Lambda_8^\vee\right)[\mathbf{2,7\supseteq 7}]+\left(-\Lambda_8^\vee\right)[\mathbf{3,7\supseteq 3}]+\left(-\Lambda_8^\vee\right)[\mathbf{3,7\supseteq 7}]+$ $\left(-\Lambda_8^\vee\right)[\mathbf{4,7\supseteq 4}]+\left(-\Lambda_8^\vee\right)[\mathbf{4,7\supseteq 7}]+\left(-\Lambda_8^\vee\right)[\mathbf{5,7\supseteq 5}]+\left(-\Lambda_8^\vee\right)[\mathbf{5,7\supseteq 7}]+\Lambda_8^\vee[\mathbf{5,6,7}]+$ $\left(-\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{1,3,8}]+\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{1,4,8}]+\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{2,4,8}]+\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{1,5,8}]+$ $\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{2,5,8}]+\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{3,5,8}]+\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{1,6,8}]+\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{2,6,8}]+$ $\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{3,6,8}]+\left(\Lambda_7^\vee-\Lambda_8^\vee\right)[\mathbf{4,6,8}]+\left(2\Lambda_7^\vee-2\Lambda_8^\vee\right)[\mathbf{1,7,8}]+\left(2\Lambda_7^\vee-2\Lambda_8^\vee\right)[\mathbf{2,7,8}]+$ $\left(2\Lambda_7^\vee-2\Lambda_8^\vee\right)[\mathbf{3,7,8}]+\left(2\Lambda_7^\vee-2\Lambda_8^\vee\right)[\mathbf{4,7,8}]+\left(2\Lambda_7^\vee-2\Lambda_8^\vee\right)[\mathbf{5,7,8}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$\Lambda_8^\vee[]$
1	3	$\Lambda_8^\vee[\mathbf{1}]+\Lambda_8^\vee[\mathbf{2}]+\Lambda_8^\vee[\mathbf{3}]+\Lambda_8^\vee[\mathbf{4}]+\Lambda_8^\vee[\mathbf{5}]+\Lambda_8^\vee[\mathbf{6}]+\Lambda_8^\vee[\mathbf{7}]$ $\Lambda_7^\vee[\mathbf{8}]$ $\Lambda_8^\vee[\mathbf{8}]$
2	8	$\Lambda_8^\vee[\mathbf{1\supseteq 1}]+\Lambda_8^\vee[\mathbf{2\supseteq 2}]+\Lambda_8^\vee[\mathbf{3\supseteq 3}]+\Lambda_8^\vee[\mathbf{4\supseteq 4}]+\Lambda_8^\vee[\mathbf{5\supseteq 5}]+\Lambda_8^\vee[\mathbf{6\supseteq 6}]+\Lambda_8^\vee[\mathbf{7\supseteq 7}]$ $\Lambda_8^\vee[\mathbf{1,3}]+\Lambda_8^\vee[\mathbf{1,4}]+\Lambda_8^\vee[\mathbf{2,4}]+\Lambda_8^\vee[\mathbf{1,5}]+\Lambda_8^\vee[\mathbf{2,5}]+\Lambda_8^\vee[\mathbf{3,5}]+\Lambda_8^\vee[\mathbf{1,6}]+\Lambda_8^\vee[\mathbf{2,6}]+\Lambda_8^\vee[\mathbf{3,6}]+$ $\Lambda_8^\vee[\mathbf{4,6}]+\Lambda_8^\vee[\mathbf{1,7}]+\Lambda_8^\vee[\mathbf{2,7}]+\Lambda_8^\vee[\mathbf{3,7}]+\Lambda_8^\vee[\mathbf{4,7}]+\Lambda_8^\vee[\mathbf{5,7}]$ $\Lambda_7^\vee[\mathbf{8\supseteq 8}]$ $\Lambda_8^\vee[\mathbf{8\supseteq 8}]$ $\Lambda_7^\vee[\mathbf{1,8}]+\Lambda_7^\vee[\mathbf{2,8}]+\Lambda_7^\vee[\mathbf{3,8}]+\Lambda_7^\vee[\mathbf{4,8}]+\Lambda_7^\vee[\mathbf{5,8}]+\Lambda_7^\vee[\mathbf{6,8}]$ $\Lambda_8^\vee[\mathbf{1,8}]+\Lambda_8^\vee[\mathbf{2,8}]+\Lambda_8^\vee[\mathbf{3,8}]+\Lambda_8^\vee[\mathbf{4,8}]+\Lambda_8^\vee[\mathbf{5,8}]+\Lambda_8^\vee[\mathbf{6,8}]$ $\Lambda_6^\vee[\mathbf{7,8}]$ $\Lambda_8^\vee[\mathbf{7,8}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
3	20	$\Lambda_8^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + \Lambda_8^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \Lambda_8^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_8^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + \Lambda_8^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + \Lambda_8^\vee[\mathbf{6 \supseteq 6 \supseteq 6}] + \Lambda_8^\vee[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$\Lambda_8^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_8^\vee[\mathbf{1,3 \supseteq 3}] + \Lambda_8^\vee[\mathbf{1,4 \supseteq 1}] + \Lambda_8^\vee[\mathbf{1,4 \supseteq 4}] + \Lambda_8^\vee[\mathbf{2,4 \supseteq 2}] + \Lambda_8^\vee[\mathbf{2,4 \supseteq 4}] + \Lambda_8^\vee[\mathbf{1,5 \supseteq 1}] + \Lambda_8^\vee[\mathbf{1,5 \supseteq 5}] + \Lambda_8^\vee[\mathbf{2,5 \supseteq 2}] + \Lambda_8^\vee[\mathbf{2,5 \supseteq 5}] + \Lambda_8^\vee[\mathbf{3,5 \supseteq 3}] + \Lambda_8^\vee[\mathbf{3,5 \supseteq 5}] + \Lambda_8^\vee[\mathbf{1,6 \supseteq 1}] + \Lambda_8^\vee[\mathbf{1,6 \supseteq 6}] + \Lambda_8^\vee[\mathbf{2,6 \supseteq 2}] + \Lambda_8^\vee[\mathbf{2,6 \supseteq 6}] + \Lambda_8^\vee[\mathbf{3,6 \supseteq 3}] + \Lambda_8^\vee[\mathbf{3,6 \supseteq 6}] + \Lambda_8^\vee[\mathbf{4,6 \supseteq 4}] + \Lambda_8^\vee[\mathbf{4,6 \supseteq 6}] + \Lambda_8^\vee[\mathbf{1,7 \supseteq 1}] + \Lambda_8^\vee[\mathbf{1,7 \supseteq 7}] + \Lambda_8^\vee[\mathbf{2,7 \supseteq 2}] + \Lambda_8^\vee[\mathbf{2,7 \supseteq 7}] + \Lambda_8^\vee[\mathbf{3,7 \supseteq 3}] + \Lambda_8^\vee[\mathbf{3,7 \supseteq 7}] + \Lambda_8^\vee[\mathbf{4,7 \supseteq 4}] + \Lambda_8^\vee[\mathbf{4,7 \supseteq 7}] + \Lambda_8^\vee[\mathbf{5,7 \supseteq 5}] + \Lambda_8^\vee[\mathbf{5,7 \supseteq 7}]$
		$\Lambda_8^\vee[\mathbf{1,2,3}] + \Lambda_8^\vee[\mathbf{2,3,4}] + \Lambda_8^\vee[\mathbf{3,4,5}] + \Lambda_8^\vee[\mathbf{4,5,6}] + \Lambda_8^\vee[\mathbf{5,6,7}]$
		$\Lambda_8^\vee[\mathbf{1,3,5}] + \Lambda_8^\vee[\mathbf{1,3,6}] + \Lambda_8^\vee[\mathbf{1,4,6}] + \Lambda_8^\vee[\mathbf{2,4,6}] + \Lambda_8^\vee[\mathbf{1,3,7}] + \Lambda_8^\vee[\mathbf{1,4,7}] + \Lambda_8^\vee[\mathbf{2,4,7}] + \Lambda_8^\vee[\mathbf{1,5,7}] + \Lambda_8^\vee[\mathbf{2,5,7}] + \Lambda_8^\vee[\mathbf{3,5,7}]$
		$\Lambda_7^\vee[\mathbf{8 \supseteq 8 \supseteq 8}]$
		$\Lambda_8^\vee[\mathbf{8 \supseteq 8 \supseteq 8}]$
		$\Lambda_7^\vee[\mathbf{1,8 \supseteq 1}] + \Lambda_7^\vee[\mathbf{2,8 \supseteq 2}] + \Lambda_7^\vee[\mathbf{3,8 \supseteq 3}] + \Lambda_7^\vee[\mathbf{4,8 \supseteq 4}] + \Lambda_7^\vee[\mathbf{5,8 \supseteq 5}] + \Lambda_7^\vee[\mathbf{6,8 \supseteq 6}]$
		$\Lambda_8^\vee[\mathbf{1,8 \supseteq 1}] + \Lambda_8^\vee[\mathbf{2,8 \supseteq 2}] + \Lambda_8^\vee[\mathbf{3,8 \supseteq 3}] + \Lambda_8^\vee[\mathbf{4,8 \supseteq 4}] + \Lambda_8^\vee[\mathbf{5,8 \supseteq 5}] + \Lambda_8^\vee[\mathbf{6,8 \supseteq 6}]$
		$\Lambda_7^\vee[\mathbf{1,8 \supseteq 8}] + \Lambda_7^\vee[\mathbf{2,8 \supseteq 8}] + \Lambda_7^\vee[\mathbf{3,8 \supseteq 8}] + \Lambda_7^\vee[\mathbf{4,8 \supseteq 8}] + \Lambda_7^\vee[\mathbf{5,8 \supseteq 8}] + \Lambda_7^\vee[\mathbf{6,8 \supseteq 8}]$
		$\Lambda_8^\vee[\mathbf{1,8 \supseteq 8}] + \Lambda_8^\vee[\mathbf{2,8 \supseteq 8}] + \Lambda_8^\vee[\mathbf{3,8 \supseteq 8}] + \Lambda_8^\vee[\mathbf{4,8 \supseteq 8}] + \Lambda_8^\vee[\mathbf{5,8 \supseteq 8}] + \Lambda_8^\vee[\mathbf{6,8 \supseteq 8}]$
		$\Lambda_7^\vee[\mathbf{1,3,8}] + \Lambda_7^\vee[\mathbf{1,4,8}] + \Lambda_7^\vee[\mathbf{2,4,8}] + \Lambda_7^\vee[\mathbf{1,5,8}] + \Lambda_7^\vee[\mathbf{2,5,8}] + \Lambda_7^\vee[\mathbf{3,5,8}] + \Lambda_7^\vee[\mathbf{1,6,8}] + \Lambda_7^\vee[\mathbf{2,6,8}] + \Lambda_7^\vee[\mathbf{3,6,8}] + \Lambda_7^\vee[\mathbf{4,6,8}]$
		$\Lambda_8^\vee[\mathbf{1,3,8}] + \Lambda_8^\vee[\mathbf{1,4,8}] + \Lambda_8^\vee[\mathbf{2,4,8}] + \Lambda_8^\vee[\mathbf{1,5,8}] + \Lambda_8^\vee[\mathbf{2,5,8}] + \Lambda_8^\vee[\mathbf{3,5,8}] + \Lambda_8^\vee[\mathbf{1,6,8}] + \Lambda_8^\vee[\mathbf{2,6,8}] + \Lambda_8^\vee[\mathbf{3,6,8}] + \Lambda_8^\vee[\mathbf{4,6,8}]$
		$\Lambda_6^\vee[\mathbf{7,8 \supseteq 7}]$
		$\Lambda_6^\vee[\mathbf{7,8 \supseteq 8}]$
		$\Lambda_7^\vee[\mathbf{7,8 \supseteq 8}]$
		$\Lambda_8^\vee[\mathbf{7,8 \supseteq 8}]$
		$\Lambda_6^\vee[\mathbf{1,7,8}] + \Lambda_6^\vee[\mathbf{2,7,8}] + \Lambda_6^\vee[\mathbf{3,7,8}] + \Lambda_6^\vee[\mathbf{4,7,8}] + \Lambda_6^\vee[\mathbf{5,7,8}]$
		$\Lambda_8^\vee[\mathbf{1,7,8}] + \Lambda_8^\vee[\mathbf{2,7,8}] + \Lambda_8^\vee[\mathbf{3,7,8}] + \Lambda_8^\vee[\mathbf{4,7,8}] + \Lambda_8^\vee[\mathbf{5,7,8}]$
		$\Lambda_5^\vee[\mathbf{6,7,8}]$
		$\Lambda_8^\vee[\mathbf{6,7,8}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

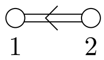
A.16.3 Cohomology with trivial coefficients

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{8} \supseteq \mathbf{8}]$
		$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)\,[\mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{5} \supseteq \mathbf{5}] + (-1)\,[\mathbf{6} \supseteq \mathbf{6}] + (-1)\,[\mathbf{7} \supseteq \mathbf{7}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{7}, \mathbf{8} \supseteq \mathbf{7}] + [\mathbf{7}, \mathbf{8} \supseteq \mathbf{8}]$
		$[\mathbf{1}, \mathbf{8} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{8} \supseteq \mathbf{8}] + (-1)\,[\mathbf{2}, \mathbf{8} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{8} \supseteq \mathbf{8}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{8}] + (-1)\,[\mathbf{3}, \mathbf{8} \supseteq \mathbf{3}] + (-1)\,[\mathbf{3}, \mathbf{8} \supseteq \mathbf{8}] + (-1)\,[\mathbf{4}, \mathbf{8} \supseteq \mathbf{4}] + (-1)\,[\mathbf{4}, \mathbf{8} \supseteq \mathbf{8}] + (-1)\,[\mathbf{5}, \mathbf{8} \supseteq \mathbf{5}] + (-1)\,[\mathbf{5}, \mathbf{8} \supseteq \mathbf{8}] + (-1)\,[\mathbf{6}, \mathbf{8} \supseteq \mathbf{6}] + (-1)\,[\mathbf{6}, \mathbf{8} \supseteq \mathbf{8}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{1}, \mathbf{3}, \mathbf{4}] + [\mathbf{2}, \mathbf{3}, \mathbf{4}] + (-1)\,[\mathbf{1}, \mathbf{5} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + (-1)\,[\mathbf{2}, \mathbf{5} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{5} \supseteq \mathbf{5}] + (-1)\,[\mathbf{3}, \mathbf{5} \supseteq \mathbf{3}] + (-1)\,[\mathbf{3}, \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{3}, \mathbf{4}, \mathbf{5}] + (-1)\,[\mathbf{1}, \mathbf{6} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{6} \supseteq \mathbf{6}] + (-1)\,[\mathbf{2}, \mathbf{6} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{6} \supseteq \mathbf{6}] + (-1)\,[\mathbf{3}, \mathbf{6} \supseteq \mathbf{3}] + (-1)\,[\mathbf{3}, \mathbf{6} \supseteq \mathbf{6}] + (-1)\,[\mathbf{4}, \mathbf{6} \supseteq \mathbf{4}] + (-1)\,[\mathbf{4}, \mathbf{6} \supseteq \mathbf{6}] + [\mathbf{4}, \mathbf{5}, \mathbf{6}] + (-1)\,[\mathbf{1}, \mathbf{7} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{7} \supseteq \mathbf{7}] + (-1)\,[\mathbf{2}, \mathbf{7} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{7} \supseteq \mathbf{7}] + (-1)\,[\mathbf{3}, \mathbf{7} \supseteq \mathbf{3}] + (-1)\,[\mathbf{3}, \mathbf{7} \supseteq \mathbf{7}] + (-1)\,[\mathbf{4}, \mathbf{7} \supseteq \mathbf{4}] + (-1)\,[\mathbf{4}, \mathbf{7} \supseteq \mathbf{7}] + (-1)\,[\mathbf{5}, \mathbf{7} \supseteq \mathbf{5}] + (-1)\,[\mathbf{5}, \mathbf{7} \supseteq \mathbf{7}] + [\mathbf{5}, \mathbf{6}, \mathbf{7}]$

k	$\mathbf{h^k}(\mathbb{F}_\mathbf{2})$	generating cocycles
0	1	[]
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}] + [\mathbf{7}]$
		$[\mathbf{8}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
2	5	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}] + [\mathbf{6 \supseteq 6}] + [\mathbf{7 \supseteq 7}]$
		$[\mathbf{1,3}]+[\mathbf{1,4}]+[\mathbf{2,4}]+[\mathbf{1,5}]+[\mathbf{2,5}]+[\mathbf{3,5}]+[\mathbf{1,6}]+[\mathbf{2,6}]+[\mathbf{3,6}]+[\mathbf{4,6}]+[\mathbf{1,7}]+[\mathbf{2,7}]+[\mathbf{3,7}]+[\mathbf{4,7}]+[\mathbf{5,7}]$
		$[\mathbf{8 \supseteq 8}]$
		$[\mathbf{1,8}] + [\mathbf{2,8}] + [\mathbf{3,8}] + [\mathbf{4,8}] + [\mathbf{5,8}] + [\mathbf{6,8}]$
		$[\mathbf{7,8}]$
3	12	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}] + [\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}] + [\mathbf{5 \supseteq 5 \supseteq 5}] + [\mathbf{6 \supseteq 6 \supseteq 6}] + [\mathbf{7 \supseteq 7 \supseteq 7}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + [\mathbf{1,4 \supseteq 1}] + [\mathbf{1,4 \supseteq 4}] + [\mathbf{2,4 \supseteq 2}] + [\mathbf{2,4 \supseteq 4}] + [\mathbf{1,5 \supseteq 1}] + [\mathbf{1,5 \supseteq 5}] + [\mathbf{2,5 \supseteq 2}] + [\mathbf{2,5 \supseteq 5}] + [\mathbf{3,5 \supseteq 3}] + [\mathbf{3,5 \supseteq 5}] + [\mathbf{1,6 \supseteq 1}] + [\mathbf{1,6 \supseteq 6}] + [\mathbf{2,6 \supseteq 2}] + [\mathbf{2,6 \supseteq 6}] + [\mathbf{3,6 \supseteq 3}] + [\mathbf{3,6 \supseteq 6}] + [\mathbf{4,6 \supseteq 4}] + [\mathbf{4,6 \supseteq 6}] + [\mathbf{1,7 \supseteq 1}] + [\mathbf{1,7 \supseteq 7}] + [\mathbf{2,7 \supseteq 2}] + [\mathbf{2,7 \supseteq 7}] + [\mathbf{3,7 \supseteq 3}] + [\mathbf{3,7 \supseteq 7}] + [\mathbf{4,7 \supseteq 4}] + [\mathbf{4,7 \supseteq 7}] + [\mathbf{5,7 \supseteq 5}] + [\mathbf{5,7 \supseteq 7}]$
		$[\mathbf{1,2,3}] + [\mathbf{2,3,4}] + [\mathbf{3,4,5}] + [\mathbf{4,5,6}] + [\mathbf{5,6,7}]$
		$[\mathbf{1,3,5}] + [\mathbf{1,3,6}] + [\mathbf{1,4,6}] + [\mathbf{2,4,6}] + [\mathbf{1,3,7}] + [\mathbf{1,4,7}] + [\mathbf{2,4,7}] + [\mathbf{1,5,7}] + [\mathbf{2,5,7}] + [\mathbf{3,5,7}]$
		$[\mathbf{8 \supseteq 8 \supseteq 8}]$
		$[\mathbf{1,8 \supseteq 1}] + [\mathbf{2,8 \supseteq 2}] + [\mathbf{3,8 \supseteq 3}] + [\mathbf{4,8 \supseteq 4}] + [\mathbf{5,8 \supseteq 5}] + [\mathbf{6,8 \supseteq 6}]$
		$[\mathbf{1,8 \supseteq 8}] + [\mathbf{2,8 \supseteq 8}] + [\mathbf{3,8 \supseteq 8}] + [\mathbf{4,8 \supseteq 8}] + [\mathbf{5,8 \supseteq 8}] + [\mathbf{6,8 \supseteq 8}]$
		$[\mathbf{1,3,8}] + [\mathbf{1,4,8}] + [\mathbf{2,4,8}] + [\mathbf{1,5,8}] + [\mathbf{2,5,8}] + [\mathbf{3,5,8}] + [\mathbf{1,6,8}] + [\mathbf{2,6,8}] + [\mathbf{3,6,8}] + [\mathbf{4,6,8}]$
		$[\mathbf{7,8 \supseteq 7}]$
		$[\mathbf{7,8 \supseteq 8}]$
		$[\mathbf{1,7,8}] + [\mathbf{2,7,8}] + [\mathbf{3,7,8}] + [\mathbf{4,7,8}] + [\mathbf{5,7,8}]$
		$[\mathbf{6,7,8}]$

A.17 Root system C_2

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_2^\vee \in P^\vee \mod Q^\vee$

A.17.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (1,1,0)$$
does not lie in the image of $\mathbf{comp}_2$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee - 2\Lambda_2^\vee\right)[\mathbf{2}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_1^\vee[\mathbf{2 \supseteq 2}] + \left(-2\Lambda_1^\vee + 2\Lambda_2^\vee\right)[\mathbf{1,2}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_1^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$
		$\left(\Lambda_1^\vee - 2\Lambda_2^\vee\right)[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_2^\vee[]$
1	2	$\Lambda_1^\vee[\mathbf{2}]$
		$2\Lambda_2^\vee[\mathbf{2}]$
2	3	$\Lambda_1^\vee[\mathbf{2} \supseteq \mathbf{2}]$
		$2\Lambda_2^\vee[\mathbf{2} \supseteq \mathbf{2}]$
		$2\Lambda_2^\vee[\mathbf{1}, \mathbf{2}]$
3	4	$\Lambda_1^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$2\Lambda_2^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_1^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$
		$2\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\\1&0\\0&0\end{pmatrix}$

A.17.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = \Lambda_2^\vee[\mathbf{2}]}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee\right)[\mathbf{1}, \mathbf{2}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}]$
		$\left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$\Lambda_1^\vee[]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
1	2	$\Lambda_1^\vee[\mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1}]$
2	3	$\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}]$
		$\Lambda_1^\vee[\mathbf{1}, \mathbf{2}]$
3	4	$\Lambda_1^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$
		$\Lambda_1^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&1\\0&0\\1&0\end{pmatrix}$

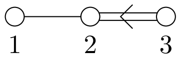
A.17.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{1} \supseteq \mathbf{1}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_\mathbf{2})$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}]$
		$[\mathbf{2}]$
2	3	$[\mathbf{1} \supseteq \mathbf{1}]$
		$[\mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{1}, \mathbf{2}]$

k	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
3	4	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$
		$[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}]$
		$[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}]$

A.18 Root system C_3

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_3^\vee \in P^\vee \mod Q^\vee$

A.18.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\boxed{\begin{array}{lll} [\phi_u] & = & (0,1,1,0,0,0) \\ & & \textbf{does not lie in the image of comp}_2 \end{array}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee - 2\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{2}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{3}] + \left(2\Lambda_2^\vee - 4\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus$ $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}] + \left(2\Lambda_1^\vee - 2\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(2\Lambda_1^\vee - 2\Lambda_2^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$\left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	1	$2\Lambda_3^\vee[]$
1	3	$2\Lambda_3^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{3}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
2	6	$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3}]$
		$\Lambda_1^\vee[\mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{2}, \mathbf{3}]$
3	10	$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$\Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_1^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}]$
		$\Lambda_1^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\\0&0\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&1&0\\0&0&0&0\\0&1&0&0\\1&0&0&0\\1&1&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}$

A.18.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{array}{rl}[\phi_u] & = \ (0,0,1) \\ & \quad \textbf{does not lie in the image of } \mathbf{comp}_2\end{array}}$$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_3^\vee[\mathbf{1}, \mathbf{2}] + \left(-\Lambda_1^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{3}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{2,3 \supseteq 3}]$
		$\Lambda_3^\vee[\mathbf{1,3 \supseteq 1}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1,3 \supseteq 3}] + \left(\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1,2,3}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	1	$\Lambda_3^\vee[\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}] + \Lambda_1^\vee[\mathbf{3}]$
2	3	$\Lambda_3^\vee[\mathbf{1 \supseteq 1}] + \Lambda_3^\vee[\mathbf{2 \supseteq 2}] + \Lambda_1^\vee[\mathbf{1,3}] + \Lambda_2^\vee[\mathbf{2,3}]$
		$\Lambda_1^\vee[\mathbf{3 \supseteq 3}] + \Lambda_3^\vee[\mathbf{1,3}]$
		$\Lambda_2^\vee[\mathbf{3 \supseteq 3}]$
3	5	$\Lambda_3^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + \Lambda_3^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \Lambda_1^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_2^\vee[\mathbf{2,3 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1,2,3}]$
		$\Lambda_1^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_3^\vee[\mathbf{1,3 \supseteq 3}]$
		$\Lambda_2^\vee[\mathbf{3 \supseteq 3 \supseteq 3}]$
		$\Lambda_3^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_1^\vee[\mathbf{1,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1,2,3}]$
		$\Lambda_3^\vee[\mathbf{2,3 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1,2,3}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\1&0\\0&1\\0&0\end{pmatrix}$

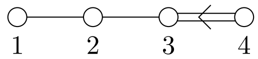
A.18.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{3 \supseteq 3}]$
		$[\mathbf{1 \supseteq 1}] + (-1)[\mathbf{2 \supseteq 2}] + [\mathbf{1,2}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{2,3 \supseteq 2}] + [\mathbf{2,3 \supseteq 3}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
1	2	$[\mathbf{1}]+[\mathbf{2}]$
		$[\mathbf{3}]$
2	4	$[\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}]$
		$[\mathbf{3}\supseteq\mathbf{3}]$
		$[\mathbf{1},\mathbf{3}]$
		$[\mathbf{2},\mathbf{3}]$
3	7	$[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]$
		$[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]$
		$[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]$
		$[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]$
		$[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]$
		$[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]$
		$[\mathbf{1},\mathbf{2},\mathbf{3}]$

A.19 Root system C_4

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_4^\vee \in P^\vee \mod Q^\vee$

A.19.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] \;=\; (0,1,1,0,0,0,0)$$
does not lie in the image of $\mathbf{comp}_2$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0},\mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee-2\Lambda_4^\vee\right)[\mathbf{4}]$
2	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_2^\vee-2\Lambda_3^\vee+2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1}\supseteq\mathbf{1}]+\left(-2\Lambda_4^\vee\right)[\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_4^\vee[\mathbf{1},\mathbf{2}]+\left(-2\Lambda_4^\vee\right)[\mathbf{3}\supseteq\mathbf{3}]+\left(-\Lambda_3^\vee+2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{4}]+\left(\Lambda_3^\vee-2\Lambda_4^\vee\right)[\mathbf{2},\mathbf{4}]+\left(2\Lambda_3^\vee-4\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4}]$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{3}] + \Lambda_3^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee - 2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}] + \left(-2\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + \left(-\Lambda_3^\vee\right)[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}] + \left(-2\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\left(\Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	1	$2\Lambda_4^\vee[]$
1	3	$2\Lambda_4^\vee[\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{4}]$
2	7	$2\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{4}] + 2\Lambda_4^\vee[\mathbf{2},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{3},\mathbf{4}]$
3	14	$2\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_4^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3},\mathbf{4}]$
		$\Lambda_1^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&0&1&1&0\\0&0&0&0&0\\0&1&0&0&0\\1&0&0&0&0\\1&1&0&0&0\\0&0&0&0&0\\0&0&1&1&0\\0&1&0&0&0\\0&1&0&0&0\end{pmatrix}$

A.19.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{array}{rl}[\phi_u] & = (0,0,0,1)\\ & \textbf{does not lie in the image of } \mathbf{comp}_2\end{array}}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{2}\supseteq\mathbf{2}] + \Lambda_3^\vee[\mathbf{1},\mathbf{2}] + \left(-\Lambda_1^\vee\right)[\mathbf{3}\supseteq\mathbf{3}] + \left(\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1},\mathbf{3}] + \left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(2\Lambda_3^\vee - 4\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}] + \Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}] + \left(-\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	0	
1	1	$\Lambda_3^\vee[\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}] + \Lambda_1^\vee[\mathbf{3}]$
2	4	$\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}\supseteq\mathbf{2}] + \Lambda_1^\vee[\mathbf{3}\supseteq\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{2}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{4}\supseteq\mathbf{4}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
3	7	$\Lambda_3^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + \Lambda_3^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \Lambda_1^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_1^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_3^\vee[\mathbf{1,3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{2,3 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2,3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + \Lambda_3^\vee[\mathbf{1,3,4}]$
		$\Lambda_3^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_1^\vee[\mathbf{1,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1,2,3}]$
		$\Lambda_4^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_4^\vee[\mathbf{1,3 \supseteq 3}] + \Lambda_3^\vee[\mathbf{1,3,4}]$
		$\Lambda_4^\vee[\mathbf{1,2,3}]$
		$\Lambda_3^\vee[\mathbf{4 \supseteq 4 \supseteq 4}]$
		$\Lambda_1^\vee[\mathbf{3,4 \supseteq 3}] + \Lambda_3^\vee[\mathbf{1,3,4}]$
		$\Lambda_2^\vee[\mathbf{3,4 \supseteq 4}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\\0&1&1\\1&0&0\\0&0&0\\0&0&0\end{pmatrix}$

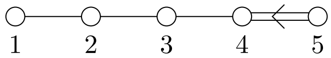
A.19.3 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{4 \supseteq 4}]$
		$[\mathbf{1 \supseteq 1}] + (-1)[\mathbf{2 \supseteq 2}] + [\mathbf{1,2}] + (-1)[\mathbf{3 \supseteq 3}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{3,4 \supseteq 3}] + [\mathbf{3,4 \supseteq 4}]$
		$[\mathbf{1,4 \supseteq 1}] + [\mathbf{1,4 \supseteq 4}] + (-1)[\mathbf{2,4 \supseteq 2}] + (-1)[\mathbf{2,4 \supseteq 4}] + (-1)[\mathbf{1,2,4}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)[\mathbf{1,2,3}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}]$
		$[\mathbf{4}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
2	5	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{3}]$
		$[\mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{4}] + [\mathbf{2}, \mathbf{4}]$
		$[\mathbf{3}, \mathbf{4}]$
3	10	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}]$
		$[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{3}, \mathbf{4} \supseteq \mathbf{3}]$
		$[\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3}, \mathbf{4}]$
		$[\mathbf{2}, \mathbf{3}, \mathbf{4}]$

A.20 Root system C_5

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_5^\vee \in P^\vee \mod Q^\vee$

A.20.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] = (0,0,1,1,0,0,0,0)$
does not lie in the image of comp$_2$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_4^\vee[\mathbf{5} \supseteq \mathbf{5}] + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{4}, \mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{1}, \mathbf{2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1}, \mathbf{5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2}, \mathbf{5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{3}, \mathbf{5}] + \left(2\Lambda_4^\vee - 4\Lambda_5^\vee\right)[\mathbf{4}, \mathbf{5}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{4,5 \supseteq 5}]$
		$2\Lambda_5^\vee[\mathbf{4,5 \supseteq 4}] + \Lambda_4^\vee[\mathbf{4,5 \supseteq 5}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$2\Lambda_5^\vee[\mathbf{1,5 \supseteq 1}] + \Lambda_4^\vee[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,2,5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 5}] + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,4,5}] + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,4,5}] + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3,4,5}]$
		$\Lambda_4^\vee[\mathbf{1,5 \supseteq 1}] + \Lambda_4^\vee[\mathbf{1,5 \supseteq 5}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,2,5}] + \left(-\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 5}] + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,4,5}] + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,4,5}] + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3,4,5}]$
		$\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{5 \supseteq 5 \supseteq 5}]$
		$2\Lambda_5^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{1,3,4}] + 2\Lambda_5^\vee[\mathbf{2,3,4}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,3,5}] + \left(2\Lambda_4^\vee - 4\Lambda_5^\vee\right)[\mathbf{1,4,5}] + \left(2\Lambda_4^\vee - 4\Lambda_5^\vee\right)[\mathbf{2,4,5}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
0	1	$2\Lambda_5^\vee[]$
1	3	$2\Lambda_5^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{5}]$
2	8	$2\Lambda_5^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4}]$
		$2\Lambda_5^\vee[\mathbf{1,3}] + 2\Lambda_5^\vee[\mathbf{1,4}] + 2\Lambda_5^\vee[\mathbf{2,4}]$
		$\Lambda_4^\vee[\mathbf{5 \supseteq 5}]$
		$2\Lambda_5^\vee[\mathbf{5 \supseteq 5}]$
		$\Lambda_4^\vee[\mathbf{1,5}] + \Lambda_4^\vee[\mathbf{2,5}] + \Lambda_4^\vee[\mathbf{3,5}]$
		$2\Lambda_5^\vee[\mathbf{1,5}] + 2\Lambda_5^\vee[\mathbf{2,5}] + 2\Lambda_5^\vee[\mathbf{3,5}]$
		$\Lambda_3^\vee[\mathbf{4,5}]$
		$2\Lambda_5^\vee[\mathbf{4,5}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
3	18	$2\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{2},\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + \Lambda_4^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + \Lambda_4^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + \Lambda_4^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_5^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{5}]$
		$\Lambda_3^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{4},\mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{4},\mathbf{5}] + \Lambda_3^\vee[\mathbf{2},\mathbf{4},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{4},\mathbf{5}] + 2\Lambda_5^\vee[\mathbf{2},\mathbf{4},\mathbf{5}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$2\Lambda_5^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

A.20.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{array}{rcl}[\phi_u] & = & (0,1) \\ & & \textbf{does not lie in the image of } \mathbf{comp}_2\end{array}}$$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{4}, \mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_5^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}] + \left(-\Lambda_5^\vee\right)[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + \left(-\Lambda_5^\vee\right)[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_5^\vee\right)[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + \left(-\Lambda_5^\vee\right)[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{3}, \mathbf{4}] + \Lambda_5^\vee[\mathbf{2}, \mathbf{3}, \mathbf{4}] + \left(-\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{5} \supseteq \mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{5} \supseteq \mathbf{2}] + \left(-\Lambda_3^\vee\right)[\mathbf{2}, \mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_1^\vee\right)[\mathbf{3}, \mathbf{5} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee\right)[\mathbf{3}, \mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_1^\vee + \Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1}, \mathbf{3}, \mathbf{5}] + \left(-\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}, \mathbf{5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{1}, \mathbf{4}, \mathbf{5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2}, \mathbf{4}, \mathbf{5}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	0	
1	0	
2	2	$\Lambda_5^\vee[\mathbf{1}, \mathbf{3}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{4}] + \Lambda_5^\vee[\mathbf{2}, \mathbf{4}] + \Lambda_3^\vee[\mathbf{1}, \mathbf{5}] + \Lambda_3^\vee[\mathbf{2}, \mathbf{5}] + \Lambda_1^\vee[\mathbf{3}, \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{5} \supseteq \mathbf{5}]$
3	6	$\Lambda_5^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + \Lambda_5^\vee[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + \Lambda_5^\vee[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + \Lambda_3^\vee[\mathbf{1}, \mathbf{5} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{2}, \mathbf{5} \supseteq \mathbf{2}] + \Lambda_1^\vee[\mathbf{3}, \mathbf{5} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{3}, \mathbf{5}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}, \mathbf{5}] + \Lambda_4^\vee[\mathbf{1}, \mathbf{4}, \mathbf{5}] + \Lambda_4^\vee[\mathbf{2}, \mathbf{4}, \mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{1}, \mathbf{2}, \mathbf{3}] + \Lambda_5^\vee[\mathbf{2}, \mathbf{3}, \mathbf{4}] + \Lambda_1^\vee[\mathbf{2}, \mathbf{3}, \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$\Lambda_3^\vee[\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + \Lambda_3^\vee[\mathbf{2}, \mathbf{5} \supseteq \mathbf{5}] + \Lambda_1^\vee[\mathbf{3}, \mathbf{5} \supseteq \mathbf{5}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{3}, \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + \Lambda_5^\vee[\mathbf{1}, \mathbf{2}, \mathbf{5}] + \Lambda_3^\vee[\mathbf{2}, \mathbf{4}, \mathbf{5}]$
		$\Lambda_3^\vee[\mathbf{4}, \mathbf{5} \supseteq \mathbf{5}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

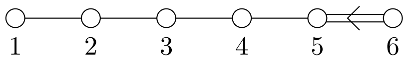
A.20.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1} \supseteq \mathbf{1}] + (-1)[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)[\mathbf{3} \supseteq \mathbf{3}] + (-1)[\mathbf{4} \supseteq \mathbf{4}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{4,5 \supseteq 4}] + [\mathbf{4,5 \supseteq 5}]$
		$[\mathbf{1,5 \supseteq 1}] + [\mathbf{1,5 \supseteq 5}] + (-1)\,[\mathbf{2,5 \supseteq 2}] + (-1)\,[\mathbf{2,5 \supseteq 5}] + (-1)\,[\mathbf{1,2,5}] + (-1)\,[\mathbf{3,5 \supseteq 3}] + (-1)\,[\mathbf{3,5 \supseteq 5}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)\,[\mathbf{1,2,3}] + (-1)\,[\mathbf{1,4 \supseteq 1}] + (-1)\,[\mathbf{1,4 \supseteq 4}] + (-1)\,[\mathbf{2,4 \supseteq 2}] + (-1)\,[\mathbf{2,4 \supseteq 4}] + [\mathbf{1,3,4}] + [\mathbf{2,3,4}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}]$
		$[\mathbf{5}]$
2	5	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}]$
		$[\mathbf{1,3}] + [\mathbf{1,4}] + [\mathbf{2,4}]$
		$[\mathbf{5 \supseteq 5}]$
		$[\mathbf{1,5}] + [\mathbf{2,5}] + [\mathbf{3,5}]$
		$[\mathbf{4,5}]$
3	11	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}] + [\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + [\mathbf{1,4 \supseteq 1}] + [\mathbf{1,4 \supseteq 4}] + [\mathbf{2,4 \supseteq 2}] + [\mathbf{2,4 \supseteq 4}]$
		$[\mathbf{1,2,3}] + [\mathbf{2,3,4}]$
		$[\mathbf{5 \supseteq 5 \supseteq 5}]$
		$[\mathbf{1,5 \supseteq 1}] + [\mathbf{2,5 \supseteq 2}] + [\mathbf{3,5 \supseteq 3}]$
		$[\mathbf{1,5 \supseteq 5}] + [\mathbf{2,5 \supseteq 5}] + [\mathbf{3,5 \supseteq 5}]$
		$[\mathbf{1,3,5}]$
		$[\mathbf{4,5 \supseteq 4}]$
		$[\mathbf{4,5 \supseteq 5}]$
		$[\mathbf{1,4,5}] + [\mathbf{2,4,5}]$
		$[\mathbf{3,4,5}]$

A.21 Root system C_6

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_6^\vee \in P^\vee \mod Q^\vee$

A.21.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\boxed{\begin{array}{ll}[\phi_u] = & (0,0,1,1,0,0,0,0) \\ & \textbf{does not lie in the image of comp}_2\end{array}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{6}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_5^\vee[\mathbf{6} \supseteq \mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{5,6}]$
		$2\Lambda_6^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_6^\vee[\mathbf{1,2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{1,6}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,6}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{3,6}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{4,6}] + \left(2\Lambda_5^\vee - 4\Lambda_6^\vee\right)[\mathbf{5,6}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{5,6} \supseteq \mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{5,6} \supseteq \mathbf{5}] + \Lambda_5^\vee[\mathbf{5,6} \supseteq \mathbf{6}] + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{4,5,6}]$
		$2\Lambda_6^\vee[\mathbf{1,6} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{1,6} \supseteq \mathbf{6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,6} \supseteq \mathbf{2}] + \left(-\Lambda_5^\vee\right)[\mathbf{2,6} \supseteq \mathbf{6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,2,6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,6} \supseteq \mathbf{3}] + \left(-\Lambda_5^\vee\right)[\mathbf{3,6} \supseteq \mathbf{6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,6} \supseteq \mathbf{4}] + \left(-\Lambda_5^\vee\right)[\mathbf{4,6} \supseteq \mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{1,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{3,5,6}] + \left(-2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{4,5,6}]$
		$\Lambda_5^\vee[\mathbf{1,6} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{1,6} \supseteq \mathbf{6}] + \left(-\Lambda_5^\vee\right)[\mathbf{2,6} \supseteq \mathbf{2}] + \left(-\Lambda_5^\vee\right)[\mathbf{2,6} \supseteq \mathbf{6}] + \left(-\Lambda_5^\vee\right)[\mathbf{1,2,6}] + \left(-\Lambda_5^\vee\right)[\mathbf{3,6} \supseteq \mathbf{3}] + \left(-\Lambda_5^\vee\right)[\mathbf{3,6} \supseteq \mathbf{6}] + \left(-\Lambda_5^\vee\right)[\mathbf{4,6} \supseteq \mathbf{4}] + \left(-\Lambda_5^\vee\right)[\mathbf{4,6} \supseteq \mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{1,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{3,5,6}] + \left(-2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{4,5,6}]$
		$\left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 2\Lambda_6^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,4} \supseteq \mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,4} \supseteq \mathbf{4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,4} \supseteq \mathbf{2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,4} \supseteq \mathbf{4}] + 2\Lambda_6^\vee[\mathbf{1,3,4}] + 2\Lambda_6^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,5} \supseteq \mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,5} \supseteq \mathbf{5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,5} \supseteq \mathbf{2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,5} \supseteq \mathbf{5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,5} \supseteq \mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,5} \supseteq \mathbf{5}] + 2\Lambda_6^\vee[\mathbf{3,4,5}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{1,3,6}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{1,4,6}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,4,6}] + \left(2\Lambda_5^\vee - 4\Lambda_6^\vee\right)[\mathbf{1,5,6}] + \left(2\Lambda_5^\vee - 4\Lambda_6^\vee\right)[\mathbf{2,5,6}] + \left(2\Lambda_5^\vee - 4\Lambda_6^\vee\right)[\mathbf{3,5,6}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_6^\vee[]$
1	3	$2\Lambda_6^\vee[\mathbf{1}] + 2\Lambda_6^\vee[\mathbf{2}] + 2\Lambda_6^\vee[\mathbf{3}] + 2\Lambda_6^\vee[\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{6}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
2	8	$2\Lambda_6^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_6^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_6^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_6^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_6^\vee[\mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{3}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{6} \supseteq \mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{6} \supseteq \mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{6}] + \Lambda_5^\vee[\mathbf{2},\mathbf{6}] + \Lambda_5^\vee[\mathbf{3},\mathbf{6}] + \Lambda_5^\vee[\mathbf{4},\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{4},\mathbf{6}]$
		$\Lambda_4^\vee[\mathbf{5},\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{5},\mathbf{6}]$
3	19	$2\Lambda_6^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_6^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_6^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_6^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_6^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] +$ $2\Lambda_6^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{2},\mathbf{3}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{3},\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{6} \supseteq \mathbf{1}] + \Lambda_5^\vee[\mathbf{2},\mathbf{6} \supseteq \mathbf{2}] + \Lambda_5^\vee[\mathbf{3},\mathbf{6} \supseteq \mathbf{3}] + \Lambda_5^\vee[\mathbf{4},\mathbf{6} \supseteq \mathbf{4}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{6} \supseteq \mathbf{1}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{6} \supseteq \mathbf{2}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{6} \supseteq \mathbf{3}] + 2\Lambda_6^\vee[\mathbf{4},\mathbf{6} \supseteq \mathbf{4}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{6} \supseteq \mathbf{6}] + \Lambda_5^\vee[\mathbf{2},\mathbf{6} \supseteq \mathbf{6}] + \Lambda_5^\vee[\mathbf{3},\mathbf{6} \supseteq \mathbf{6}] + \Lambda_5^\vee[\mathbf{4},\mathbf{6} \supseteq \mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_6^\vee[\mathbf{4},\mathbf{6} \supseteq \mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{6}] + \Lambda_5^\vee[\mathbf{1},\mathbf{4},\mathbf{6}] + \Lambda_5^\vee[\mathbf{2},\mathbf{4},\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{3},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{4},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{4},\mathbf{6}]$
		$\Lambda_4^\vee[\mathbf{5},\mathbf{6} \supseteq \mathbf{5}]$
		$\Lambda_4^\vee[\mathbf{5},\mathbf{6} \supseteq \mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{5},\mathbf{6} \supseteq \mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{5},\mathbf{6} \supseteq \mathbf{6}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{5},\mathbf{6}] + \Lambda_4^\vee[\mathbf{2},\mathbf{5},\mathbf{6}] + \Lambda_4^\vee[\mathbf{3},\mathbf{5},\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{1},\mathbf{5},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{5},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{5},\mathbf{6}]$
		$\Lambda_3^\vee[\mathbf{4},\mathbf{5},\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{4},\mathbf{5},\mathbf{6}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

A.21.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$[\phi_u] = (0,1)$$
does not lie in the image of $\mathbf{comp}_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[5,6 \supseteq 6]}$
		$\Lambda_5^\vee\mathbf{[1,3 \supseteq 1]} + \Lambda_5^\vee\mathbf{[1,3 \supseteq 3]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,2,3]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,4 \supseteq 1]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,4 \supseteq 4]} + \left(-\Lambda_5^\vee\right)\mathbf{[2,4 \supseteq 2]} + \left(-\Lambda_5^\vee\right)\mathbf{[2,4 \supseteq 4]} + \Lambda_5^\vee\mathbf{[1,3,4]} + \Lambda_5^\vee\mathbf{[2,3,4]} + \left(-\Lambda_3^\vee\right)\mathbf{[1,5 \supseteq 1]} + \left(-\Lambda_3^\vee\right)\mathbf{[1,5 \supseteq 5]} + \left(-\Lambda_3^\vee\right)\mathbf{[2,5 \supseteq 2]} + \left(-\Lambda_3^\vee\right)\mathbf{[2,5 \supseteq 5]} + \left(-\Lambda_1^\vee\right)\mathbf{[3,5 \supseteq 3]} + \left(-\Lambda_1^\vee\right)\mathbf{[3,5 \supseteq 5]} + \left(-\Lambda_1^\vee + \Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[1,3,5]} + \left(-\Lambda_2^\vee + \Lambda_3^\vee\right)\mathbf{[2,3,5]} + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[1,4,5]} + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[2,4,5]} + \Lambda_1^\vee\mathbf{[3,4,5]}$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	2	$\Lambda_5^\vee\mathbf{[1,3]} + \Lambda_5^\vee\mathbf{[1,4]} + \Lambda_5^\vee\mathbf{[2,4]} + \Lambda_3^\vee\mathbf{[1,5]} + \Lambda_3^\vee\mathbf{[2,5]} + \Lambda_1^\vee\mathbf{[3,5]}$
		$\Lambda_5^\vee\mathbf{[6 \supseteq 6]}$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
3	6	$\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\Lambda_5^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\Lambda_5^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+\Lambda_5^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+\Lambda_5^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+\Lambda_5^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+\Lambda_3^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+\Lambda_3^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_3^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+\Lambda_3^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+\Lambda_1^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+\Lambda_1^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{5}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+\Lambda_5^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+\Lambda_1^\vee[\mathbf{2},\mathbf{3},\mathbf{5}]+\Lambda_3^\vee[\mathbf{1},\mathbf{4},\mathbf{5}]+\Lambda_3^\vee[\mathbf{2},\mathbf{4},\mathbf{5}]+\Lambda_1^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]$
		$\Lambda_6^\vee[\mathbf{1},\mathbf{3},\mathbf{5}]$
		$\Lambda_5^\vee[\mathbf{6}\supseteq\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{1},\mathbf{6}\supseteq\mathbf{6}]+\Lambda_6^\vee[\mathbf{1},\mathbf{2},\mathbf{6}]+\Lambda_4^\vee[\mathbf{2},\mathbf{5},\mathbf{6}]+\Lambda_4^\vee[\mathbf{3},\mathbf{5},\mathbf{6}]$
		$\Lambda_4^\vee[\mathbf{5},\mathbf{6}\supseteq\mathbf{6}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}0&1\\0&1\\0&1\\1&0\\0&0\\0&0\end{pmatrix}$

A.21.3 Cohomology with trivial coefficients

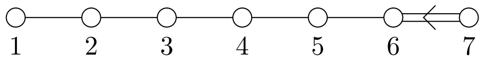
$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0},\mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{6}\supseteq\mathbf{6}]$
		$[\mathbf{1}\supseteq\mathbf{1}]+(-1)[\mathbf{2}\supseteq\mathbf{2}]+[\mathbf{1},\mathbf{2}]+(-1)[\mathbf{3}\supseteq\mathbf{3}]+(-1)[\mathbf{4}\supseteq\mathbf{4}]+(-1)[\mathbf{5}\supseteq\mathbf{5}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{5},\mathbf{6}\supseteq\mathbf{5}]+[\mathbf{5},\mathbf{6}\supseteq\mathbf{6}]$
		$[\mathbf{1},\mathbf{6}\supseteq\mathbf{1}]+[\mathbf{1},\mathbf{6}\supseteq\mathbf{6}]+(-1)[\mathbf{2},\mathbf{6}\supseteq\mathbf{2}]+(-1)[\mathbf{2},\mathbf{6}\supseteq\mathbf{6}]+(-1)[\mathbf{1},\mathbf{2},\mathbf{6}]+(-1)[\mathbf{3},\mathbf{6}\supseteq\mathbf{3}]+(-1)[\mathbf{3},\mathbf{6}\supseteq\mathbf{6}]+(-1)[\mathbf{4},\mathbf{6}\supseteq\mathbf{4}]+(-1)[\mathbf{4},\mathbf{6}\supseteq\mathbf{6}]$
		$[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+(-1)[\mathbf{1},\mathbf{2},\mathbf{3}]+(-1)[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+(-1)[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+(-1)[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+(-1)[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+[\mathbf{1},\mathbf{3},\mathbf{4}]+[\mathbf{2},\mathbf{3},\mathbf{4}]+(-1)[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+(-1)[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+(-1)[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+(-1)[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+(-1)[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+(-1)[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+[\mathbf{3},\mathbf{4},\mathbf{5}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}]+[\mathbf{2}]+[\mathbf{3}]+[\mathbf{4}]+[\mathbf{5}]$
		$[\mathbf{6}]$

k	$h^k(\mathbb{F}_2)$	generating cocycles
2	5	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4}] + [\mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1},\mathbf{3}] + [\mathbf{1},\mathbf{4}] + [\mathbf{2},\mathbf{4}] + [\mathbf{1},\mathbf{5}] + [\mathbf{2},\mathbf{5}] + [\mathbf{3},\mathbf{5}]$
		$[\mathbf{6} \supseteq \mathbf{6}]$
		$[\mathbf{1},\mathbf{6}] + [\mathbf{2},\mathbf{6}] + [\mathbf{3},\mathbf{6}] + [\mathbf{4},\mathbf{6}]$
		$[\mathbf{5},\mathbf{6}]$
3	12	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + [\mathbf{2},\mathbf{4} \supseteq \mathbf{2}] + [\mathbf{2},\mathbf{4} \supseteq \mathbf{4}] + [\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + [\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + [\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + [\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + [\mathbf{3},\mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1},\mathbf{2},\mathbf{3}] + [\mathbf{2},\mathbf{3},\mathbf{4}] + [\mathbf{3},\mathbf{4},\mathbf{5}]$
		$[\mathbf{1},\mathbf{3},\mathbf{5}]$
		$[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}]$
		$[\mathbf{1},\mathbf{6} \supseteq \mathbf{1}] + [\mathbf{2},\mathbf{6} \supseteq \mathbf{2}] + [\mathbf{3},\mathbf{6} \supseteq \mathbf{3}] + [\mathbf{4},\mathbf{6} \supseteq \mathbf{4}]$
		$[\mathbf{1},\mathbf{6} \supseteq \mathbf{6}] + [\mathbf{2},\mathbf{6} \supseteq \mathbf{6}] + [\mathbf{3},\mathbf{6} \supseteq \mathbf{6}] + [\mathbf{4},\mathbf{6} \supseteq \mathbf{6}]$
		$[\mathbf{1},\mathbf{3},\mathbf{6}] + [\mathbf{1},\mathbf{4},\mathbf{6}] + [\mathbf{2},\mathbf{4},\mathbf{6}]$
		$[\mathbf{5},\mathbf{6} \supseteq \mathbf{5}]$
		$[\mathbf{5},\mathbf{6} \supseteq \mathbf{6}]$
		$[\mathbf{1},\mathbf{5},\mathbf{6}] + [\mathbf{2},\mathbf{5},\mathbf{6}] + [\mathbf{3},\mathbf{5},\mathbf{6}]$
		$[\mathbf{4},\mathbf{5},\mathbf{6}]$

A.22 Root system C_7

Dynkin diagram



Fundamental group

$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$
generated by $\Lambda_7^\vee \in P^\vee \mod Q^\vee$

A.22.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (0,0,1,1,0,0,0,0)$$
does not lie in the image of comp_2

k	$H^k(W_0, X^\vee)$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_6^\vee[\mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{6},\mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5},\mathbf{7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{6},\mathbf{7}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{6,7 \supseteq 7}]$
		$2\Lambda_7^\vee[\mathbf{6,7 \supseteq 6}] + \Lambda_6^\vee[\mathbf{6,7 \supseteq 7}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$2\Lambda_7^\vee[\mathbf{1,7 \supseteq 1}] + \Lambda_6^\vee[\mathbf{1,7 \supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(-2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{5,6,7}]$
		$\Lambda_6^\vee[\mathbf{1,7 \supseteq 1}] + \Lambda_6^\vee[\mathbf{1,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(-2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{5,6,7}]$
		$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$2\Lambda_7^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{4,6,7}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_7^\vee[]$
1	3	$2\Lambda_7^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}]$
		$\Lambda_6^\vee[\mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{7}]$
2	8	$2\Lambda_7^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{6 \supseteq 6}]$
		$2\Lambda_7^\vee[\mathbf{1,3}] + 2\Lambda_7^\vee[\mathbf{1,4}] + 2\Lambda_7^\vee[\mathbf{2,4}] + 2\Lambda_7^\vee[\mathbf{1,5}] + 2\Lambda_7^\vee[\mathbf{2,5}] + 2\Lambda_7^\vee[\mathbf{3,5}] + 2\Lambda_7^\vee[\mathbf{1,6}] + 2\Lambda_7^\vee[\mathbf{2,6}] + 2\Lambda_7^\vee[\mathbf{3,6}] + 2\Lambda_7^\vee[\mathbf{4,6}]$
		$\Lambda_6^\vee[\mathbf{7 \supseteq 7}]$
		$2\Lambda_7^\vee[\mathbf{7 \supseteq 7}]$
		$\Lambda_6^\vee[\mathbf{1,7}] + \Lambda_6^\vee[\mathbf{2,7}] + \Lambda_6^\vee[\mathbf{3,7}] + \Lambda_6^\vee[\mathbf{4,7}] + \Lambda_6^\vee[\mathbf{5,7}]$
		$2\Lambda_7^\vee[\mathbf{1,7}] + 2\Lambda_7^\vee[\mathbf{2,7}] + 2\Lambda_7^\vee[\mathbf{3,7}] + 2\Lambda_7^\vee[\mathbf{4,7}] + 2\Lambda_7^\vee[\mathbf{5,7}]$
		$\Lambda_5^\vee[\mathbf{6,7}]$
		$2\Lambda_7^\vee[\mathbf{6,7}]$

k	$h^k(\overline{X}^\vee)$	generating cocycles
3	20	$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}]$
		$2\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{1,4} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{2,4} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{2,4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{1,5} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{2,5} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{2,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3,5} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{3,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{1,6} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{2,6} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{2,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{3,6} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{3,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4,6} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{4,6} \supseteq \mathbf{6}]$
		$2\Lambda_7^\vee[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}]$
		$2\Lambda_7^\vee[\mathbf{1,3,5}] + 2\Lambda_7^\vee[\mathbf{1,3,6}] + 2\Lambda_7^\vee[\mathbf{1,4,6}] + 2\Lambda_7^\vee[\mathbf{2,4,6}]$
		$\Lambda_6^\vee[\mathbf{7} \supseteq \mathbf{7} \supseteq \mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7} \supseteq \mathbf{7}]$
		$\Lambda_6^\vee[\mathbf{1,7} \supseteq \mathbf{1}] + \Lambda_6^\vee[\mathbf{2,7} \supseteq \mathbf{2}] + \Lambda_6^\vee[\mathbf{3,7} \supseteq \mathbf{3}] + \Lambda_6^\vee[\mathbf{4,7} \supseteq \mathbf{4}] + \Lambda_6^\vee[\mathbf{5,7} \supseteq \mathbf{5}]$
		$2\Lambda_7^\vee[\mathbf{1,7} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2,7} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3,7} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4,7} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5,7} \supseteq \mathbf{5}]$
		$\Lambda_6^\vee[\mathbf{1,7} \supseteq \mathbf{7}] + \Lambda_6^\vee[\mathbf{2,7} \supseteq \mathbf{7}] + \Lambda_6^\vee[\mathbf{3,7} \supseteq \mathbf{7}] + \Lambda_6^\vee[\mathbf{4,7} \supseteq \mathbf{7}] + \Lambda_6^\vee[\mathbf{5,7} \supseteq \mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{1,7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{2,7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{3,7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{4,7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{5,7} \supseteq \mathbf{7}]$
		$\Lambda_6^\vee[\mathbf{1,3,7}] + \Lambda_6^\vee[\mathbf{1,4,7}] + \Lambda_6^\vee[\mathbf{2,4,7}] + \Lambda_6^\vee[\mathbf{1,5,7}] + \Lambda_6^\vee[\mathbf{2,5,7}] + \Lambda_6^\vee[\mathbf{3,5,7}]$
		$2\Lambda_7^\vee[\mathbf{1,3,7}] + 2\Lambda_7^\vee[\mathbf{1,4,7}] + 2\Lambda_7^\vee[\mathbf{2,4,7}] + 2\Lambda_7^\vee[\mathbf{1,5,7}] + 2\Lambda_7^\vee[\mathbf{2,5,7}] + 2\Lambda_7^\vee[\mathbf{3,5,7}]$
		$\Lambda_5^\vee[\mathbf{6,7} \supseteq \mathbf{6}]$
		$\Lambda_5^\vee[\mathbf{6,7} \supseteq \mathbf{7}]$
		$\Lambda_6^\vee[\mathbf{6,7} \supseteq \mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{6,7} \supseteq \mathbf{7}]$
		$\Lambda_5^\vee[\mathbf{1,6,7}] + \Lambda_5^\vee[\mathbf{2,6,7}] + \Lambda_5^\vee[\mathbf{3,6,7}] + \Lambda_5^\vee[\mathbf{4,6,7}]$
		$2\Lambda_7^\vee[\mathbf{1,6,7}] + 2\Lambda_7^\vee[\mathbf{2,6,7}] + 2\Lambda_7^\vee[\mathbf{3,6,7}] + 2\Lambda_7^\vee[\mathbf{4,6,7}]$
		$\Lambda_4^\vee[\mathbf{5,6,7}]$
		$2\Lambda_7^\vee[\mathbf{5,6,7}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

A.22.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{array}{rl}[\phi_u] & = (1) \\ & \textbf{does not lie in the image of comp}_2\end{array}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{6,7 \supseteq 7}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	0	
1	0	
2	1	$\Lambda_6^\vee[\mathbf{7 \supseteq 7}]$
3	4	$\Lambda_7^\vee[\mathbf{1,3,5}] + \Lambda_7^\vee[\mathbf{1,3,6}] + \Lambda_7^\vee[\mathbf{1,4,6}] + \Lambda_7^\vee[\mathbf{2,4,6}] + \Lambda_5^\vee[\mathbf{1,3,7}] + \Lambda_5^\vee[\mathbf{1,4,7}] + \Lambda_5^\vee[\mathbf{2,4,7}] + \Lambda_3^\vee[\mathbf{1,5,7}] + \Lambda_3^\vee[\mathbf{2,5,7}] + \Lambda_1^\vee[\mathbf{3,5,7}]$
		$\Lambda_6^\vee[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$\Lambda_6^\vee[\mathbf{1,7 \supseteq 7}] + \Lambda_7^\vee[\mathbf{1,2,7}] + \Lambda_5^\vee[\mathbf{2,6,7}] + \Lambda_5^\vee[\mathbf{3,6,7}] + \Lambda_5^\vee[\mathbf{4,6,7}]$
		$\Lambda_5^\vee[\mathbf{6,7 \supseteq 7}]$

k	0	1	2	3
$\mathbf{comp}_k$	()	()	()	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

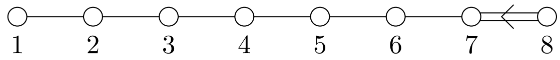
A.22.3 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{7 \supseteq 7}]$
		$[\mathbf{1 \supseteq 1}] + (-1)[\mathbf{2 \supseteq 2}] + [\mathbf{1,2}] + (-1)[\mathbf{3 \supseteq 3}] + (-1)[\mathbf{4 \supseteq 4}] + (-1)[\mathbf{5 \supseteq 5}] + (-1)[\mathbf{6 \supseteq 6}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{6,7 \supseteq 6}] + [\mathbf{6,7 \supseteq 7}]$
		$[\mathbf{1,7 \supseteq 1}] + [\mathbf{1,7 \supseteq 7}] + (-1)\,[\mathbf{2,7 \supseteq 2}] + (-1)\,[\mathbf{2,7 \supseteq 7}] + (-1)\,[\mathbf{1,2,7}] + (-1)\,[\mathbf{3,7 \supseteq 3}] + (-1)\,[\mathbf{3,7 \supseteq 7}] + (-1)\,[\mathbf{4,7 \supseteq 4}] + (-1)\,[\mathbf{4,7 \supseteq 7}] + (-1)\,[\mathbf{5,7 \supseteq 5}] + (-1)\,[\mathbf{5,7 \supseteq 7}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)\,[\mathbf{1,2,3}] + (-1)\,[\mathbf{1,4 \supseteq 1}] + (-1)\,[\mathbf{1,4 \supseteq 4}] + (-1)\,[\mathbf{2,4 \supseteq 2}] + (-1)\,[\mathbf{2,4 \supseteq 4}] + [\mathbf{1,3,4}] + [\mathbf{2,3,4}] + (-1)\,[\mathbf{1,5 \supseteq 1}] + (-1)\,[\mathbf{1,5 \supseteq 5}] + (-1)\,[\mathbf{2,5 \supseteq 2}] + (-1)\,[\mathbf{2,5 \supseteq 5}] + (-1)\,[\mathbf{3,5 \supseteq 3}] + (-1)\,[\mathbf{3,5 \supseteq 5}] + [\mathbf{3,4,5}] + (-1)\,[\mathbf{1,6 \supseteq 1}] + (-1)\,[\mathbf{1,6 \supseteq 6}] + (-1)\,[\mathbf{2,6 \supseteq 2}] + (-1)\,[\mathbf{2,6 \supseteq 6}] + (-1)\,[\mathbf{3,6 \supseteq 3}] + (-1)\,[\mathbf{3,6 \supseteq 6}] + (-1)\,[\mathbf{4,6 \supseteq 4}] + (-1)\,[\mathbf{4,6 \supseteq 6}] + [\mathbf{4,5,6}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}]$
		$[\mathbf{7}]$
2	5	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}] + [\mathbf{6 \supseteq 6}]$
		$[\mathbf{1,3}] + [\mathbf{1,4}] + [\mathbf{2,4}] + [\mathbf{1,5}] + [\mathbf{2,5}] + [\mathbf{3,5}] + [\mathbf{1,6}] + [\mathbf{2,6}] + [\mathbf{3,6}] + [\mathbf{4,6}]$
		$[\mathbf{7 \supseteq 7}]$
		$[\mathbf{1,7}] + [\mathbf{2,7}] + [\mathbf{3,7}] + [\mathbf{4,7}] + [\mathbf{5,7}]$
		$[\mathbf{6,7}]$
3	12	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}] + [\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}] + [\mathbf{5 \supseteq 5 \supseteq 5}] + [\mathbf{6 \supseteq 6 \supseteq 6}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + [\mathbf{1,4 \supseteq 1}] + [\mathbf{1,4 \supseteq 4}] + [\mathbf{2,4 \supseteq 2}] + [\mathbf{2,4 \supseteq 4}] + [\mathbf{1,5 \supseteq 1}] + [\mathbf{1,5 \supseteq 5}] + [\mathbf{2,5 \supseteq 2}] + [\mathbf{2,5 \supseteq 5}] + [\mathbf{3,5 \supseteq 3}] + [\mathbf{3,5 \supseteq 5}] + [\mathbf{1,6 \supseteq 1}] + [\mathbf{1,6 \supseteq 6}] + [\mathbf{2,6 \supseteq 2}] + [\mathbf{2,6 \supseteq 6}] + [\mathbf{3,6 \supseteq 3}] + [\mathbf{3,6 \supseteq 6}] + [\mathbf{4,6 \supseteq 4}] + [\mathbf{4,6 \supseteq 6}]$
		$[\mathbf{1,2,3}] + [\mathbf{2,3,4}] + [\mathbf{3,4,5}] + [\mathbf{4,5,6}]$
		$[\mathbf{1,3,5}] + [\mathbf{1,3,6}] + [\mathbf{1,4,6}] + [\mathbf{2,4,6}]$
		$[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$[\mathbf{1,7 \supseteq 1}] + [\mathbf{2,7 \supseteq 2}] + [\mathbf{3,7 \supseteq 3}] + [\mathbf{4,7 \supseteq 4}] + [\mathbf{5,7 \supseteq 5}]$
		$[\mathbf{1,7 \supseteq 7}] + [\mathbf{2,7 \supseteq 7}] + [\mathbf{3,7 \supseteq 7}] + [\mathbf{4,7 \supseteq 7}] + [\mathbf{5,7 \supseteq 7}]$
		$[\mathbf{1,3,7}] + [\mathbf{1,4,7}] + [\mathbf{2,4,7}] + [\mathbf{1,5,7}] + [\mathbf{2,5,7}] + [\mathbf{3,5,7}]$
		$[\mathbf{6,7 \supseteq 6}]$
		$[\mathbf{6,7 \supseteq 7}]$
		$[\mathbf{1,6,7}] + [\mathbf{2,6,7}] + [\mathbf{3,6,7}] + [\mathbf{4,6,7}]$
		$[\mathbf{5,6,7}]$

A.23 Root system C_8

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_8^\vee \in P^\vee \mod Q^\vee$

A.23.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] \;=\; (0,0,1,1,0,0,0,0)$
does not lie in the image of comp$_2$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{8}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_7^\vee[\mathbf{8} \supseteq \mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{7},\mathbf{8}]$
		$2\Lambda_8^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_8^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(-\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1},\mathbf{8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{3},\mathbf{8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{4},\mathbf{8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{6},\mathbf{8}] + \left(2\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{7},\mathbf{8}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus$ $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{7,8\supseteq 8}]$ $2\Lambda_8^\vee[\mathbf{7,8\supseteq 7}] + \Lambda_7^\vee[\mathbf{7,8\supseteq 8}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{6,7,8}]$ $2\Lambda_8^\vee[\mathbf{1,8\supseteq 1}] + \Lambda_7^\vee[\mathbf{1,8\supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,8\supseteq 2}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,8\supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,2,8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,8\supseteq 3}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,8\supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,8\supseteq 4}] + \left(-\Lambda_7^\vee\right)[\mathbf{4,8\supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,8\supseteq 5}] + \left(-\Lambda_7^\vee\right)[\mathbf{5,8\supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{6,8\supseteq 6}] + \left(-\Lambda_7^\vee\right)[\mathbf{6,8\supseteq 8}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,7,8}] + \left(-2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{6,7,8}]$ $\Lambda_7^\vee[\mathbf{1,8\supseteq 1}] + \Lambda_7^\vee[\mathbf{1,8\supseteq 8}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,8\supseteq 2}] + \left(-\Lambda_7^\vee\right)[\mathbf{2,8\supseteq 8}] + \left(-\Lambda_7^\vee\right)[\mathbf{1,2,8}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,8\supseteq 3}] + \left(-\Lambda_7^\vee\right)[\mathbf{3,8\supseteq 8}] + \left(-\Lambda_7^\vee\right)[\mathbf{4,8\supseteq 4}] + \left(-\Lambda_7^\vee\right)[\mathbf{4,8\supseteq 8}] + \left(-\Lambda_7^\vee\right)[\mathbf{5,8\supseteq 5}] + \left(-\Lambda_7^\vee\right)[\mathbf{5,8\supseteq 8}] + \left(-\Lambda_7^\vee\right)[\mathbf{6,8\supseteq 6}] + \left(-\Lambda_7^\vee\right)[\mathbf{6,8\supseteq 8}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,7,8}] + \left(-2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{6,7,8}]$ $\left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{8\supseteq 8\supseteq 8}]$ $2\Lambda_8^\vee[\mathbf{1,3\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,3\supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,4\supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,4\supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4\supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{1,3,4}] + 2\Lambda_8^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5\supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5\supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,5\supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,5\supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5\supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,6\supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,6\supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6\supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6\supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,6\supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,6\supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6\supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,7\supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,7\supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7\supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7\supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,7\supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,7\supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7\supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{5,6,7}] + \left(-\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,3,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,4,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,4,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,5,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,5,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,5,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,6,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,6,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,6,8}] + \left(\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,6,8}] + \left(2\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(2\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{2,7,8}] + \left(2\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(2\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(2\Lambda_7^\vee - 4\Lambda_8^\vee\right)[\mathbf{5,7,8}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$2\Lambda_8^\vee[]$
1	3	$2\Lambda_8^\vee[\mathbf{1}] + 2\Lambda_8^\vee[\mathbf{2}] + 2\Lambda_8^\vee[\mathbf{3}] + 2\Lambda_8^\vee[\mathbf{4}] + 2\Lambda_8^\vee[\mathbf{5}] + 2\Lambda_8^\vee[\mathbf{6}] + 2\Lambda_8^\vee[\mathbf{7}]$ $\Lambda_7^\vee[\mathbf{8}]$ $2\Lambda_8^\vee[\mathbf{8}]$
2	8	$2\Lambda_8^\vee[\mathbf{1\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{2\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{3\supseteq 3}] + 2\Lambda_8^\vee[\mathbf{4\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{5\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{7\supseteq 7}]$ $2\Lambda_8^\vee[\mathbf{1,3}] + 2\Lambda_8^\vee[\mathbf{1,4}] + 2\Lambda_8^\vee[\mathbf{2,4}] + 2\Lambda_8^\vee[\mathbf{1,5}] + 2\Lambda_8^\vee[\mathbf{2,5}] + 2\Lambda_8^\vee[\mathbf{3,5}] + 2\Lambda_8^\vee[\mathbf{1,6}] + 2\Lambda_8^\vee[\mathbf{2,6}] + 2\Lambda_8^\vee[\mathbf{3,6}] + 2\Lambda_8^\vee[\mathbf{4,6}] + 2\Lambda_8^\vee[\mathbf{1,7}] + 2\Lambda_8^\vee[\mathbf{2,7}] + 2\Lambda_8^\vee[\mathbf{3,7}] + 2\Lambda_8^\vee[\mathbf{4,7}] + 2\Lambda_8^\vee[\mathbf{5,7}]$ $\Lambda_7^\vee[\mathbf{8\supseteq 8}]$ $2\Lambda_8^\vee[\mathbf{8\supseteq 8}]$ $\Lambda_7^\vee[\mathbf{1,8}] + \Lambda_7^\vee[\mathbf{2,8}] + \Lambda_7^\vee[\mathbf{3,8}] + \Lambda_7^\vee[\mathbf{4,8}] + \Lambda_7^\vee[\mathbf{5,8}] + \Lambda_7^\vee[\mathbf{6,8}]$ $2\Lambda_8^\vee[\mathbf{1,8}] + 2\Lambda_8^\vee[\mathbf{2,8}] + 2\Lambda_8^\vee[\mathbf{3,8}] + 2\Lambda_8^\vee[\mathbf{4,8}] + 2\Lambda_8^\vee[\mathbf{5,8}] + 2\Lambda_8^\vee[\mathbf{6,8}]$ $\Lambda_6^\vee[\mathbf{7,8}]$ $2\Lambda_8^\vee[\mathbf{7,8}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
3	20	$2\Lambda_8^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_8^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_8^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_8^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_8^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_8^\vee[\mathbf{6}\supseteq\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_8^\vee[\mathbf{7}\supseteq\mathbf{7}\supseteq\mathbf{7}]$
		$2\Lambda_8^\vee[\mathbf{1,3}\supseteq\mathbf{1}]+2\Lambda_8^\vee[\mathbf{1,3}\supseteq\mathbf{3}]+2\Lambda_8^\vee[\mathbf{1,4}\supseteq\mathbf{1}]+2\Lambda_8^\vee[\mathbf{1,4}\supseteq\mathbf{4}]+2\Lambda_8^\vee[\mathbf{2,4}\supseteq\mathbf{2}]+2\Lambda_8^\vee[\mathbf{2,4}\supseteq\mathbf{4}]+2\Lambda_8^\vee[\mathbf{1,5}\supseteq\mathbf{1}]+2\Lambda_8^\vee[\mathbf{1,5}\supseteq\mathbf{5}]+2\Lambda_8^\vee[\mathbf{2,5}\supseteq\mathbf{2}]+2\Lambda_8^\vee[\mathbf{2,5}\supseteq\mathbf{5}]+2\Lambda_8^\vee[\mathbf{3,5}\supseteq\mathbf{3}]+2\Lambda_8^\vee[\mathbf{3,5}\supseteq\mathbf{5}]+2\Lambda_8^\vee[\mathbf{1,6}\supseteq\mathbf{1}]+2\Lambda_8^\vee[\mathbf{1,6}\supseteq\mathbf{6}]+2\Lambda_8^\vee[\mathbf{2,6}\supseteq\mathbf{2}]+2\Lambda_8^\vee[\mathbf{2,6}\supseteq\mathbf{6}]+2\Lambda_8^\vee[\mathbf{3,6}\supseteq\mathbf{3}]+2\Lambda_8^\vee[\mathbf{3,6}\supseteq\mathbf{6}]+2\Lambda_8^\vee[\mathbf{4,6}\supseteq\mathbf{4}]+2\Lambda_8^\vee[\mathbf{4,6}\supseteq\mathbf{6}]+2\Lambda_8^\vee[\mathbf{1,7}\supseteq\mathbf{1}]+2\Lambda_8^\vee[\mathbf{1,7}\supseteq\mathbf{7}]+2\Lambda_8^\vee[\mathbf{2,7}\supseteq\mathbf{2}]+2\Lambda_8^\vee[\mathbf{2,7}\supseteq\mathbf{7}]+2\Lambda_8^\vee[\mathbf{3,7}\supseteq\mathbf{3}]+2\Lambda_8^\vee[\mathbf{3,7}\supseteq\mathbf{7}]+2\Lambda_8^\vee[\mathbf{4,7}\supseteq\mathbf{4}]+2\Lambda_8^\vee[\mathbf{4,7}\supseteq\mathbf{7}]+2\Lambda_8^\vee[\mathbf{5,7}\supseteq\mathbf{5}]+2\Lambda_8^\vee[\mathbf{5,7}\supseteq\mathbf{7}]$
		$2\Lambda_8^\vee[\mathbf{1,2,3}]+2\Lambda_8^\vee[\mathbf{2,3,4}]+2\Lambda_8^\vee[\mathbf{3,4,5}]+2\Lambda_8^\vee[\mathbf{4,5,6}]+2\Lambda_8^\vee[\mathbf{5,6,7}]$
		$2\Lambda_8^\vee[\mathbf{1,3,5}]+2\Lambda_8^\vee[\mathbf{1,3,6}]+2\Lambda_8^\vee[\mathbf{1,4,6}]+2\Lambda_8^\vee[\mathbf{2,4,6}]+2\Lambda_8^\vee[\mathbf{1,3,7}]+2\Lambda_8^\vee[\mathbf{1,4,7}]+2\Lambda_8^\vee[\mathbf{2,4,7}]+2\Lambda_8^\vee[\mathbf{1,5,7}]+2\Lambda_8^\vee[\mathbf{2,5,7}]+2\Lambda_8^\vee[\mathbf{3,5,7}]$
		$\Lambda_7^\vee[\mathbf{8}\supseteq\mathbf{8}\supseteq\mathbf{8}]$
		$2\Lambda_8^\vee[\mathbf{8}\supseteq\mathbf{8}\supseteq\mathbf{8}]$
		$\Lambda_7^\vee[\mathbf{1,8}\supseteq\mathbf{1}]+\Lambda_7^\vee[\mathbf{2,8}\supseteq\mathbf{2}]+\Lambda_7^\vee[\mathbf{3,8}\supseteq\mathbf{3}]+\Lambda_7^\vee[\mathbf{4,8}\supseteq\mathbf{4}]+\Lambda_7^\vee[\mathbf{5,8}\supseteq\mathbf{5}]+\Lambda_7^\vee[\mathbf{6,8}\supseteq\mathbf{6}]$
		$2\Lambda_8^\vee[\mathbf{1,8}\supseteq\mathbf{1}]+2\Lambda_8^\vee[\mathbf{2,8}\supseteq\mathbf{2}]+2\Lambda_8^\vee[\mathbf{3,8}\supseteq\mathbf{3}]+2\Lambda_8^\vee[\mathbf{4,8}\supseteq\mathbf{4}]+2\Lambda_8^\vee[\mathbf{5,8}\supseteq\mathbf{5}]+2\Lambda_8^\vee[\mathbf{6,8}\supseteq\mathbf{6}]$
		$\Lambda_7^\vee[\mathbf{1,8}\supseteq\mathbf{8}]+\Lambda_7^\vee[\mathbf{2,8}\supseteq\mathbf{8}]+\Lambda_7^\vee[\mathbf{3,8}\supseteq\mathbf{8}]+\Lambda_7^\vee[\mathbf{4,8}\supseteq\mathbf{8}]+\Lambda_7^\vee[\mathbf{5,8}\supseteq\mathbf{8}]+\Lambda_7^\vee[\mathbf{6,8}\supseteq\mathbf{8}]$
		$2\Lambda_8^\vee[\mathbf{1,8}\supseteq\mathbf{8}]+2\Lambda_8^\vee[\mathbf{2,8}\supseteq\mathbf{8}]+2\Lambda_8^\vee[\mathbf{3,8}\supseteq\mathbf{8}]+2\Lambda_8^\vee[\mathbf{4,8}\supseteq\mathbf{8}]+2\Lambda_8^\vee[\mathbf{5,8}\supseteq\mathbf{8}]+2\Lambda_8^\vee[\mathbf{6,8}\supseteq\mathbf{8}]$
		$\Lambda_7^\vee[\mathbf{1,3,8}]+\Lambda_7^\vee[\mathbf{1,4,8}]+\Lambda_7^\vee[\mathbf{2,4,8}]+\Lambda_7^\vee[\mathbf{1,5,8}]+\Lambda_7^\vee[\mathbf{2,5,8}]+\Lambda_7^\vee[\mathbf{3,5,8}]+\Lambda_7^\vee[\mathbf{1,6,8}]+\Lambda_7^\vee[\mathbf{2,6,8}]+\Lambda_7^\vee[\mathbf{3,6,8}]+\Lambda_7^\vee[\mathbf{4,6,8}]$
		$2\Lambda_8^\vee[\mathbf{1,3,8}]+2\Lambda_8^\vee[\mathbf{1,4,8}]+2\Lambda_8^\vee[\mathbf{2,4,8}]+2\Lambda_8^\vee[\mathbf{1,5,8}]+2\Lambda_8^\vee[\mathbf{2,5,8}]+2\Lambda_8^\vee[\mathbf{3,5,8}]+2\Lambda_8^\vee[\mathbf{1,6,8}]+2\Lambda_8^\vee[\mathbf{2,6,8}]+2\Lambda_8^\vee[\mathbf{3,6,8}]+2\Lambda_8^\vee[\mathbf{4,6,8}]$
		$\Lambda_6^\vee[\mathbf{7,8}\supseteq\mathbf{7}]$
		$\Lambda_6^\vee[\mathbf{7,8}\supseteq\mathbf{8}]$
		$\Lambda_7^\vee[\mathbf{7,8}\supseteq\mathbf{8}]$
		$2\Lambda_8^\vee[\mathbf{7,8}\supseteq\mathbf{8}]$
		$\Lambda_6^\vee[\mathbf{1,7,8}]+\Lambda_6^\vee[\mathbf{2,7,8}]+\Lambda_6^\vee[\mathbf{3,7,8}]+\Lambda_6^\vee[\mathbf{4,7,8}]+\Lambda_6^\vee[\mathbf{5,7,8}]$
		$2\Lambda_8^\vee[\mathbf{1,7,8}]+2\Lambda_8^\vee[\mathbf{2,7,8}]+2\Lambda_8^\vee[\mathbf{3,7,8}]+2\Lambda_8^\vee[\mathbf{4,7,8}]+2\Lambda_8^\vee[\mathbf{5,7,8}]$
		$\Lambda_5^\vee[\mathbf{6,7,8}]$
		$2\Lambda_8^\vee[\mathbf{6,7,8}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\1&0\\0&0\\0&1\\0&1\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&1&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}$

A.23.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$[\phi_u] = (1)$$
does not lie in the image of $\mathbf{comp}_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{7}, \mathbf{8} \supseteq \mathbf{8}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	1	$\Lambda_7^\vee[\mathbf{8} \supseteq \mathbf{8}]$
3	4	$\Lambda_7^\vee[\mathbf{1},\mathbf{3},\mathbf{5}] + \Lambda_7^\vee[\mathbf{1},\mathbf{3},\mathbf{6}] + \Lambda_7^\vee[\mathbf{1},\mathbf{4},\mathbf{6}] + \Lambda_7^\vee[\mathbf{2},\mathbf{4},\mathbf{6}] + \Lambda_5^\vee[\mathbf{1},\mathbf{3},\mathbf{7}] + \Lambda_5^\vee[\mathbf{1},\mathbf{4},\mathbf{7}] + \Lambda_5^\vee[\mathbf{2},\mathbf{4},\mathbf{7}] + \Lambda_3^\vee[\mathbf{1},\mathbf{5},\mathbf{7}] + \Lambda_3^\vee[\mathbf{2},\mathbf{5},\mathbf{7}] + \Lambda_1^\vee[\mathbf{3},\mathbf{5},\mathbf{7}]$
		$\Lambda_7^\vee[\mathbf{8} \supseteq \mathbf{8} \supseteq \mathbf{8}]$
		$\Lambda_7^\vee[\mathbf{1},\mathbf{8} \supseteq \mathbf{8}] + \Lambda_8^\vee[\mathbf{1},\mathbf{2},\mathbf{8}] + \Lambda_6^\vee[\mathbf{2},\mathbf{7},\mathbf{8}] + \Lambda_6^\vee[\mathbf{3},\mathbf{7},\mathbf{8}] + \Lambda_6^\vee[\mathbf{4},\mathbf{7},\mathbf{8}] + \Lambda_6^\vee[\mathbf{5},\mathbf{7},\mathbf{8}]$
		$\Lambda_6^\vee[\mathbf{7},\mathbf{8} \supseteq \mathbf{8}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

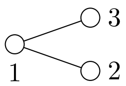
A.23.3 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{8 \supseteq 8}]$
		$[\mathbf{1 \supseteq 1}] + (-1)\,[\mathbf{2 \supseteq 2}] + [\mathbf{1, 2}] + (-1)\,[\mathbf{3 \supseteq 3}] + (-1)\,[\mathbf{4 \supseteq 4}] + (-1)\,[\mathbf{5 \supseteq 5}] + (-1)\,[\mathbf{6 \supseteq 6}] + (-1)\,[\mathbf{7 \supseteq 7}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{7, 8 \supseteq 7}] + [\mathbf{7, 8 \supseteq 8}]$
		$[\mathbf{1, 8 \supseteq 1}] + [\mathbf{1, 8 \supseteq 8}] + (-1)\,[\mathbf{2, 8 \supseteq 2}] + (-1)\,[\mathbf{2, 8 \supseteq 8}] + (-1)\,[\mathbf{1, 2, 8}] + (-1)\,[\mathbf{3, 8 \supseteq 3}] + (-1)\,[\mathbf{3, 8 \supseteq 8}] + (-1)\,[\mathbf{4, 8 \supseteq 4}] + (-1)\,[\mathbf{4, 8 \supseteq 8}] + (-1)\,[\mathbf{5, 8 \supseteq 5}] + (-1)\,[\mathbf{5, 8 \supseteq 8}] + (-1)\,[\mathbf{6, 8 \supseteq 6}] + (-1)\,[\mathbf{6, 8 \supseteq 8}]$
		$[\mathbf{1, 3 \supseteq 1}] + [\mathbf{1, 3 \supseteq 3}] + (-1)\,[\mathbf{1, 2, 3}] + (-1)\,[\mathbf{1, 4 \supseteq 1}] + (-1)\,[\mathbf{1, 4 \supseteq 4}] + (-1)\,[\mathbf{2, 4 \supseteq 2}] + (-1)\,[\mathbf{2, 4 \supseteq 4}] + [\mathbf{1, 3, 4}] + [\mathbf{2, 3, 4}] + (-1)\,[\mathbf{1, 5 \supseteq 1}] + (-1)\,[\mathbf{1, 5 \supseteq 5}] + (-1)\,[\mathbf{2, 5 \supseteq 2}] + (-1)\,[\mathbf{2, 5 \supseteq 5}] + (-1)\,[\mathbf{3, 5 \supseteq 3}] + (-1)\,[\mathbf{3, 5 \supseteq 5}] + [\mathbf{3, 4, 5}] + (-1)\,[\mathbf{1, 6 \supseteq 1}] + (-1)\,[\mathbf{1, 6 \supseteq 6}] + (-1)\,[\mathbf{2, 6 \supseteq 2}] + (-1)\,[\mathbf{2, 6 \supseteq 6}] + (-1)\,[\mathbf{3, 6 \supseteq 3}] + (-1)\,[\mathbf{3, 6 \supseteq 6}] + (-1)\,[\mathbf{4, 6 \supseteq 4}] + (-1)\,[\mathbf{4, 6 \supseteq 6}] + [\mathbf{4, 5, 6}] + (-1)\,[\mathbf{1, 7 \supseteq 1}] + (-1)\,[\mathbf{1, 7 \supseteq 7}] + (-1)\,[\mathbf{2, 7 \supseteq 2}] + (-1)\,[\mathbf{2, 7 \supseteq 7}] + (-1)\,[\mathbf{3, 7 \supseteq 3}] + (-1)\,[\mathbf{3, 7 \supseteq 7}] + (-1)\,[\mathbf{4, 7 \supseteq 4}] + (-1)\,[\mathbf{4, 7 \supseteq 7}] + (-1)\,[\mathbf{5, 7 \supseteq 5}] + (-1)\,[\mathbf{5, 7 \supseteq 7}] + [\mathbf{5, 6, 7}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	2	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}] + [\mathbf{7}]$
		$[\mathbf{8}]$
2	5	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}] + [\mathbf{6 \supseteq 6}] + [\mathbf{7 \supseteq 7}]$
		$[\mathbf{1, 3}] + [\mathbf{1, 4}] + [\mathbf{2, 4}] + [\mathbf{1, 5}] + [\mathbf{2, 5}] + [\mathbf{3, 5}] + [\mathbf{1, 6}] + [\mathbf{2, 6}] + [\mathbf{3, 6}] + [\mathbf{4, 6}] + [\mathbf{1, 7}] + [\mathbf{2, 7}] + [\mathbf{3, 7}] + [\mathbf{4, 7}] + [\mathbf{5, 7}]$
		$[\mathbf{8 \supseteq 8}]$
		$[\mathbf{1, 8}] + [\mathbf{2, 8}] + [\mathbf{3, 8}] + [\mathbf{4, 8}] + [\mathbf{5, 8}] + [\mathbf{6, 8}]$
		$[\mathbf{7, 8}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
3	12	$[1 \supseteq 1 \supseteq 1] + [2 \supseteq 2 \supseteq 2] + [3 \supseteq 3 \supseteq 3] + [4 \supseteq 4 \supseteq 4] + [5 \supseteq 5 \supseteq 5] + [6 \supseteq 6 \supseteq 6] + [7 \supseteq 7 \supseteq 7]$
		$[1,3 \supseteq 1] + [1,3 \supseteq 3] + [1,4 \supseteq 1] + [1,4 \supseteq 4] + [2,4 \supseteq 2] + [2,4 \supseteq 4] + [1,5 \supseteq 1] + [1,5 \supseteq 5] + [2,5 \supseteq 2] + [2,5 \supseteq 5] + [3,5 \supseteq 3] + [3,5 \supseteq 5] + [1,6 \supseteq 1] + [1,6 \supseteq 6] + [2,6 \supseteq 2] + [2,6 \supseteq 6] + [3,6 \supseteq 3] + [3,6 \supseteq 6] + [4,6 \supseteq 4] + [4,6 \supseteq 6] + [1,7 \supseteq 1] + [1,7 \supseteq 7] + [2,7 \supseteq 2] + [2,7 \supseteq 7] + [3,7 \supseteq 3] + [3,7 \supseteq 7] + [4,7 \supseteq 4] + [4,7 \supseteq 7] + [5,7 \supseteq 5] + [5,7 \supseteq 7]$
		$[1,2,3] + [2,3,4] + [3,4,5] + [4,5,6] + [5,6,7]$
		$[1,3,5] + [1,3,6] + [1,4,6] + [2,4,6] + [1,3,7] + [1,4,7] + [2,4,7] + [1,5,7] + [2,5,7] + [3,5,7]$
		$[8 \supseteq 8 \supseteq 8]$
		$[1,8 \supseteq 1] + [2,8 \supseteq 2] + [3,8 \supseteq 3] + [4,8 \supseteq 4] + [5,8 \supseteq 5] + [6,8 \supseteq 6]$
		$[1,8 \supseteq 8] + [2,8 \supseteq 8] + [3,8 \supseteq 8] + [4,8 \supseteq 8] + [5,8 \supseteq 8] + [6,8 \supseteq 8]$
		$[1,3,8] + [1,4,8] + [2,4,8] + [1,5,8] + [2,5,8] + [3,5,8] + [1,6,8] + [2,6,8] + [3,6,8] + [4,6,8]$
		$[7,8 \supseteq 7]$
		$[7,8 \supseteq 8]$
		$[1,7,8] + [2,7,8] + [3,7,8] + [4,7,8] + [5,7,8]$
		$[6,7,8]$

A.24 Root system D_3

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/4\mathbb{Z}$ generated by $\Lambda_3^\vee \in P^\vee \mod Q^\vee$

A.24.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (0,1)$$

does not lie in the image of comp$_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/4\mathbb{Z}$	$\left(\Lambda_1^\vee - 2\Lambda_3^\vee\right)[3]$
2	$\mathbb{Z}/2\mathbb{Z}$	$4\Lambda_3^\vee[1 \supseteq 1] + \left(-4\Lambda_3^\vee\right)[2 \supseteq 2] + 4\Lambda_3^\vee[1,2] + \left(-4\Lambda_2^\vee\right)[3 \supseteq 3] + \left(-4\Lambda_1^\vee + 4\Lambda_2^\vee + 4\Lambda_3^\vee\right)[1,3] + \left(4\Lambda_2^\vee - 4\Lambda_3^\vee\right)[2,3]$
3	$\mathbb{Z}/2\mathbb{Z}$	$4\Lambda_3^\vee[2,3 \supseteq 2] + 4\Lambda_2^\vee[2,3 \supseteq 3] + \left(-4\Lambda_1^\vee + 4\Lambda_2^\vee + 4\Lambda_3^\vee\right)[1,2,3]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$4\Lambda_3^\vee[]$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	generating cocycles
1	2	$\left(\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{1}]+4\Lambda_3^\vee[\mathbf{2}]+4\Lambda_3^\vee[\mathbf{3}]$
		$\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{2}]$
2	2	$\left(\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{1}\supseteq\mathbf{1}]+4\Lambda_3^\vee[\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+3\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}]+4\Lambda_3^\vee[\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+3\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}]$
		$\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+3\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}]+\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
3	3	$\left(\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+4\Lambda_3^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+5\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+4\Lambda_3^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]$
		$\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+5\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+4\Lambda_3^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$
		$\left(\Lambda_1^\vee+6\Lambda_3^\vee\right)[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+4\Lambda_3^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_2^\vee+\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\end{pmatrix}$

A.24.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega=\langle(2)\rangle$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau=(\Lambda_1^\vee+2\Lambda_3^\vee)\,[\mathbf{1}]+(\Lambda_2^\vee+\Lambda_3^\vee)\,[\mathbf{2}]+(\Lambda_2^\vee+\Lambda_3^\vee)\,[\mathbf{3}]}$$

k	$\mathbf{H^k}(\mathbf{W_0},\mathbf{X}^\vee)$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee-2\Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}]+\left(-2\Lambda_3^\vee\right)[\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_3^\vee[\mathbf{1},\mathbf{2}]+\left(-2\Lambda_2^\vee\right)[\mathbf{3}\supseteq\mathbf{3}]+\left(-2\Lambda_1^\vee+2\Lambda_2^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}]+\left(2\Lambda_2^\vee-2\Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_1^\vee-2\Lambda_2^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee-\Lambda_2^\vee-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+2\Lambda_2^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(-2\Lambda_1^\vee+2\Lambda_2^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	generating cocycles
0	1	$2\Lambda_3^\vee[]$
1	2	$2\Lambda_3^\vee[\mathbf{1}]+2\Lambda_3^\vee[\mathbf{2}]+2\Lambda_3^\vee[\mathbf{3}]$
		$\left(\Lambda_1^\vee+2\Lambda_3^\vee\right)[\mathbf{2}]$

k	$h^k(\overline{X^\vee})$	generating cocycles
2	3	$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}]$
		$\left(\Lambda_1^\vee + 2\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2}]$
		$\Lambda_1^\vee[\mathbf{2},\mathbf{3}]$
3	4	$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$\left(\Lambda_1^\vee + 2\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \Lambda_1^\vee[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 3\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$
		$\left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}] + \Lambda_1^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]$
		$\Lambda_1^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\1&1\\1&1\\0&1\end{pmatrix}$

A.24.3 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_2^\vee)[\mathbf{1}] + \Lambda_2^\vee[\mathbf{2}] + \Lambda_3^\vee[\mathbf{3}]}$$

k	$H^k(W_0, X^\vee)$	generating cocycles
0	0	
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_3^\vee[\mathbf{1},\mathbf{2}] + \left(-\Lambda_2^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}] + \left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
3	$\mathbb{Z}/4\mathbb{Z}$	$\Lambda_3^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	$h^k(\overline{X^\vee})$	generating cocycles
0	0	
1	1	$\Lambda_2^\vee[\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}] + \Lambda_2^\vee[\mathbf{3}]$
2	2	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1},\mathbf{2}] + \Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{2}]$
3	2	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \Lambda_1^\vee[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{2}] + \Lambda_3^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{3}] + \Lambda_1^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]$
		$\Lambda_3^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{2},\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

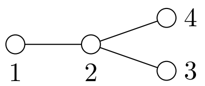
$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$

A.24.4 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)\,[\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{1}, \mathbf{3}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + [\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{1}, \mathbf{2}, \mathbf{3}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}]$
2	2	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{2}, \mathbf{3}]$
3	3	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + [\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

A.25 Root system D_4

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_4^\vee, \Lambda_3^\vee \in P^\vee \mod Q^\vee$

A.25.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (1,1,1,1,0,0,1,0)$$
does not lie in the image of $\mathbf{comp}_2$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	

k	$H^k(W_0, X^\vee)$	generating cocycles
1	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_4^\vee\right)[\mathbf{4}]$
		$\left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_4^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_2^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{4}] + \left(-\Lambda_1^\vee + 3\Lambda_2^\vee - \Lambda_3^\vee - 3\Lambda_4^\vee\right)[\mathbf{2},\mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{1},\mathbf{2}] + \left(-\Lambda_2^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}] + \left(-\Lambda_1^\vee + 3\Lambda_2^\vee - 3\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}] + \left(-2\Lambda_3^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{3},\mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_4^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{3}] + \left(-\Lambda_2^\vee + 4\Lambda_3^\vee\right)[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee - 3\Lambda_2^\vee + 3\Lambda_3^\vee + 3\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_2^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(-\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{1}] + \Lambda_2^\vee[\mathbf{1},\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	$h^k(\overline{X^\vee})$	generating cocycles
0	2	$2\Lambda_3^\vee[]$
		$2\Lambda_4^\vee[]$
1	4	$\Lambda_2^\vee[\mathbf{1}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{2}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3}] + 2\Lambda_3^\vee[\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}]$
		$\left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{3}]$
2	8	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{1},\mathbf{3}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \Lambda_2^\vee[\mathbf{1},\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$\left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{1},\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}] + \Lambda_2^\vee[\mathbf{3},\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1},\mathbf{3}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3}]$
		$\left(\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \Lambda_2^\vee[\mathbf{1},\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{4}] + \Lambda_2^\vee[\mathbf{3},\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1},\mathbf{4}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
3	17	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1,2} \supseteq \mathbf{2}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_4^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,3}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_3^\vee[\mathbf{1,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,4}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1,2} \supseteq \mathbf{1}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1,2} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{1,2,3}] + \Lambda_2^\vee[\mathbf{1,2,4}]$
		$\left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_4^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + \Lambda_2^\vee[\mathbf{2,3} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + 3\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,3}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3,4}]$
		$\Lambda_2^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1,3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,3}]$
		$2\Lambda_3^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1,3} \supseteq \mathbf{3}]$
		$2\Lambda_4^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{1,3} \supseteq \mathbf{3}]$
		$\left(\Lambda_2^\vee + 2\Lambda_3^\vee\right)[\mathbf{1,3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,3}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{1,2,3}] + \Lambda_2^\vee[\mathbf{2,3,4}]$
		$2\Lambda_3^\vee[\mathbf{1,2,3}]$
		$\left(\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_3^\vee[\mathbf{1,4} \supseteq \mathbf{4}] + \Lambda_2^\vee[\mathbf{2,4} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + 3\Lambda_4^\vee\right)[\mathbf{2,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,4}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{3,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3,4}]$
		$\Lambda_2^\vee[\mathbf{1,4} \supseteq \mathbf{1}] + \left(2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,4}]$
		$2\Lambda_3^\vee[\mathbf{1,4} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1,4} \supseteq \mathbf{4}]$
		$\left(\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{1,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,4}]$
		$\Lambda_2^\vee[\mathbf{3,4} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{3,4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3,4}]$
		$2\Lambda_3^\vee[\mathbf{3,4} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{3,4} \supseteq \mathbf{4}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1&0\\1&0\\1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\\0&1\\1&1\\0&0\\1&0\\1&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1&1&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\\1&0&1&1&0&0\\1&0&0&0&0&0\\0&0&1&1&1&0\\0&0&1&1&1&1\\0&0&0&0&0&0\\1&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&1&1&0&0\\1&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$

A.25.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(0,1)\rangle$

$$\phi_u = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_4^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 3\Lambda_4^\vee)\,[\mathbf{2}] + (\Lambda_3^\vee + 2\Lambda_4^\vee)\,[\mathbf{3}] + (\Lambda_1^\vee + \Lambda_4^\vee)\,[\mathbf{4}]$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_4^\vee\right)[\mathbf{4}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_4^\vee[\mathbf{1}\supseteq\mathbf{1}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{3}\supseteq\mathbf{3}] + \left(-\Lambda_2^\vee\right)[\mathbf{4}\supseteq\mathbf{4}] + \left(-\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{4}] + \left(-\Lambda_1^\vee + 3\Lambda_2^\vee - \Lambda_3^\vee - 3\Lambda_4^\vee\right)[\mathbf{2},\mathbf{4}] + \left(\Lambda_2^\vee - 2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_4^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{3}] + \left(-\Lambda_2^\vee + 4\Lambda_3^\vee\right)[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}] + \left(\Lambda_1^\vee - 3\Lambda_2^\vee + 3\Lambda_3^\vee + 3\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\left(2\Lambda_1^\vee - \Lambda_2^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}] + \Lambda_2^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_4^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}]$
2	4	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}\supseteq\mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}\supseteq\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}\supseteq\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1}\supseteq\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}\supseteq\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}\supseteq\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{4}]$
3	7	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}] + \Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}] + \Lambda_3^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}] + \Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}] + \Lambda_2^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + 3\Lambda_4^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}] + 2\Lambda_4^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}] + \Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}] + \Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}] + \Lambda_3^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1},\mathbf{2},\mathbf{4}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\1&0&0\\0&1&0\\1&0&0\\0&0&1\\0&0&1\end{pmatrix}$

A.25.3 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(1,1)\rangle$

$$\phi_u = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + 2\Lambda_4^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 2\Lambda_4^\vee)\,[\mathbf{2}] + (\Lambda_3^\vee + \Lambda_4^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee)\,[\mathbf{4}]$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_4^\vee\right)[\mathbf{4}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_4^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_3^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{1},\mathbf{4}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee - 2\Lambda_4^\vee\right)[\mathbf{2},\mathbf{4}] + \left(2\Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{3},\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee - 2\Lambda_2^\vee + 2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + \Lambda_2^\vee[\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{4}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
0	1	$2\Lambda_4^\vee[]$
1	2	$2\Lambda_4^\vee[\mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4}]$
		$\left(\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{3}]$
2	4	$2\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_4^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_4^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_4^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$\left(\Lambda_2^\vee + 2\Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4}]$

k	$h^k(\overline{X}^\vee)$	generating cocycles
3	7	$2\Lambda_4^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_4^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_4^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_4^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]$
		$\left(\Lambda_2^\vee+2\Lambda_4^\vee\right)[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\left(\Lambda_3^\vee+3\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\left(\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\Lambda_1^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+\Lambda_2^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\Lambda_1^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+\Lambda_2^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$2\Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\Lambda_1^\vee[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+\Lambda_2^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{3}]+2\Lambda_4^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\Lambda_1^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&0\\0&0&1\\0&0&1\\0&0&0\\0&1&0\\0&1&0\end{pmatrix}$

A.25.4 Cohomology of lattice $X^\vee$ corresponding to $\Omega=\langle(1,0)\rangle$

$$\boxed{\begin{aligned}\phi_u \;&=\; \partial\tau \text{ with } \tau = \\ &(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee)\,[\mathbf{1}]+(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee)\,[\mathbf{2}]+(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee)\,[\mathbf{3}]+\Lambda_4^\vee[\mathbf{4}]\end{aligned}}$$

k	$\mathbf{H^k(W_0,X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee-2\Lambda_3^\vee\right)[\mathbf{3}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_3^\vee[\mathbf{1}\supseteq\mathbf{1}]+\left(-2\Lambda_3^\vee\right)[\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_3^\vee[\mathbf{1},\mathbf{2}]+\left(-\Lambda_2^\vee\right)[\mathbf{3}\supseteq\mathbf{3}]+\left(-\Lambda_2^\vee+2\Lambda_3^\vee\right)[\mathbf{1},\mathbf{3}]+$ $\left(-\Lambda_1^\vee+3\Lambda_2^\vee-3\Lambda_3^\vee-\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3}]+\left(-2\Lambda_3^\vee\right)[\mathbf{4}\supseteq\mathbf{4}]+\left(-\Lambda_2^\vee+2\Lambda_3^\vee\right)[\mathbf{3},\mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\Lambda_2^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{3}]+2\Lambda_3^\vee[\mathbf{3},\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_1^\vee-\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_4^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]$
		$\left(2\Lambda_1^\vee-\Lambda_2^\vee\right)[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(-\Lambda_1^\vee+\Lambda_2^\vee-\Lambda_3^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\Lambda_2^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(-\Lambda_1^\vee+\Lambda_2^\vee-\Lambda_3^\vee-\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]$

k	$h^k(\overline{X}^\vee)$	generating cocycles
0	1	$2\Lambda_3^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}]+2\Lambda_3^\vee[\mathbf{2}]+2\Lambda_3^\vee[\mathbf{3}]+2\Lambda_3^\vee[\mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1}]+2\Lambda_3^\vee[\mathbf{2}]+2\Lambda_3^\vee[\mathbf{3}]+2\Lambda_3^\vee[\mathbf{4}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
2	4	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{4}]$
3	7	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{2}, \mathbf{4}]$
		$2\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_3^\vee[\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_3^\vee[\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \left(\Lambda_2^\vee + 2\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{2}, \mathbf{3} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + 3\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + \Lambda_4^\vee[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{2}, \mathbf{4}] + 2\Lambda_3^\vee[\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2}, \mathbf{3}, \mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{2}, \mathbf{3}] + \Lambda_2^\vee[\mathbf{1}, \mathbf{2}, \mathbf{4}]$
		$\Lambda_2^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + 2\Lambda_3^\vee[\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$\Lambda_4^\vee[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$2\Lambda_3^\vee[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + \Lambda_4^\vee[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{2}, \mathbf{4}] + 2\Lambda_3^\vee[\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2}, \mathbf{3}, \mathbf{4}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\1&0&0\\0&1&0\\0&0&1\\0&0&1\\0&0&0\end{pmatrix}$

A.25.5 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u \quad = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + \Lambda_3^\vee)[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee)[\mathbf{2}] + (\Lambda_1^\vee + \Lambda_3^\vee)[\mathbf{3}] + \Lambda_4^\vee[\mathbf{4}]}$$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_4^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_4^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_4^\vee[\mathbf{1}, \mathbf{2}] + \left(-\Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-\Lambda_1^\vee - \Lambda_3^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1}, \mathbf{4}] + \left(\Lambda_2^\vee - \Lambda_4^\vee\right)[\mathbf{2}, \mathbf{4}] + \left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{3}, \mathbf{4}]$
		$\Lambda_3^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-\Lambda_3^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \Lambda_3^\vee[\mathbf{1}, \mathbf{2}] + \left(-\Lambda_1^\vee - \Lambda_4^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(\Lambda_1^\vee - \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1}, \mathbf{3}] + \left(\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{2}, \mathbf{3}] + \left(-\Lambda_3^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{3}, \mathbf{4}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_2^\vee - 2\Lambda_3^\vee\right)[\mathbf{3,4 \supseteq 4}] + \left(\Lambda_2^\vee - \Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{2,3,4}]$
		$\Lambda_4^\vee[\mathbf{2,4 \supseteq 2}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{2,4 \supseteq 4}] + \left(-\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,4}] + \left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{2,3,4}]$
		$\Lambda_3^\vee[\mathbf{1,4 \supseteq 1}] + \Lambda_3^\vee[\mathbf{1,4 \supseteq 4}] + \left(-\Lambda_3^\vee\right)[\mathbf{1,2,4}] + \left(-\Lambda_1^\vee\right)[\mathbf{3,4 \supseteq 3}] + \left(-\Lambda_1^\vee\right)[\mathbf{3,4 \supseteq 4}] + \left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee\right)[\mathbf{1,3,4}] + \left(-\Lambda_1^\vee\right)[\mathbf{2,3,4}]$
		$\Lambda_3^\vee[\mathbf{2,3 \supseteq 2}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee - \Lambda_4^\vee\right)[\mathbf{2,3 \supseteq 3}] + \left(-\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_3^\vee - \Lambda_4^\vee\right)[\mathbf{2,3,4}]$
		$\Lambda_4^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_4^\vee[\mathbf{1,3 \supseteq 3}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,2,3}] + \left(-\Lambda_1^\vee\right)[\mathbf{3,4 \supseteq 3}] + \left(-\Lambda_1^\vee\right)[\mathbf{3,4 \supseteq 4}] + \left(-\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1,3,4}] + \left(-\Lambda_1^\vee\right)[\mathbf{2,3,4}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
0	0	
1	2	$\Lambda_3^\vee[\mathbf{1}] + \Lambda_3^\vee[\mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{3}] + \Lambda_3^\vee[\mathbf{4}]$
		$\Lambda_4^\vee[\mathbf{1}] + \Lambda_4^\vee[\mathbf{2}] + \Lambda_4^\vee[\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{4}]$
2	7	$\Lambda_3^\vee[\mathbf{1 \supseteq 1}] + \Lambda_3^\vee[\mathbf{2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1,3}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{2,3}] + \Lambda_3^\vee[\mathbf{4 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{3,4}]$
		$\Lambda_4^\vee[\mathbf{1 \supseteq 1}] + \Lambda_4^\vee[\mathbf{2 \supseteq 2}] + \Lambda_4^\vee[\mathbf{3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{4 \supseteq 4}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1,4}] + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{2,4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{3,4}]$
		$\Lambda_3^\vee[\mathbf{1,2}]$
		$\Lambda_4^\vee[\mathbf{1,2}]$
		$\Lambda_4^\vee[\mathbf{1,3}] + \Lambda_1^\vee[\mathbf{3,4}]$
		$\Lambda_4^\vee[\mathbf{2,3}]$
		$\Lambda_3^\vee[\mathbf{1,4}] + \Lambda_1^\vee[\mathbf{3,4}]$
3	11	$\Lambda_3^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + \Lambda_3^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_1^\vee[\mathbf{1,3 \supseteq 1}] + \left(\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{2,3 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + \Lambda_3^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + \Lambda_3^\vee[\mathbf{3,4 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{3,4 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,3,4}] + \Lambda_2^\vee[\mathbf{2,3,4}]$
		$\Lambda_4^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + \Lambda_4^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \Lambda_4^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{4 \supseteq 4 \supseteq 4}] + \Lambda_1^\vee[\mathbf{1,4 \supseteq 1}] + \left(\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,4 \supseteq 4}] + \Lambda_2^\vee[\mathbf{2,4 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,4 \supseteq 4}] + \Lambda_2^\vee[\mathbf{1,2,4}] + \Lambda_3^\vee[\mathbf{3,4 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{3,4 \supseteq 4}] + \Lambda_2^\vee[\mathbf{2,3,4}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2 \supseteq 2 \supseteq 2}] + \Lambda_1^\vee[\mathbf{1,2 \supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,2 \supseteq 2}] + \Lambda_2^\vee[\mathbf{2,3 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1,2,3}] + \Lambda_2^\vee[\mathbf{2,4 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,4 \supseteq 4}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,4}]$
		$\Lambda_3^\vee[\mathbf{1,2 \supseteq 1}] + \Lambda_3^\vee[\mathbf{1,2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1,2,3}]$
		$\Lambda_2^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{2,3 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3,4}]$
		$\Lambda_3^\vee[\mathbf{1,3 \supseteq 1}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1,2,3}] + \Lambda_3^\vee[\mathbf{1,3,4}]$
		$\Lambda_4^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_4^\vee[\mathbf{1,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1,3,4}]$
		$\Lambda_3^\vee[\mathbf{1,4 \supseteq 1}] + \Lambda_3^\vee[\mathbf{1,4 \supseteq 4}] + \left(\Lambda_1^\vee + \Lambda_4^\vee\right)[\mathbf{1,3,4}]$
		$\Lambda_4^\vee[\mathbf{1,4 \supseteq 1}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1,2,4}] + \Lambda_4^\vee[\mathbf{1,3,4}]$
		$\Lambda_1^\vee[\mathbf{3,4 \supseteq 3}] + \Lambda_1^\vee[\mathbf{3,4 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{1,3,4}]$
		$\Lambda_4^\vee[\mathbf{3,4 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{3,4 \supseteq 4}] + \Lambda_4^\vee[\mathbf{1,3,4}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_4^\vee\right)[\mathbf{2,3,4}]$

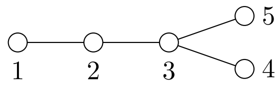
$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

A.25.6 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)\,[\mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{4} \supseteq \mathbf{4}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{3}, \mathbf{4} \supseteq \mathbf{3}] + [\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{2}, \mathbf{3}, \mathbf{4}]$
		$[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{3}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
0	1	$[]$
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}]$
2	4	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3}]$
		$[\mathbf{1}, \mathbf{4}]$
		$[\mathbf{3}, \mathbf{4}]$
3	8	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{3}]$
		$[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{4}]$
		$[\mathbf{3}, \mathbf{4} \supseteq \mathbf{3}] + [\mathbf{3}, \mathbf{4} \supseteq \mathbf{4}]$
		$[\mathbf{1}, \mathbf{3}, \mathbf{4}]$
		$[\mathbf{2}, \mathbf{3}, \mathbf{4}]$

A.26 Root system D_5

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/4\mathbb{Z}$ generated by $\Lambda_5^\vee \in P^\vee \mod Q^\vee$

A.26.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] \quad = \quad (0,0,1,0)$$
does not lie in the image of comp$_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/4\mathbb{Z}$	$\left(\Lambda_3^\vee - 2\Lambda_5^\vee\right)\mathbf{[5]}$
2	$\mathbb{Z}/2\mathbb{Z}$	$4\Lambda_5^\vee\mathbf{[1 \supseteq 1]} + \left(-4\Lambda_5^\vee\right)\mathbf{[2 \supseteq 2]} + 4\Lambda_5^\vee\mathbf{[1,2]} + \left(-4\Lambda_5^\vee\right)\mathbf{[3 \supseteq 3]} + \left(-4\Lambda_5^\vee\right)\mathbf{[4 \supseteq 4]} +$ $\left(-4\Lambda_4^\vee\right)\mathbf{[5 \supseteq 5]} + \left(-2\Lambda_3^\vee + 4\Lambda_5^\vee\right)\mathbf{[1,5]} + \left(2\Lambda_3^\vee - 4\Lambda_5^\vee\right)\mathbf{[2,5]} +$ $\left(-2\Lambda_2^\vee + 4\Lambda_3^\vee - 4\Lambda_5^\vee\right)\mathbf{[3,5]} + \left(4\Lambda_4^\vee - 4\Lambda_5^\vee\right)\mathbf{[4,5]}$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$4\Lambda_5^\vee\mathbf{[4,5 \supseteq 4]} + 4\Lambda_4^\vee\mathbf{[4,5 \supseteq 5]} + \left(2\Lambda_2^\vee - 4\Lambda_3^\vee + 4\Lambda_4^\vee + 4\Lambda_5^\vee\right)\mathbf{[3,4,5]}$
		$\left(\Lambda_3^\vee - 2\Lambda_5^\vee\right)\mathbf{[1,5 \supseteq 1]} + \left(-\Lambda_3^\vee - 6\Lambda_5^\vee\right)\mathbf{[2,5 \supseteq 2]} + \left(-4\Lambda_3^\vee\right)\mathbf{[2,5 \supseteq 5]} +$ $\left(-\Lambda_3^\vee - 2\Lambda_5^\vee\right)\mathbf{[1,2,5]} + \left(-\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[3,5 \supseteq 3]} + \left(-\Lambda_2^\vee + \Lambda_3^\vee - 2\Lambda_4^\vee\right)\mathbf{[3,5 \supseteq 5]} +$ $\left(\Lambda_1^\vee + 2\Lambda_2^\vee - 2\Lambda_3^\vee + 4\Lambda_4^\vee + 2\Lambda_5^\vee\right)\mathbf{[2,3,5]} + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)\mathbf{[3,4,5]}$
		$4\Lambda_5^\vee\mathbf{[1,3 \supseteq 1]} + 4\Lambda_5^\vee\mathbf{[1,3 \supseteq 3]} + \left(-4\Lambda_5^\vee\right)\mathbf{[1,2,3]} + \left(-4\Lambda_5^\vee\right)\mathbf{[1,4 \supseteq 1]} +$ $\left(-4\Lambda_5^\vee\right)\mathbf{[1,4 \supseteq 4]} + \left(-4\Lambda_5^\vee\right)\mathbf{[2,4 \supseteq 2]} + \left(-4\Lambda_5^\vee\right)\mathbf{[2,4 \supseteq 4]} + 4\Lambda_5^\vee\mathbf{[1,3,4]} +$ $4\Lambda_5^\vee\mathbf{[2,3,4]} + \left(-4\Lambda_5^\vee\right)\mathbf{[1,5 \supseteq 1]} + \left(-2\Lambda_3^\vee\right)\mathbf{[1,5 \supseteq 5]} + \left(-4\Lambda_5^\vee\right)\mathbf{[2,5 \supseteq 2]} +$ $\left(-2\Lambda_3^\vee\right)\mathbf{[2,5 \supseteq 5]} + \left(2\Lambda_2^\vee - 2\Lambda_3^\vee + 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)\mathbf{[1,3,5]} +$ $\left(2\Lambda_2^\vee - 2\Lambda_3^\vee + 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)\mathbf{[2,3,5]} + \left(2\Lambda_3^\vee - 4\Lambda_5^\vee\right)\mathbf{[1,4,5]} + \left(2\Lambda_3^\vee - 4\Lambda_5^\vee\right)\mathbf{[2,4,5]}$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$4\Lambda_5^\vee\mathbf{[]}$
1	2	$\Lambda_2^\vee\mathbf{[1]} + 4\Lambda_5^\vee\mathbf{[2]} + 4\Lambda_5^\vee\mathbf{[3]} + 4\Lambda_5^\vee\mathbf{[4]} + 4\Lambda_5^\vee\mathbf{[5]}$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)\mathbf{[4]}$
2	4	$\Lambda_2^\vee\mathbf{[1 \supseteq 1]} + 4\Lambda_5^\vee\mathbf{[2 \supseteq 2]} + \left(\Lambda_1^\vee + \Lambda_2^\vee + 2\Lambda_5^\vee\right)\mathbf{[1,2]} + 4\Lambda_5^\vee\mathbf{[3 \supseteq 3]} + 4\Lambda_5^\vee\mathbf{[4 \supseteq 4]} + 4\Lambda_5^\vee\mathbf{[5 \supseteq 5]}$
		$\Lambda_2^\vee\mathbf{[1,3]} + \left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)\mathbf{[2,3]} + 4\Lambda_5^\vee\mathbf{[1,4]} + 4\Lambda_5^\vee\mathbf{[2,4]} + \left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)\mathbf{[3,4]} + 4\Lambda_5^\vee\mathbf{[1,5]} + 4\Lambda_5^\vee\mathbf{[2,5]} +$ $\left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)\mathbf{[3,5]}$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)\mathbf{[4 \supseteq 4]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)\mathbf{[3,4]} + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)\mathbf{[4,5]}$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)\mathbf{[1,4]} + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)\mathbf{[2,4]} + \Lambda_2^\vee\mathbf{[3,4]}$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
3	8	$\Lambda_2^\vee\,[\mathbf{1 \supseteq 1 \supseteq 1}] + 4\Lambda_5^\vee\,[\mathbf{2 \supseteq 2 \supseteq 2}] + \left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 2}] + 4\Lambda_5^\vee\,[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_2^\vee\,[\mathbf{1, 2, 3}] + 4\Lambda_5^\vee\,[\mathbf{4 \supseteq 4 \supseteq 4}] + 4\Lambda_5^\vee\,[\mathbf{2, 3, 4}] + 4\Lambda_5^\vee\,[\mathbf{5 \supseteq 5 \supseteq 5}] + 4\Lambda_5^\vee\,[\mathbf{2, 3, 5}]$
		$4\Lambda_5^\vee\,[\mathbf{1 \supseteq 1 \supseteq 1}] + 4\Lambda_5^\vee\,[\mathbf{2 \supseteq 2 \supseteq 2}] + 4\Lambda_5^\vee\,[\mathbf{3 \supseteq 3 \supseteq 3}] + 4\Lambda_5^\vee\,[\mathbf{4 \supseteq 4 \supseteq 4}] + 4\Lambda_5^\vee\,[\mathbf{5 \supseteq 5 \supseteq 5}]$
		$\Lambda_2^\vee\,[\mathbf{1, 3 \supseteq 1}] + 4\Lambda_5^\vee\,[\mathbf{1, 3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 4\Lambda_5^\vee\right)[\mathbf{1, 2, 3}] + 4\Lambda_5^\vee\,[\mathbf{1, 4 \supseteq 1}] + 4\Lambda_5^\vee\,[\mathbf{1, 4 \supseteq 4}] + 4\Lambda_5^\vee\,[\mathbf{2, 4 \supseteq 2}] + 4\Lambda_5^\vee\,[\mathbf{2, 4 \supseteq 4}] + \left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 3, 4}] + 4\Lambda_5^\vee\,[\mathbf{1, 5 \supseteq 1}] + 4\Lambda_5^\vee\,[\mathbf{1, 5 \supseteq 5}] + 4\Lambda_5^\vee\,[\mathbf{2, 5 \supseteq 2}] + 4\Lambda_5^\vee\,[\mathbf{2, 5 \supseteq 5}] + \left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 3, 5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{4 \supseteq 4 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{3, 4 \supseteq 3}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3, 4 \supseteq 4}] + 4\Lambda_5^\vee\,[\mathbf{4, 5 \supseteq 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{3, 4, 5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 4 \supseteq 1}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{2, 4 \supseteq 2}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3, 4 \supseteq 3}] + \left(\Lambda_2^\vee + 4\Lambda_5^\vee\right)[\mathbf{3, 4 \supseteq 4}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{2, 3, 4}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{3, 4, 5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 4 \supseteq 4}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2, 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{1, 3, 4}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 4, 5}] + 4\Lambda_5^\vee\,[\mathbf{2, 4, 5}]$
		$\left(\Lambda_3^\vee + 6\Lambda_5^\vee\right)[\mathbf{5 \supseteq 5 \supseteq 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{3, 5 \supseteq 3}] + \left(\Lambda_4^\vee + 5\Lambda_5^\vee\right)[\mathbf{3, 5 \supseteq 5}] + 4\Lambda_5^\vee\,[\mathbf{4, 5 \supseteq 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{3, 4, 5}]$
		$\Lambda_2^\vee\,[\mathbf{3, 4, 5}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\\1&0&0\\0&1&0\\0&0&0\\1&0&0\\0&0&0\end{pmatrix}$

A.26.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(2)\rangle$

$$\phi_u = \partial\tau \text{ with } \tau = \left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)[\mathbf{1}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{2}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{3}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{4}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{5}]$$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_5^\vee\,[\mathbf{1 \supseteq 1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2 \supseteq 2}] + 2\Lambda_5^\vee\,[\mathbf{1, 2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3 \supseteq 3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{4 \supseteq 4}] + \left(-2\Lambda_4^\vee\right)[\mathbf{5 \supseteq 5}] + \left(-\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 5}] + \left(\Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{2, 5}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{3, 5}] + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{4, 5}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{4,5 \supseteq 5}] + \left(\Lambda_3^\vee - \Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$2\Lambda_5^\vee[\mathbf{4,5 \supseteq 4}] + 2\Lambda_4^\vee[\mathbf{4,5 \supseteq 5}] + \left(\Lambda_2^\vee - 2\Lambda_3^\vee + 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$\left(\Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-\Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_3^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-\Lambda_3^\vee\right)[\mathbf{1,2,5}] + \left(-\Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_2^\vee + \Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 5}] + \left(\Lambda_1^\vee + 2\Lambda_4^\vee\right)[\mathbf{2,3,5}] + \left(-2\Lambda_3^\vee + 2\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$2\Lambda_5^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{1,3,4}] + 2\Lambda_5^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-\Lambda_3^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-\Lambda_3^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(\Lambda_2^\vee - \Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1,3,5}] + \left(\Lambda_2^\vee - \Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{2,3,5}] + \left(\Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{1,4,5}] + \left(\Lambda_3^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,4,5}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_5^\vee[]$
1	2	$2\Lambda_5^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{4}]$
2	5	$2\Lambda_5^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5}]$
		$2\Lambda_5^\vee[\mathbf{1,3}] + 2\Lambda_5^\vee[\mathbf{1,4}] + 2\Lambda_5^\vee[\mathbf{2,4}] + 2\Lambda_5^\vee[\mathbf{1,5}] + 2\Lambda_5^\vee[\mathbf{2,5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{4 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3,4}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,4}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,4}] + \Lambda_2^\vee[\mathbf{3,4}]$
		$\Lambda_3^\vee[\mathbf{4,5}]$
3	10	$2\Lambda_5^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5 \supseteq 5}]$
		$2\Lambda_5^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1,3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{1,4 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1,4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{2,4 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{2,4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{1,5 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1,5 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{2,5 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{2,5 \supseteq 5}]$
		$2\Lambda_5^\vee[\mathbf{1,2,3}] + 2\Lambda_5^\vee[\mathbf{2,3,4}] + 2\Lambda_5^\vee[\mathbf{2,3,5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{4 \supseteq 4 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3,4 \supseteq 3}] + \left(\Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{3,4 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3,4 \supseteq 3}] + \Lambda_2^\vee[\mathbf{3,4 \supseteq 4}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,3,4}] + \Lambda_3^\vee[\mathbf{3,4,5}]$
		$\left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1,2,4}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1,3,4}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,4,5}]$
		$\left(\Lambda_1^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 4}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1,2,4}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3,4 \supseteq 3}] + \Lambda_2^\vee[\mathbf{3,4 \supseteq 4}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{2,3,4}] + \left(\Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,4,5}] + \Lambda_3^\vee[\mathbf{3,4,5}]$
		$\Lambda_3^\vee[\mathbf{4,5 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{4,5 \supseteq 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$\Lambda_3^\vee[\mathbf{1,4,5}] + \Lambda_3^\vee[\mathbf{2,4,5}]$
		$\Lambda_2^\vee[\mathbf{3,4,5}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\1&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}$

A.26.3 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{aligned}\phi_u \quad &= \partial\tau \text{ with } \tau = \\ &\Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee)[\mathbf{3}] + (\Lambda_4^\vee + \Lambda_5^\vee)[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee)[\mathbf{5}]\end{aligned}}$$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_3^\vee - 2\Lambda_4^\vee\right)[\mathbf{4,5 \supseteq 5}] + \left(\Lambda_3^\vee - \Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$\Lambda_5^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_5^\vee[\mathbf{1,3 \supseteq 3}] + \left(-\Lambda_5^\vee\right)[\mathbf{1,2,3}] + \left(-\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 4}] +$ $\left(-\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 4}] + \Lambda_5^\vee[\mathbf{1,3,4}] + \Lambda_5^\vee[\mathbf{2,3,4}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,5 \supseteq 1}] +$ $\left(-\Lambda_4^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-\Lambda_1^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_1^\vee\right)[\mathbf{3,5 \supseteq 5}] +$ $\left(-\Lambda_1^\vee + \Lambda_2^\vee - \Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1,3,5}] + \Lambda_4^\vee[\mathbf{2,3,5}] + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{1,4,5}] + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)[\mathbf{2,4,5}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	0	
1	0	
2	2	$\Lambda_4^\vee[\mathbf{1,3}] + \Lambda_5^\vee[\mathbf{1,4}] + \Lambda_5^\vee[\mathbf{2,4}] + \Lambda_1^\vee[\mathbf{3,4}] + \Lambda_4^\vee[\mathbf{1,5}] + \Lambda_4^\vee[\mathbf{2,5}]$
		$\Lambda_5^\vee[\mathbf{3,4}]$
3	4	$\Lambda_4^\vee[\mathbf{1,2 \supseteq 1}] + \Lambda_4^\vee[\mathbf{1,2 \supseteq 2}] + \Lambda_3^\vee[\mathbf{2,4 \supseteq 2}] + \Lambda_4^\vee[\mathbf{3,4 \supseteq 3}] + \left(\Lambda_2^\vee + \Lambda_5^\vee\right)[\mathbf{3,4 \supseteq 4}] + \Lambda_3^\vee[\mathbf{2,3,4}] +$ $\left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{3,4,5}]$
		$\Lambda_4^\vee[\mathbf{1,3 \supseteq 1}] + \Lambda_5^\vee[\mathbf{1,3 \supseteq 3}] + \Lambda_3^\vee[\mathbf{2,3 \supseteq 2}] + \Lambda_1^\vee[\mathbf{2,3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{1,2,3}] + \Lambda_5^\vee[\mathbf{1,4 \supseteq 1}] +$ $\Lambda_5^\vee[\mathbf{1,4 \supseteq 4}] + \Lambda_5^\vee[\mathbf{2,4 \supseteq 2}] + \Lambda_5^\vee[\mathbf{2,4 \supseteq 4}] + \Lambda_1^\vee[\mathbf{1,3,4}] + \Lambda_3^\vee[\mathbf{2,3,4}] + \Lambda_4^\vee[\mathbf{1,5 \supseteq 1}] + \Lambda_4^\vee[\mathbf{1,5 \supseteq 5}] +$ $\Lambda_4^\vee[\mathbf{2,5 \supseteq 2}] + \Lambda_4^\vee[\mathbf{2,5 \supseteq 5}] + \Lambda_1^\vee[\mathbf{3,5 \supseteq 3}] + \Lambda_1^\vee[\mathbf{3,5 \supseteq 5}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{1,3,5}] + \Lambda_3^\vee[\mathbf{2,3,5}] +$ $\left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{1,4,5}] + \left(\Lambda_4^\vee + \Lambda_5^\vee\right)[\mathbf{2,4,5}]$
		$\Lambda_4^\vee[\mathbf{1,2,3}] + \Lambda_5^\vee[\mathbf{2,3,4}] + \Lambda_4^\vee[\mathbf{2,3,5}]$
		$\Lambda_5^\vee[\mathbf{1,3,4}]$

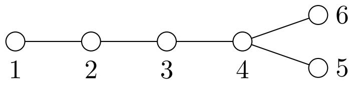
$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	$\begin{pmatrix}1&0\\0&1\\0&1\\0&0\end{pmatrix}$

A.26.4 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1} \supseteq \mathbf{1}] + (-1)\,[\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{1}, \mathbf{2}] + (-1)\,[\mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{5} \supseteq \mathbf{5}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{4}, \mathbf{5} \supseteq \mathbf{4}] + [\mathbf{4}, \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{3}, \mathbf{4}, \mathbf{5}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{2}, \mathbf{3}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{1}, \mathbf{3}, \mathbf{4}] + [\mathbf{2}, \mathbf{3}, \mathbf{4}] + (-1)\,[\mathbf{1}, \mathbf{5} \supseteq \mathbf{1}] + (-1)\,[\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + (-1)\,[\mathbf{2}, \mathbf{5} \supseteq \mathbf{2}] + (-1)\,[\mathbf{2}, \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{1}, \mathbf{3}, \mathbf{5}] + [\mathbf{2}, \mathbf{3}, \mathbf{5}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_\mathbf{2})$	generating cocycles
0	1	[]
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}]$
2	3	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4}] + [\mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{3}] + [\mathbf{1}, \mathbf{4}] + [\mathbf{2}, \mathbf{4}] + [\mathbf{1}, \mathbf{5}] + [\mathbf{2}, \mathbf{5}]$
		$[\mathbf{4}, \mathbf{5}]$
3	6	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{3} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{1}, \mathbf{4} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{2}] + [\mathbf{2}, \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{1}, \mathbf{5} \supseteq \mathbf{1}] + [\mathbf{1}, \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{2}, \mathbf{5} \supseteq \mathbf{2}] + [\mathbf{2}, \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{2}, \mathbf{3}] + [\mathbf{2}, \mathbf{3}, \mathbf{4}] + [\mathbf{2}, \mathbf{3}, \mathbf{5}]$
		$[\mathbf{4}, \mathbf{5} \supseteq \mathbf{4}] + [\mathbf{4}, \mathbf{5} \supseteq \mathbf{5}]$
		$[\mathbf{1}, \mathbf{4}, \mathbf{5}] + [\mathbf{2}, \mathbf{4}, \mathbf{5}]$
		$[\mathbf{3}, \mathbf{4}, \mathbf{5}]$

A.27 Root system D_6

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_6^\vee, \Lambda_5^\vee \in P^\vee \mod Q^\vee$

A.27.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] \;=\; (1,1,1,0,0,1,0,1)$
does not lie in the image of $\mathbf{comp_2}$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{6}]$ $\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_6^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_6^\vee[\mathbf{1,2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{1,6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{3,6}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{4,6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{5,6}]$ $2\Lambda_5^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_5^\vee[\mathbf{1,2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-\Lambda_4^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,5}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{3,5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{4,5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{5,6}]$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_6^\vee[\mathbf{5,6 \supseteq 5}] + \left(-\Lambda_4^\vee + 4\Lambda_5^\vee\right)[\mathbf{5,6 \supseteq 6}] + \left(\Lambda_3^\vee - 3\Lambda_4^\vee + 3\Lambda_5^\vee + 3\Lambda_6^\vee\right)[\mathbf{4,5,6}]$
		$\Lambda_4^\vee[\mathbf{5,6 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{5,6 \supseteq 6}] + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{4,5,6}]$
		$\left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,2,6}] + \left(-\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 3\Lambda_4^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(2\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,4,6}] + \left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{3,4,6}]$
		$\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,2,5}] + \left(-\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 3\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 5}] + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,4,5}] + \left(\Lambda_2^\vee + 2\Lambda_6^\vee\right)[\mathbf{3,4,5}]$
		$2\Lambda_6^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_6^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_6^\vee[\mathbf{1,3,4}] + 2\Lambda_6^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,5 \supseteq 5}] + 2\Lambda_6^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-\Lambda_4^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{1,3,6}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{1,4,6}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{2,4,6}] + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3,4,6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{1,5,6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2,5,6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{3,5,6}]$
		$2\Lambda_5^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{1,3,4}] + 2\Lambda_5^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,3,5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{1,4,5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{2,4,5}] + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3,4,5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,6 \supseteq 6}] + 2\Lambda_5^\vee[\mathbf{3,4,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3,5,6}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	2	$2\Lambda_5^\vee[]$
		$2\Lambda_6^\vee[]$
1	4	$\Lambda_2^\vee[\mathbf{1}] + \left(2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{2}] + \left(2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{3}] + \left(2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{4}] + \left(2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{5}] + \left(2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{6}]$
		$2\Lambda_5^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5}] + 2\Lambda_5^\vee[\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{1}] + 2\Lambda_6^\vee[\mathbf{2}] + 2\Lambda_6^\vee[\mathbf{3}] + 2\Lambda_6^\vee[\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{6}]$
		$\left(\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{5}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
2	8	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{1},\mathbf{2}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{3}\supseteq\mathbf{3}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{4}\supseteq\mathbf{4}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}]+\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{6}\supseteq\mathbf{6}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{6}]+\left(\Lambda_2^\vee+2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{6}]$
		$2\Lambda_5^\vee[\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_5^\vee[\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_5^\vee[\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_5^\vee[\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_5^\vee[\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_5^\vee[\mathbf{6}\supseteq\mathbf{6}]$
		$2\Lambda_6^\vee[\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_6^\vee[\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_6^\vee[\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_6^\vee[\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_6^\vee[\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_6^\vee[\mathbf{6}\supseteq\mathbf{6}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{2},\mathbf{3}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{4}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{4}]+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{3},\mathbf{4}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{6}]+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{6}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{6}]$
		$2\Lambda_5^\vee[\mathbf{1},\mathbf{3}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{4}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{5}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{5}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{5}]+2\Lambda_5^\vee[\mathbf{1},\mathbf{6}]+2\Lambda_5^\vee[\mathbf{2},\mathbf{6}]+2\Lambda_5^\vee[\mathbf{3},\mathbf{6}]$
		$\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{5}\supseteq\mathbf{5}]+\left(\Lambda_2^\vee+\Lambda_4^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{4},\mathbf{5}]+\Lambda_4^\vee[\mathbf{5},\mathbf{6}]$
		$\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{1},\mathbf{5}]+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{2},\mathbf{5}]+\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}]$
		$\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{6}\supseteq\mathbf{6}]+\left(\Lambda_2^\vee+\Lambda_4^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{6}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{4},\mathbf{6}]+\Lambda_4^\vee[\mathbf{5},\mathbf{6}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
3	19	$\Lambda_2^\vee\mathbf{[1\supseteq 1\supseteq 1]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2\supseteq 2\supseteq 2]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,2\supseteq 1]}+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,2\supseteq 2]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3\supseteq 3\supseteq 3]}+\Lambda_2^\vee\mathbf{[1,2,3]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[4\supseteq 4\supseteq 4]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,3,4]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[5\supseteq 5\supseteq 5]}+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[2,5\supseteq 2]}+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[3,5\supseteq 3]}+\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]}+\Lambda_2^\vee\mathbf{[1,3,5]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,5]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+3\Lambda_6^\vee\right)\mathbf{[3,4,5]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[6\supseteq 6\supseteq 6]}+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[2,6\supseteq 2]}+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[3,6\supseteq 3]}+\left(\Lambda_2^\vee+2\Lambda_5^\vee\right)\mathbf{[3,6\supseteq 6]}+\Lambda_2^\vee\mathbf{[1,3,6]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,6]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[3,4,6]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5,6]}$
		$2\Lambda_5^\vee\mathbf{[1\supseteq 1\supseteq 1]}+2\Lambda_5^\vee\mathbf{[2\supseteq 2\supseteq 2]}+2\Lambda_5^\vee\mathbf{[3\supseteq 3\supseteq 3]}+2\Lambda_5^\vee\mathbf{[4\supseteq 4\supseteq 4]}+2\Lambda_5^\vee\mathbf{[5\supseteq 5\supseteq 5]}+2\Lambda_5^\vee\mathbf{[6\supseteq 6\supseteq 6]}$
		$2\Lambda_6^\vee\mathbf{[1\supseteq 1\supseteq 1]}+2\Lambda_6^\vee\mathbf{[2\supseteq 2\supseteq 2]}+2\Lambda_6^\vee\mathbf{[3\supseteq 3\supseteq 3]}+2\Lambda_6^\vee\mathbf{[4\supseteq 4\supseteq 4]}+2\Lambda_6^\vee\mathbf{[5\supseteq 5\supseteq 5]}+2\Lambda_6^\vee\mathbf{[6\supseteq 6\supseteq 6]}$
		$\left(\Lambda_1^\vee+\Lambda_3^\vee+4\Lambda_5^\vee+4\Lambda_6^\vee\right)\mathbf{[2\supseteq 2\supseteq 2]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,2\supseteq 1]}+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,2\supseteq 2]}+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3\supseteq 2]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_5^\vee+3\Lambda_6^\vee\right)\mathbf{[2,3\supseteq 3]}+\Lambda_2^\vee\mathbf{[1,2,3]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,3,4]}+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[2,5\supseteq 2]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,4,5]}+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[2,6\supseteq 2]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,4,6]}$
		$\Lambda_2^\vee\mathbf{[1,3\supseteq 1]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,3\supseteq 3]}+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,2,3]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,4\supseteq 1]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,4\supseteq 4]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,4\supseteq 2]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,4\supseteq 4]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,3,4]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 1]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 5]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,5\supseteq 2]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,5\supseteq 5]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 3]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]}+\Lambda_4^\vee\mathbf{[1,3,5]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,6\supseteq 1]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,6\supseteq 6]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,6\supseteq 2]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,6\supseteq 6]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,6\supseteq 3]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,6\supseteq 6]}+\Lambda_4^\vee\mathbf{[1,3,6]}$
		$\Lambda_4^\vee\mathbf{[1,3\supseteq 1]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3\supseteq 2]}+\left(\Lambda_1^\vee+3\Lambda_5^\vee+3\Lambda_6^\vee\right)\mathbf{[2,3\supseteq 3]}+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,2,3]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,3,4]}+\left(\Lambda_3^\vee+3\Lambda_5^\vee+3\Lambda_6^\vee\right)\mathbf{[2,3,4]}+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[3,5\supseteq 3]}+\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]}+\left(\Lambda_2^\vee+\Lambda_4^\vee\right)\mathbf{[1,3,5]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,5]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+3\Lambda_6^\vee\right)\mathbf{[3,4,5]}+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[3,6\supseteq 3]}+\left(\Lambda_2^\vee+2\Lambda_5^\vee\right)\mathbf{[3,6\supseteq 6]}+\left(\Lambda_2^\vee+\Lambda_4^\vee\right)\mathbf{[1,3,6]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,6]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[3,4,6]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5,6]}$
		$2\Lambda_5^\vee\mathbf{[1,3\supseteq 1]}+2\Lambda_5^\vee\mathbf{[1,3\supseteq 3]}+2\Lambda_5^\vee\mathbf{[1,4\supseteq 1]}+2\Lambda_5^\vee\mathbf{[1,4\supseteq 4]}+2\Lambda_5^\vee\mathbf{[2,4\supseteq 2]}+2\Lambda_5^\vee\mathbf{[2,4\supseteq 4]}+2\Lambda_5^\vee\mathbf{[1,5\supseteq 1]}+2\Lambda_5^\vee\mathbf{[1,5\supseteq 5]}+2\Lambda_5^\vee\mathbf{[2,5\supseteq 2]}+2\Lambda_5^\vee\mathbf{[2,5\supseteq 5]}+2\Lambda_5^\vee\mathbf{[3,5\supseteq 3]}+2\Lambda_5^\vee\mathbf{[3,5\supseteq 5]}+2\Lambda_5^\vee\mathbf{[1,6\supseteq 1]}+2\Lambda_5^\vee\mathbf{[1,6\supseteq 6]}+2\Lambda_5^\vee\mathbf{[2,6\supseteq 2]}+2\Lambda_5^\vee\mathbf{[2,6\supseteq 6]}+2\Lambda_5^\vee\mathbf{[3,6\supseteq 3]}+2\Lambda_5^\vee\mathbf{[3,6\supseteq 6]}$
		$2\Lambda_5^\vee\mathbf{[1,2,3]}+2\Lambda_5^\vee\mathbf{[2,3,4]}+2\Lambda_5^\vee\mathbf{[3,4,5]}+2\Lambda_5^\vee\mathbf{[3,4,6]}$
		$\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[5\supseteq 5\supseteq 5]}+\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]}+\Lambda_2^\vee\mathbf{[1,3,5]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,5]}+\Lambda_4^\vee\mathbf{[4,5\supseteq 4]}+\left(\Lambda_3^\vee+3\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[4,5\supseteq 5]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[3,4,5]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[5,6\supseteq 6]}+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5,6]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[4,5,6]}$
		$\left(\Lambda_2^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 1]}+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,2,5]}+\left(\Lambda_2^\vee+\Lambda_4^\vee\right)\mathbf{[1,3,5]}+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,4,5]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,4,5]}$
		$\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[1,5\supseteq 1]}+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[2,5\supseteq 2]}+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[3,5\supseteq 3]}+\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]}+\Lambda_2^\vee\mathbf{[1,3,5]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,5]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[3,4,5]}+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5,6]}$
		$\left(\Lambda_2^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 5]}+\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]}+\Lambda_2^\vee\mathbf{[1,3,5]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,5]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[4,5\supseteq 4]}+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[4,5\supseteq 5]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[3,4,5]}+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[3,5,6]}+\Lambda_4^\vee\mathbf{[4,5,6]}$
		$\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[1,5\supseteq 5]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,2,5]}+\left(\Lambda_2^\vee+\Lambda_4^\vee\right)\mathbf{[1,3,5]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,4,5]}+2\Lambda_5^\vee\mathbf{[2,4,5]}+\Lambda_4^\vee\mathbf{[1,5,6]}+2\Lambda_6^\vee\mathbf{[2,5,6]}+2\Lambda_6^\vee\mathbf{[3,5,6]}$
		$2\Lambda_5^\vee\mathbf{[1,3,5]}$
		$\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[6\supseteq 6\supseteq 6]}+\left(\Lambda_2^\vee+2\Lambda_5^\vee\right)\mathbf{[3,6\supseteq 6]}+\Lambda_2^\vee\mathbf{[1,3,6]}+\left(\Lambda_1^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[2,3,6]}+\Lambda_4^\vee\mathbf{[4,6\supseteq 4]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+3\Lambda_6^\vee\right)\mathbf{[4,6\supseteq 6]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[3,4,6]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[5,6\supseteq 6]}+\left(\Lambda_4^\vee+2\Lambda_5^\vee\right)\mathbf{[3,5,6]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[4,5,6]}$
		$\left(\Lambda_2^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,6\supseteq 1]}+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,2,6]}+\left(\Lambda_2^\vee+\Lambda_4^\vee\right)\mathbf{[1,3,6]}+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[1,4,6]}+\left(2\Lambda_5^\vee+2\Lambda_6^\vee\right)\mathbf{[2,4,6]}$
		$\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)\mathbf{[1,6\supseteq 6]}+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,2,6]}+\left(\Lambda_2^\vee+\Lambda_4^\vee\right)\mathbf{[1,3,6]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[1,4,6]}+2\Lambda_6^\vee\mathbf{[2,4,6]}+\Lambda_4^\vee\mathbf{[1,5,6]}+2\Lambda_5^\vee\mathbf{[2,5,6]}+2\Lambda_5^\vee\mathbf{[3,5,6]}$
		$\Lambda_4^\vee\mathbf{[5,6\supseteq 5]}+2\Lambda_5^\vee\mathbf{[5,6\supseteq 6]}+\left(\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)\mathbf{[4,5,6]}$
		$2\Lambda_5^\vee\mathbf{[5,6\supseteq 5]}+2\Lambda_5^\vee\mathbf{[5,6\supseteq 6]}$

k	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1&0\\1&0\\1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\\0&0\\0&0\\0&0\\1&1\\1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&0&1&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&1&1\\0&0&0&0&1&1\\1&0&0&0&0&0\\1&0&0&1&1&1\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&1&1\\0&0&0&0&1&1\\1&0&0&0&0&0\\1&0&1&0&1&1\\0&0&0&0&1&1\\1&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$

A.27.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(0,1)\rangle$

$$\boxed{\begin{array}{rl}[\phi_u] & = \ (1,1,0,0)\\ & \ \ \textbf{does not lie in the image of } \mathbf{comp}_2\end{array}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[6]}$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_6^\vee\mathbf{[1\supseteq 1]} + \left(-2\Lambda_6^\vee\right)\mathbf{[2\supseteq 2]} + 2\Lambda_6^\vee\mathbf{[1,2]} + \left(-2\Lambda_6^\vee\right)\mathbf{[3\supseteq 3]} + \left(-2\Lambda_6^\vee\right)\mathbf{[4\supseteq 4]} + \left(-2\Lambda_6^\vee\right)\mathbf{[5\supseteq 5]} + \left(-\Lambda_4^\vee\right)\mathbf{[6\supseteq 6]} + \left(-\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[1,6]} + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[2,6]} + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[3,6]} + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)\mathbf{[4,6]} + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[5,6]}$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_6^\vee\mathbf{[5,6\supseteq 5]} + \left(-\Lambda_4^\vee + 4\Lambda_5^\vee\right)\mathbf{[5,6\supseteq 6]} + \left(\Lambda_3^\vee - 3\Lambda_4^\vee + 3\Lambda_5^\vee + 3\Lambda_6^\vee\right)\mathbf{[4,5,6]}$ $\left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[1,6\supseteq 1]} + \left(-\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[2,6\supseteq 2]} + \left(-2\Lambda_4^\vee\right)\mathbf{[2,6\supseteq 6]} + \left(-\Lambda_4^\vee\right)\mathbf{[1,2,6]} + \left(-\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[3,6\supseteq 3]} + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 3\Lambda_4^\vee\right)\mathbf{[3,6\supseteq 6]} + \left(2\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[2,4,6]} + \left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)\mathbf{[3,4,6]}$ $2\Lambda_6^\vee\mathbf{[1,3\supseteq 1]} + 2\Lambda_6^\vee\mathbf{[1,3\supseteq 3]} + \left(-2\Lambda_6^\vee\right)\mathbf{[1,2,3]} + \left(-2\Lambda_6^\vee\right)\mathbf{[1,4\supseteq 1]} + \left(-2\Lambda_6^\vee\right)\mathbf{[1,4\supseteq 4]} + \left(-2\Lambda_6^\vee\right)\mathbf{[2,4\supseteq 2]} + \left(-2\Lambda_6^\vee\right)\mathbf{[2,4\supseteq 4]} + 2\Lambda_6^\vee\mathbf{[1,3,4]} + 2\Lambda_6^\vee\mathbf{[2,3,4]} + \left(-2\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 1]} + \left(-2\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 5]} + \left(-2\Lambda_6^\vee\right)\mathbf{[2,5\supseteq 2]} + \left(-2\Lambda_6^\vee\right)\mathbf{[2,5\supseteq 5]} + \left(-2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 3]} + \left(-2\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]} + 2\Lambda_6^\vee\mathbf{[3,4,5]} + \left(-2\Lambda_6^\vee\right)\mathbf{[1,6\supseteq 1]} + \left(-\Lambda_4^\vee\right)\mathbf{[1,6\supseteq 6]} + \left(-2\Lambda_6^\vee\right)\mathbf{[2,6\supseteq 2]} + \left(-\Lambda_4^\vee\right)\mathbf{[2,6\supseteq 6]} + \left(-2\Lambda_6^\vee\right)\mathbf{[3,6\supseteq 3]} + \left(-\Lambda_4^\vee\right)\mathbf{[3,6\supseteq 6]} + \left(-\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[1,3,6]} + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)\mathbf{[1,4,6]} + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)\mathbf{[2,4,6]} + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[3,4,6]} + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[1,5,6]} + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[2,5,6]} + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)\mathbf{[3,5,6]}$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
0	1	$2\Lambda_6^\vee\,[\,]$
1	2	$\Lambda_2^\vee\,\mathbf{[1]} + 2\Lambda_6^\vee\,\mathbf{[2]} + 2\Lambda_6^\vee\,\mathbf{[3]} + 2\Lambda_6^\vee\,\mathbf{[4]} + 2\Lambda_6^\vee\,\mathbf{[5]} + 2\Lambda_6^\vee\,\mathbf{[6]}$
		$2\Lambda_6^\vee\,\mathbf{[1]} + 2\Lambda_6^\vee\,\mathbf{[2]} + 2\Lambda_6^\vee\,\mathbf{[3]} + 2\Lambda_6^\vee\,\mathbf{[4]} + 2\Lambda_6^\vee\,\mathbf{[5]} + 2\Lambda_6^\vee\,\mathbf{[6]}$
2	4	$\Lambda_2^\vee\,\mathbf{[1 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[2 \supseteq 2]} + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_6^\vee\right)\mathbf{[1,2]} + 2\Lambda_6^\vee\,\mathbf{[3 \supseteq 3]} + 2\Lambda_6^\vee\,\mathbf{[4 \supseteq 4]} + 2\Lambda_6^\vee\,\mathbf{[5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[6 \supseteq 6]} + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[2,6]} + \Lambda_2^\vee\,\mathbf{[3,6]}$
		$2\Lambda_6^\vee\,\mathbf{[1 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[2 \supseteq 2]} + 2\Lambda_6^\vee\,\mathbf{[3 \supseteq 3]} + 2\Lambda_6^\vee\,\mathbf{[4 \supseteq 4]} + 2\Lambda_6^\vee\,\mathbf{[5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[6 \supseteq 6]}$
		$\Lambda_2^\vee\,\mathbf{[1,3]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[2,3]} + 2\Lambda_6^\vee\,\mathbf{[1,4]} + 2\Lambda_6^\vee\,\mathbf{[2,4]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[3,4]} + 2\Lambda_6^\vee\,\mathbf{[1,5]} + 2\Lambda_6^\vee\,\mathbf{[2,5]} + 2\Lambda_6^\vee\,\mathbf{[3,5]} + 2\Lambda_6^\vee\,\mathbf{[1,6]} + 2\Lambda_6^\vee\,\mathbf{[2,6]} + \Lambda_4^\vee\,\mathbf{[3,6]}$
		$\left(\Lambda_2^\vee + 2\Lambda_6^\vee\right)\mathbf{[1,5]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[4,5]} + \Lambda_4^\vee\,\mathbf{[5,6]}$
3	9	$\Lambda_2^\vee\,\mathbf{[1 \supseteq 1 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[2 \supseteq 2 \supseteq 2]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[1,2 \supseteq 1]} + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[1,2 \supseteq 2]} + 2\Lambda_6^\vee\,\mathbf{[3 \supseteq 3 \supseteq 3]} + \Lambda_2^\vee\,\mathbf{[1,2,3]} + 2\Lambda_6^\vee\,\mathbf{[4 \supseteq 4 \supseteq 4]} + 2\Lambda_6^\vee\,\mathbf{[2,3,4]} + 2\Lambda_6^\vee\,\mathbf{[5 \supseteq 5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[3,4,5]} + 2\Lambda_6^\vee\,\mathbf{[6 \supseteq 6 \supseteq 6]} + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[2,6 \supseteq 2]} + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[3,6 \supseteq 3]} + \Lambda_2^\vee\,\mathbf{[3,6 \supseteq 6]} + \Lambda_2^\vee\,\mathbf{[1,3,6]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[2,3,6]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[3,4,6]}$
		$2\Lambda_6^\vee\,\mathbf{[1 \supseteq 1 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[2 \supseteq 2 \supseteq 2]} + 2\Lambda_6^\vee\,\mathbf{[3 \supseteq 3 \supseteq 3]} + 2\Lambda_6^\vee\,\mathbf{[4 \supseteq 4 \supseteq 4]} + 2\Lambda_6^\vee\,\mathbf{[5 \supseteq 5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[6 \supseteq 6 \supseteq 6]}$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + 4\Lambda_6^\vee\right)\mathbf{[2 \supseteq 2 \supseteq 2]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[1,2 \supseteq 1]} + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[1,2 \supseteq 2]} + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[2,3 \supseteq 2]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_6^\vee\right)\mathbf{[2,3 \supseteq 3]} + \Lambda_2^\vee\,\mathbf{[1,2,3]} + 2\Lambda_6^\vee\,\mathbf{[2,3,4]} + 2\Lambda_6^\vee\,\mathbf{[3,4,5]} + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[2,6 \supseteq 2]} + 2\Lambda_6^\vee\,\mathbf{[3,4,6]}$
		$\Lambda_5^\vee\,\mathbf{[1,2 \supseteq 1]} + \Lambda_5^\vee\,\mathbf{[1,2 \supseteq 2]} + \Lambda_4^\vee\,\mathbf{[2,5 \supseteq 2]} + \Lambda_4^\vee\,\mathbf{[3,5 \supseteq 3]} + \Lambda_5^\vee\,\mathbf{[4,5 \supseteq 4]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[4,5 \supseteq 5]} + \Lambda_4^\vee\,\mathbf{[3,4,5]} + 2\Lambda_6^\vee\,\mathbf{[5,6 \supseteq 6]} + \left(\Lambda_3^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[4,5,6]}$
		$\Lambda_2^\vee\,\mathbf{[1,3 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[1,3 \supseteq 3]} + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_6^\vee\right)\mathbf{[1,2,3]} + 2\Lambda_6^\vee\,\mathbf{[1,4 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[1,4 \supseteq 4]} + 2\Lambda_6^\vee\,\mathbf{[2,4 \supseteq 2]} + 2\Lambda_6^\vee\,\mathbf{[2,4 \supseteq 4]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[1,3,4]} + 2\Lambda_6^\vee\,\mathbf{[1,5 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[1,5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[2,5 \supseteq 2]} + 2\Lambda_6^\vee\,\mathbf{[2,5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[3,5 \supseteq 3]} + 2\Lambda_6^\vee\,\mathbf{[3,5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[1,6 \supseteq 1]} + 2\Lambda_6^\vee\,\mathbf{[1,6 \supseteq 6]} + 2\Lambda_6^\vee\,\mathbf{[2,6 \supseteq 2]} + 2\Lambda_6^\vee\,\mathbf{[2,6 \supseteq 6]} + 2\Lambda_6^\vee\,\mathbf{[3,6 \supseteq 3]} + 2\Lambda_6^\vee\,\mathbf{[3,6 \supseteq 6]} + \Lambda_4^\vee\,\mathbf{[1,3,6]}$
		$\Lambda_4^\vee\,\mathbf{[1,3 \supseteq 1]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[2,3 \supseteq 2]} + \left(\Lambda_1^\vee + 3\Lambda_6^\vee\right)\mathbf{[2,3 \supseteq 3]} + \left(\Lambda_1^\vee + \Lambda_3^\vee + 2\Lambda_6^\vee\right)\mathbf{[1,2,3]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[1,3,4]} + \left(\Lambda_3^\vee + \Lambda_5^\vee + 3\Lambda_6^\vee\right)\mathbf{[2,3,4]} + 2\Lambda_6^\vee\,\mathbf{[3,4,5]} + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[3,6 \supseteq 3]} + \Lambda_2^\vee\,\mathbf{[3,6 \supseteq 6]} + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)\mathbf{[1,3,6]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[2,3,6]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[3,4,6]}$
		$\Lambda_5^\vee\,\mathbf{[1,2,3]} + \Lambda_5^\vee\,\mathbf{[2,3,4]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[2,3,5]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[1,4,5]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[2,4,5]} + \left(\Lambda_1^\vee + 3\Lambda_6^\vee\right)\mathbf{[3,4,5]} + \Lambda_5^\vee\,\mathbf{[3,4,6]} + \Lambda_4^\vee\,\mathbf{[1,5,6]} + \Lambda_4^\vee\,\mathbf{[2,5,6]} + 2\Lambda_6^\vee\,\mathbf{[3,5,6]}$
		$\left(\Lambda_2^\vee + 2\Lambda_6^\vee\right)\mathbf{[1,5 \supseteq 5]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[4,5 \supseteq 4]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[4,5 \supseteq 5]} + 2\Lambda_6^\vee\,\mathbf{[5,6 \supseteq 6]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[4,5,6]}$
		$\Lambda_2^\vee\,\mathbf{[1,3,5]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[2,3,5]} + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)\mathbf{[3,5,6]}$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&1&1\\1&0&0\\0&0&1\\0&1&0\\0&0&0\\1&0&0\\0&0&0\end{pmatrix}$

A.27.3 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(1,1)\rangle$

$$\boxed{\phi_u = \partial\tau \text{ with } \tau = (\Lambda_1^\vee + 2\Lambda_6^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 2\Lambda_6^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_6^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee + 2\Lambda_6^\vee)\,[\mathbf{4}] + (\Lambda_5^\vee + \Lambda_6^\vee)\,[\mathbf{5}] + (\Lambda_5^\vee + \Lambda_6^\vee)\,[\mathbf{6}]}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{6}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_6^\vee[\mathbf{1}\supseteq\mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3}\supseteq\mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4}\supseteq\mathbf{4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{5}\supseteq\mathbf{5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{6}\supseteq\mathbf{6}] + \left(-\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{6}] + \left(-\Lambda_3^\vee + 2\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{4},\mathbf{6}] + \left(2\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{5},\mathbf{6}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{5},\mathbf{6}\supseteq\mathbf{6}] + \left(\Lambda_4^\vee - \Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{4},\mathbf{5},\mathbf{6}]$ $2\Lambda_6^\vee[\mathbf{5},\mathbf{6}\supseteq\mathbf{5}] + 2\Lambda_5^\vee[\mathbf{5},\mathbf{6}\supseteq\mathbf{6}] + \left(\Lambda_3^\vee - 2\Lambda_4^\vee + 2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{4},\mathbf{5},\mathbf{6}]$ $\left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{6}\supseteq\mathbf{1}] + \left(-\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{6}\supseteq\mathbf{2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2},\mathbf{6}\supseteq\mathbf{6}] + \left(-\Lambda_4^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{6}] + \left(-\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{6}\supseteq\mathbf{3}] + \left(-2\Lambda_4^\vee\right)[\mathbf{3},\mathbf{6}\supseteq\mathbf{6}] + \left(-\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{4},\mathbf{6}\supseteq\mathbf{4}] + \left(-\Lambda_3^\vee + \Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{4},\mathbf{6}\supseteq\mathbf{6}] + \left(2\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{6}] + \left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{6}] + \left(-2\Lambda_4^\vee + 2\Lambda_5^\vee + 2\Lambda_6^\vee\right)[\mathbf{4},\mathbf{5},\mathbf{6}]$ $2\Lambda_6^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{3},\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{3},\mathbf{4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{4},\mathbf{5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{6}\supseteq\mathbf{1}] + \left(-\Lambda_4^\vee\right)[\mathbf{1},\mathbf{6}\supseteq\mathbf{6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{6}\supseteq\mathbf{2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2},\mathbf{6}\supseteq\mathbf{6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{6}\supseteq\mathbf{3}] + \left(-\Lambda_4^\vee\right)[\mathbf{3},\mathbf{6}\supseteq\mathbf{6}] + \left(-\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{6}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{6}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - \Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{6}] + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3},\mathbf{4},\mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5},\mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5},\mathbf{6}] + \left(\Lambda_4^\vee - 2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5},\mathbf{6}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$2\Lambda_6^\vee[]$
1	2	$2\Lambda_6^\vee[\mathbf{1}] + 2\Lambda_6^\vee[\mathbf{2}] + 2\Lambda_6^\vee[\mathbf{3}] + 2\Lambda_6^\vee[\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{6}]$ $\left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{5}]$
2	5	$2\Lambda_6^\vee[\mathbf{1}\supseteq\mathbf{1}] + 2\Lambda_6^\vee[\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_6^\vee[\mathbf{3}\supseteq\mathbf{3}] + 2\Lambda_6^\vee[\mathbf{4}\supseteq\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{5}\supseteq\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{6}\supseteq\mathbf{6}]$ $2\Lambda_6^\vee[\mathbf{1},\mathbf{3}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{4}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{5}] + 2\Lambda_6^\vee[\mathbf{1},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{2},\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{3},\mathbf{6}]$ $\left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{5}\supseteq\mathbf{5}] + \left(\Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{4},\mathbf{5}]$ $\left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{1},\mathbf{5}] + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{2},\mathbf{5}] + \left(\Lambda_4^\vee + 2\Lambda_6^\vee\right)[\mathbf{3},\mathbf{5}] + \Lambda_3^\vee[\mathbf{4},\mathbf{5}]$ $\Lambda_4^\vee[\mathbf{5},\mathbf{6}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
3	11	$2\Lambda_6^\vee[\mathbf{1\supseteq 1\supseteq 1}]+2\Lambda_6^\vee[\mathbf{2\supseteq 2\supseteq 2}]+2\Lambda_6^\vee[\mathbf{3\supseteq 3\supseteq 3}]+2\Lambda_6^\vee[\mathbf{4\supseteq 4\supseteq 4}]+2\Lambda_6^\vee[\mathbf{5\supseteq 5\supseteq 5}]+2\Lambda_6^\vee[\mathbf{6\supseteq 6\supseteq 6}]$
		$2\Lambda_6^\vee[\mathbf{1,3\supseteq 1}]+2\Lambda_6^\vee[\mathbf{1,3\supseteq 3}]+2\Lambda_6^\vee[\mathbf{1,4\supseteq 1}]+2\Lambda_6^\vee[\mathbf{1,4\supseteq 4}]+2\Lambda_6^\vee[\mathbf{2,4\supseteq 2}]+2\Lambda_6^\vee[\mathbf{2,4\supseteq 4}]+2\Lambda_6^\vee[\mathbf{1,5\supseteq 1}]+2\Lambda_6^\vee[\mathbf{1,5\supseteq 5}]+2\Lambda_6^\vee[\mathbf{2,5\supseteq 2}]+2\Lambda_6^\vee[\mathbf{2,5\supseteq 5}]+2\Lambda_6^\vee[\mathbf{3,5\supseteq 3}]+2\Lambda_6^\vee[\mathbf{3,5\supseteq 5}]+2\Lambda_6^\vee[\mathbf{1,6\supseteq 1}]+2\Lambda_6^\vee[\mathbf{1,6\supseteq 6}]+2\Lambda_6^\vee[\mathbf{2,6\supseteq 2}]+2\Lambda_6^\vee[\mathbf{2,6\supseteq 6}]+2\Lambda_6^\vee[\mathbf{3,6\supseteq 3}]+2\Lambda_6^\vee[\mathbf{3,6\supseteq 6}]$
		$2\Lambda_6^\vee[\mathbf{1,2,3}]+2\Lambda_6^\vee[\mathbf{2,3,4}]+2\Lambda_6^\vee[\mathbf{3,4,5}]+2\Lambda_6^\vee[\mathbf{3,4,6}]$
		$\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{5\supseteq 5\supseteq 5}]+\left(\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{4,5\supseteq 4}]+\left(\Lambda_5^\vee+3\Lambda_6^\vee\right)[\mathbf{4,5\supseteq 5}]+\left(\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{4,5,6}]$
		$\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{1,5\supseteq 1}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{2,5\supseteq 2}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{3,5\supseteq 3}]+\left(\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{4,5\supseteq 4}]+\Lambda_3^\vee[\mathbf{4,5\supseteq 5}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{3,4,5}]+\Lambda_4^\vee[\mathbf{4,5,6}]$
		$\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{1,5\supseteq 5}]+\left(\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{1,2,5}]+\left(\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{1,4,5}]+2\Lambda_6^\vee[\mathbf{2,4,5}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{2,5,6}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{3,5,6}]$
		$\left(\Lambda_2^\vee+2\Lambda_6^\vee\right)[\mathbf{3,5\supseteq 5}]+\left(\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{2,3,5}]+\left(\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{4,5\supseteq 4}]+\Lambda_3^\vee[\mathbf{4,5\supseteq 5}]+\left(\Lambda_3^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{3,4,5}]+\left(\Lambda_4^\vee+2\Lambda_6^\vee\right)[\mathbf{3,5,6}]+\Lambda_4^\vee[\mathbf{4,5,6}]$
		$2\Lambda_6^\vee[\mathbf{1,3,5}]$
		$\Lambda_4^\vee[\mathbf{5,6\supseteq 5}]+2\Lambda_6^\vee[\mathbf{5,6\supseteq 6}]+\left(\Lambda_4^\vee+\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{4,5,6}]$
		$\Lambda_4^\vee[\mathbf{1,5,6}]+\Lambda_4^\vee[\mathbf{2,5,6}]+\Lambda_4^\vee[\mathbf{3,5,6}]$
		$\Lambda_3^\vee[\mathbf{4,5,6}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\1&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}$

A.27.4 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(1,0)\rangle$

$$[\phi_u] = (1,1,0,0)$$
does not lie in the image of $\mathbf{comp}_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{5}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_5^\vee[\mathbf{1\supseteq 1}]+\left(-2\Lambda_5^\vee\right)[\mathbf{2\supseteq 2}]+2\Lambda_5^\vee[\mathbf{1,2}]+\left(-2\Lambda_5^\vee\right)[\mathbf{3\supseteq 3}]+\left(-2\Lambda_5^\vee\right)[\mathbf{4\supseteq 4}]+\left(-\Lambda_4^\vee\right)[\mathbf{5\supseteq 5}]+\left(-\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{1,5}]+\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{2,5}]+\left(\Lambda_4^\vee-2\Lambda_5^\vee\right)[\mathbf{3,5}]+\left(-\Lambda_3^\vee+3\Lambda_4^\vee-3\Lambda_5^\vee-\Lambda_6^\vee\right)[\mathbf{4,5}]+\left(-2\Lambda_5^\vee\right)[\mathbf{6\supseteq 6}]+\left(-\Lambda_4^\vee+2\Lambda_5^\vee\right)[\mathbf{5,6}]$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_4^\vee[\mathbf{5,6 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{5,6 \supseteq 6}] + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{4,5,6}]$
		$\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,2,5}] + \left(-\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_2^\vee + 2\Lambda_3^\vee - 3\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 5}] + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee\right)[\mathbf{2,4,5}] + \left(\Lambda_2^\vee + 2\Lambda_6^\vee\right)[\mathbf{3,4,5}]$
		$2\Lambda_5^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{1,3,4}] + 2\Lambda_5^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-\Lambda_4^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-\Lambda_4^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-\Lambda_4^\vee\right)[\mathbf{3,5 \supseteq 5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,3,5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{1,4,5}] + \left(-\Lambda_3^\vee + 3\Lambda_4^\vee - 3\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{2,4,5}] + \left(\Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3,4,5}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,6 \supseteq 6}] + 2\Lambda_5^\vee[\mathbf{3,4,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3,5,6}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
0	1	$2\Lambda_5^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5}] + 2\Lambda_5^\vee[\mathbf{6}]$
		$2\Lambda_5^\vee[\mathbf{1}] + 2\Lambda_5^\vee[\mathbf{2}] + 2\Lambda_5^\vee[\mathbf{3}] + 2\Lambda_5^\vee[\mathbf{4}] + 2\Lambda_5^\vee[\mathbf{5}] + 2\Lambda_5^\vee[\mathbf{6}]$
2	4	$\Lambda_2^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_5^\vee\right)[\mathbf{1,2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{2,5}] + \Lambda_2^\vee[\mathbf{3,5}] + 2\Lambda_5^\vee[\mathbf{6 \supseteq 6}]$
		$2\Lambda_5^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{6 \supseteq 6}]$
		$\Lambda_2^\vee[\mathbf{1,3}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2,3}] + 2\Lambda_5^\vee[\mathbf{1,4}] + 2\Lambda_5^\vee[\mathbf{2,4}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{3,4}] + 2\Lambda_5^\vee[\mathbf{1,5}] + 2\Lambda_5^\vee[\mathbf{2,5}] + \Lambda_4^\vee[\mathbf{3,5}] + 2\Lambda_5^\vee[\mathbf{1,6}] + 2\Lambda_5^\vee[\mathbf{2,6}] + 2\Lambda_5^\vee[\mathbf{3,6}]$
		$\left(\Lambda_2^\vee + 2\Lambda_5^\vee\right)[\mathbf{1,6}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{4,6}] + \Lambda_4^\vee[\mathbf{5,6}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
3	9	$\Lambda_2^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1, 2, 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{2, 3, 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{2, 5 \supseteq 2}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3, 5 \supseteq 3}] + \Lambda_2^\vee[\mathbf{3, 5 \supseteq 5}] + \Lambda_2^\vee[\mathbf{1, 3, 5}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3, 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3, 4, 5}] + 2\Lambda_5^\vee[\mathbf{6 \supseteq 6 \supseteq 6}] + 2\Lambda_5^\vee[\mathbf{3, 4, 6}]$
		$2\Lambda_5^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{6 \supseteq 6 \supseteq 6}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee + 4\Lambda_5^\vee\right)[\mathbf{2 \supseteq 2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{1, 2 \supseteq 2}] + \left(\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 2}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1, 2, 3}] + 2\Lambda_5^\vee[\mathbf{2, 3, 4}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{2, 5 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{3, 4, 5}] + 2\Lambda_5^\vee[\mathbf{3, 4, 6}]$
		$\Lambda_6^\vee[\mathbf{1, 2 \supseteq 1}] + \Lambda_6^\vee[\mathbf{1, 2 \supseteq 2}] + \Lambda_4^\vee[\mathbf{2, 6 \supseteq 2}] + \Lambda_4^\vee[\mathbf{3, 6 \supseteq 3}] + \Lambda_6^\vee[\mathbf{4, 6 \supseteq 4}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{4, 6 \supseteq 6}] + \Lambda_4^\vee[\mathbf{3, 4, 6}] + 2\Lambda_5^\vee[\mathbf{5, 6 \supseteq 6}] + \left(\Lambda_3^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{4, 5, 6}]$
		$\Lambda_2^\vee[\mathbf{1, 3 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1, 3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 2, 3}] + 2\Lambda_5^\vee[\mathbf{1, 4 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1, 4 \supseteq 4}] + 2\Lambda_5^\vee[\mathbf{2, 4 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{2, 4 \supseteq 4}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 3, 4}] + 2\Lambda_5^\vee[\mathbf{1, 5 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1, 5 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{2, 5 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{2, 5 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{3, 5 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{3, 5 \supseteq 5}] + \Lambda_4^\vee[\mathbf{1, 3, 5}] + 2\Lambda_5^\vee[\mathbf{1, 6 \supseteq 1}] + 2\Lambda_5^\vee[\mathbf{1, 6 \supseteq 6}] + 2\Lambda_5^\vee[\mathbf{2, 6 \supseteq 2}] + 2\Lambda_5^\vee[\mathbf{2, 6 \supseteq 6}] + 2\Lambda_5^\vee[\mathbf{3, 6 \supseteq 3}] + 2\Lambda_5^\vee[\mathbf{3, 6 \supseteq 6}]$
		$\Lambda_4^\vee[\mathbf{1, 3 \supseteq 1}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 2}] + \left(\Lambda_1^\vee + 3\Lambda_5^\vee\right)[\mathbf{2, 3 \supseteq 3}] + \left(\Lambda_1^\vee + \Lambda_3^\vee + 2\Lambda_5^\vee\right)[\mathbf{1, 2, 3}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{1, 3, 4}] + \left(\Lambda_3^\vee + 3\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2, 3, 4}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3, 5 \supseteq 3}] + \Lambda_2^\vee[\mathbf{3, 5 \supseteq 5}] + \left(\Lambda_2^\vee + \Lambda_4^\vee\right)[\mathbf{1, 3, 5}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3, 5}] + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{3, 4, 5}] + 2\Lambda_5^\vee[\mathbf{3, 4, 6}]$
		$\Lambda_6^\vee[\mathbf{1, 2, 3}] + \Lambda_6^\vee[\mathbf{2, 3, 4}] + \Lambda_6^\vee[\mathbf{3, 4, 5}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3, 6}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{1, 4, 6}] + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)[\mathbf{2, 4, 6}] + \left(\Lambda_1^\vee + 3\Lambda_5^\vee\right)[\mathbf{3, 4, 6}] + \Lambda_4^\vee[\mathbf{1, 5, 6}] + \Lambda_4^\vee[\mathbf{2, 5, 6}] + 2\Lambda_5^\vee[\mathbf{3, 5, 6}]$
		$\Lambda_2^\vee[\mathbf{1, 3, 6}] + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)[\mathbf{2, 3, 6}] + \left(\Lambda_4^\vee + 2\Lambda_5^\vee\right)[\mathbf{3, 5, 6}]$
		$2\Lambda_5^\vee[\mathbf{5, 6 \supseteq 5}] + 2\Lambda_5^\vee[\mathbf{5, 6 \supseteq 6}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&1&1\\1&0&0\\0&0&1\\0&1&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

A.27.5 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\phi_u = \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee)[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee)[\mathbf{4}] + (\Lambda_5^\vee + \Lambda_6^\vee)[\mathbf{5}] + (\Lambda_5^\vee + \Lambda_6^\vee)[\mathbf{6}]}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	0	

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_4^\vee - 2\Lambda_5^\vee\right)\mathbf{[5,6\supseteq 6]} + \left(\Lambda_4^\vee - \Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[4,5,6]}$
		$\Lambda_6^\vee\mathbf{[1,3\supseteq 1]} + \Lambda_6^\vee\mathbf{[1,3\supseteq 3]} + \left(-\Lambda_6^\vee\right)\mathbf{[1,2,3]} + \left(-\Lambda_6^\vee\right)\mathbf{[1,4\supseteq 1]} + \left(-\Lambda_6^\vee\right)\mathbf{[1,4\supseteq 4]} + \left(-\Lambda_6^\vee\right)\mathbf{[2,4\supseteq 2]} + \left(-\Lambda_6^\vee\right)\mathbf{[2,4\supseteq 4]} + \Lambda_6^\vee\mathbf{[1,3,4]} + \Lambda_6^\vee\mathbf{[2,3,4]} + \left(-\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 1]} + \left(-\Lambda_6^\vee\right)\mathbf{[1,5\supseteq 5]} + \left(-\Lambda_6^\vee\right)\mathbf{[2,5\supseteq 2]} + \left(-\Lambda_6^\vee\right)\mathbf{[2,5\supseteq 5]} + \left(-\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 3]} + \left(-\Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]} + \Lambda_6^\vee\mathbf{[3,4,5]} + \left(-\Lambda_3^\vee - \Lambda_5^\vee\right)\mathbf{[1,6\supseteq 1]} + \left(-\Lambda_3^\vee - \Lambda_5^\vee\right)\mathbf{[1,6\supseteq 6]} + \left(-\Lambda_3^\vee - \Lambda_5^\vee\right)\mathbf{[2,6\supseteq 2]} + \left(-\Lambda_3^\vee - \Lambda_5^\vee\right)\mathbf{[2,6\supseteq 6]} + \left(-\Lambda_1^\vee - \Lambda_5^\vee\right)\mathbf{[3,6\supseteq 3]} + \left(-\Lambda_1^\vee - \Lambda_5^\vee\right)\mathbf{[3,6\supseteq 6]} + \left(-\Lambda_1^\vee + \Lambda_3^\vee - \Lambda_4^\vee + \Lambda_6^\vee\right)\mathbf{[1,3,6]} + \left(-\Lambda_2^\vee + \Lambda_3^\vee\right)\mathbf{[2,3,6]} + \left(\Lambda_4^\vee - \Lambda_6^\vee\right)\mathbf{[1,4,6]} + \left(\Lambda_4^\vee - \Lambda_6^\vee\right)\mathbf{[2,4,6]} + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)\mathbf{[3,4,6]} + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[1,5,6]} + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[2,5,6]} + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[3,5,6]}$
		$\Lambda_5^\vee\mathbf{[1,3\supseteq 1]} + \Lambda_5^\vee\mathbf{[1,3\supseteq 3]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,2,3]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,4\supseteq 1]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,4\supseteq 4]} + \left(-\Lambda_5^\vee\right)\mathbf{[2,4\supseteq 2]} + \left(-\Lambda_5^\vee\right)\mathbf{[2,4\supseteq 4]} + \Lambda_5^\vee\mathbf{[1,3,4]} + \Lambda_5^\vee\mathbf{[2,3,4]} + \left(-\Lambda_3^\vee - \Lambda_6^\vee\right)\mathbf{[1,5\supseteq 1]} + \left(-\Lambda_3^\vee - \Lambda_6^\vee\right)\mathbf{[1,5\supseteq 5]} + \left(-\Lambda_3^\vee - \Lambda_6^\vee\right)\mathbf{[2,5\supseteq 2]} + \left(-\Lambda_3^\vee - \Lambda_6^\vee\right)\mathbf{[2,5\supseteq 5]} + \left(-\Lambda_1^\vee - \Lambda_6^\vee\right)\mathbf{[3,5\supseteq 3]} + \left(-\Lambda_1^\vee - \Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]} + \left(-\Lambda_1^\vee + \Lambda_3^\vee - \Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[1,3,5]} + \left(-\Lambda_2^\vee + \Lambda_3^\vee\right)\mathbf{[2,3,5]} + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[1,4,5]} + \left(\Lambda_4^\vee - \Lambda_5^\vee\right)\mathbf{[2,4,5]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[3,4,5]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,6\supseteq 1]} + \left(-\Lambda_5^\vee\right)\mathbf{[1,6\supseteq 6]} + \left(-\Lambda_5^\vee\right)\mathbf{[2,6\supseteq 2]} + \left(-\Lambda_5^\vee\right)\mathbf{[2,6\supseteq 6]} + \left(-\Lambda_5^\vee\right)\mathbf{[3,6\supseteq 3]} + \left(-\Lambda_5^\vee\right)\mathbf{[3,6\supseteq 6]} + \Lambda_5^\vee\mathbf{[3,4,6]} + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[1,5,6]} + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[2,5,6]} + \left(\Lambda_5^\vee - \Lambda_6^\vee\right)\mathbf{[3,5,6]}$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	3	$\Lambda_5^\vee\mathbf{[1,3]} + \Lambda_5^\vee\mathbf{[1,4]} + \Lambda_5^\vee\mathbf{[2,4]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[1,5]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[2,5]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[3,5]} + \Lambda_5^\vee\mathbf{[1,6]} + \Lambda_5^\vee\mathbf{[2,6]} + \Lambda_5^\vee\mathbf{[3,6]}$
		$\Lambda_6^\vee\mathbf{[1,3]} + \Lambda_6^\vee\mathbf{[1,4]} + \Lambda_6^\vee\mathbf{[2,4]} + \Lambda_6^\vee\mathbf{[1,5]} + \Lambda_6^\vee\mathbf{[2,5]} + \Lambda_6^\vee\mathbf{[3,5]} + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)\mathbf{[1,6]} + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)\mathbf{[2,6]} + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)\mathbf{[3,6]}$
		$\Lambda_6^\vee\mathbf{[4,5]}$
3	8	$\Lambda_5^\vee\mathbf{[1,2\supseteq 1]} + \Lambda_5^\vee\mathbf{[1,2\supseteq 2]} + \Lambda_4^\vee\mathbf{[2,5\supseteq 2]} + \Lambda_4^\vee\mathbf{[3,5\supseteq 3]} + \Lambda_5^\vee\mathbf{[4,5\supseteq 4]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[4,5\supseteq 5]} + \Lambda_4^\vee\mathbf{[3,4,5]} + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[4,5,6]}$
		$\Lambda_5^\vee\mathbf{[1,3\supseteq 1]} + \Lambda_5^\vee\mathbf{[1,3\supseteq 3]} + \Lambda_5^\vee\mathbf{[1,4\supseteq 1]} + \Lambda_5^\vee\mathbf{[1,4\supseteq 4]} + \Lambda_5^\vee\mathbf{[2,4\supseteq 2]} + \Lambda_5^\vee\mathbf{[2,4\supseteq 4]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[1,5\supseteq 1]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[1,5\supseteq 5]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[2,5\supseteq 2]} + \left(\Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[2,5\supseteq 5]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[3,5\supseteq 3]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[3,5\supseteq 5]} + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_5^\vee\right)\mathbf{[1,3,5]} + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)\mathbf{[2,3,5]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[1,4,5]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_5^\vee\right)\mathbf{[2,4,5]} + \Lambda_5^\vee\mathbf{[1,6\supseteq 1]} + \Lambda_5^\vee\mathbf{[1,6\supseteq 6]} + \Lambda_5^\vee\mathbf{[2,6\supseteq 2]} + \Lambda_5^\vee\mathbf{[2,6\supseteq 6]} + \Lambda_5^\vee\mathbf{[3,6\supseteq 3]} + \Lambda_5^\vee\mathbf{[3,6\supseteq 6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[1,5,6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[2,5,6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[3,5,6]}$
		$\Lambda_6^\vee\mathbf{[1,3\supseteq 1]} + \Lambda_6^\vee\mathbf{[1,3\supseteq 3]} + \Lambda_6^\vee\mathbf{[1,4\supseteq 1]} + \Lambda_6^\vee\mathbf{[1,4\supseteq 4]} + \Lambda_6^\vee\mathbf{[2,4\supseteq 2]} + \Lambda_6^\vee\mathbf{[2,4\supseteq 4]} + \Lambda_6^\vee\mathbf{[1,5\supseteq 1]} + \Lambda_6^\vee\mathbf{[1,5\supseteq 5]} + \Lambda_6^\vee\mathbf{[2,5\supseteq 2]} + \Lambda_6^\vee\mathbf{[2,5\supseteq 5]} + \Lambda_6^\vee\mathbf{[3,5\supseteq 3]} + \Lambda_6^\vee\mathbf{[3,5\supseteq 5]} + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)\mathbf{[1,6\supseteq 1]} + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)\mathbf{[1,6\supseteq 6]} + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)\mathbf{[2,6\supseteq 2]} + \left(\Lambda_3^\vee + \Lambda_5^\vee\right)\mathbf{[2,6\supseteq 6]} + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)\mathbf{[3,6\supseteq 3]} + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)\mathbf{[3,6\supseteq 6]} + \left(\Lambda_1^\vee + \Lambda_3^\vee + \Lambda_6^\vee\right)\mathbf{[1,3,6]} + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee\right)\mathbf{[2,3,6]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_6^\vee\right)\mathbf{[1,4,6]} + \left(\Lambda_3^\vee + \Lambda_4^\vee + \Lambda_6^\vee\right)\mathbf{[2,4,6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[1,5,6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[2,5,6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)\mathbf{[3,5,6]}$
		$\Lambda_5^\vee\mathbf{[1,2,3]} + \Lambda_5^\vee\mathbf{[2,3,4]} + \Lambda_1^\vee\mathbf{[2,3,5]} + \Lambda_3^\vee\mathbf{[1,4,5]} + \Lambda_3^\vee\mathbf{[2,4,5]} + \left(\Lambda_1^\vee + \Lambda_6^\vee\right)\mathbf{[3,4,5]} + \Lambda_5^\vee\mathbf{[3,4,6]}$
		$\Lambda_6^\vee\mathbf{[1,2,3]} + \Lambda_6^\vee\mathbf{[2,3,4]} + \Lambda_6^\vee\mathbf{[3,4,5]} + \Lambda_1^\vee\mathbf{[2,3,6]} + \Lambda_3^\vee\mathbf{[1,4,6]} + \Lambda_3^\vee\mathbf{[2,4,6]} + \left(\Lambda_1^\vee + \Lambda_5^\vee\right)\mathbf{[3,4,6]}$
		$\Lambda_6^\vee\mathbf{[1,3,5]} + \Lambda_3^\vee\mathbf{[1,5,6]} + \Lambda_3^\vee\mathbf{[2,5,6]} + \Lambda_1^\vee\mathbf{[3,5,6]}$
		$\Lambda_6^\vee\mathbf{[1,4,5]} + \Lambda_6^\vee\mathbf{[2,4,5]}$
		$\Lambda_5^\vee\mathbf{[1,3,6]} + \Lambda_3^\vee\mathbf{[1,5,6]} + \Lambda_3^\vee\mathbf{[2,5,6]} + \Lambda_1^\vee\mathbf{[3,5,6]}$

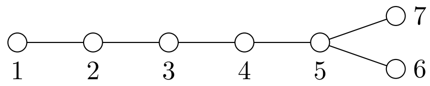
k	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\\0&0&1\\0&1&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

A.27.6 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1 \supseteq 1}] + (-1)\,[\mathbf{2 \supseteq 2}] + [\mathbf{1, 2}] + (-1)\,[\mathbf{3 \supseteq 3}] + (-1)\,[\mathbf{4 \supseteq 4}] + (-1)\,[\mathbf{5 \supseteq 5}] + (-1)\,[\mathbf{6 \supseteq 6}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{5, 6 \supseteq 5}] + [\mathbf{5, 6 \supseteq 6}] + [\mathbf{4, 5, 6}]$
		$[\mathbf{1, 3 \supseteq 1}] + [\mathbf{1, 3 \supseteq 3}] + (-1)\,[\mathbf{1, 2, 3}] + (-1)\,[\mathbf{1, 4 \supseteq 1}] + (-1)\,[\mathbf{1, 4 \supseteq 4}] + (-1)\,[\mathbf{2, 4 \supseteq 2}] + (-1)\,[\mathbf{2, 4 \supseteq 4}] + [\mathbf{1, 3, 4}] + [\mathbf{2, 3, 4}] + (-1)\,[\mathbf{1, 5 \supseteq 1}] + (-1)\,[\mathbf{1, 5 \supseteq 5}] + (-1)\,[\mathbf{2, 5 \supseteq 2}] + (-1)\,[\mathbf{2, 5 \supseteq 5}] + (-1)\,[\mathbf{3, 5 \supseteq 3}] + (-1)\,[\mathbf{3, 5 \supseteq 5}] + [\mathbf{3, 4, 5}] + (-1)\,[\mathbf{1, 6 \supseteq 1}] + (-1)\,[\mathbf{1, 6 \supseteq 6}] + (-1)\,[\mathbf{2, 6 \supseteq 2}] + (-1)\,[\mathbf{2, 6 \supseteq 6}] + (-1)\,[\mathbf{3, 6 \supseteq 3}] + (-1)\,[\mathbf{3, 6 \supseteq 6}] + [\mathbf{3, 4, 6}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}]$
2	3	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}] + [\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}] + [\mathbf{5 \supseteq 5}] + [\mathbf{6 \supseteq 6}]$
		$[\mathbf{1, 3}] + [\mathbf{1, 4}] + [\mathbf{2, 4}] + [\mathbf{1, 5}] + [\mathbf{2, 5}] + [\mathbf{3, 5}] + [\mathbf{1, 6}] + [\mathbf{2, 6}] + [\mathbf{3, 6}]$
		$[\mathbf{5, 6}]$
3	8	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}] + [\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}] + [\mathbf{5 \supseteq 5 \supseteq 5}] + [\mathbf{6 \supseteq 6 \supseteq 6}]$
		$[\mathbf{1, 3 \supseteq 1}] + [\mathbf{1, 3 \supseteq 3}] + [\mathbf{1, 4 \supseteq 1}] + [\mathbf{1, 4 \supseteq 4}] + [\mathbf{2, 4 \supseteq 2}] + [\mathbf{2, 4 \supseteq 4}] + [\mathbf{1, 5 \supseteq 1}] + [\mathbf{1, 5 \supseteq 5}] + [\mathbf{2, 5 \supseteq 2}] + [\mathbf{2, 5 \supseteq 5}] + [\mathbf{3, 5 \supseteq 3}] + [\mathbf{3, 5 \supseteq 5}] + [\mathbf{1, 6 \supseteq 1}] + [\mathbf{1, 6 \supseteq 6}] + [\mathbf{2, 6 \supseteq 2}] + [\mathbf{2, 6 \supseteq 6}] + [\mathbf{3, 6 \supseteq 3}] + [\mathbf{3, 6 \supseteq 6}]$
		$[\mathbf{1, 2, 3}] + [\mathbf{2, 3, 4}] + [\mathbf{3, 4, 5}] + [\mathbf{3, 4, 6}]$
		$[\mathbf{1, 3, 5}]$
		$[\mathbf{1, 3, 6}]$
		$[\mathbf{5, 6 \supseteq 5}] + [\mathbf{5, 6 \supseteq 6}]$
		$[\mathbf{1, 5, 6}] + [\mathbf{2, 5, 6}] + [\mathbf{3, 5, 6}]$
		$[\mathbf{4, 5, 6}]$

A.28 Root system D_7

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/4\mathbb{Z}$ generated by $\Lambda_7^\vee \in P^\vee \mod Q^\vee$

A.28.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (0,0,1,0)$$
does not lie in the image of comp$_2$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	$\mathbb{Z}/4\mathbb{Z}$	$\left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$4\Lambda_7^\vee[\mathbf{1}\supseteq\mathbf{1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2}\supseteq\mathbf{2}] + 4\Lambda_7^\vee[\mathbf{1,2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3}\supseteq\mathbf{3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4}\supseteq\mathbf{4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{5}\supseteq\mathbf{5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{6}\supseteq\mathbf{6}] + \left(-4\Lambda_6^\vee\right)[\mathbf{7}\supseteq\mathbf{7}] + \left(-2\Lambda_5^\vee + 4\Lambda_7^\vee\right)[\mathbf{1,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{4,7}] + \left(-2\Lambda_4^\vee + 4\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{5,7}] + \left(4\Lambda_6^\vee - 4\Lambda_7^\vee\right)[\mathbf{6,7}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$4\Lambda_7^\vee[\mathbf{6,7}\supseteq\mathbf{6}] + 4\Lambda_6^\vee[\mathbf{6,7}\supseteq\mathbf{7}] + \left(2\Lambda_4^\vee - 4\Lambda_5^\vee + 4\Lambda_6^\vee + 4\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $\left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,7}\supseteq\mathbf{1}] + \left(-\Lambda_5^\vee - 6\Lambda_7^\vee\right)[\mathbf{2,7}\supseteq\mathbf{2}] + \left(-4\Lambda_5^\vee\right)[\mathbf{2,7}\supseteq\mathbf{7}] + \left(-\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_5^\vee - 6\Lambda_7^\vee\right)[\mathbf{3,7}\supseteq\mathbf{3}] + \left(-4\Lambda_5^\vee\right)[\mathbf{3,7}\supseteq\mathbf{7}] + \left(-\Lambda_5^\vee - 6\Lambda_7^\vee\right)[\mathbf{4,7}\supseteq\mathbf{4}] + \left(-4\Lambda_5^\vee\right)[\mathbf{4,7}\supseteq\mathbf{7}] + \left(-\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{5,7}\supseteq\mathbf{5}] + \left(-\Lambda_4^\vee + \Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{5,7}\supseteq\mathbf{7}] + \left(4\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(4\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + 2\Lambda_4^\vee - 2\Lambda_5^\vee + 4\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(-2\Lambda_5^\vee + 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $4\Lambda_7^\vee[\mathbf{1,3}\supseteq\mathbf{1}] + 4\Lambda_7^\vee[\mathbf{1,3}\supseteq\mathbf{3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,4}\supseteq\mathbf{1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,4}\supseteq\mathbf{4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,4}\supseteq\mathbf{2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,4}\supseteq\mathbf{4}] + 4\Lambda_7^\vee[\mathbf{1,3,4}] + 4\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,5}\supseteq\mathbf{1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,5}\supseteq\mathbf{5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,5}\supseteq\mathbf{2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,5}\supseteq\mathbf{5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,5}\supseteq\mathbf{3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,5}\supseteq\mathbf{5}] + 4\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,6}\supseteq\mathbf{1}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,6}\supseteq\mathbf{6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,6}\supseteq\mathbf{2}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,6}\supseteq\mathbf{6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,6}\supseteq\mathbf{3}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,6}\supseteq\mathbf{6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4,6}\supseteq\mathbf{4}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4,6}\supseteq\mathbf{6}] + 4\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-4\Lambda_7^\vee\right)[\mathbf{1,7}\supseteq\mathbf{1}] + \left(-2\Lambda_5^\vee\right)[\mathbf{1,7}\supseteq\mathbf{7}] + \left(-4\Lambda_7^\vee\right)[\mathbf{2,7}\supseteq\mathbf{2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,7}\supseteq\mathbf{7}] + \left(-4\Lambda_7^\vee\right)[\mathbf{3,7}\supseteq\mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,7}\supseteq\mathbf{7}] + \left(-2\Lambda_5^\vee + 4\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(-4\Lambda_7^\vee\right)[\mathbf{4,7}\supseteq\mathbf{4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{4,7}\supseteq\mathbf{7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(-2\Lambda_4^\vee + 6\Lambda_5^\vee - 2\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(-2\Lambda_4^\vee + 6\Lambda_5^\vee - 2\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(-2\Lambda_4^\vee + 6\Lambda_5^\vee - 2\Lambda_6^\vee - 6\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(2\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_5^\vee - 4\Lambda_7^\vee\right)[\mathbf{4,6,7}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	1	$4\Lambda_7^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 4\Lambda_7^\vee[\mathbf{2}] + 4\Lambda_7^\vee[\mathbf{3}] + 4\Lambda_7^\vee[\mathbf{4}] + 4\Lambda_7^\vee[\mathbf{5}] + 4\Lambda_7^\vee[\mathbf{6}] + 4\Lambda_7^\vee[\mathbf{7}]$ $\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{6}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
2	4	$\Lambda_2^\vee\mathbf{[1 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[2 \supseteq 2]} + \left(\Lambda_1^\vee + \Lambda_2^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,2]} + 4\Lambda_7^\vee\mathbf{[3 \supseteq 3]} + 4\Lambda_7^\vee\mathbf{[4 \supseteq 4]} + 4\Lambda_7^\vee\mathbf{[5 \supseteq 5]} + 4\Lambda_7^\vee\mathbf{[6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[7 \supseteq 7]}$
		$\Lambda_2^\vee\mathbf{[1,3]} + \left(\Lambda_1^\vee + 2\Lambda_7^\vee\right)\mathbf{[2,3]} + 4\Lambda_7^\vee\mathbf{[1,4]} + 4\Lambda_7^\vee\mathbf{[2,4]} + \left(\Lambda_1^\vee + 2\Lambda_7^\vee\right)\mathbf{[3,4]} + 4\Lambda_7^\vee\mathbf{[1,5]} + 4\Lambda_7^\vee\mathbf{[2,5]} + 4\Lambda_7^\vee\mathbf{[3,5]} + 4\Lambda_7^\vee\mathbf{[1,6]} + 4\Lambda_7^\vee\mathbf{[2,6]} + 4\Lambda_7^\vee\mathbf{[3,6]} + 4\Lambda_7^\vee\mathbf{[4,6]} + 4\Lambda_7^\vee\mathbf{[1,7]} + 4\Lambda_7^\vee\mathbf{[2,7]} + 4\Lambda_7^\vee\mathbf{[3,7]} + 4\Lambda_7^\vee\mathbf{[4,7]}$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[6 \supseteq 6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)\mathbf{[5,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[6,7]}$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[2,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[3,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[4,6]} + \Lambda_4^\vee\mathbf{[5,6]}$
3	10	$\Lambda_2^\vee\mathbf{[1 \supseteq 1 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[2 \supseteq 2 \supseteq 2]} + \left(\Lambda_1^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,2 \supseteq 1]} + \left(\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,2 \supseteq 2]} + 4\Lambda_7^\vee\mathbf{[3 \supseteq 3 \supseteq 3]} + \Lambda_2^\vee\mathbf{[1,2,3]} + 4\Lambda_7^\vee\mathbf{[4 \supseteq 4 \supseteq 4]} + 4\Lambda_7^\vee\mathbf{[2,3,4]} + 4\Lambda_7^\vee\mathbf{[5 \supseteq 5 \supseteq 5]} + 4\Lambda_7^\vee\mathbf{[3,4,5]} + 4\Lambda_7^\vee\mathbf{[6 \supseteq 6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[4,5,6]} + 4\Lambda_7^\vee\mathbf{[7 \supseteq 7 \supseteq 7]} + 4\Lambda_7^\vee\mathbf{[4,5,7]}$
		$4\Lambda_7^\vee\mathbf{[1 \supseteq 1 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[2 \supseteq 2 \supseteq 2]} + 4\Lambda_7^\vee\mathbf{[3 \supseteq 3 \supseteq 3]} + 4\Lambda_7^\vee\mathbf{[4 \supseteq 4 \supseteq 4]} + 4\Lambda_7^\vee\mathbf{[5 \supseteq 5 \supseteq 5]} + 4\Lambda_7^\vee\mathbf{[6 \supseteq 6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[7 \supseteq 7 \supseteq 7]}$
		$\Lambda_2^\vee\mathbf{[1,3 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[1,3 \supseteq 3]} + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 4\Lambda_7^\vee\right)\mathbf{[1,2,3]} + 4\Lambda_7^\vee\mathbf{[1,4 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[1,4 \supseteq 4]} + 4\Lambda_7^\vee\mathbf{[2,4 \supseteq 2]} + 4\Lambda_7^\vee\mathbf{[2,4 \supseteq 4]} + \left(\Lambda_1^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,3,4]} + 4\Lambda_7^\vee\mathbf{[1,5 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[1,5 \supseteq 5]} + 4\Lambda_7^\vee\mathbf{[2,5 \supseteq 2]} + 4\Lambda_7^\vee\mathbf{[2,5 \supseteq 5]} + 4\Lambda_7^\vee\mathbf{[3,5 \supseteq 3]} + 4\Lambda_7^\vee\mathbf{[3,5 \supseteq 5]} + 4\Lambda_7^\vee\mathbf{[1,6 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[1,6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[2,6 \supseteq 2]} + 4\Lambda_7^\vee\mathbf{[2,6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[3,6 \supseteq 3]} + 4\Lambda_7^\vee\mathbf{[3,6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[4,6 \supseteq 4]} + 4\Lambda_7^\vee\mathbf{[4,6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[1,7 \supseteq 1]} + 4\Lambda_7^\vee\mathbf{[1,7 \supseteq 7]} + 4\Lambda_7^\vee\mathbf{[2,7 \supseteq 2]} + 4\Lambda_7^\vee\mathbf{[2,7 \supseteq 7]} + 4\Lambda_7^\vee\mathbf{[3,7 \supseteq 3]} + 4\Lambda_7^\vee\mathbf{[3,7 \supseteq 7]} + 4\Lambda_7^\vee\mathbf{[4,7 \supseteq 4]} + 4\Lambda_7^\vee\mathbf{[4,7 \supseteq 7]}$
		$\Lambda_2^\vee\mathbf{[1,3,5]} + \left(\Lambda_1^\vee + 2\Lambda_7^\vee\right)\mathbf{[2,3,5]} + 4\Lambda_7^\vee\mathbf{[1,3,6]} + 4\Lambda_7^\vee\mathbf{[1,4,6]} + 4\Lambda_7^\vee\mathbf{[2,4,6]} + \left(\Lambda_1^\vee + 2\Lambda_7^\vee\right)\mathbf{[3,5,6]} + 4\Lambda_7^\vee\mathbf{[1,3,7]} + 4\Lambda_7^\vee\mathbf{[1,4,7]} + 4\Lambda_7^\vee\mathbf{[2,4,7]} + \left(\Lambda_1^\vee + 2\Lambda_7^\vee\right)\mathbf{[3,5,7]}$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[6 \supseteq 6 \supseteq 6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)\mathbf{[5,6 \supseteq 5]} + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)\mathbf{[5,6 \supseteq 6]} + 4\Lambda_7^\vee\mathbf{[6,7 \supseteq 7]} + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)\mathbf{[5,6,7]}$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,6 \supseteq 1]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[2,6 \supseteq 2]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[3,6 \supseteq 3]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[4,6 \supseteq 4]} + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)\mathbf{[5,6 \supseteq 5]} + \left(\Lambda_4^\vee + 4\Lambda_7^\vee\right)\mathbf{[5,6 \supseteq 6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[4,5,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[5,6,7]}$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,6 \supseteq 6]} + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)\mathbf{[1,2,6]} + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)\mathbf{[1,5,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,6,7]} + 4\Lambda_7^\vee\mathbf{[2,6,7]} + 4\Lambda_7^\vee\mathbf{[3,6,7]} + 4\Lambda_7^\vee\mathbf{[4,6,7]}$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,3,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,4,6]} + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)\mathbf{[2,4,6]} + \Lambda_4^\vee\mathbf{[1,5,6]} + \Lambda_4^\vee\mathbf{[2,5,6]} + \Lambda_4^\vee\mathbf{[3,5,6]}$
		$\left(\Lambda_5^\vee + 6\Lambda_7^\vee\right)\mathbf{[7 \supseteq 7 \supseteq 7]} + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)\mathbf{[5,7 \supseteq 5]} + \left(\Lambda_6^\vee + 5\Lambda_7^\vee\right)\mathbf{[5,7 \supseteq 7]} + 4\Lambda_7^\vee\mathbf{[6,7 \supseteq 7]} + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee\right)\mathbf{[5,6,7]}$
		$\Lambda_4^\vee\mathbf{[5,6,7]}$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\\0&0&0\\1&0&0\\0&1&0\\0&0&0\\0&0&0\\1&0&0\\0&0&0\end{pmatrix}$

A.28.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(2)\rangle$

$$\boxed{\begin{aligned}\phi_u \;=\; \partial\tau \text{ with } \tau = {} & (\Lambda_1^\vee + 2\Lambda_7^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 2\Lambda_7^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_7^\vee)\,[\mathbf{3}] + \\ & (\Lambda_3^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee)\,[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee + 2\Lambda_7^\vee)\,[\mathbf{5}] + (\Lambda_6^\vee + \Lambda_7^\vee)\,[\mathbf{6}] + (\Lambda_6^\vee + \Lambda_7^\vee)\,[\mathbf{7}]\end{aligned}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_7^\vee[\mathbf{1}\supseteq\mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{1,2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3}\supseteq\mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4}\supseteq\mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5}\supseteq\mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{6}\supseteq\mathbf{6}] + \left(-2\Lambda_6^\vee\right)[\mathbf{7}\supseteq\mathbf{7}] + \left(-\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,7}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{5,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{6,7}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{6,7}\supseteq\mathbf{7}] + \left(\Lambda_5^\vee - \Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$2\Lambda_7^\vee[\mathbf{6,7}\supseteq\mathbf{6}] + 2\Lambda_6^\vee[\mathbf{6,7}\supseteq\mathbf{7}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$\left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,7}\supseteq\mathbf{1}] + \left(-\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,7}\supseteq\mathbf{2}] + \left(-2\Lambda_5^\vee\right)[\mathbf{2,7}\supseteq\mathbf{7}] + \left(-\Lambda_5^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,7}\supseteq\mathbf{3}] + \left(-2\Lambda_5^\vee\right)[\mathbf{3,7}\supseteq\mathbf{7}] + \left(-\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,7}\supseteq\mathbf{4}] + \left(-2\Lambda_5^\vee\right)[\mathbf{4,7}\supseteq\mathbf{7}] + \left(-\Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{5,7}\supseteq\mathbf{5}] + \left(-\Lambda_4^\vee + \Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{5,7}\supseteq\mathbf{7}] + \left(2\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(2\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + 2\Lambda_6^\vee\right)[\mathbf{4,5,7}] + \left(-2\Lambda_5^\vee + 2\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$2\Lambda_7^\vee[\mathbf{1,3}\supseteq\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,3}\supseteq\mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4}\supseteq\mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4}\supseteq\mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4}\supseteq\mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4}\supseteq\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5}\supseteq\mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5}\supseteq\mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5}\supseteq\mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5}\supseteq\mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5}\supseteq\mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5}\supseteq\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6}\supseteq\mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6}\supseteq\mathbf{6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6}\supseteq\mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6}\supseteq\mathbf{6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6}\supseteq\mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6}\supseteq\mathbf{6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6}\supseteq\mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6}\supseteq\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,7}\supseteq\mathbf{1}] + \left(-\Lambda_5^\vee\right)[\mathbf{1,7}\supseteq\mathbf{7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,7}\supseteq\mathbf{2}] + \left(-\Lambda_5^\vee\right)[\mathbf{2,7}\supseteq\mathbf{7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,7}\supseteq\mathbf{3}] + \left(-\Lambda_5^\vee\right)[\mathbf{3,7}\supseteq\mathbf{7}] + \left(-\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,7}\supseteq\mathbf{4}] + \left(-\Lambda_5^\vee\right)[\mathbf{4,7}\supseteq\mathbf{7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(-\Lambda_4^\vee + 3\Lambda_5^\vee - \Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(-\Lambda_4^\vee + 3\Lambda_5^\vee - \Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(-\Lambda_4^\vee + 3\Lambda_5^\vee - \Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_4^\vee - \Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(\Lambda_5^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_7^\vee[]$
1	2	$2\Lambda_7^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}]$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{6}]$
2	5	$2\Lambda_7^\vee[\mathbf{1}\supseteq\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}\supseteq\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}\supseteq\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}\supseteq\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}\supseteq\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}\supseteq\mathbf{7}]$
		$2\Lambda_7^\vee[\mathbf{1,3}] + 2\Lambda_7^\vee[\mathbf{1,4}] + 2\Lambda_7^\vee[\mathbf{2,4}] + 2\Lambda_7^\vee[\mathbf{1,5}] + 2\Lambda_7^\vee[\mathbf{2,5}] + 2\Lambda_7^\vee[\mathbf{3,5}] + 2\Lambda_7^\vee[\mathbf{1,6}] + 2\Lambda_7^\vee[\mathbf{2,6}] + 2\Lambda_7^\vee[\mathbf{3,6}] + 2\Lambda_7^\vee[\mathbf{4,6}] + 2\Lambda_7^\vee[\mathbf{1,7}] + 2\Lambda_7^\vee[\mathbf{2,7}] + 2\Lambda_7^\vee[\mathbf{3,7}] + 2\Lambda_7^\vee[\mathbf{4,7}]$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{6}\supseteq\mathbf{6}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6}]$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,6}] + \Lambda_4^\vee[\mathbf{5,6}]$
		$\Lambda_5^\vee[\mathbf{6,7}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
3	12	$2\Lambda_7^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{6 \supseteq 6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{7 \supseteq 7 \supseteq 7}]$
		$2\Lambda_7^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{1,4 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{2,4 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,5 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{2,5 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,5 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{1,6 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{2,6 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{3,6 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,6 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{1,7 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{2,7 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{3,7 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{4,7 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,7 \supseteq 7}]$
		$2\Lambda_7^\vee[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + 2\Lambda_7^\vee[\mathbf{4,5,7}]$
		$2\Lambda_7^\vee[\mathbf{1,3,5}] + 2\Lambda_7^\vee[\mathbf{1,3,6}] + 2\Lambda_7^\vee[\mathbf{1,4,6}] + 2\Lambda_7^\vee[\mathbf{2,4,6}] + 2\Lambda_7^\vee[\mathbf{1,3,7}] + 2\Lambda_7^\vee[\mathbf{1,4,7}] + 2\Lambda_7^\vee[\mathbf{2,4,7}]$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{6 \supseteq 6 \supseteq 6}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6 \supseteq 5}] + \left(\Lambda_6^\vee + 3\Lambda_7^\vee\right)[\mathbf{5,6 \supseteq 6}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6 \supseteq 5}] + \Lambda_4^\vee[\mathbf{5,6 \supseteq 6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,5,6}] + \Lambda_5^\vee[\mathbf{5,6,7}]$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1,2,6}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{1,5,6}] + 2\Lambda_7^\vee[\mathbf{2,5,6}] + 2\Lambda_7^\vee[\mathbf{3,5,6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,6,7}]$
		$\left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3,6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,4,6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,4,6}] + \Lambda_4^\vee[\mathbf{1,5,6}] + \Lambda_4^\vee[\mathbf{2,5,6}] + \Lambda_4^\vee[\mathbf{3,5,6}]$
		$\left(\Lambda_3^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 6}] + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{3,4,6}] + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6 \supseteq 5}] + \Lambda_4^\vee[\mathbf{5,6 \supseteq 6}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{4,5,6}] + \left(\Lambda_5^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \Lambda_5^\vee[\mathbf{5,6,7}]$
		$\Lambda_5^\vee[\mathbf{6,7 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{6,7 \supseteq 7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$\Lambda_5^\vee[\mathbf{1,6,7}] + \Lambda_5^\vee[\mathbf{2,6,7}] + \Lambda_5^\vee[\mathbf{3,6,7}] + \Lambda_5^\vee[\mathbf{4,6,7}]$
		$\Lambda_4^\vee[\mathbf{5,6,7}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}$

A.28.3 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\phi_u = \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee)[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee)[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee)[\mathbf{5}] + (\Lambda_6^\vee + \Lambda_7^\vee)[\mathbf{6}] + (\Lambda_6^\vee + \Lambda_7^\vee)[\mathbf{7}]$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{6,7 \supseteq 7}] + \left(\Lambda_5^\vee - \Lambda_6^\vee - \Lambda_7^\vee\right)[\mathbf{5,6,7}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	1	$\Lambda_7^\vee[\mathbf{5,6}]$
3	3	$\Lambda_6^\vee[\mathbf{1,2 \supseteq 1}] + \Lambda_6^\vee[\mathbf{1,2 \supseteq 2}] + \Lambda_5^\vee[\mathbf{2,6 \supseteq 2}] + \Lambda_5^\vee[\mathbf{3,6 \supseteq 3}] + \Lambda_5^\vee[\mathbf{4,6 \supseteq 4}] + \Lambda_6^\vee[\mathbf{5,6 \supseteq 5}] + \left(\Lambda_4^\vee + \Lambda_7^\vee\right)[\mathbf{5,6 \supseteq 6}] + \Lambda_5^\vee[\mathbf{4,5,6}] + \left(\Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$\Lambda_6^\vee[\mathbf{1,3,5}] + \Lambda_7^\vee[\mathbf{1,3,6}] + \Lambda_7^\vee[\mathbf{1,4,6}] + \Lambda_7^\vee[\mathbf{2,4,6}] + \Lambda_3^\vee[\mathbf{1,5,6}] + \Lambda_3^\vee[\mathbf{2,5,6}] + \Lambda_1^\vee[\mathbf{3,5,6}] + \Lambda_6^\vee[\mathbf{1,3,7}] + \Lambda_6^\vee[\mathbf{1,4,7}] + \Lambda_6^\vee[\mathbf{2,4,7}]$
		$\Lambda_7^\vee[\mathbf{1,5,6}] + \Lambda_7^\vee[\mathbf{2,5,6}] + \Lambda_7^\vee[\mathbf{3,5,6}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$

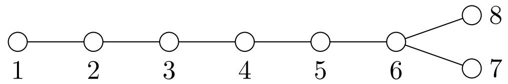
A.28.4 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1 \supseteq 1}] + (-1)[\mathbf{2 \supseteq 2}] + [\mathbf{1,2}] + (-1)[\mathbf{3 \supseteq 3}] + (-1)[\mathbf{4 \supseteq 4}] + (-1)[\mathbf{5 \supseteq 5}] + (-1)[\mathbf{6 \supseteq 6}] + (-1)[\mathbf{7 \supseteq 7}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{6,7 \supseteq 6}] + [\mathbf{6,7 \supseteq 7}] + [\mathbf{5,6,7}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)[\mathbf{1,2,3}] + (-1)[\mathbf{1,4 \supseteq 1}] + (-1)[\mathbf{1,4 \supseteq 4}] + (-1)[\mathbf{2,4 \supseteq 2}] + (-1)[\mathbf{2,4 \supseteq 4}] + [\mathbf{1,3,4}] + [\mathbf{2,3,4}] + (-1)[\mathbf{1,5 \supseteq 1}] + (-1)[\mathbf{1,5 \supseteq 5}] + (-1)[\mathbf{2,5 \supseteq 2}] + (-1)[\mathbf{2,5 \supseteq 5}] + (-1)[\mathbf{3,5 \supseteq 3}] + (-1)[\mathbf{3,5 \supseteq 5}] + [\mathbf{3,4,5}] + (-1)[\mathbf{1,6 \supseteq 1}] + (-1)[\mathbf{1,6 \supseteq 6}] + (-1)[\mathbf{2,6 \supseteq 2}] + (-1)[\mathbf{2,6 \supseteq 6}] + (-1)[\mathbf{3,6 \supseteq 3}] + (-1)[\mathbf{3,6 \supseteq 6}] + (-1)[\mathbf{4,6 \supseteq 4}] + (-1)[\mathbf{4,6 \supseteq 6}] + [\mathbf{4,5,6}] + (-1)[\mathbf{1,7 \supseteq 1}] + (-1)[\mathbf{1,7 \supseteq 7}] + (-1)[\mathbf{2,7 \supseteq 2}] + (-1)[\mathbf{2,7 \supseteq 7}] + (-1)[\mathbf{3,7 \supseteq 3}] + (-1)[\mathbf{3,7 \supseteq 7}] + (-1)[\mathbf{4,7 \supseteq 4}] + (-1)[\mathbf{4,7 \supseteq 7}] + [\mathbf{4,5,7}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}] + [\mathbf{7}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
2	3	$\mathbf{[1 \supseteq 1] + [2 \supseteq 2] + [3 \supseteq 3] + [4 \supseteq 4] + [5 \supseteq 5] + [6 \supseteq 6] + [7 \supseteq 7]}$ $\mathbf{[1,3] + [1,4] + [2,4] + [1,5] + [2,5] + [3,5] + [1,6] + [2,6] + [3,6] + [4,6] + [1,7] + [2,7] + [3,7] + [4,7]}$ $\mathbf{[6,7]}$
3	7	$\mathbf{[1 \supseteq 1 \supseteq 1] + [2 \supseteq 2 \supseteq 2] + [3 \supseteq 3 \supseteq 3] + [4 \supseteq 4 \supseteq 4] + [5 \supseteq 5 \supseteq 5] + [6 \supseteq 6 \supseteq 6] + [7 \supseteq 7 \supseteq 7]}$ $\mathbf{[1,3 \supseteq 1] + [1,3 \supseteq 3] + [1,4 \supseteq 1] + [1,4 \supseteq 4] + [2,4 \supseteq 2] + [2,4 \supseteq 4] + [1,5 \supseteq 1] + [1,5 \supseteq 5] + [2,5 \supseteq 2] + [2,5 \supseteq 5] + [3,5 \supseteq 3] + [3,5 \supseteq 5] + [1,6 \supseteq 1] + [1,6 \supseteq 6] + [2,6 \supseteq 2] + [2,6 \supseteq 6] + [3,6 \supseteq 3] + [3,6 \supseteq 6] + [4,6 \supseteq 4] + [4,6 \supseteq 6] + [1,7 \supseteq 1] + [1,7 \supseteq 7] + [2,7 \supseteq 2] + [2,7 \supseteq 7] + [3,7 \supseteq 3] + [3,7 \supseteq 7] + [4,7 \supseteq 4] + [4,7 \supseteq 7]}$ $\mathbf{[1,2,3] + [2,3,4] + [3,4,5] + [4,5,6] + [4,5,7]}$ $\mathbf{[1,3,5] + [1,3,6] + [1,4,6] + [2,4,6] + [1,3,7] + [1,4,7] + [2,4,7]}$ $\mathbf{[6,7 \supseteq 6] + [6,7 \supseteq 7]}$ $\mathbf{[1,6,7] + [2,6,7] + [3,6,7] + [4,6,7]}$ $\mathbf{[5,6,7]}$

A.29 Root system D_8

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_8^\vee, \Lambda_7^\vee \in P^\vee \mod Q^\vee$

A.29.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] \;=\; (1,1,1,0,0,1,0,1)$
does not lie in the image of $\mathbf{comp}_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)\mathbf{[8]}$ $\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)\mathbf{[7]}$
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_8^\vee\mathbf{[1 \supseteq 1]} + \left(-2\Lambda_8^\vee\right)\mathbf{[2 \supseteq 2]} + 2\Lambda_8^\vee\mathbf{[1,2]} + \left(-2\Lambda_8^\vee\right)\mathbf{[3 \supseteq 3]} + \left(-2\Lambda_8^\vee\right)\mathbf{[4 \supseteq 4]} + \left(-2\Lambda_8^\vee\right)\mathbf{[5 \supseteq 5]} + \left(-2\Lambda_8^\vee\right)\mathbf{[6 \supseteq 6]} + \left(-2\Lambda_8^\vee\right)\mathbf{[7 \supseteq 7]} + \left(-\Lambda_6^\vee\right)\mathbf{[8 \supseteq 8]} + \left(-\Lambda_6^\vee + 2\Lambda_8^\vee\right)\mathbf{[1,8]} + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)\mathbf{[2,8]} + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)\mathbf{[3,8]} + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)\mathbf{[4,8]} + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)\mathbf{[5,8]} + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)\mathbf{[6,8]} + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)\mathbf{[7,8]}$ $2\Lambda_7^\vee\mathbf{[1 \supseteq 1]} + \left(-2\Lambda_7^\vee\right)\mathbf{[2 \supseteq 2]} + 2\Lambda_7^\vee\mathbf{[1,2]} + \left(-2\Lambda_7^\vee\right)\mathbf{[3 \supseteq 3]} + \left(-2\Lambda_7^\vee\right)\mathbf{[4 \supseteq 4]} + \left(-2\Lambda_7^\vee\right)\mathbf{[5 \supseteq 5]} + \left(-2\Lambda_7^\vee\right)\mathbf{[6 \supseteq 6]} + \left(-\Lambda_6^\vee\right)\mathbf{[7 \supseteq 7]} + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)\mathbf{[1,7]} + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)\mathbf{[2,7]} + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)\mathbf{[3,7]} + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)\mathbf{[4,7]} + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)\mathbf{[5,7]} + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)\mathbf{[6,7]} + \left(-2\Lambda_7^\vee\right)\mathbf{[8 \supseteq 8]} + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)\mathbf{[7,8]}$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_8^\vee[\mathbf{7,8 \supseteq 7}] + \left(-\Lambda_6^\vee + 4\Lambda_7^\vee\right)[\mathbf{7,8 \supseteq 8}] + \left(\Lambda_5^\vee - 3\Lambda_6^\vee + 3\Lambda_7^\vee + 3\Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$\Lambda_6^\vee[\mathbf{7,8 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{7,8 \supseteq 8}] + \left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,8 \supseteq 1}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,8 \supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,8 \supseteq 8}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,8 \supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,8 \supseteq 8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,8 \supseteq 4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,8 \supseteq 8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,8 \supseteq 5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{5,8 \supseteq 8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,6,8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,6,8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,6,8}] + \left(\Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,8}]$
		$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,6,7}]$
		$2\Lambda_8^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_8^\vee[\mathbf{1,3,4}] + 2\Lambda_8^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5 \supseteq 5}] + 2\Lambda_8^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6 \supseteq 6}] + 2\Lambda_8^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7 \supseteq 7}] + 2\Lambda_8^\vee[\mathbf{5,6,7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,8 \supseteq 1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,8 \supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,8 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,8 \supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,8 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,8 \supseteq 8}] + \left(-\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,3,8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,8 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,8 \supseteq 8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,4,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,4,8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,8 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,8 \supseteq 8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,5,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,5,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,5,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{1,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{2,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{3,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{4,6,8}] + \left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,7,8}]$
		$2\Lambda_7^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,7 \supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{4,6,7}] + \left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,8 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,8 \supseteq 8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,8 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,8 \supseteq 8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,8 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,8 \supseteq 8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,8 \supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,8 \supseteq 8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,8 \supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,8 \supseteq 8}] + 2\Lambda_7^\vee[\mathbf{5,6,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,7,8}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	2	$2\Lambda_7^\vee[]$
		$2\Lambda_8^\vee[]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
1	4	$\Lambda_2^\vee[\mathbf{1}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{4}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{5}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{8}]$
		$2\Lambda_7^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{8}]$
		$2\Lambda_8^\vee[\mathbf{1}] + 2\Lambda_8^\vee[\mathbf{2}] + 2\Lambda_8^\vee[\mathbf{3}] + 2\Lambda_8^\vee[\mathbf{4}] + 2\Lambda_8^\vee[\mathbf{5}] + 2\Lambda_8^\vee[\mathbf{6}] + 2\Lambda_8^\vee[\mathbf{7}] + 2\Lambda_8^\vee[\mathbf{8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	8	$\Lambda_2^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1},\mathbf{2}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{8} \supseteq \mathbf{8}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{8}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{3},\mathbf{8}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{4},\mathbf{8}] + \left(\Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{5},\mathbf{8}]$
		$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{8} \supseteq \mathbf{8}]$
		$2\Lambda_8^\vee[\mathbf{1} \supseteq \mathbf{1}] + 2\Lambda_8^\vee[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_8^\vee[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_8^\vee[\mathbf{4} \supseteq \mathbf{4}] + 2\Lambda_8^\vee[\mathbf{5} \supseteq \mathbf{5}] + 2\Lambda_8^\vee[\mathbf{6} \supseteq \mathbf{6}] + 2\Lambda_8^\vee[\mathbf{7} \supseteq \mathbf{7}] + 2\Lambda_8^\vee[\mathbf{8} \supseteq \mathbf{8}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}] + \left(\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2},\mathbf{3}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1},\mathbf{4}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{4}] + \left(\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3},\mathbf{4}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1},\mathbf{5}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{5}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3},\mathbf{5}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1},\mathbf{6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3},\mathbf{6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{4},\mathbf{6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1},\mathbf{7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{3},\mathbf{7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{4},\mathbf{7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1},\mathbf{8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{8}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{4},\mathbf{8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{8}]$
		$2\Lambda_7^\vee[\mathbf{1},\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{1},\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{2},\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{1},\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{2},\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3},\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{1},\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{2},\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{3},\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4},\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{1},\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{2},\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{3},\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{4},\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{5},\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{1},\mathbf{8}] + 2\Lambda_7^\vee[\mathbf{2},\mathbf{8}] + 2\Lambda_7^\vee[\mathbf{3},\mathbf{8}] + 2\Lambda_7^\vee[\mathbf{4},\mathbf{8}] + 2\Lambda_7^\vee[\mathbf{5},\mathbf{8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6},\mathbf{7}] + \Lambda_6^\vee[\mathbf{7},\mathbf{8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{7}]$
		$\left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{8} \supseteq \mathbf{8}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{8}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6},\mathbf{8}] + \Lambda_6^\vee[\mathbf{7},\mathbf{8}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
3	19	$\Lambda_2^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2 \supseteq 2 \supseteq 2}] + (\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{1,2 \supseteq 1}] + (\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{1,2 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3 \supseteq 3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4 \supseteq 4 \supseteq 4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,3,4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5 \supseteq 5 \supseteq 5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,4,5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{6 \supseteq 6 \supseteq 6}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,5,6}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{7 \supseteq 7 \supseteq 7}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{2,7 \supseteq 2}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{3,7 \supseteq 3}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{4,7 \supseteq 4}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{5,7 \supseteq 5}] + (\Lambda_4^\vee + 2\Lambda_8^\vee)[\mathbf{5,7 \supseteq 7}] + (\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,5,7}] + (\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{4,5,7}] + (\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + 3\Lambda_8^\vee)[\mathbf{5,6,7}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{8 \supseteq 8 \supseteq 8}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{2,8 \supseteq 2}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{3,8 \supseteq 3}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{4,8 \supseteq 4}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{5,8 \supseteq 5}] + (\Lambda_4^\vee + 2\Lambda_7^\vee)[\mathbf{5,8 \supseteq 8}] + (\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,5,8}] + (\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{4,5,8}] + (\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{5,6,8}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,7,8}]$
		$2\Lambda_7^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{6 \supseteq 6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{7 \supseteq 7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{8 \supseteq 8 \supseteq 8}]$
		$2\Lambda_8^\vee[\mathbf{1 \supseteq 1 \supseteq 1}] + 2\Lambda_8^\vee[\mathbf{2 \supseteq 2 \supseteq 2}] + 2\Lambda_8^\vee[\mathbf{3 \supseteq 3 \supseteq 3}] + 2\Lambda_8^\vee[\mathbf{4 \supseteq 4 \supseteq 4}] + 2\Lambda_8^\vee[\mathbf{5 \supseteq 5 \supseteq 5}] + 2\Lambda_8^\vee[\mathbf{6 \supseteq 6 \supseteq 6}] + 2\Lambda_8^\vee[\mathbf{7 \supseteq 7 \supseteq 7}] + 2\Lambda_8^\vee[\mathbf{8 \supseteq 8 \supseteq 8}]$
		$(\Lambda_1^\vee + \Lambda_3^\vee + 4\Lambda_7^\vee + 4\Lambda_8^\vee)[\mathbf{2 \supseteq 2 \supseteq 2}] + (\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{1,2 \supseteq 1}] + (\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{1,2 \supseteq 2}] + (\Lambda_2^\vee + \Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{2,3 \supseteq 2}] + (\Lambda_3^\vee + \Lambda_4^\vee + 3\Lambda_7^\vee + 3\Lambda_8^\vee)[\mathbf{2,3 \supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,3,4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,4,5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,5,6}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{2,7 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,6,7}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{2,8 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,6,8}]$
		$\Lambda_2^\vee[\mathbf{1,3 \supseteq 1}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,3 \supseteq 3}] + (\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,2,3}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,4 \supseteq 1}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,4 \supseteq 4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,4 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,4 \supseteq 4}] + (\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{1,3,4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,5 \supseteq 1}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,5 \supseteq 5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,5 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,5 \supseteq 5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,5 \supseteq 3}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,5 \supseteq 5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,6 \supseteq 1}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,6 \supseteq 6}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,6 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,6 \supseteq 6}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,6 \supseteq 3}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,6 \supseteq 6}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,6 \supseteq 4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,6 \supseteq 6}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,7 \supseteq 1}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,7 \supseteq 7}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,7 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,7 \supseteq 7}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,7 \supseteq 3}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,7 \supseteq 7}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,7 \supseteq 4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,7 \supseteq 7}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{1,4,7}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{2,4,7}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,7 \supseteq 5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,7 \supseteq 7}] + (\Lambda_4^\vee + 2\Lambda_8^\vee)[\mathbf{1,5,7}] + (\Lambda_4^\vee + 2\Lambda_8^\vee)[\mathbf{2,5,7}] + (\Lambda_2^\vee + 2\Lambda_7^\vee)[\mathbf{3,5,7}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,8 \supseteq 1}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,8 \supseteq 8}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,8 \supseteq 2}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,8 \supseteq 8}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,8 \supseteq 3}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,8 \supseteq 8}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,8 \supseteq 4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,8 \supseteq 8}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{1,4,8}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{2,4,8}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,8 \supseteq 5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,8 \supseteq 8}] + (\Lambda_4^\vee + 2\Lambda_7^\vee)[\mathbf{1,5,8}] + (\Lambda_4^\vee + 2\Lambda_7^\vee)[\mathbf{2,5,8}] + (\Lambda_2^\vee + 2\Lambda_8^\vee)[\mathbf{3,5,8}]$
		$\Lambda_4^\vee[\mathbf{1,3 \supseteq 1}] + (\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{2,3 \supseteq 2}] + (\Lambda_1^\vee + 3\Lambda_7^\vee + 3\Lambda_8^\vee)[\mathbf{2,3 \supseteq 3}] + (\Lambda_1^\vee + \Lambda_3^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{1,2,3}] + (\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{1,3,4}] + (\Lambda_3^\vee + \Lambda_5^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{2,3,4}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{3,4,5}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{4,5,6}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{3,7 \supseteq 3}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{4,7 \supseteq 4}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{1,4,7}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{2,4,7}] + (\Lambda_6^\vee + 2\Lambda_7^\vee)[\mathbf{5,7 \supseteq 5}] + (\Lambda_4^\vee + 2\Lambda_8^\vee)[\mathbf{5,7 \supseteq 7}] + (\Lambda_4^\vee + 2\Lambda_8^\vee)[\mathbf{1,5,7}] + (\Lambda_4^\vee + 2\Lambda_8^\vee)[\mathbf{2,5,7}] + (\Lambda_4^\vee + 2\Lambda_8^\vee)[\mathbf{3,5,7}] + (\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{4,5,7}] + (\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + 3\Lambda_8^\vee)[\mathbf{5,6,7}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{3,8 \supseteq 3}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{4,8 \supseteq 4}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{1,4,8}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{2,4,8}] + (\Lambda_6^\vee + 2\Lambda_8^\vee)[\mathbf{5,8 \supseteq 5}] + (\Lambda_4^\vee + 2\Lambda_7^\vee)[\mathbf{5,8 \supseteq 8}] + (\Lambda_4^\vee + 2\Lambda_7^\vee)[\mathbf{1,5,8}] + (\Lambda_4^\vee + 2\Lambda_7^\vee)[\mathbf{2,5,8}] + (\Lambda_4^\vee + 2\Lambda_7^\vee)[\mathbf{3,5,8}] + (\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{4,5,8}] + (\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee + \Lambda_8^\vee)[\mathbf{5,6,8}] + (2\Lambda_7^\vee + 2\Lambda_8^\vee)[\mathbf{5,7,8}]$
		$2\Lambda_7^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,3 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{1,4 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{2,4 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,5 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{2,5 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,5 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{1,6 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{2,6 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{3,6 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,6 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{1,7 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{2,7 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{3,7 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{4,7 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{5,7 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{5,7 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{1,8 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,8 \supseteq 8}] + 2\Lambda_7^\vee[\mathbf{2,8 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,8 \supseteq 8}] + 2\Lambda_7^\vee[\mathbf{3,8 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,8 \supseteq 8}] + 2\Lambda_7^\vee[\mathbf{4,8 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,8 \supseteq 8}] + 2\Lambda_7^\vee[\mathbf{5,8 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{5,8 \supseteq 8}]$
		$2\Lambda_7^\vee[\mathbf{1,2,3}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + 2\Lambda_7^\vee[\mathbf{5,6,7}] + 2\Lambda_7^\vee[\mathbf{5,6,8}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	**generating cocycles**
		$\Lambda_2^\vee[\mathbf{1,3,5}] + \left(\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{2,3,5}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,3,6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,4,6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,4,6}] + \left(\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{3,5,6}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,3,7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,4,7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,4,7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,5,7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{3,5,7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,3,8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,4,8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,4,8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,5,8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,5,8}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,5,8}]$
		$2\Lambda_7^\vee[\mathbf{1,3,5}] + 2\Lambda_7^\vee[\mathbf{1,3,6}] + 2\Lambda_7^\vee[\mathbf{1,4,6}] + 2\Lambda_7^\vee[\mathbf{2,4,6}] + 2\Lambda_7^\vee[\mathbf{1,3,7}] + 2\Lambda_7^\vee[\mathbf{1,4,7}] + 2\Lambda_7^\vee[\mathbf{2,4,7}] + 2\Lambda_7^\vee[\mathbf{1,5,7}] + 2\Lambda_7^\vee[\mathbf{2,5,7}] + 2\Lambda_7^\vee[\mathbf{3,5,7}] + 2\Lambda_7^\vee[\mathbf{1,3,8}] + 2\Lambda_7^\vee[\mathbf{1,4,8}] + 2\Lambda_7^\vee[\mathbf{2,4,8}] + 2\Lambda_7^\vee[\mathbf{1,5,8}] + 2\Lambda_7^\vee[\mathbf{2,5,8}] + 2\Lambda_7^\vee[\mathbf{3,5,8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{7 \supseteq 7 \supseteq 7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,5,7}] + \Lambda_6^\vee[\mathbf{6,7 \supseteq 6}] + \left(\Lambda_5^\vee + 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7 \supseteq 7}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{7,8 \supseteq 8}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,7,8}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + 3\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,7,8}]$
		$\left(\Lambda_2^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,7 \supseteq 7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3,5,7}] + \left(\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_1^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7 \supseteq 6}] + \left(\Lambda_1^\vee + \Lambda_5^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{6,7 \supseteq 7}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,7,8}] + \Lambda_6^\vee[\mathbf{6,7,8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 7}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,2,7}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,6,7}] + 2\Lambda_7^\vee[\mathbf{2,6,7}] + 2\Lambda_7^\vee[\mathbf{3,6,7}] + 2\Lambda_7^\vee[\mathbf{4,6,7}] + \Lambda_6^\vee[\mathbf{1,7,8}] + 2\Lambda_8^\vee[\mathbf{2,7,8}] + 2\Lambda_8^\vee[\mathbf{3,7,8}] + 2\Lambda_8^\vee[\mathbf{4,7,8}] + 2\Lambda_8^\vee[\mathbf{5,7,8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_2^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,5,7}]$
		$\left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{8 \supseteq 8 \supseteq 8}] + \left(\Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,8 \supseteq 8}] + \left(\Lambda_2^\vee + \Lambda_4^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{3,5,8}] + \left(\Lambda_3^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{4,5,8}] + \Lambda_6^\vee[\mathbf{6,8 \supseteq 6}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + 3\Lambda_8^\vee\right)[\mathbf{6,8 \supseteq 8}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,8}] + \left(2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{7,8 \supseteq 8}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,7,8}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$\left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,8 \supseteq 8}] + \left(\Lambda_5^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,2,8}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,5,8}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{1,6,8}] + 2\Lambda_8^\vee[\mathbf{2,6,8}] + 2\Lambda_8^\vee[\mathbf{3,6,8}] + 2\Lambda_8^\vee[\mathbf{4,6,8}] + \Lambda_6^\vee[\mathbf{1,7,8}] + 2\Lambda_7^\vee[\mathbf{2,7,8}] + 2\Lambda_7^\vee[\mathbf{3,7,8}] + 2\Lambda_7^\vee[\mathbf{4,7,8}] + 2\Lambda_7^\vee[\mathbf{5,7,8}]$
		$\Lambda_6^\vee[\mathbf{7,8 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{7,8 \supseteq 8}] + \left(\Lambda_5^\vee + \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$2\Lambda_7^\vee[\mathbf{7,8 \supseteq 7}] + 2\Lambda_7^\vee[\mathbf{7,8 \supseteq 8}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1&0\\1&0\\1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\\0&0\\0&0\\0&0\\1&1\\1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&0&1&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&1&1\\0&0&0&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&0&0\\0&0&0&0&1&1\\0&0&0&0&1&1\\1&0&0&0&0&0\\0&0&0&0&1&1\\1&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$

A.29.2 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(0,1)\rangle$

$$\boxed{\begin{array}{rl}[\phi_u] & = \ (1,1,0,0)\\ & \textbf{does not lie in the image of comp}_2\end{array}}$$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{8}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_8^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_8^\vee[\mathbf{1},\mathbf{2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee\right)[\mathbf{8} \supseteq \mathbf{8}] + \left(-\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{1},\mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2},\mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3},\mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4},\mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5},\mathbf{8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{6},\mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{7},\mathbf{8}]$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$2\Lambda_8^\vee[\mathbf{7,8 \supseteq 7}] + \left(-\Lambda_6^\vee + 4\Lambda_7^\vee\right)[\mathbf{7,8 \supseteq 8}] + \left(\Lambda_5^\vee - 3\Lambda_6^\vee + 3\Lambda_7^\vee + 3\Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,8 \supseteq 1}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,8 \supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,8 \supseteq 8}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,8 \supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,8 \supseteq 8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,8 \supseteq 4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,8 \supseteq 8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,8 \supseteq 5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{5,8 \supseteq 8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,6,8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,6,8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,6,8}] + \left(\Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,8}]$
		$2\Lambda_8^\vee[\mathbf{1,3 \supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,3 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,4 \supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4 \supseteq 4}] + 2\Lambda_8^\vee[\mathbf{1,3,4}] + 2\Lambda_8^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5 \supseteq 5}] + 2\Lambda_8^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6 \supseteq 6}] + 2\Lambda_8^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7 \supseteq 7}] + 2\Lambda_8^\vee[\mathbf{5,6,7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,8 \supseteq 1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,8 \supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,8 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,8 \supseteq 8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,8 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,8 \supseteq 8}] + \left(-\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,3,8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,8 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,8 \supseteq 8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,4,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,4,8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,8 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,8 \supseteq 8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,5,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,5,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,5,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{1,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{2,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{3,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{4,6,8}] + \left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,7,8}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$2\Lambda_8^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_8^\vee[\mathbf{2}] + 2\Lambda_8^\vee[\mathbf{3}] + 2\Lambda_8^\vee[\mathbf{4}] + 2\Lambda_8^\vee[\mathbf{5}] + 2\Lambda_8^\vee[\mathbf{6}] + 2\Lambda_8^\vee[\mathbf{7}] + 2\Lambda_8^\vee[\mathbf{8}]$
		$2\Lambda_8^\vee[\mathbf{1}] + 2\Lambda_8^\vee[\mathbf{2}] + 2\Lambda_8^\vee[\mathbf{3}] + 2\Lambda_8^\vee[\mathbf{4}] + 2\Lambda_8^\vee[\mathbf{5}] + 2\Lambda_8^\vee[\mathbf{6}] + 2\Lambda_8^\vee[\mathbf{7}] + 2\Lambda_8^\vee[\mathbf{8}]$
2	4	$\Lambda_2^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_8^\vee[\mathbf{2 \supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee + \Lambda_8^\vee\right)[\mathbf{1,2}] + 2\Lambda_8^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_8^\vee[\mathbf{4 \supseteq 4}] + 2\Lambda_8^\vee[\mathbf{5 \supseteq 5}] + 2\Lambda_8^\vee[\mathbf{6 \supseteq 6}] + 2\Lambda_8^\vee[\mathbf{7 \supseteq 7}] + 2\Lambda_8^\vee[\mathbf{8 \supseteq 8}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{2,8}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{3,8}] + \left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{4,8}] + \Lambda_4^\vee[\mathbf{5,8}]$
		$2\Lambda_8^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_8^\vee[\mathbf{2 \supseteq 2}] + 2\Lambda_8^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_8^\vee[\mathbf{4 \supseteq 4}] + 2\Lambda_8^\vee[\mathbf{5 \supseteq 5}] + 2\Lambda_8^\vee[\mathbf{6 \supseteq 6}] + 2\Lambda_8^\vee[\mathbf{7 \supseteq 7}] + 2\Lambda_8^\vee[\mathbf{8 \supseteq 8}]$
		$\Lambda_2^\vee[\mathbf{1,3}] + \left(\Lambda_1^\vee + \Lambda_8^\vee\right)[\mathbf{2,3}] + 2\Lambda_8^\vee[\mathbf{1,4}] + 2\Lambda_8^\vee[\mathbf{2,4}] + \left(\Lambda_1^\vee + \Lambda_8^\vee\right)[\mathbf{3,4}] + 2\Lambda_8^\vee[\mathbf{1,5}] + 2\Lambda_8^\vee[\mathbf{2,5}] + 2\Lambda_8^\vee[\mathbf{3,5}] + 2\Lambda_8^\vee[\mathbf{1,6}] + 2\Lambda_8^\vee[\mathbf{2,6}] + 2\Lambda_8^\vee[\mathbf{3,6}] + 2\Lambda_8^\vee[\mathbf{4,6}] + 2\Lambda_8^\vee[\mathbf{1,7}] + 2\Lambda_8^\vee[\mathbf{2,7}] + 2\Lambda_8^\vee[\mathbf{3,7}] + 2\Lambda_8^\vee[\mathbf{4,7}] + 2\Lambda_8^\vee[\mathbf{5,7}] + 2\Lambda_8^\vee[\mathbf{1,8}] + 2\Lambda_8^\vee[\mathbf{2,8}] + \Lambda_6^\vee[\mathbf{3,8}] + 2\Lambda_8^\vee[\mathbf{4,8}] + 2\Lambda_8^\vee[\mathbf{5,8}]$
		$\left(\Lambda_2^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,7}] + \left(\Lambda_1^\vee + \Lambda_8^\vee\right)[\mathbf{6,7}] + \Lambda_6^\vee[\mathbf{7,8}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}}^\vee)$	**generating cocycles**
3	9	$\Lambda_2^\vee[\mathbf{1\supseteq 1\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{2\supseteq 2\supseteq 2}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{1,2\supseteq 1}] + \left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_8^\vee\right)[\mathbf{1,2\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{3\supseteq 3\supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + 2\Lambda_8^\vee[\mathbf{4\supseteq 4\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{2,3,4}] + 2\Lambda_8^\vee[\mathbf{5\supseteq 5\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{3,4,5}] + 2\Lambda_8^\vee[\mathbf{6\supseteq 6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{4,5,6}] + 2\Lambda_8^\vee[\mathbf{7\supseteq 7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{5,6,7}] + 2\Lambda_8^\vee[\mathbf{8\supseteq 8\supseteq 8}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,8\supseteq 2}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{3,8\supseteq 3}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{4,8\supseteq 4}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{5,8\supseteq 5}] + \Lambda_4^\vee[\mathbf{5,8\supseteq 8}] + \left(\Lambda_2^\vee+\Lambda_4^\vee+2\Lambda_8^\vee\right)[\mathbf{3,5,8}] + \left(\Lambda_3^\vee+\Lambda_8^\vee\right)[\mathbf{4,5,8}] + \left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{5,6,8}]$ $2\Lambda_8^\vee[\mathbf{1\supseteq 1\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{2\supseteq 2\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{3\supseteq 3\supseteq 3}] + 2\Lambda_8^\vee[\mathbf{4\supseteq 4\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{5\supseteq 5\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{6\supseteq 6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{7\supseteq 7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{8\supseteq 8\supseteq 8}]$ $\left(\Lambda_1^\vee+\Lambda_3^\vee+4\Lambda_8^\vee\right)[\mathbf{2\supseteq 2\supseteq 2}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{1,2\supseteq 1}] + \left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_8^\vee\right)[\mathbf{1,2\supseteq 2}] + \left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_8^\vee\right)[\mathbf{2,3\supseteq 2}] + \left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_8^\vee\right)[\mathbf{2,3\supseteq 3}] + \Lambda_2^\vee[\mathbf{1,2,3}] + 2\Lambda_8^\vee[\mathbf{2,3,4}] + 2\Lambda_8^\vee[\mathbf{3,4,5}] + 2\Lambda_8^\vee[\mathbf{4,5,6}] + 2\Lambda_8^\vee[\mathbf{5,6,7}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,8\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{5,6,8}]$ $\Lambda_7^\vee[\mathbf{1,2\supseteq 1}] + \Lambda_7^\vee[\mathbf{1,2\supseteq 2}] + \Lambda_6^\vee[\mathbf{2,7\supseteq 2}] + \Lambda_6^\vee[\mathbf{3,7\supseteq 3}] + \Lambda_6^\vee[\mathbf{4,7\supseteq 4}] + \Lambda_6^\vee[\mathbf{5,7\supseteq 5}] + \Lambda_7^\vee[\mathbf{6,7\supseteq 6}] + \left(\Lambda_5^\vee+\Lambda_8^\vee\right)[\mathbf{6,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{5,6,7}] + 2\Lambda_8^\vee[\mathbf{7,8\supseteq 8}] + \left(\Lambda_5^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7,8}]$ $\Lambda_2^\vee[\mathbf{1,3\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,3\supseteq 3}] + \left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_8^\vee\right)[\mathbf{1,2,3}] + 2\Lambda_8^\vee[\mathbf{1,4\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,4\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{2,4\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{2,4\supseteq 4}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{1,3,4}] + 2\Lambda_8^\vee[\mathbf{1,5\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,5\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{2,5\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{2,5\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{3,5\supseteq 3}] + 2\Lambda_8^\vee[\mathbf{3,5\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{1,6\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{2,6\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{2,6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{3,6\supseteq 3}] + 2\Lambda_8^\vee[\mathbf{3,6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{4,6\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{4,6\supseteq 6}] + 2\Lambda_8^\vee[\mathbf{1,7\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{2,7\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{2,7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{3,7\supseteq 3}] + 2\Lambda_8^\vee[\mathbf{3,7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{4,7\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{4,7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{5,7\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{5,7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{1,8\supseteq 1}] + 2\Lambda_8^\vee[\mathbf{1,8\supseteq 8}] + 2\Lambda_8^\vee[\mathbf{2,8\supseteq 2}] + 2\Lambda_8^\vee[\mathbf{2,8\supseteq 8}] + 2\Lambda_8^\vee[\mathbf{3,8\supseteq 3}] + 2\Lambda_8^\vee[\mathbf{3,8\supseteq 8}] + 2\Lambda_8^\vee[\mathbf{4,8\supseteq 4}] + 2\Lambda_8^\vee[\mathbf{4,8\supseteq 8}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,4,8}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,4,8}] + 2\Lambda_8^\vee[\mathbf{5,8\supseteq 5}] + 2\Lambda_8^\vee[\mathbf{5,8\supseteq 8}] + \Lambda_4^\vee[\mathbf{1,5,8}] + \Lambda_4^\vee[\mathbf{2,5,8}] + \left(\Lambda_2^\vee+2\Lambda_8^\vee\right)[\mathbf{3,5,8}]$ $\Lambda_4^\vee[\mathbf{1,3\supseteq 1}] + \left(\Lambda_3^\vee+\Lambda_8^\vee\right)[\mathbf{2,3\supseteq 2}] + \left(\Lambda_1^\vee+3\Lambda_8^\vee\right)[\mathbf{2,3\supseteq 3}] + \left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_8^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{1,3,4}] + \left(\Lambda_3^\vee+\Lambda_5^\vee+2\Lambda_8^\vee\right)[\mathbf{2,3,4}] + 2\Lambda_8^\vee[\mathbf{3,4,5}] + 2\Lambda_8^\vee[\mathbf{4,5,6}] + 2\Lambda_8^\vee[\mathbf{5,6,7}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{3,8\supseteq 3}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{4,8\supseteq 4}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,4,8}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,4,8}] + \left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{5,8\supseteq 5}] + \Lambda_4^\vee[\mathbf{5,8\supseteq 8}] + \Lambda_4^\vee[\mathbf{1,5,8}] + \Lambda_4^\vee[\mathbf{2,5,8}] + \Lambda_4^\vee[\mathbf{3,5,8}] + \left(\Lambda_3^\vee+\Lambda_8^\vee\right)[\mathbf{4,5,8}] + \left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{5,6,8}]$ $\Lambda_2^\vee[\mathbf{1,3,5}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{2,3,5}] + 2\Lambda_8^\vee[\mathbf{1,3,6}] + 2\Lambda_8^\vee[\mathbf{1,4,6}] + 2\Lambda_8^\vee[\mathbf{2,4,6}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{3,5,6}] + 2\Lambda_8^\vee[\mathbf{1,3,7}] + 2\Lambda_8^\vee[\mathbf{1,4,7}] + 2\Lambda_8^\vee[\mathbf{2,4,7}] + 2\Lambda_8^\vee[\mathbf{1,5,7}] + 2\Lambda_8^\vee[\mathbf{2,5,7}] + 2\Lambda_8^\vee[\mathbf{3,5,7}] + 2\Lambda_8^\vee[\mathbf{1,3,8}] + 2\Lambda_8^\vee[\mathbf{1,4,8}] + 2\Lambda_8^\vee[\mathbf{2,4,8}] + 2\Lambda_8^\vee[\mathbf{1,5,8}] + 2\Lambda_8^\vee[\mathbf{2,5,8}] + \Lambda_6^\vee[\mathbf{3,5,8}]$ $\left(\Lambda_2^\vee+2\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 1}] + \left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_8^\vee\right)[\mathbf{1,2,7}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{1,6,7}] + 2\Lambda_8^\vee[\mathbf{2,6,7}] + 2\Lambda_8^\vee[\mathbf{3,6,7}] + 2\Lambda_8^\vee[\mathbf{4,6,7}] + \Lambda_6^\vee[\mathbf{1,7,8}] + 2\Lambda_8^\vee[\mathbf{2,7,8}] + 2\Lambda_8^\vee[\mathbf{3,7,8}] + 2\Lambda_8^\vee[\mathbf{4,7,8}] + 2\Lambda_8^\vee[\mathbf{5,7,8}]$ $\left(\Lambda_2^\vee+2\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 7}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{6,7\supseteq 6}] + \left(\Lambda_1^\vee+\Lambda_8^\vee\right)[\mathbf{6,7\supseteq 7}] + 2\Lambda_8^\vee[\mathbf{7,8\supseteq 8}] + \left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7,8}]$

k	0	1	2	3
$\mathbf{comp}_k$	$\begin{pmatrix}\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\0&1&1\\1&0&0\\0&0&1\\0&0&0\\0&0&0\\0&0&1\\1&0&0\end{pmatrix}$

A.29.3 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(1,1)\rangle$

$$\begin{aligned}\phi_u \;=\; \partial\tau \text{ with } \tau = {}&(\Lambda_1^\vee + 2\Lambda_8^\vee)\,[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee + 2\Lambda_8^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee + 2\Lambda_8^\vee)\,[\mathbf{3}] + \\ &(\Lambda_3^\vee + \Lambda_4^\vee + 2\Lambda_8^\vee)\,[\mathbf{4}] + (\Lambda_4^\vee + \Lambda_5^\vee + 2\Lambda_8^\vee)\,[\mathbf{5}] + (\Lambda_5^\vee + \Lambda_6^\vee + 2\Lambda_8^\vee)\,[\mathbf{6}] + \\ &(\Lambda_7^\vee + \Lambda_8^\vee)\,[\mathbf{7}] + (\Lambda_7^\vee + \Lambda_8^\vee)\,[\mathbf{8}]\end{aligned}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{8}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_8^\vee[\mathbf{1}\supseteq\mathbf{1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2}\supseteq\mathbf{2}] + 2\Lambda_8^\vee[\mathbf{1,2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3}\supseteq\mathbf{3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4}\supseteq\mathbf{4}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{5}\supseteq\mathbf{5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{6}\supseteq\mathbf{6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{7}\supseteq\mathbf{7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{8}\supseteq\mathbf{8}] +$ $\left(-\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,8}] +$ $\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,8}] + \left(-\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{6,8}] + \left(2\Lambda_7^\vee - 2\Lambda_8^\vee\right)[\mathbf{7,8}]$
3	$\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus$ $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7,8}\supseteq\mathbf{8}] + \left(\Lambda_6^\vee - \Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{6,7,8}]$ $2\Lambda_8^\vee[\mathbf{7,8}\supseteq\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{7,8}\supseteq\mathbf{8}] + \left(\Lambda_5^\vee - 2\Lambda_6^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{6,7,8}]$ $\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,8}\supseteq\mathbf{1}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,8}\supseteq\mathbf{2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,8}\supseteq\mathbf{8}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,8}] +$ $\left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,8}\supseteq\mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,8}\supseteq\mathbf{8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,8}\supseteq\mathbf{4}] +$ $\left(-2\Lambda_6^\vee\right)[\mathbf{4,8}\supseteq\mathbf{8}] + \left(-\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,8}\supseteq\mathbf{5}] + \left(-2\Lambda_6^\vee\right)[\mathbf{5,8}\supseteq\mathbf{8}] +$ $\left(-\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{6,8}\supseteq\mathbf{6}] + \left(-\Lambda_5^\vee + \Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{6,8}\supseteq\mathbf{8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,6,8}] +$ $\left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,6,8}] + \left(2\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,6,8}] + \left(\Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,6,8}] +$ $\left(-2\Lambda_6^\vee + 2\Lambda_7^\vee + 2\Lambda_8^\vee\right)[\mathbf{6,7,8}]$ $2\Lambda_8^\vee[\mathbf{1,3}\supseteq\mathbf{1}] + 2\Lambda_8^\vee[\mathbf{1,3}\supseteq\mathbf{3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,4}\supseteq\mathbf{1}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{1,4}\supseteq\mathbf{4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4}\supseteq\mathbf{2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,4}\supseteq\mathbf{4}] + 2\Lambda_8^\vee[\mathbf{1,3,4}] +$ $2\Lambda_8^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5}\supseteq\mathbf{1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,5}\supseteq\mathbf{5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,5}\supseteq\mathbf{2}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{2,5}\supseteq\mathbf{5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5}\supseteq\mathbf{3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,5}\supseteq\mathbf{5}] + 2\Lambda_8^\vee[\mathbf{3,4,5}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{1,6}\supseteq\mathbf{1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,6}\supseteq\mathbf{6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6}\supseteq\mathbf{2}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,6}\supseteq\mathbf{6}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{3,6}\supseteq\mathbf{3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,6}\supseteq\mathbf{6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6}\supseteq\mathbf{4}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,6}\supseteq\mathbf{6}] +$ $2\Lambda_8^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7}\supseteq\mathbf{1}] + \left(-2\Lambda_8^\vee\right)[\mathbf{1,7}\supseteq\mathbf{7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,7}\supseteq\mathbf{2}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{2,7}\supseteq\mathbf{7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7}\supseteq\mathbf{3}] + \left(-2\Lambda_8^\vee\right)[\mathbf{3,7}\supseteq\mathbf{7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,7}\supseteq\mathbf{4}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{4,7}\supseteq\mathbf{7}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7}\supseteq\mathbf{5}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,7}\supseteq\mathbf{7}] + 2\Lambda_8^\vee[\mathbf{5,6,7}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{1,8}\supseteq\mathbf{1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,8}\supseteq\mathbf{8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{2,8}\supseteq\mathbf{2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,8}\supseteq\mathbf{8}] +$ $\left(-2\Lambda_8^\vee\right)[\mathbf{3,8}\supseteq\mathbf{3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,8}\supseteq\mathbf{8}] + \left(-\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{1,3,8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{4,8}\supseteq\mathbf{4}] +$ $\left(-\Lambda_6^\vee\right)[\mathbf{4,8}\supseteq\mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,4,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,4,8}] + \left(-2\Lambda_8^\vee\right)[\mathbf{5,8}\supseteq\mathbf{5}] +$ $\left(-\Lambda_6^\vee\right)[\mathbf{5,8}\supseteq\mathbf{8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,5,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,5,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,5,8}] +$ $\left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{1,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{2,6,8}] +$ $\left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{3,6,8}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - \Lambda_7^\vee - 3\Lambda_8^\vee\right)[\mathbf{4,6,8}] +$ $\left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{1,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{2,7,8}] +$ $\left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{3,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{4,7,8}] + \left(\Lambda_6^\vee - 2\Lambda_8^\vee\right)[\mathbf{5,7,8}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$2\Lambda_8^\vee[]$
1	2	$2\Lambda_8^\vee[\mathbf{1}] + 2\Lambda_8^\vee[\mathbf{2}] + 2\Lambda_8^\vee[\mathbf{3}] + 2\Lambda_8^\vee[\mathbf{4}] + 2\Lambda_8^\vee[\mathbf{5}] + 2\Lambda_8^\vee[\mathbf{6}] + 2\Lambda_8^\vee[\mathbf{7}] + 2\Lambda_8^\vee[\mathbf{8}]$ $\left(\Lambda_6^\vee + 2\Lambda_8^\vee\right)[\mathbf{7}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	**generating cocycles**
2	5	$2\Lambda_8^\vee[\mathbf{1\supseteq 1}]+2\Lambda_8^\vee[\mathbf{2\supseteq 2}]+2\Lambda_8^\vee[\mathbf{3\supseteq 3}]+2\Lambda_8^\vee[\mathbf{4\supseteq 4}]+2\Lambda_8^\vee[\mathbf{5\supseteq 5}]+2\Lambda_8^\vee[\mathbf{6\supseteq 6}]+2\Lambda_8^\vee[\mathbf{7\supseteq 7}]+2\Lambda_8^\vee[\mathbf{8\supseteq 8}]$
		$2\Lambda_8^\vee[\mathbf{1,3}]+2\Lambda_8^\vee[\mathbf{1,4}]+2\Lambda_8^\vee[\mathbf{2,4}]+2\Lambda_8^\vee[\mathbf{1,5}]+2\Lambda_8^\vee[\mathbf{2,5}]+2\Lambda_8^\vee[\mathbf{3,5}]+2\Lambda_8^\vee[\mathbf{1,6}]+2\Lambda_8^\vee[\mathbf{2,6}]+2\Lambda_8^\vee[\mathbf{3,6}]+2\Lambda_8^\vee[\mathbf{4,6}]+2\Lambda_8^\vee[\mathbf{1,7}]+2\Lambda_8^\vee[\mathbf{2,7}]+2\Lambda_8^\vee[\mathbf{3,7}]+2\Lambda_8^\vee[\mathbf{4,7}]+2\Lambda_8^\vee[\mathbf{5,7}]+2\Lambda_8^\vee[\mathbf{1,8}]+2\Lambda_8^\vee[\mathbf{2,8}]+2\Lambda_8^\vee[\mathbf{3,8}]+2\Lambda_8^\vee[\mathbf{4,8}]+2\Lambda_8^\vee[\mathbf{5,8}]$
		$\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{7\supseteq 7}]+\left(\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7}]$
		$\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{3,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{4,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{5,7}]+\Lambda_5^\vee[\mathbf{6,7}]$
		$\Lambda_6^\vee[\mathbf{7,8}]$
3	12	$2\Lambda_8^\vee[\mathbf{1\supseteq 1\supseteq 1}]+2\Lambda_8^\vee[\mathbf{2\supseteq 2\supseteq 2}]+2\Lambda_8^\vee[\mathbf{3\supseteq 3\supseteq 3}]+2\Lambda_8^\vee[\mathbf{4\supseteq 4\supseteq 4}]+2\Lambda_8^\vee[\mathbf{5\supseteq 5\supseteq 5}]+2\Lambda_8^\vee[\mathbf{6\supseteq 6\supseteq 6}]+2\Lambda_8^\vee[\mathbf{7\supseteq 7\supseteq 7}]+2\Lambda_8^\vee[\mathbf{8\supseteq 8\supseteq 8}]$
		$2\Lambda_8^\vee[\mathbf{1,3\supseteq 1}]+2\Lambda_8^\vee[\mathbf{1,3\supseteq 3}]+2\Lambda_8^\vee[\mathbf{1,4\supseteq 1}]+2\Lambda_8^\vee[\mathbf{1,4\supseteq 4}]+2\Lambda_8^\vee[\mathbf{2,4\supseteq 2}]+2\Lambda_8^\vee[\mathbf{2,4\supseteq 4}]+2\Lambda_8^\vee[\mathbf{1,5\supseteq 1}]+2\Lambda_8^\vee[\mathbf{1,5\supseteq 5}]+2\Lambda_8^\vee[\mathbf{2,5\supseteq 2}]+2\Lambda_8^\vee[\mathbf{2,5\supseteq 5}]+2\Lambda_8^\vee[\mathbf{3,5\supseteq 3}]+2\Lambda_8^\vee[\mathbf{3,5\supseteq 5}]+2\Lambda_8^\vee[\mathbf{1,6\supseteq 1}]+2\Lambda_8^\vee[\mathbf{1,6\supseteq 6}]+2\Lambda_8^\vee[\mathbf{2,6\supseteq 2}]+2\Lambda_8^\vee[\mathbf{2,6\supseteq 6}]+2\Lambda_8^\vee[\mathbf{3,6\supseteq 3}]+2\Lambda_8^\vee[\mathbf{3,6\supseteq 6}]+2\Lambda_8^\vee[\mathbf{4,6\supseteq 4}]+2\Lambda_8^\vee[\mathbf{4,6\supseteq 6}]+2\Lambda_8^\vee[\mathbf{1,7\supseteq 1}]+2\Lambda_8^\vee[\mathbf{1,7\supseteq 7}]+2\Lambda_8^\vee[\mathbf{2,7\supseteq 2}]+2\Lambda_8^\vee[\mathbf{2,7\supseteq 7}]+2\Lambda_8^\vee[\mathbf{3,7\supseteq 3}]+2\Lambda_8^\vee[\mathbf{3,7\supseteq 7}]+2\Lambda_8^\vee[\mathbf{4,7\supseteq 4}]+2\Lambda_8^\vee[\mathbf{4,7\supseteq 7}]+2\Lambda_8^\vee[\mathbf{5,7\supseteq 5}]+2\Lambda_8^\vee[\mathbf{5,7\supseteq 7}]+2\Lambda_8^\vee[\mathbf{1,8\supseteq 1}]+2\Lambda_8^\vee[\mathbf{1,8\supseteq 8}]+2\Lambda_8^\vee[\mathbf{2,8\supseteq 2}]+2\Lambda_8^\vee[\mathbf{2,8\supseteq 8}]+2\Lambda_8^\vee[\mathbf{3,8\supseteq 3}]+2\Lambda_8^\vee[\mathbf{3,8\supseteq 8}]+2\Lambda_8^\vee[\mathbf{4,8\supseteq 4}]+2\Lambda_8^\vee[\mathbf{4,8\supseteq 8}]+2\Lambda_8^\vee[\mathbf{5,8\supseteq 5}]+2\Lambda_8^\vee[\mathbf{5,8\supseteq 8}]$
		$2\Lambda_8^\vee[\mathbf{1,2,3}]+2\Lambda_8^\vee[\mathbf{2,3,4}]+2\Lambda_8^\vee[\mathbf{3,4,5}]+2\Lambda_8^\vee[\mathbf{4,5,6}]+2\Lambda_8^\vee[\mathbf{5,6,7}]+2\Lambda_8^\vee[\mathbf{5,6,8}]$
		$2\Lambda_8^\vee[\mathbf{1,3,5}]+2\Lambda_8^\vee[\mathbf{1,3,6}]+2\Lambda_8^\vee[\mathbf{1,4,6}]+2\Lambda_8^\vee[\mathbf{2,4,6}]+2\Lambda_8^\vee[\mathbf{1,3,7}]+2\Lambda_8^\vee[\mathbf{1,4,7}]+2\Lambda_8^\vee[\mathbf{2,4,7}]+2\Lambda_8^\vee[\mathbf{1,5,7}]+2\Lambda_8^\vee[\mathbf{2,5,7}]+2\Lambda_8^\vee[\mathbf{3,5,7}]+2\Lambda_8^\vee[\mathbf{1,3,8}]+2\Lambda_8^\vee[\mathbf{1,4,8}]+2\Lambda_8^\vee[\mathbf{2,4,8}]+2\Lambda_8^\vee[\mathbf{1,5,8}]+2\Lambda_8^\vee[\mathbf{2,5,8}]+2\Lambda_8^\vee[\mathbf{3,5,8}]$
		$\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{7\supseteq 7\supseteq 7}]+\left(\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7\supseteq 6}]+\left(\Lambda_7^\vee+3\Lambda_8^\vee\right)[\mathbf{6,7\supseteq 7}]+\left(\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 1}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,7\supseteq 2}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{3,7\supseteq 3}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{4,7\supseteq 4}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{5,7\supseteq 5}]+\left(\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7\supseteq 6}]+\Lambda_5^\vee[\mathbf{6,7\supseteq 7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{5,6,7}]+\Lambda_6^\vee[\mathbf{6,7,8}]$
		$\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,7\supseteq 7}]+\left(\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{1,2,7}]+\left(\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{1,6,7}]+2\Lambda_8^\vee[\mathbf{2,6,7}]+2\Lambda_8^\vee[\mathbf{3,6,7}]+2\Lambda_8^\vee[\mathbf{4,6,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,7,8}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{3,7,8}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{4,7,8}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{5,7,8}]$
		$\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,3,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,4,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,4,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{1,5,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{2,5,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{3,5,7}]+\Lambda_5^\vee[\mathbf{1,6,7}]+\Lambda_5^\vee[\mathbf{2,6,7}]+\Lambda_5^\vee[\mathbf{3,6,7}]+\Lambda_5^\vee[\mathbf{4,6,7}]$
		$\left(\Lambda_4^\vee+2\Lambda_8^\vee\right)[\mathbf{5,7\supseteq 7}]+\left(\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{4,5,7}]+\left(\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7\supseteq 6}]+\Lambda_5^\vee[\mathbf{6,7\supseteq 7}]+\left(\Lambda_5^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{5,6,7}]+\left(\Lambda_6^\vee+2\Lambda_8^\vee\right)[\mathbf{5,7,8}]+\Lambda_6^\vee[\mathbf{6,7,8}]$
		$\Lambda_6^\vee[\mathbf{7,8\supseteq 7}]+2\Lambda_8^\vee[\mathbf{7,8\supseteq 8}]+\left(\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6,7,8}]$
		$\Lambda_6^\vee[\mathbf{1,7,8}]+\Lambda_6^\vee[\mathbf{2,7,8}]+\Lambda_6^\vee[\mathbf{3,7,8}]+\Lambda_6^\vee[\mathbf{4,7,8}]+\Lambda_6^\vee[\mathbf{5,7,8}]$
		$\Lambda_5^\vee[\mathbf{6,7,8}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix}$

A.29.4 Cohomology of lattice $X^\vee$ corresponding to $\Omega = \langle(1,0)\rangle$

$$\boxed{\begin{array}{rcl}[\phi_u] & = & (1,1,0,0) \\ & & \textbf{does not lie in the image of comp}_2\end{array}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + 2\Lambda_7^\vee[\mathbf{1,2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_6^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{6,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{8} \supseteq \mathbf{8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{7,8}]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\Lambda_6^\vee[\mathbf{7,8} \supseteq \mathbf{7}] + 2\Lambda_7^\vee[\mathbf{7,8} \supseteq \mathbf{8}] + \left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6,7,8}]$ $\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,7} \supseteq \mathbf{1}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,7} \supseteq \mathbf{2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,2,7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,7} \supseteq \mathbf{3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,7} \supseteq \mathbf{4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5,7} \supseteq \mathbf{5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(\Lambda_4^\vee + 2\Lambda_8^\vee\right)[\mathbf{5,6,7}]$ $2\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{1}] + 2\Lambda_7^\vee[\mathbf{1,3} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4} \supseteq \mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,4} \supseteq \mathbf{4}] + 2\Lambda_7^\vee[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5} \supseteq \mathbf{5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6} \supseteq \mathbf{6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6} \supseteq \mathbf{6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6} \supseteq \mathbf{6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6} \supseteq \mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6} \supseteq \mathbf{6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,7} \supseteq \mathbf{1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,7} \supseteq \mathbf{7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,7} \supseteq \mathbf{2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7} \supseteq \mathbf{7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,7} \supseteq \mathbf{3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,3,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,7} \supseteq \mathbf{4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7} \supseteq \mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,4,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,7} \supseteq \mathbf{5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7} \supseteq \mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{4,6,7}] + \left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{5,6,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,8} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,8} \supseteq \mathbf{8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,8} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,8} \supseteq \mathbf{8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,8} \supseteq \mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,8} \supseteq \mathbf{8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,8} \supseteq \mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,8} \supseteq \mathbf{8}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,8} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,8} \supseteq \mathbf{8}] + 2\Lambda_7^\vee[\mathbf{5,6,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,7,8}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,7,8}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	1	$2\Lambda_7^\vee[]$
1	2	$\Lambda_2^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{8}]$ $2\Lambda_7^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}] + 2\Lambda_7^\vee[\mathbf{8}]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
2	4	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2}]+2\Lambda_7^\vee[\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_7^\vee[\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{7}\supseteq\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}]+\Lambda_4^\vee[\mathbf{5},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{8}\supseteq\mathbf{8}]$
		$2\Lambda_7^\vee[\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_7^\vee[\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{7}\supseteq\mathbf{7}]+2\Lambda_7^\vee[\mathbf{8}\supseteq\mathbf{8}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{2},\mathbf{3}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{4}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{4}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{3},\mathbf{4}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{5}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{5}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{5}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{7}]+\Lambda_6^\vee[\mathbf{3},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{8}]$
		$\left(\Lambda_2^\vee+2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{8}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{6},\mathbf{8}]+\Lambda_6^\vee[\mathbf{7},\mathbf{8}]$
3	9	$\Lambda_2^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_7^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+2\Lambda_7^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]+2\Lambda_7^\vee[\mathbf{6}\supseteq\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{5},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{7}\supseteq\mathbf{7}\supseteq\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}\supseteq\mathbf{2}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}\supseteq\mathbf{3}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}\supseteq\mathbf{4}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{5},\mathbf{7}\supseteq\mathbf{5}]+\Lambda_4^\vee[\mathbf{5},\mathbf{7}\supseteq\mathbf{7}]+\left(\Lambda_2^\vee+\Lambda_4^\vee+2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{5},\mathbf{7}]+\left(\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{4},\mathbf{5},\mathbf{7}]+\left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{5},\mathbf{6},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{8}\supseteq\mathbf{8}\supseteq\mathbf{8}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{6},\mathbf{8}]$
		$2\Lambda_7^\vee[\mathbf{1}\supseteq\mathbf{1}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{3}\supseteq\mathbf{3}\supseteq\mathbf{3}]+2\Lambda_7^\vee[\mathbf{4}\supseteq\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{5}\supseteq\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{6}\supseteq\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{7}\supseteq\mathbf{7}\supseteq\mathbf{7}]+2\Lambda_7^\vee[\mathbf{8}\supseteq\mathbf{8}\supseteq\mathbf{8}]$
		$\left(\Lambda_1^\vee+\Lambda_3^\vee+4\Lambda_7^\vee\right)[\mathbf{2}\supseteq\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+\left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\left(\Lambda_3^\vee+\Lambda_4^\vee+3\Lambda_7^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\Lambda_2^\vee[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{3},\mathbf{4}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{5},\mathbf{6}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{6},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{6},\mathbf{8}]$
		$\Lambda_8^\vee[\mathbf{1},\mathbf{2}\supseteq\mathbf{1}]+\Lambda_8^\vee[\mathbf{1},\mathbf{2}\supseteq\mathbf{2}]+\Lambda_6^\vee[\mathbf{2},\mathbf{8}\supseteq\mathbf{2}]+\Lambda_6^\vee[\mathbf{3},\mathbf{8}\supseteq\mathbf{3}]+\Lambda_6^\vee[\mathbf{4},\mathbf{8}\supseteq\mathbf{4}]+\Lambda_6^\vee[\mathbf{5},\mathbf{8}\supseteq\mathbf{5}]+\Lambda_8^\vee[\mathbf{6},\mathbf{8}\supseteq\mathbf{6}]+\left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{6},\mathbf{8}\supseteq\mathbf{8}]+\Lambda_6^\vee[\mathbf{5},\mathbf{6},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{7},\mathbf{8}\supseteq\mathbf{8}]+\left(\Lambda_5^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{6},\mathbf{7},\mathbf{8}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_3^\vee+2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{4}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{4}\supseteq\mathbf{4}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{3}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{5}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{6}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{6}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{6}\supseteq\mathbf{3}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{6}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{6}\supseteq\mathbf{6}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{7}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{7}\supseteq\mathbf{7}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{7}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{7}\supseteq\mathbf{7}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{7}\supseteq\mathbf{3}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{7}\supseteq\mathbf{7}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{7}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{7}\supseteq\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{7}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{7}\supseteq\mathbf{7}]+\Lambda_4^\vee[\mathbf{1},\mathbf{5},\mathbf{7}]+\Lambda_4^\vee[\mathbf{2},\mathbf{5},\mathbf{7}]+\left(\Lambda_2^\vee+2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{5},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{8}\supseteq\mathbf{1}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{8}\supseteq\mathbf{8}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{8}\supseteq\mathbf{2}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{8}\supseteq\mathbf{8}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{8}\supseteq\mathbf{3}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{8}\supseteq\mathbf{8}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{8}\supseteq\mathbf{4}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{8}\supseteq\mathbf{8}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{8}\supseteq\mathbf{5}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{8}\supseteq\mathbf{8}]$
		$\Lambda_4^\vee[\mathbf{1},\mathbf{3}\supseteq\mathbf{1}]+\left(\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{2}]+\left(\Lambda_1^\vee+3\Lambda_7^\vee\right)[\mathbf{2},\mathbf{3}\supseteq\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_3^\vee+2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{3}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{3},\mathbf{4}]+\left(\Lambda_3^\vee+\Lambda_5^\vee+2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{4}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{4},\mathbf{5}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{5},\mathbf{6}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}\supseteq\mathbf{3}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}\supseteq\mathbf{4}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{4},\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{4},\mathbf{7}]+\left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{5},\mathbf{7}\supseteq\mathbf{5}]+\Lambda_4^\vee[\mathbf{5},\mathbf{7}\supseteq\mathbf{7}]+\Lambda_4^\vee[\mathbf{1},\mathbf{5},\mathbf{7}]+\Lambda_4^\vee[\mathbf{2},\mathbf{5},\mathbf{7}]+\Lambda_4^\vee[\mathbf{3},\mathbf{5},\mathbf{7}]+\left(\Lambda_3^\vee+\Lambda_7^\vee\right)[\mathbf{4},\mathbf{5},\mathbf{7}]+\left(\Lambda_5^\vee+\Lambda_6^\vee+\Lambda_7^\vee+\Lambda_8^\vee\right)[\mathbf{5},\mathbf{6},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{6},\mathbf{8}]$
		$\Lambda_2^\vee[\mathbf{1},\mathbf{3},\mathbf{5}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{2},\mathbf{3},\mathbf{5}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{3},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{4},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{4},\mathbf{6}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{3},\mathbf{5},\mathbf{6}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{3},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{4},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{4},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{5},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{5},\mathbf{7}]+\Lambda_6^\vee[\mathbf{3},\mathbf{5},\mathbf{7}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{3},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{4},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{4},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{1},\mathbf{5},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{5},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{5},\mathbf{8}]$
		$\left(\Lambda_2^\vee+2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{8}\supseteq\mathbf{1}]+\left(\Lambda_1^\vee+\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{2},\mathbf{8}]+\left(\Lambda_1^\vee+\Lambda_7^\vee\right)[\mathbf{1},\mathbf{6},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{6},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{6},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{6},\mathbf{8}]+\Lambda_6^\vee[\mathbf{1},\mathbf{7},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{2},\mathbf{7},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{3},\mathbf{7},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{4},\mathbf{7},\mathbf{8}]+2\Lambda_7^\vee[\mathbf{5},\mathbf{7},\mathbf{8}]$
		$2\Lambda_7^\vee[\mathbf{7},\mathbf{8}\supseteq\mathbf{7}]+2\Lambda_7^\vee[\mathbf{7},\mathbf{8}\supseteq\mathbf{8}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\0&1&1\\1&0&0\\0&0&1\\0&0&0\\0&0&0\\0&0&1\\0&0&0\end{pmatrix}$

A.29.5 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{aligned}\phi_u \quad = \;& \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + (\Lambda_1^\vee + \Lambda_2^\vee)\,[\mathbf{2}] + (\Lambda_2^\vee + \Lambda_3^\vee)\,[\mathbf{3}] + (\Lambda_3^\vee + \Lambda_4^\vee)\,[\mathbf{4}] + \\ & (\Lambda_4^\vee + \Lambda_5^\vee)\,[\mathbf{5}] + (\Lambda_5^\vee + \Lambda_6^\vee)\,[\mathbf{6}] + (\Lambda_7^\vee + \Lambda_8^\vee)\,[\mathbf{7}] + (\Lambda_7^\vee + \Lambda_8^\vee)\,[\mathbf{8}]\end{aligned}}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}, \mathbf{8} \supseteq \mathbf{8}] + \left(\Lambda_6^\vee - \Lambda_7^\vee - \Lambda_8^\vee\right)[\mathbf{6}, \mathbf{7}, \mathbf{8}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	0	
1	0	
2	1	$\Lambda_8^\vee[\mathbf{6}, \mathbf{7}]$
3	4	$\Lambda_7^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{1}] + \Lambda_7^\vee[\mathbf{1}, \mathbf{2} \supseteq \mathbf{2}] + \Lambda_6^\vee[\mathbf{2}, \mathbf{7} \supseteq \mathbf{2}] + \Lambda_6^\vee[\mathbf{3}, \mathbf{7} \supseteq \mathbf{3}] + \Lambda_6^\vee[\mathbf{4}, \mathbf{7} \supseteq \mathbf{4}] + \Lambda_6^\vee[\mathbf{5}, \mathbf{7} \supseteq \mathbf{5}] +$ $\Lambda_7^\vee[\mathbf{6}, \mathbf{7} \supseteq \mathbf{6}] + \left(\Lambda_5^\vee + \Lambda_8^\vee\right)[\mathbf{6}, \mathbf{7} \supseteq \mathbf{7}] + \Lambda_6^\vee[\mathbf{5}, \mathbf{6}, \mathbf{7}] + \left(\Lambda_7^\vee + \Lambda_8^\vee\right)[\mathbf{6}, \mathbf{7}, \mathbf{8}]$
		$\Lambda_7^\vee[\mathbf{1}, \mathbf{3}, \mathbf{5}] + \Lambda_7^\vee[\mathbf{1}, \mathbf{3}, \mathbf{6}] + \Lambda_7^\vee[\mathbf{1}, \mathbf{4}, \mathbf{6}] + \Lambda_7^\vee[\mathbf{2}, \mathbf{4}, \mathbf{6}] + \left(\Lambda_5^\vee + \Lambda_8^\vee\right)[\mathbf{1}, \mathbf{3}, \mathbf{7}] + \left(\Lambda_5^\vee + \Lambda_8^\vee\right)[\mathbf{1}, \mathbf{4}, \mathbf{7}] +$ $\left(\Lambda_5^\vee + \Lambda_8^\vee\right)[\mathbf{2}, \mathbf{4}, \mathbf{7}] + \left(\Lambda_3^\vee + \Lambda_8^\vee\right)[\mathbf{1}, \mathbf{5}, \mathbf{7}] + \left(\Lambda_3^\vee + \Lambda_8^\vee\right)[\mathbf{2}, \mathbf{5}, \mathbf{7}] + \left(\Lambda_1^\vee + \Lambda_8^\vee\right)[\mathbf{3}, \mathbf{5}, \mathbf{7}] + \Lambda_7^\vee[\mathbf{1}, \mathbf{3}, \mathbf{8}] +$ $\Lambda_7^\vee[\mathbf{1}, \mathbf{4}, \mathbf{8}] + \Lambda_7^\vee[\mathbf{2}, \mathbf{4}, \mathbf{8}] + \Lambda_7^\vee[\mathbf{1}, \mathbf{5}, \mathbf{8}] + \Lambda_7^\vee[\mathbf{2}, \mathbf{5}, \mathbf{8}] + \Lambda_7^\vee[\mathbf{3}, \mathbf{5}, \mathbf{8}]$
		$\Lambda_8^\vee[\mathbf{1}, \mathbf{3}, \mathbf{5}] + \Lambda_8^\vee[\mathbf{1}, \mathbf{3}, \mathbf{6}] + \Lambda_8^\vee[\mathbf{1}, \mathbf{4}, \mathbf{6}] + \Lambda_8^\vee[\mathbf{2}, \mathbf{4}, \mathbf{6}] + \Lambda_8^\vee[\mathbf{1}, \mathbf{3}, \mathbf{7}] + \Lambda_8^\vee[\mathbf{1}, \mathbf{4}, \mathbf{7}] + \Lambda_8^\vee[\mathbf{2}, \mathbf{4}, \mathbf{7}] + \Lambda_8^\vee[\mathbf{1}, \mathbf{5}, \mathbf{7}] +$ $\Lambda_8^\vee[\mathbf{2}, \mathbf{5}, \mathbf{7}] + \Lambda_8^\vee[\mathbf{3}, \mathbf{5}, \mathbf{7}] + \left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{1}, \mathbf{3}, \mathbf{8}] + \left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{1}, \mathbf{4}, \mathbf{8}] + \left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{2}, \mathbf{4}, \mathbf{8}] +$ $\left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{1}, \mathbf{5}, \mathbf{8}] + \left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{2}, \mathbf{5}, \mathbf{8}] + \left(\Lambda_1^\vee + \Lambda_7^\vee\right)[\mathbf{3}, \mathbf{5}, \mathbf{8}]$
		$\Lambda_8^\vee[\mathbf{1}, \mathbf{6}, \mathbf{7}] + \Lambda_8^\vee[\mathbf{2}, \mathbf{6}, \mathbf{7}] + \Lambda_8^\vee[\mathbf{3}, \mathbf{6}, \mathbf{7}] + \Lambda_8^\vee[\mathbf{4}, \mathbf{6}, \mathbf{7}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

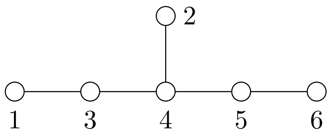
A.29.6 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[1 \supseteq 1] + (-1)\,[2 \supseteq 2] + [1,2] + (-1)\,[3 \supseteq 3] + (-1)\,[4 \supseteq 4] + (-1)\,[5 \supseteq 5] + (-1)\,[6 \supseteq 6] + (-1)\,[7 \supseteq 7] + (-1)\,[8 \supseteq 8]$
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[7,8 \supseteq 7] + [7,8 \supseteq 8] + [6,7,8]$ $[1,3 \supseteq 1] + [1,3 \supseteq 3] + (-1)\,[1,2,3] + (-1)\,[1,4 \supseteq 1] + (-1)\,[1,4 \supseteq 4] + (-1)\,[2,4 \supseteq 2] + (-1)\,[2,4 \supseteq 4] + [1,3,4] + [2,3,4] + (-1)\,[1,5 \supseteq 1] + (-1)\,[1,5 \supseteq 5] + (-1)\,[2,5 \supseteq 2] + (-1)\,[2,5 \supseteq 5] + (-1)\,[3,5 \supseteq 3] + (-1)\,[3,5 \supseteq 5] + [3,4,5] + (-1)\,[1,6 \supseteq 1] + (-1)\,[1,6 \supseteq 6] + (-1)\,[2,6 \supseteq 2] + (-1)\,[2,6 \supseteq 6] + (-1)\,[3,6 \supseteq 3] + (-1)\,[3,6 \supseteq 6] + (-1)\,[4,6 \supseteq 4] + (-1)\,[4,6 \supseteq 6] + [4,5,6] + (-1)\,[1,7 \supseteq 1] + (-1)\,[1,7 \supseteq 7] + (-1)\,[2,7 \supseteq 2] + (-1)\,[2,7 \supseteq 7] + (-1)\,[3,7 \supseteq 3] + (-1)\,[3,7 \supseteq 7] + (-1)\,[4,7 \supseteq 4] + (-1)\,[4,7 \supseteq 7] + (-1)\,[5,7 \supseteq 5] + (-1)\,[5,7 \supseteq 7] + [5,6,7] + (-1)\,[1,8 \supseteq 1] + (-1)\,[1,8 \supseteq 8] + (-1)\,[2,8 \supseteq 2] + (-1)\,[2,8 \supseteq 8] + (-1)\,[3,8 \supseteq 3] + (-1)\,[3,8 \supseteq 8] + (-1)\,[4,8 \supseteq 4] + (-1)\,[4,8 \supseteq 8] + (-1)\,[5,8 \supseteq 5] + (-1)\,[5,8 \supseteq 8] + [5,6,8]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	$[]$
1	1	$[1] + [2] + [3] + [4] + [5] + [6] + [7] + [8]$
2	3	$[1 \supseteq 1] + [2 \supseteq 2] + [3 \supseteq 3] + [4 \supseteq 4] + [5 \supseteq 5] + [6 \supseteq 6] + [7 \supseteq 7] + [8 \supseteq 8]$ $[1,3] + [1,4] + [2,4] + [1,5] + [2,5] + [3,5] + [1,6] + [2,6] + [3,6] + [4,6] + [1,7] + [2,7] + [3,7] + [4,7] + [5,7] + [1,8] + [2,8] + [3,8] + [4,8] + [5,8]$ $[7,8]$
3	7	$[1 \supseteq 1 \supseteq 1] + [2 \supseteq 2 \supseteq 2] + [3 \supseteq 3 \supseteq 3] + [4 \supseteq 4 \supseteq 4] + [5 \supseteq 5 \supseteq 5] + [6 \supseteq 6 \supseteq 6] + [7 \supseteq 7 \supseteq 7] + [8 \supseteq 8 \supseteq 8]$ $[1,3 \supseteq 1] + [1,3 \supseteq 3] + [1,4 \supseteq 1] + [1,4 \supseteq 4] + [2,4 \supseteq 2] + [2,4 \supseteq 4] + [1,5 \supseteq 1] + [1,5 \supseteq 5] + [2,5 \supseteq 2] + [2,5 \supseteq 5] + [3,5 \supseteq 3] + [3,5 \supseteq 5] + [1,6 \supseteq 1] + [1,6 \supseteq 6] + [2,6 \supseteq 2] + [2,6 \supseteq 6] + [3,6 \supseteq 3] + [3,6 \supseteq 6] + [4,6 \supseteq 4] + [4,6 \supseteq 6] + [1,7 \supseteq 1] + [1,7 \supseteq 7] + [2,7 \supseteq 2] + [2,7 \supseteq 7] + [3,7 \supseteq 3] + [3,7 \supseteq 7] + [4,7 \supseteq 4] + [4,7 \supseteq 7] + [5,7 \supseteq 5] + [5,7 \supseteq 7] + [1,8 \supseteq 1] + [1,8 \supseteq 8] + [2,8 \supseteq 2] + [2,8 \supseteq 8] + [3,8 \supseteq 3] + [3,8 \supseteq 8] + [4,8 \supseteq 4] + [4,8 \supseteq 8] + [5,8 \supseteq 5] + [5,8 \supseteq 8]$ $[1,2,3] + [2,3,4] + [3,4,5] + [4,5,6] + [5,6,7] + [5,6,8]$ $[1,3,5] + [1,3,6] + [1,4,6] + [2,4,6] + [1,3,7] + [1,4,7] + [2,4,7] + [1,5,7] + [2,5,7] + [3,5,7] + [1,3,8] + [1,4,8] + [2,4,8] + [1,5,8] + [2,5,8] + [3,5,8]$ $[7,8 \supseteq 7] + [7,8 \supseteq 8]$ $[1,7,8] + [2,7,8] + [3,7,8] + [4,7,8] + [5,7,8]$ $[6,7,8]$

A.30 Root system E_6

Dynkin diagram



Fundamental group $P^\vee/Q^\vee \simeq \mathbb{Z}/3\mathbb{Z}$ generated by $\Lambda_6^\vee \in P^\vee \mod Q^\vee$

A.30.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] = (1)$
does not lie in the image of comp$_2$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	$\mathbb{Z}/3\mathbb{Z}$	$\left(\Lambda_5^\vee - 2\Lambda_6^\vee\right)[\mathbf{6}]$
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$3\Lambda_6^\vee[\mathbf{2,5}\supseteq\mathbf{2}] + 3\Lambda_6^\vee[\mathbf{2,5}\supseteq\mathbf{5}] + \left(-3\Lambda_6^\vee\right)[\mathbf{2,4,5}] + \left(-\Lambda_2^\vee\right)[\mathbf{5,6}\supseteq\mathbf{5}] + \left(-\Lambda_2^\vee\right)[\mathbf{5,6}\supseteq\mathbf{6}] + \left(-\Lambda_2^\vee + 2\Lambda_4^\vee - 3\Lambda_5^\vee + 3\Lambda_6^\vee\right)[\mathbf{2,5,6}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	1	$\left(\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1,3}] + \left(\Lambda_4^\vee + 3\Lambda_6^\vee\right)[\mathbf{3,5}] + \left(\Lambda_3^\vee + 2\Lambda_6^\vee\right)[\mathbf{4,5}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{2,6}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{3,6}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{4,6}] + \Lambda_4^\vee[\mathbf{5,6}]$
3	2	$\left(\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1,3}\supseteq\mathbf{1}] + \left(\Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{1,3}\supseteq\mathbf{3}] + \left(\Lambda_4^\vee + 3\Lambda_6^\vee\right)[\mathbf{3,5}\supseteq\mathbf{3}] + \left(\Lambda_3^\vee + 2\Lambda_6^\vee\right)[\mathbf{4,5}\supseteq\mathbf{4}] + \left(\Lambda_3^\vee + 2\Lambda_6^\vee\right)[\mathbf{4,5}\supseteq\mathbf{5}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{2,6}\supseteq\mathbf{2}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{3,6}\supseteq\mathbf{3}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{4,6}\supseteq\mathbf{4}] + 3\Lambda_6^\vee[\mathbf{5,6}\supseteq\mathbf{5}] + \Lambda_4^\vee[\mathbf{5,6}\supseteq\mathbf{6}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{4,5,6}] $\Lambda_2^\vee[\mathbf{1,3,4}] + \left(\Lambda_5^\vee + 4\Lambda_6^\vee\right)[\mathbf{2,3,4}] + \left(\Lambda_3^\vee + 2\Lambda_6^\vee\right)[\mathbf{2,4,5}] + \Lambda_2^\vee[\mathbf{3,4,5}] + \left(\Lambda_3^\vee + 2\Lambda_6^\vee\right)[\mathbf{4,5,6}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$

A.30.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$$\boxed{\begin{array}{rcl}[\phi_u] & = & (1) \\ & & \textbf{does not lie in the image of } \mathbf{comp}_2\end{array}}$$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_6^\vee[\mathbf{2,5\supseteq 2}] + \Lambda_6^\vee[\mathbf{2,5\supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,4,5}] + \left(-\Lambda_2^\vee\right)[\mathbf{5,6\supseteq 5}] + \left(-\Lambda_2^\vee\right)[\mathbf{5,6\supseteq 6}] + \left(-\Lambda_2^\vee + \Lambda_4^\vee - \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2,5,6}]$

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	0	
1	0	
2	1	$\Lambda_5^\vee[\mathbf{1,3}] + \left(\Lambda_4^\vee + \Lambda_6^\vee\right)[\mathbf{3,5}] + \Lambda_3^\vee[\mathbf{4,5}]$
3	2	$\Lambda_5^\vee[\mathbf{1,3\supseteq 1}] + \Lambda_5^\vee[\mathbf{1,3\supseteq 3}] + \left(\Lambda_4^\vee + \Lambda_6^\vee\right)[\mathbf{3,5\supseteq 3}] + \Lambda_3^\vee[\mathbf{4,5\supseteq 4}] + \Lambda_3^\vee[\mathbf{4,5\supseteq 5}]$
		$\Lambda_2^\vee[\mathbf{1,3,4}] + \Lambda_5^\vee[\mathbf{2,3,4}] + \Lambda_3^\vee[\mathbf{2,4,5}] + \Lambda_2^\vee[\mathbf{3,4,5}] + \Lambda_3^\vee[\mathbf{4,5,6}]$

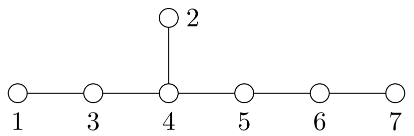
k	0	1	2	3
$\mathbf{comp}_k$	()	()	()	$\begin{pmatrix}1\\0\end{pmatrix}$

A.30.3 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1\supseteq 1}] + (-1)[\mathbf{2\supseteq 2}] + (-1)[\mathbf{3\supseteq 3}] + [\mathbf{1,3}] + (-1)[\mathbf{4\supseteq 4}] + (-1)[\mathbf{5\supseteq 5}] + (-1)[\mathbf{6\supseteq 6}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1,2\supseteq 1}] + [\mathbf{1,2\supseteq 2}] + (-1)[\mathbf{2,3\supseteq 2}] + (-1)[\mathbf{2,3\supseteq 3}] + (-1)[\mathbf{1,4\supseteq 1}] + (-1)[\mathbf{1,4\supseteq 4}] + [\mathbf{1,2,4}] + [\mathbf{1,3,4}] + (-1)[\mathbf{2,3,4}] + (-1)[\mathbf{1,5\supseteq 1}] + (-1)[\mathbf{1,5\supseteq 5}] + (-1)[\mathbf{2,5\supseteq 2}] + (-1)[\mathbf{2,5\supseteq 5}] + (-1)[\mathbf{3,5\supseteq 3}] + (-1)[\mathbf{3,5\supseteq 5}] + [\mathbf{2,4,5}] + [\mathbf{3,4,5}] + (-1)[\mathbf{1,6\supseteq 1}] + (-1)[\mathbf{1,6\supseteq 6}] + (-1)[\mathbf{2,6\supseteq 2}] + (-1)[\mathbf{2,6\supseteq 6}] + (-1)[\mathbf{3,6\supseteq 3}] + (-1)[\mathbf{3,6\supseteq 6}] + (-1)[\mathbf{4,6\supseteq 4}] + (-1)[\mathbf{4,6\supseteq 6}] + [\mathbf{4,5,6}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
0	1	$[]$
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}]$
2	2	$[\mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4}] + [\mathbf{5} \supseteq \mathbf{5}] + [\mathbf{6} \supseteq \mathbf{6}]$
		$[\mathbf{1},\mathbf{2}] + [\mathbf{2},\mathbf{3}] + [\mathbf{1},\mathbf{4}] + [\mathbf{1},\mathbf{5}] + [\mathbf{2},\mathbf{5}] + [\mathbf{3},\mathbf{5}] + [\mathbf{1},\mathbf{6}] + [\mathbf{2},\mathbf{6}] + [\mathbf{3},\mathbf{6}] + [\mathbf{4},\mathbf{6}]$
3	4	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}] + [\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}] + [\mathbf{3} \supseteq \mathbf{3} \supseteq \mathbf{3}] + [\mathbf{4} \supseteq \mathbf{4} \supseteq \mathbf{4}] + [\mathbf{5} \supseteq \mathbf{5} \supseteq \mathbf{5}] + [\mathbf{6} \supseteq \mathbf{6} \supseteq \mathbf{6}]$
		$[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{2} \supseteq \mathbf{2}] + [\mathbf{2},\mathbf{3} \supseteq \mathbf{2}] + [\mathbf{2},\mathbf{3} \supseteq \mathbf{3}] + [\mathbf{1},\mathbf{4} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{4} \supseteq \mathbf{4}] + [\mathbf{1},\mathbf{5} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{5} \supseteq \mathbf{5}] + [\mathbf{2},\mathbf{5} \supseteq \mathbf{2}] + [\mathbf{2},\mathbf{5} \supseteq \mathbf{5}] + [\mathbf{3},\mathbf{5} \supseteq \mathbf{3}] + [\mathbf{3},\mathbf{5} \supseteq \mathbf{5}] + [\mathbf{1},\mathbf{6} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{6} \supseteq \mathbf{6}] + [\mathbf{2},\mathbf{6} \supseteq \mathbf{2}] + [\mathbf{2},\mathbf{6} \supseteq \mathbf{6}] + [\mathbf{3},\mathbf{6} \supseteq \mathbf{3}] + [\mathbf{3},\mathbf{6} \supseteq \mathbf{6}] + [\mathbf{4},\mathbf{6} \supseteq \mathbf{4}] + [\mathbf{4},\mathbf{6} \supseteq \mathbf{6}]$
		$[\mathbf{1},\mathbf{3},\mathbf{4}] + [\mathbf{2},\mathbf{3},\mathbf{4}] + [\mathbf{2},\mathbf{4},\mathbf{5}] + [\mathbf{3},\mathbf{4},\mathbf{5}] + [\mathbf{4},\mathbf{5},\mathbf{6}]$
		$[\mathbf{1},\mathbf{2},\mathbf{5}] + [\mathbf{2},\mathbf{3},\mathbf{5}] + [\mathbf{1},\mathbf{2},\mathbf{6}] + [\mathbf{2},\mathbf{3},\mathbf{6}] + [\mathbf{1},\mathbf{4},\mathbf{6}]$

A.31 Root system E_7

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq \mathbb{Z}/2\mathbb{Z}$ generated by $\Lambda_7^\vee \in P^\vee \mod Q^\vee$

A.31.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] \;=\; (0,0,0,1)$$
does not lie in the image of $\mathbf{comp}_2$

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0},\mathbf{X}^\vee)$	**generating cocycles**
0	0	
1	$\mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{7}]$
2	$\mathbb{Z}/2\mathbb{Z}$	$2\Lambda_7^\vee[\mathbf{1} \supseteq \mathbf{1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2} \supseteq \mathbf{2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3} \supseteq \mathbf{3}] + 2\Lambda_7^\vee[\mathbf{1},\mathbf{3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4} \supseteq \mathbf{4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5} \supseteq \mathbf{5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{6} \supseteq \mathbf{6}] + \left(-\Lambda_6^\vee\right)[\mathbf{7} \supseteq \mathbf{7}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4},\mathbf{7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5},\mathbf{7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{6},\mathbf{7}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-2\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-2\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,3,7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-2\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(-\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - 3\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \Lambda_4^\vee[\mathbf{5,6,7}]$ $\Lambda_6^\vee[\mathbf{2,5 \supseteq 2}] + \Lambda_6^\vee[\mathbf{2,5 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,4,5}] + \left(-\Lambda_2^\vee - \Lambda_7^\vee\right)[\mathbf{5,6 \supseteq 5}] + \left(-\Lambda_2^\vee - \Lambda_7^\vee\right)[\mathbf{5,6 \supseteq 6}] + \left(-\Lambda_2^\vee + \Lambda_4^\vee - \Lambda_5^\vee + \Lambda_6^\vee\right)[\mathbf{2,5,6}] + \left(-\Lambda_4^\vee + 2\Lambda_5^\vee - \Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_4^\vee - 2\Lambda_5^\vee + 2\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{5,6,7}]$ $2\Lambda_7^\vee[\mathbf{1,2 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,2 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,3 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,3 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,2,4}] + 2\Lambda_7^\vee[\mathbf{1,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,3,4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,5 \supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,5 \supseteq 5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{2,4,5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 1}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 2}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 3}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,6 \supseteq 6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 4}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + \left(-2\Lambda_7^\vee\right)[\mathbf{1,7 \supseteq 1}] + \left(-\Lambda_6^\vee\right)[\mathbf{1,7 \supseteq 7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{2,7 \supseteq 2}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,7 \supseteq 7}] + \left(-\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{1,2,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{3,7 \supseteq 3}] + \left(-\Lambda_6^\vee\right)[\mathbf{3,7 \supseteq 7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,3,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{4,7 \supseteq 4}] + \left(-\Lambda_6^\vee\right)[\mathbf{4,7 \supseteq 7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \left(-2\Lambda_7^\vee\right)[\mathbf{5,7 \supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{5,7 \supseteq 7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{1,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{2,5,7}] + \left(\Lambda_6^\vee - 2\Lambda_7^\vee\right)[\mathbf{3,5,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{1,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{2,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{3,6,7}] + \left(-\Lambda_5^\vee + 3\Lambda_6^\vee - 3\Lambda_7^\vee\right)[\mathbf{4,6,7}] + \left(\Lambda_5^\vee - \Lambda_6^\vee + \Lambda_7^\vee\right)[\mathbf{5,6,7}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	1	$2\Lambda_7^\vee[]$
1	2	$2\Lambda_7^\vee[\mathbf{1}] + 2\Lambda_7^\vee[\mathbf{2}] + 2\Lambda_7^\vee[\mathbf{3}] + 2\Lambda_7^\vee[\mathbf{4}] + 2\Lambda_7^\vee[\mathbf{5}] + 2\Lambda_7^\vee[\mathbf{6}] + 2\Lambda_7^\vee[\mathbf{7}]$ $\left(\Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{2}]$
2	4	$2\Lambda_7^\vee[\mathbf{1 \supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2 \supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3 \supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4 \supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5 \supseteq 5}] + 2\Lambda_7^\vee[\mathbf{6 \supseteq 6}] + 2\Lambda_7^\vee[\mathbf{7 \supseteq 7}]$ $\Lambda_4^\vee[\mathbf{1,2}] + \Lambda_4^\vee[\mathbf{2,3}] + 2\Lambda_7^\vee[\mathbf{1,4}] + \left(\Lambda_2^\vee + \Lambda_7^\vee\right)[\mathbf{3,4}] + 2\Lambda_7^\vee[\mathbf{1,5}] + 2\Lambda_7^\vee[\mathbf{2,5}] + 2\Lambda_7^\vee[\mathbf{3,5}] + 2\Lambda_7^\vee[\mathbf{1,6}] + 2\Lambda_7^\vee[\mathbf{2,6}] + 2\Lambda_7^\vee[\mathbf{3,6}] + 2\Lambda_7^\vee[\mathbf{4,6}] + 2\Lambda_7^\vee[\mathbf{1,7}] + \Lambda_6^\vee[\mathbf{2,7}] + 2\Lambda_7^\vee[\mathbf{3,7}] + \Lambda_6^\vee[\mathbf{4,7}] + \left(\Lambda_4^\vee + 2\Lambda_7^\vee\right)[\mathbf{5,7}]$ $2\Lambda_7^\vee[\mathbf{1,2}] + 2\Lambda_7^\vee[\mathbf{2,3}] + 2\Lambda_7^\vee[\mathbf{1,4}] + 2\Lambda_7^\vee[\mathbf{1,5}] + 2\Lambda_7^\vee[\mathbf{2,5}] + 2\Lambda_7^\vee[\mathbf{3,5}] + 2\Lambda_7^\vee[\mathbf{1,6}] + 2\Lambda_7^\vee[\mathbf{2,6}] + 2\Lambda_7^\vee[\mathbf{3,6}] + 2\Lambda_7^\vee[\mathbf{4,6}] + 2\Lambda_7^\vee[\mathbf{1,7}] + 2\Lambda_7^\vee[\mathbf{2,7}] + 2\Lambda_7^\vee[\mathbf{3,7}] + 2\Lambda_7^\vee[\mathbf{4,7}] + 2\Lambda_7^\vee[\mathbf{5,7}]$ $\left(\Lambda_5^\vee + \Lambda_7^\vee\right)[\mathbf{1,3}] + \left(\Lambda_4^\vee + \Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,5}] + \Lambda_3^\vee[\mathbf{4,5}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{2,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{3,7}] + \left(\Lambda_6^\vee + 2\Lambda_7^\vee\right)[\mathbf{4,7}] + \Lambda_4^\vee[\mathbf{5,7}]$

k	$\mathbf{h^k(\overline{X}^\vee)}$	generating cocycles
3	8	$2\Lambda_7^\vee[\mathbf{1\supseteq 1\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{2\supseteq 2\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{3\supseteq 3\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{4\supseteq 4\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{5\supseteq 5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{6\supseteq 6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{7\supseteq 7\supseteq 7}]$
		$\Lambda_4^\vee[\mathbf{1,2\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,2\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,3\supseteq 2}] + \Lambda_4^\vee[\mathbf{2,3\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{1,4\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,4\supseteq 4}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{3,4\supseteq 3}] + \left(\Lambda_3^\vee+\Lambda_5^\vee+3\Lambda_7^\vee\right)[\mathbf{3,4\supseteq 4}] + \Lambda_3^\vee[\mathbf{1,3,4}] + \left(\Lambda_4^\vee+2\Lambda_7^\vee\right)[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{1,5\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{2,5\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,5\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,5\supseteq 5}] + \left(\Lambda_4^\vee+\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{1,6\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{2,6\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{3,6\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,6\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{1,7\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{2,7\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,7\supseteq 7}] + 2\Lambda_7^\vee[\mathbf{3,7\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{4,7\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,7\supseteq 7}] + \Lambda_6^\vee[\mathbf{5,7\supseteq 5}] + \left(\Lambda_4^\vee+2\Lambda_7^\vee\right)[\mathbf{5,7\supseteq 7}] + \Lambda_4^\vee[\mathbf{2,5,7}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee+\Lambda_6^\vee+3\Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$2\Lambda_7^\vee[\mathbf{1,2\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,2\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,3\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,3\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{1,4\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,4\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{1,5\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{2,5\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{3,5\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,5\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{1,6\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{2,6\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{3,6\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{4,6\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,6\supseteq 6}] + 2\Lambda_7^\vee[\mathbf{1,7\supseteq 1}] + 2\Lambda_7^\vee[\mathbf{1,7\supseteq 7}] + 2\Lambda_7^\vee[\mathbf{2,7\supseteq 2}] + 2\Lambda_7^\vee[\mathbf{2,7\supseteq 7}] + 2\Lambda_7^\vee[\mathbf{3,7\supseteq 3}] + 2\Lambda_7^\vee[\mathbf{3,7\supseteq 7}] + 2\Lambda_7^\vee[\mathbf{4,7\supseteq 4}] + 2\Lambda_7^\vee[\mathbf{4,7\supseteq 7}] + 2\Lambda_7^\vee[\mathbf{5,7\supseteq 5}] + 2\Lambda_7^\vee[\mathbf{5,7\supseteq 7}]$
		$\left(\Lambda_4^\vee+2\Lambda_7^\vee\right)[\mathbf{1,2\supseteq 2}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{1,2,3}] + \left(\Lambda_2^\vee+\Lambda_4^\vee+\Lambda_7^\vee\right)[\mathbf{1,2,4}] + \left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,3,7}] + \left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{1,4,7}] + \Lambda_4^\vee[\mathbf{1,5,7}] + \Lambda_4^\vee[\mathbf{3,5,7}] + \Lambda_3^\vee[\mathbf{4,5,7}]$
		$\left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{1,3\supseteq 1}] + \left(\Lambda_5^\vee+\Lambda_7^\vee\right)[\mathbf{1,3\supseteq 3}] + \left(\Lambda_4^\vee+\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3,5\supseteq 3}] + \Lambda_3^\vee[\mathbf{4,5\supseteq 4}] + \Lambda_3^\vee[\mathbf{4,5\supseteq 5}] + \left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{2,7\supseteq 2}] + \left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{3,7\supseteq 3}] + \left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{4,7\supseteq 4}] + \left(\Lambda_6^\vee+2\Lambda_7^\vee\right)[\mathbf{5,7\supseteq 5}] + \Lambda_4^\vee[\mathbf{5,7\supseteq 7}] + \Lambda_4^\vee[\mathbf{2,5,7}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{4,5,7}] + \left(\Lambda_5^\vee+\Lambda_6^\vee+3\Lambda_7^\vee\right)[\mathbf{5,6,7}]$
		$\left(\Lambda_2^\vee+\Lambda_5^\vee+2\Lambda_7^\vee\right)[\mathbf{1,3,4}] + 2\Lambda_7^\vee[\mathbf{2,3,4}] + 2\Lambda_7^\vee[\mathbf{2,4,5}] + 2\Lambda_7^\vee[\mathbf{3,4,5}] + 2\Lambda_7^\vee[\mathbf{4,5,6}] + 2\Lambda_7^\vee[\mathbf{5,6,7}]$
		$\left(\Lambda_4^\vee+2\Lambda_7^\vee\right)[\mathbf{2,5\supseteq 2}] + \left(\Lambda_2^\vee+\Lambda_3^\vee+\Lambda_4^\vee+\Lambda_5^\vee+2\Lambda_7^\vee\right)[\mathbf{2,4,5}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{2,5,6}] + \Lambda_6^\vee[\mathbf{2,5,7}]$
		$\Lambda_4^\vee[\mathbf{1,2,5}] + \Lambda_4^\vee[\mathbf{2,3,5}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{1,4,5}] + 2\Lambda_7^\vee[\mathbf{1,2,6}] + 2\Lambda_7^\vee[\mathbf{2,3,6}] + 2\Lambda_7^\vee[\mathbf{1,4,6}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{1,5,6}] + \left(\Lambda_2^\vee+\Lambda_7^\vee\right)[\mathbf{3,5,6}] + 2\Lambda_7^\vee[\mathbf{1,2,7}] + 2\Lambda_7^\vee[\mathbf{2,3,7}] + 2\Lambda_7^\vee[\mathbf{1,4,7}] + \Lambda_6^\vee[\mathbf{1,5,7}] + \Lambda_6^\vee[\mathbf{3,5,7}]$

k	0	1	2	3
$\mathbf{comp}_k$	$()$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&0\\1&0&1\\0&0&1\\0&1&0\\0&0&1\\1&1&1\\0&0&0\end{pmatrix}$

A.31.2 Cohomology of coweight lattice $X^\vee = P^\vee$

$[\phi_u] = (1)$
does not lie in the image of comp$_2$

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_6^\vee[\mathbf{2,5\supseteq 2}] + \Lambda_6^\vee[\mathbf{2,5\supseteq 5}] + \left(-\Lambda_6^\vee\right)[\mathbf{2,4,5}] + \left(-\Lambda_2^\vee\right)[\mathbf{5,6\supseteq 5}] + \left(-\Lambda_2^\vee\right)[\mathbf{5,6\supseteq 6}] + \left(-\Lambda_2^\vee+\Lambda_4^\vee-\Lambda_5^\vee+\Lambda_6^\vee\right)[\mathbf{2,5,6}]$

$\mathbf{k}$	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	0	
1	0	
2	1	$\Lambda_5^\vee[\mathbf{1,3}] + \left(\Lambda_4^\vee + \Lambda_6^\vee\right)[\mathbf{3,5}] + \Lambda_3^\vee[\mathbf{4,5}]$
3	3	$\Lambda_5^\vee[\mathbf{1,3\supseteq 1}] + \Lambda_5^\vee[\mathbf{1,3\supseteq 3}] + \left(\Lambda_4^\vee + \Lambda_6^\vee\right)[\mathbf{3,5\supseteq 3}] + \Lambda_3^\vee[\mathbf{4,5\supseteq 4}] + \Lambda_3^\vee[\mathbf{4,5\supseteq 5}]$
		$\Lambda_2^\vee[\mathbf{1,3,4}] + \Lambda_5^\vee[\mathbf{2,3,4}] + \left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{2,4,5}] + \Lambda_2^\vee[\mathbf{3,4,5}] + \left(\Lambda_3^\vee + \Lambda_7^\vee\right)[\mathbf{4,5,6}] + \Lambda_6^\vee[\mathbf{2,5,7}] + \Lambda_2^\vee[\mathbf{4,5,7}] + \Lambda_2^\vee[\mathbf{5,6,7}]$
		$\Lambda_7^\vee[\mathbf{1,2,5}] + \Lambda_7^\vee[\mathbf{2,3,5}] + \Lambda_7^\vee[\mathbf{1,2,6}] + \Lambda_7^\vee[\mathbf{2,3,6}] + \Lambda_7^\vee[\mathbf{1,4,6}] + \Lambda_5^\vee[\mathbf{1,2,7}] + \Lambda_5^\vee[\mathbf{2,3,7}] + \Lambda_5^\vee[\mathbf{1,4,7}] + \Lambda_2^\vee[\mathbf{1,5,7}] + \Lambda_2^\vee[\mathbf{3,5,7}] + \Lambda_1^\vee[\mathbf{4,5,7}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	$\begin{pmatrix}1\\0\\0\end{pmatrix}$

A.31.3 Cohomology with trivial coefficients

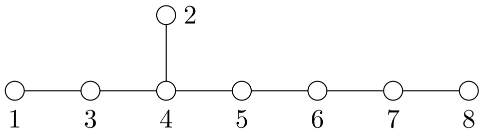
$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0},\mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1\supseteq 1}] + (-1)[\mathbf{2\supseteq 2}] + (-1)[\mathbf{3\supseteq 3}] + [\mathbf{1,3}] + (-1)[\mathbf{4\supseteq 4}] + (-1)[\mathbf{5\supseteq 5}] + (-1)[\mathbf{6\supseteq 6}] + (-1)[\mathbf{7\supseteq 7}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1,2\supseteq 1}] + [\mathbf{1,2\supseteq 2}] + (-1)[\mathbf{2,3\supseteq 2}] + (-1)[\mathbf{2,3\supseteq 3}] + (-1)[\mathbf{1,4\supseteq 1}] + (-1)[\mathbf{1,4\supseteq 4}] + [\mathbf{1,2,4}] + [\mathbf{1,3,4}] + (-1)[\mathbf{2,3,4}] + (-1)[\mathbf{1,5\supseteq 1}] + (-1)[\mathbf{1,5\supseteq 5}] + (-1)[\mathbf{2,5\supseteq 2}] + (-1)[\mathbf{2,5\supseteq 5}] + (-1)[\mathbf{3,5\supseteq 3}] + (-1)[\mathbf{3,5\supseteq 5}] + [\mathbf{2,4,5}] + [\mathbf{3,4,5}] + (-1)[\mathbf{1,6\supseteq 1}] + (-1)[\mathbf{1,6\supseteq 6}] + (-1)[\mathbf{2,6\supseteq 2}] + (-1)[\mathbf{2,6\supseteq 6}] + (-1)[\mathbf{3,6\supseteq 3}] + (-1)[\mathbf{3,6\supseteq 6}] + (-1)[\mathbf{4,6\supseteq 4}] + (-1)[\mathbf{4,6\supseteq 6}] + [\mathbf{4,5,6}] + (-1)[\mathbf{1,7\supseteq 1}] + (-1)[\mathbf{1,7\supseteq 7}] + (-1)[\mathbf{2,7\supseteq 2}] + (-1)[\mathbf{2,7\supseteq 7}] + (-1)[\mathbf{3,7\supseteq 3}] + (-1)[\mathbf{3,7\supseteq 7}] + (-1)[\mathbf{4,7\supseteq 4}] + (-1)[\mathbf{4,7\supseteq 7}] + (-1)[\mathbf{5,7\supseteq 5}] + (-1)[\mathbf{5,7\supseteq 7}] + [\mathbf{5,6,7}]$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_\mathbf{2})$	generating cocycles
0	1	[]
1	1	$[\mathbf{1}] + [\mathbf{2}] + [\mathbf{3}] + [\mathbf{4}] + [\mathbf{5}] + [\mathbf{6}] + [\mathbf{7}]$
2	2	$[\mathbf{1\supseteq 1}] + [\mathbf{2\supseteq 2}] + [\mathbf{3\supseteq 3}] + [\mathbf{4\supseteq 4}] + [\mathbf{5\supseteq 5}] + [\mathbf{6\supseteq 6}] + [\mathbf{7\supseteq 7}]$
		$[\mathbf{1,2}]+[\mathbf{2,3}]+[\mathbf{1,4}]+[\mathbf{1,5}]+[\mathbf{2,5}]+[\mathbf{3,5}]+[\mathbf{1,6}]+[\mathbf{2,6}]+[\mathbf{3,6}]+[\mathbf{4,6}]+[\mathbf{1,7}]+[\mathbf{2,7}]+[\mathbf{3,7}]+[\mathbf{4,7}]+[\mathbf{5,7}]$

k	$\mathbf{h^k}(\mathbb{F}_2)$	generating cocycles
3	5	$[1 \supseteq 1 \supseteq 1] + [2 \supseteq 2 \supseteq 2] + [3 \supseteq 3 \supseteq 3] + [4 \supseteq 4 \supseteq 4] + [5 \supseteq 5 \supseteq 5] + [6 \supseteq 6 \supseteq 6] + [7 \supseteq 7 \supseteq 7]$
		$[1,2 \supseteq 1] + [1,2 \supseteq 2] + [2,3 \supseteq 2] + [2,3 \supseteq 3] + [1,4 \supseteq 1] + [1,4 \supseteq 4] + [1,5 \supseteq 1] + [1,5 \supseteq 5] + [2,5 \supseteq 2] + [2,5 \supseteq 5] + [3,5 \supseteq 3] + [3,5 \supseteq 5] + [1,6 \supseteq 1] + [1,6 \supseteq 6] + [2,6 \supseteq 2] + [2,6 \supseteq 6] + [3,6 \supseteq 3] + [3,6 \supseteq 6] + [4,6 \supseteq 4] + [4,6 \supseteq 6] + [1,7 \supseteq 1] + [1,7 \supseteq 7] + [2,7 \supseteq 2] + [2,7 \supseteq 7] + [3,7 \supseteq 3] + [3,7 \supseteq 7] + [4,7 \supseteq 4] + [4,7 \supseteq 7] + [5,7 \supseteq 5] + [5,7 \supseteq 7]$
		$[1,3,4] + [2,3,4] + [2,4,5] + [3,4,5] + [4,5,6] + [5,6,7]$
		$[1,2,5] + [2,3,5] + [1,2,6] + [2,3,6] + [1,4,6] + [1,2,7] + [2,3,7] + [1,4,7] + [1,5,7] + [3,5,7]$
		$[2,5,7]$

A.32 Root system E_8

Dynkin diagram



Fundamental group $P^\vee/Q^\vee \simeq 0$

A.32.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$[\phi_u] = (1)$
does not lie in the image of comp$_2$

k	$\mathbf{H^k}(\mathbf{W_0}, \mathbf{X}^\vee)$	generating cocycles
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_6^\vee[2,5 \supseteq 2] + \Lambda_6^\vee[2,5 \supseteq 5] + \left(-\Lambda_6^\vee\right)[2,4,5] + \left(-\Lambda_2^\vee\right)[5,6 \supseteq 5] + \left(-\Lambda_2^\vee\right)[5,6 \supseteq 6] + \left(-\Lambda_2^\vee + \Lambda_4^\vee - \Lambda_5^\vee + \Lambda_6^\vee\right)[2,5,6]$

k	$\mathbf{h^k}(\overline{\mathbf{X}^\vee})$	generating cocycles
0	0	
1	0	
2	1	$\Lambda_5^\vee[1,3] + \left(\Lambda_4^\vee + \Lambda_6^\vee\right)[3,5] + \Lambda_3^\vee[4,5]$
3	2	$\Lambda_5^\vee[1,3 \supseteq 1] + \Lambda_5^\vee[1,3 \supseteq 3] + \left(\Lambda_4^\vee + \Lambda_6^\vee\right)[3,5 \supseteq 3] + \Lambda_3^\vee[4,5 \supseteq 4] + \Lambda_3^\vee[4,5 \supseteq 5]$
		$\Lambda_7^\vee[1,2,4] + \left(\Lambda_6^\vee + \Lambda_8^\vee\right)[1,4,7] + \left(\Lambda_6^\vee + \Lambda_8^\vee\right)[1,5,7] + \Lambda_6^\vee[2,5,7] + \left(\Lambda_6^\vee + \Lambda_8^\vee\right)[3,5,7] + \Lambda_5^\vee[1,6,7] + \Lambda_5^\vee[2,6,7] + \Lambda_5^\vee[3,6,7] + \Lambda_5^\vee[4,6,7] + \Lambda_6^\vee[2,5,8] + \Lambda_2^\vee[5,6,8]$

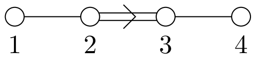
$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	$()$	$()$	$()$	$\begin{pmatrix}1\\0\end{pmatrix}$

A.32.2 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k}(\mathbf{W_0}, \mathbb{Z})$	generating cocycles
0	$\mathbb{Z}$	$[]$
1	0	
2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbf{[1 \supseteq 1]} + (-1)\,\mathbf{[2 \supseteq 2]} + (-1)\,\mathbf{[3 \supseteq 3]} + \mathbf{[1,3]} + (-1)\,\mathbf{[4 \supseteq 4]} + (-1)\,\mathbf{[5 \supseteq 5]} + (-1)\,\mathbf{[6 \supseteq 6]} + (-1)\,\mathbf{[7 \supseteq 7]} + (-1)\,\mathbf{[8 \supseteq 8]}$
3	$\mathbb{Z}/2\mathbb{Z}$	$\mathbf{[1,2 \supseteq 1]} + \mathbf{[1,2 \supseteq 2]} + (-1)\,\mathbf{[2,3 \supseteq 2]} + (-1)\,\mathbf{[2,3 \supseteq 3]} + (-1)\,\mathbf{[1,4 \supseteq 1]} + (-1)\,\mathbf{[1,4 \supseteq 4]} + \mathbf{[1,2,4]} + \mathbf{[1,3,4]} + (-1)\,\mathbf{[2,3,4]} + (-1)\,\mathbf{[1,5 \supseteq 1]} + (-1)\,\mathbf{[1,5 \supseteq 5]} + (-1)\,\mathbf{[2,5 \supseteq 2]} + (-1)\,\mathbf{[2,5 \supseteq 5]} + (-1)\,\mathbf{[3,5 \supseteq 3]} + (-1)\,\mathbf{[3,5 \supseteq 5]} + \mathbf{[2,4,5]} + \mathbf{[3,4,5]} + (-1)\,\mathbf{[1,6 \supseteq 1]} + (-1)\,\mathbf{[1,6 \supseteq 6]} + (-1)\,\mathbf{[2,6 \supseteq 2]} + (-1)\,\mathbf{[2,6 \supseteq 6]} + (-1)\,\mathbf{[3,6 \supseteq 3]} + (-1)\,\mathbf{[3,6 \supseteq 6]} + (-1)\,\mathbf{[4,6 \supseteq 4]} + (-1)\,\mathbf{[4,6 \supseteq 6]} + \mathbf{[4,5,6]} + (-1)\,\mathbf{[1,7 \supseteq 1]} + (-1)\,\mathbf{[1,7 \supseteq 7]} + (-1)\,\mathbf{[2,7 \supseteq 2]} + (-1)\,\mathbf{[2,7 \supseteq 7]} + (-1)\,\mathbf{[3,7 \supseteq 3]} + (-1)\,\mathbf{[3,7 \supseteq 7]} + (-1)\,\mathbf{[4,7 \supseteq 4]} + (-1)\,\mathbf{[4,7 \supseteq 7]} + (-1)\,\mathbf{[5,7 \supseteq 5]} + (-1)\,\mathbf{[5,7 \supseteq 7]} + \mathbf{[5,6,7]} + (-1)\,\mathbf{[1,8 \supseteq 1]} + (-1)\,\mathbf{[1,8 \supseteq 8]} + (-1)\,\mathbf{[2,8 \supseteq 2]} + (-1)\,\mathbf{[2,8 \supseteq 8]} + (-1)\,\mathbf{[3,8 \supseteq 3]} + (-1)\,\mathbf{[3,8 \supseteq 8]} + (-1)\,\mathbf{[4,8 \supseteq 4]} + (-1)\,\mathbf{[4,8 \supseteq 8]} + (-1)\,\mathbf{[5,8 \supseteq 5]} + (-1)\,\mathbf{[5,8 \supseteq 8]} + (-1)\,\mathbf{[6,8 \supseteq 6]} + (-1)\,\mathbf{[6,8 \supseteq 8]} + \mathbf{[6,7,8]}$

$\mathbf{k}$	$\mathbf{h^k}(\mathbb{F}_\mathbf{2})$	generating cocycles
0	1	$[]$
1	1	$\mathbf{[1]} + \mathbf{[2]} + \mathbf{[3]} + \mathbf{[4]} + \mathbf{[5]} + \mathbf{[6]} + \mathbf{[7]} + \mathbf{[8]}$
2	2	$\mathbf{[1 \supseteq 1]} + \mathbf{[2 \supseteq 2]} + \mathbf{[3 \supseteq 3]} + \mathbf{[4 \supseteq 4]} + \mathbf{[5 \supseteq 5]} + \mathbf{[6 \supseteq 6]} + \mathbf{[7 \supseteq 7]} + \mathbf{[8 \supseteq 8]}$
		$\mathbf{[1,2]} + \mathbf{[2,3]} + \mathbf{[1,4]} + \mathbf{[1,5]} + \mathbf{[2,5]} + \mathbf{[3,5]} + \mathbf{[1,6]} + \mathbf{[2,6]} + \mathbf{[3,6]} + \mathbf{[4,6]} + \mathbf{[1,7]} + \mathbf{[2,7]} + \mathbf{[3,7]} + \mathbf{[4,7]} + \mathbf{[5,7]} + \mathbf{[1,8]} + \mathbf{[2,8]} + \mathbf{[3,8]} + \mathbf{[4,8]} + \mathbf{[5,8]} + \mathbf{[6,8]}$
3	4	$\mathbf{[1 \supseteq 1 \supseteq 1]} + \mathbf{[2 \supseteq 2 \supseteq 2]} + \mathbf{[3 \supseteq 3 \supseteq 3]} + \mathbf{[4 \supseteq 4 \supseteq 4]} + \mathbf{[5 \supseteq 5 \supseteq 5]} + \mathbf{[6 \supseteq 6 \supseteq 6]} + \mathbf{[7 \supseteq 7 \supseteq 7]} + \mathbf{[8 \supseteq 8 \supseteq 8]}$
		$\mathbf{[1,2 \supseteq 1]} + \mathbf{[1,2 \supseteq 2]} + \mathbf{[2,3 \supseteq 2]} + \mathbf{[2,3 \supseteq 3]} + \mathbf{[1,4 \supseteq 1]} + \mathbf{[1,4 \supseteq 4]} + \mathbf{[1,5 \supseteq 1]} + \mathbf{[1,5 \supseteq 5]} + \mathbf{[2,5 \supseteq 2]} + \mathbf{[2,5 \supseteq 5]} + \mathbf{[3,5 \supseteq 3]} + \mathbf{[3,5 \supseteq 5]} + \mathbf{[1,6 \supseteq 1]} + \mathbf{[1,6 \supseteq 6]} + \mathbf{[2,6 \supseteq 2]} + \mathbf{[2,6 \supseteq 6]} + \mathbf{[3,6 \supseteq 3]} + \mathbf{[3,6 \supseteq 6]} + \mathbf{[4,6 \supseteq 4]} + \mathbf{[4,6 \supseteq 6]} + \mathbf{[1,7 \supseteq 1]} + \mathbf{[1,7 \supseteq 7]} + \mathbf{[2,7 \supseteq 2]} + \mathbf{[2,7 \supseteq 7]} + \mathbf{[3,7 \supseteq 3]} + \mathbf{[3,7 \supseteq 7]} + \mathbf{[4,7 \supseteq 4]} + \mathbf{[4,7 \supseteq 7]} + \mathbf{[5,7 \supseteq 5]} + \mathbf{[5,7 \supseteq 7]} + \mathbf{[1,8 \supseteq 1]} + \mathbf{[1,8 \supseteq 8]} + \mathbf{[2,8 \supseteq 2]} + \mathbf{[2,8 \supseteq 8]} + \mathbf{[3,8 \supseteq 3]} + \mathbf{[3,8 \supseteq 8]} + \mathbf{[4,8 \supseteq 4]} + \mathbf{[4,8 \supseteq 8]} + \mathbf{[5,8 \supseteq 5]} + \mathbf{[5,8 \supseteq 8]} + \mathbf{[6,8 \supseteq 6]} + \mathbf{[6,8 \supseteq 8]}$
		$\mathbf{[1,3,4]} + \mathbf{[2,3,4]} + \mathbf{[2,4,5]} + \mathbf{[3,4,5]} + \mathbf{[4,5,6]} + \mathbf{[5,6,7]} + \mathbf{[6,7,8]}$
		$\mathbf{[1,2,5]} + \mathbf{[2,3,5]} + \mathbf{[1,2,6]} + \mathbf{[2,3,6]} + \mathbf{[1,4,6]} + \mathbf{[1,2,7]} + \mathbf{[2,3,7]} + \mathbf{[1,4,7]} + \mathbf{[1,5,7]} + \mathbf{[2,5,7]} + \mathbf{[3,5,7]} + \mathbf{[1,2,8]} + \mathbf{[2,3,8]} + \mathbf{[1,4,8]} + \mathbf{[1,5,8]} + \mathbf{[2,5,8]} + \mathbf{[3,5,8]} + \mathbf{[1,6,8]} + \mathbf{[2,6,8]} + \mathbf{[3,6,8]} + \mathbf{[4,6,8]}$

A.33 Root system F_4

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq 0$

A.33.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$[\phi_u] = (1)$$
does not lie in the image of comp$_2$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	0	
2	0	
3	$\mathbb{Z}/2\mathbb{Z}$	$\Lambda_4^\vee[\mathbf{2,3\supseteq 2}] + \Lambda_4^\vee[\mathbf{2,3\supseteq 3}] + \left(-\Lambda_1^\vee + 2\Lambda_2^\vee - 2\Lambda_3^\vee + 2\Lambda_4^\vee\right)[\mathbf{2,3,4}]$

$\mathbf{k}$	$\mathbf{h^k(\overline{X^\vee})}$	**generating cocycles**
0	0	
1	0	
2	1	$\left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2\supseteq 2}] + \left(\Lambda_1^\vee + \Lambda_2^\vee\right)[\mathbf{1,2}]$
3	3	$\left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2\supseteq 2\supseteq 2}] + \Lambda_1^\vee[\mathbf{1,2\supseteq 1}] + \left(\Lambda_2^\vee + \Lambda_3^\vee\right)[\mathbf{1,2\supseteq 2}]$
		$\left(\Lambda_1^\vee + \Lambda_3^\vee\right)[\mathbf{2,3\supseteq 2}]$
		$\Lambda_1^\vee[\mathbf{2,3\supseteq 3}]$

$\mathbf{k}$	0	1	2	3
$\mathbf{comp}_k$	()	()	()	$\begin{pmatrix}1\\0\\0\end{pmatrix}$

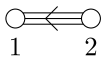
A.33.2 Cohomology with trivial coefficients

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{3\supseteq 3}] + (-1)[\mathbf{4\supseteq 4}] + [\mathbf{3,4}]$
		$[\mathbf{1\supseteq 1}] + (-1)[\mathbf{2\supseteq 2}] + [\mathbf{1,2}]$

$\mathbf{k}$	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
3	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{2,3 \supseteq 2}] + [\mathbf{2,3 \supseteq 3}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,3 \supseteq 3}] + (-1)\,[\mathbf{1,4 \supseteq 1}] + (-1)\,[\mathbf{1,4 \supseteq 4}] + (-1)\,[\mathbf{2,4 \supseteq 2}] + (-1)\,[\mathbf{2,4 \supseteq 4}] + [\mathbf{1,3,4}]$

$\mathbf{k}$	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	2	$[\mathbf{1}] + [\mathbf{2}]$
		$[\mathbf{3}] + [\mathbf{4}]$
2	4	$[\mathbf{1 \supseteq 1}] + [\mathbf{2 \supseteq 2}]$
		$[\mathbf{3 \supseteq 3}] + [\mathbf{4 \supseteq 4}]$
		$[\mathbf{1,3}] + [\mathbf{1,4}] + [\mathbf{2,4}]$
		$[\mathbf{2,3}]$
3	8	$[\mathbf{1 \supseteq 1 \supseteq 1}] + [\mathbf{2 \supseteq 2 \supseteq 2}]$
		$[\mathbf{3 \supseteq 3 \supseteq 3}] + [\mathbf{4 \supseteq 4 \supseteq 4}]$
		$[\mathbf{1,3 \supseteq 1}] + [\mathbf{1,4 \supseteq 1}] + [\mathbf{2,4 \supseteq 2}]$
		$[\mathbf{1,3 \supseteq 3}] + [\mathbf{1,4 \supseteq 4}] + [\mathbf{2,4 \supseteq 4}]$
		$[\mathbf{2,3 \supseteq 2}]$
		$[\mathbf{2,3 \supseteq 3}]$
		$[\mathbf{1,2,3}]$
		$[\mathbf{2,3,4}]$

A.34 Root system G_2

Dynkin diagram	
Fundamental group	$P^\vee/Q^\vee \simeq 0$

A.34.1 Cohomology of coroot lattice $X^\vee = Q^\vee$

$$\boxed{\phi_u \quad = \quad \partial\tau \text{ with } \tau = \Lambda_1^\vee[\mathbf{1}] + \Lambda_2^\vee[\mathbf{2}]}$$

$\mathbf{k}$	$\mathbf{H^k(W_0, X^\vee)}$	**generating cocycles**
0	0	
1	0	
2	0	

k	$\mathbf{H^k(W_0, X^\vee)}$	generating cocycles
3	0	

k	$\mathbf{h^k(\overline{X^\vee})}$	generating cocycles
0	0	
1	0	
2	0	
3	0	

k	0	1	2	3
$\mathbf{comp}_k$	()	()	()	()

A.34.2 Cohomology with trivial coefficients

k	$\mathbf{H^k(W_0, \mathbb{Z})}$	generating cocycles
0	$\mathbb{Z}$	[]
1	0	
2	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{1} \supseteq \mathbf{1}]$
3	$\mathbb{Z}/2\mathbb{Z}$	$[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}] + [\mathbf{1},\mathbf{2} \supseteq \mathbf{2}]$

k	$\mathbf{h^k(\mathbb{F}_2)}$	generating cocycles
0	1	[]
1	2	$[\mathbf{1}]$
		$[\mathbf{2}]$
2	3	$[\mathbf{1} \supseteq \mathbf{1}]$
		$[\mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{1},\mathbf{2}]$
3	4	$[\mathbf{1} \supseteq \mathbf{1} \supseteq \mathbf{1}]$
		$[\mathbf{2} \supseteq \mathbf{2} \supseteq \mathbf{2}]$
		$[\mathbf{1},\mathbf{2} \supseteq \mathbf{1}]$
		$[\mathbf{1},\mathbf{2} \supseteq \mathbf{2}]$

B Program listings

B.1 DeConciniSalvetti.py

```python
from itertools import chain
from sage.all import ZZ
from sage.combinat.free_module import CombinatorialFreeModule

def subsets_of_cardinality_atmost(S,k):
    """Returns an iterable of all the subsets of cardinality <= k of the (finite) set S."""
    assert k >= 0
    if k > 0 and len(S) > 0:
        for SS in subsets_of_cardinality_atmost(S[1:], k):
            yield SS
            if len(SS) < k:
                yield (S[0],)+SS
    else:
        yield ()

# TODO: Return only those flags with \Gamma_1 generating a finite subgroup, i.e. such that
# order(st) < \infty for all s,t \in S, in order to support finitely generated infinite Coxeter groups.
def flags_of_cardinality(S,k):
    """Returns an iterable over all the flags of cardinality k over the (finite) set S.

    Returns an iterable over all tuples (Gamma_1, Gamma_2, ...) s.t. S \supseteq Gamma_1 \supseteq Gamma_2
    and \sum_{i \geq 1} \# Gamma_i = k
    """
    assert k >= 0
    if k == 0:
        yield ()
    elif len(S) > 0:
        for Gamma_1 in subsets_of_cardinality_atmost(S,k):
            m = len(Gamma_1)
            for flag in flags_of_cardinality(Gamma_1, k-m):
                yield (Gamma_1,) + flag

def minimal_coset_representatives(W, TT, T):
    """Returns the representatives of the left <TT>-cosets in <T>.

    Given a Coxeter group W and subsets TT \subseteq T of the set S of distinguished generators of W,
    returns the representatives of the left <TT>-cosets in the subgroups <T> of W.
    """
    reps = [W.one()]
    yield W.one()
    n = 0
    while len(reps) > 0:
        n = n+1
        longer_reps = []
        for g in reps:
            for s in T:
                gg = s*g
                l = gg.length()
                if l == n and not gg in longer_reps and all( (gg*t).length() > l for t in TT ):
                    yield gg
                    longer_reps.append(gg)
        reps = longer_reps

def mu(Gamma, tau):
    """Returns \# { \gamma \in \Gamma : \gamma \leq \tau \}.

    Given a set Gamma of integers and an integer tau in Gamma, returns the
    number of elements of Gamma that are smaller or equal to tau.
    """
    return sum(1 for x in Gamma if x <= tau)

def number_of_inversions(f, X):
    return sum(chain.from_iterable((1 for y in X if x < y and f[x] > f[y]) for x in X))

def alpha(W, Gamma, i, tau, beta, conj_map):
    return i*beta.length() + sum(len(Gamma_j) for Gamma_j in Gamma[:i-1]) + mu(Gamma[i-1], tau)
```

```
        + sum(number_of_inversions(conj_map, Gamma_j) for Gamma_j in Gamma[i:])

def compute_conj_map(W, beta, X, Y):
    """Returns the dictionary describing the mapping Gamma_i_plus_one -> Gamma_i_minus_tau, x |-> \beta.inverse() * x * beta if well-defined, otherwise None.

    Given a Coxeter group W and an element beta of W, and sequences X, Y of elements of W.simple_reflections().keys(),
    returns a dictionary f such that
        f[x] = beta.inverse() * x * beta
    for all x in X, if the right hand side is an element of Y for all x, otherwise returns None.
    """
    beta_inv = beta.inverse()
    conj_map = {}
    for x in X:
        gg = beta_inv * W.simple_reflections()[x] * beta
        for y in W.simple_reflections().keys():
            if W.simple_reflections()[y] == gg:
                if y in Y:
                    conj_map[x] = y
                    break
                else:
                    return None
        if not x in conj_map: # beta^(-1) * x * beta isn't even a simple reflection
            return None
    return conj_map

class DeConciniSalvettiResolution:
    """Class representing the free resolution of the trivial R[W]-module R as constructed by DeConcini-Salvetti"""
    def __init__(self, W, R=ZZ):
        """Constructs the DeConcini-Salvetti resolution of W.

        Given a finite Coxeter group W (actually, W can be finitely or even countably generated, but that's not implemented right now),
        returns the DeConcini-Salvetti resolution of the trivial R[W]-module R.
        """
        self.W = W
        self.R = R
        self._modules = {}
        self._flags = {}
        self._morphisms = {}

    def S(self, k):
        """Returns the canonical basis of C(k), given by the set of flags of cardinality k."""
        assert k >= 0
        if not k in self._flags:
            self._flags[k] = list(flags_of_cardinality(self.W.simple_reflections().keys(), k)) # TODO: I'm using the keys instead of the generators themselves; is this necessary?
        return self._flags[k]

    def C(self, k):
        """The k-dimensional piece of the deConcini-Salvetti complex (C(k) = 0 for k < 0)."""
        if not k in self._modules:
            if k >= 0:
                self._modules[k] = CombinatorialFreeModule(self.W.algebra(self.R), self.S(k))
            else:
                self._modules[k] = CombinatorialFreeModule(self.W.algebra(self.R), [])
        return self._modules[k]

    def delta(self, k):
        """The differential delta(k): C(k) -> C(k-1).

        Given an integer k, returns the differential delta(k): C(k) -> C(k-1) in degree k.
        """
        Ck_minus_1_basis = self.C(k-1).basis()
        def index_to_reflection(k):
            return self.W.simple_reflections()[k]
        def delta_terms(Gamma):
            for i in [ i for i in range(len(Gamma)) if len(Gamma[i]) > (len(Gamma[i+1]) if i+1 < len(Gamma) else 0) ]:
                Gamma_i_plus_one = Gamma[i+1] if i+1 < len(Gamma) else ()
```

```
                for tau in Gamma[i]:
                    Gamma_i_minus_tau = tuple( x for x in Gamma[i] if not x == tau )
                    for beta in minimal_coset_representatives(self.W, map(index_to_reflection, Gamma_i_minus_tau), map(index_to_reflection, Gamma[i])):
                        beta_inv = beta.inverse()
                        conj_map = compute_conj_map(self.W, beta, Gamma_i_plus_one, Gamma_i_minus_tau)
                        if conj_map == None: # x |-> beta^{-1} * x * beta does not define a map from Gamma_i_plus_one to Gamma_i_minus_tau
                            continue
                        if len(Gamma_i_minus_tau) > 0:
                            Gamma_prime = Gamma[:i] + (Gamma_i_minus_tau,) + tuple(tuple(sorted(conj_map[x] for x in Gamma_j)) for Gamma_j in Gamma[i+1:])
                        else:
                            Gamma_prime = Gamma[:i]
                        yield (-1)**alpha(self.W, Gamma, i+1, tau, beta, conj_map) * (beta * Ck_minus_1_basis[Gamma_prime])

        def delta(Gamma):
            return sum(delta_terms(Gamma))
        if not k in self._morphisms:
            if k >= 1:
                self._morphisms[k] = self.C(k).module_morphism(on_basis=delta, codomain=self.C(k-1))
            else:
                self._morphisms[k] = self.C(k).module_morphism(on_basis=lambda x: self.C(k).zero(), codomain=self.C(k-1))
        return self._morphisms[k]
```

B.2 CoxeterCohomology.py

```
import itertools
from itertools import chain
from sage.modules.free_module_morphism import FreeModuleMorphism
from sage.modules.free_module import FreeModule, FreeModule_generic
from sage.groups.abelian_gps.abelian_group import AbelianGroup
from sage.all import vector, matrix, divisors, GF, Hom, ZZ, QQ
from DeConciniSalvetti import *

def finite_direct_sum_of_constant_family(I, R, M):
    '''Given a finite set I, and a FreeModule M over a ring R, returns the copower M^{(I)} = \bigoplus_{i \in I} M.

    More precisely, returns a triple (N, components, from_components), where
       N - instance of FreeModule
       components - function that given an element n of N, returns a dict { i: i-th component of m }
       from_components - function that given a dict { i: n[i] }, returns an element n of N such that n[i] is the i-th component of N
    '''
    assert isinstance(M, FreeModule_generic)
    m = len(M.gens())
    assert m == M.rank()
    index_to_I = list(I)
    len_I = len(index_to_I)
    N = FreeModule(R, len_I*m)
    I_to_index = {}
    for k in range(len_I):
        I_to_index[index_to_I[k]] = k
    def components(x):
        return { i : M.linear_combination_of_basis(x[m*I_to_index[i]:m*(I_to_index[i]+1)]) for i in I }
    def from_components(comp):
        return N.linear_combination_of_basis(list(chain.from_iterable(( M.coordinate_vector(comp[i]) if i in comp.keys() else M.zero_vector() for i in I))))
    return (N, components, from_components)

# M should be a free (combinatorial ?) R[G]-module and N should be a FreeModule over a PID R (must be = ZZ) at the moment
# and action is a function of two variables that given (g, x) as input, where g is an element of G
# and x in an element of N, returns another element of N.
# More precisely: x in an element of N.V(), and given any two elements x,x' of N.V() that define the same element of N,
```

```
# i.e. such that x-x' lies in N.W(), the elements action(g,x) and action(g,x') should define the same element of N
def hom_module(M, N, action):
    M_gens = M.gens()
    M_basis = M.basis()
    M_basis_keys = M_basis.keys()
    hom_M_N, components, from_components = finite_direct_sum_of_constant_family(M_basis_keys, N.base_ring(), N)
    def new_action(g, x):
        comp = components(x)
        return from_components({ i: action(g,comp[i]) for i in M_basis_keys })
    return (hom_M_N, M_basis, M_basis_keys, components, from_components, new_action)

def sum_in_module(M, iterable):
    x = M.zero()
    for val in iterable:
        x = x+val
    return x

# Given an action of a monoid on a module,
# this computes the linearly extended action
def linearly_extended_action(M, action, a, x):
    return sum_in_module(M, map(lambda g: a[g]*action(g, x), a.support()))

# Given f: M' --> M, computes the induced homomorphism
#     f^\ast: Hom(M, N) ---> Hom(M', N)
# not very elegant to make this function take hom_M_prime_N_data and hom_M_N_data as additional arguments
# , but I want to avoid unnecessary computations and the problem of nonunique representations, i.e. we shouldn't rely
# on hom_module returning the same objects given the same input.
def hom_module_induced_morphism(f, N, action, hom_M_prime_N_data, hom_M_N_data):
    M_prime = f.domain()
    M = f.codomain()
    hom_M_prime_N, M_prime_basis, M_prime_basis_keys, prime_components, prime_from_components, prime_hom_action = hom_M_prime_N_data
    hom_M_N, M_basis, M_basis_keys, components, from_components, hom_action = hom_M_N_data
    f_values = {}
    for i_prime in M_prime_basis_keys:
        f_values[i_prime] = f(M_prime_basis[i_prime])
    def f_star_components(x):
        c_x = components(x)
        c = {}
        for i_prime in M_prime_basis_keys:
            a = f_values[i_prime]
            for c in N.coordinates(sum_in_module(N, [ linearly_extended_action(N, action, a[i], c_x[i]) for i in a.support() ])):
                yield c
    matrix_representing_f_star = matrix([tuple(f_star_components(x)) for x in hom_M_N.basis()], ncols=hom_M_prime_N.rank())
    return FreeModuleMorphism(Hom(hom_M_N, hom_M_prime_N), matrix_representing_f_star)

def coroot_lattice_as_fg_Z_W_module(cartan_type):
    R = RootSystem(cartan_type)
    L = R.coroot_lattice()
    basis = L.basis()
    basis_keys = basis.keys()
    M = FreeModule(ZZ, len(basis_keys)) # Instances of CombinatorialFreeModule aren't instances of FreeModule: WTF?!?
    W = L.weyl_group()
    def M_to_L(x):
        d = x.dict()
        return sum_in_module(L, [ d[i]*basis[basis_keys[i]] for i in d.keys() ])
    def L_to_M(x): # this is slows a hell!
        return vector(list([ x[k] for k in basis_keys ]))
    def new_action(g, x): # TODO this is slow, *really*, *really* slow. Fix this!
        return L_to_M(g.action(M_to_L(x)))
    return (W, L, M, M_to_L, L_to_M, new_action)

# given a FreeModule M over a ring R' with a coerce_map to R,
# returns the base change of M to R, i.e. M\otimes_{R'} R
def base_change_module(M,R):
    assert R.has_coerce_map_from(M.base_ring())
    MM = FreeModule(R, len(M.basis()))
```

```
    # given an element x of M, returns the image of x in MM under the canonical map
    # M ---> MM = M \otimes_{R'} R
    def base_change_map(x):
        return MM.from_vector(vector(map(R.coerce_map_from(M.base_ring()), M.coordinates(x))))
    MM.base_change_map = base_change_map
    return MM

# given a f homomorphism between FreeModule's over a ring R' that is
# endowed with a coerce map to a ring R, returns the base change of f:
#     f\otimes_{R'} R: M\otimes_{R'} R ---> N\otimes_{R'} R
def base_change_morphism(f, R):
    M = f.domain()
    N = f.codomain()
    assert R.has_coerce_map_from(f.base_ring())
    MM = base_change_module(M, R)
    NN = base_change_module(N, R)
    return FreeModuleMorphism(Hom(MM,NN), f.matrix().change_ring(R))

# class generating the lattices Q(R^vee) <= X_\Omega <= P(R^vee)
# , corresponding to subgroups \Omega <= P(R^vee)/Q(R^vee), as
# free ZZ-modules endowed with action of the Weyl Group
#
# TODO: At the moment, only the computation of the cocharacter lattice is implemented,
# as it's the only thing we need. Maybe one fine day I shall implement the character lattice
# too.
class RootDatumGenerator:
    def __init__(self, R):
        self.R = R # don't really need to keep this, but might as well
        self.Pvee = R.coweight_lattice()
        basis = self.Pvee.basis()
        basis_keys = tuple(basis.keys())
        dim = len(basis_keys)
        self.MPvee = FreeModule(ZZ, dim)
        self.MQvee = self.MPvee.submodule([ vector([ alpha_vee[j] for j in basis_keys ]) for alpha_vee in self.Pvee.simple_roots() ])
        self.fundamental_group = self.MPvee/self.MQvee
        def Pvee_to_MPvee(x):
            return vector([ x[k] for k in basis_keys ]) # using x[k] is way faster than x.coefficient(k)
        def MPvee_to_Pvee(x):
            return sum_in_module(self.Pvee, [ x[i]*basis[basis_keys[i]] for i in range(dim) ])
        def action52(g, x):
            return Pvee_to_MPvee(g.action(MPvee_to_Pvee(x)))
        self.action = action52
        self.Pvee_to_MPvee = Pvee_to_MPvee
        self.MPvee_to_Pvee = MPvee_to_Pvee

    # returns an element of Pvee that maps to x under Pvee -->> fundamental_group
    def lift_to_Pvee(self, x):
        return self.MPvee_to_Pvee(x.lift())

    # returns the image of the element x under the map Pvee -->> fundamental_group
    def map_to_fundamental_group(self, x):
        return self.Pvee_to_MPvee(x)

    # returns the sublattice X of Pvee corresponding to the subgroup Omega of the fundamental_group
    def cocharacter_lattice(self, Omega):
        return self.MPvee.submodule(list(self.MQvee.gens()) + [ x.lift() for x in Omega.gens() ])

class CohomologyOfRootData(RootDatumGenerator):
    def __init__(self, R):
        RootDatumGenerator.__init__(self, R)
        self.W = R.coweight_lattice().weyl_group()

    def cohomology_of_cocharacter_lattice(self, Omega):
        WC = WeylCohomology(self.W, self.cocharacter_lattice(Omega), self.action)
        WC_mod2, comparison = WC.base_change(GF(2))
        def universal_2_cocycle():
            assert tuple(self.W.simple_reflections().keys()) == tuple(self.Pvee.simple_roots().keys())
            assert list(self.W.simple_reflections().keys()) == list(self.Pvee.basis().keys())
            indices = self.W.simple_reflections().keys()
```

```
            assert all(self.W.simple_reflections()[i].action(self.Pvee.simple_roots()[i]) == -self.Pvee.simple_roots()[i] for i in indices) # make sure numberings match up
            K2, K2_basis, K2_basis_keys, K2_components, K2_from_components, K2_action = WC._K(2)
            phi_u_components = {}
            for i in indices:
                Gamma = ((i,), (i,))
                assert Gamma in K2_basis_keys
                phi_u_components[Gamma] = self.Pvee.simple_roots()[i] # again, L.simple_roots() actually consists of coroots (weird convention)
            return K2_from_components({Gamma: self.Pvee_to_MPvee(phi_u_components[Gamma]) for Gamma in phi_u_components.keys()})
        phi_u = WC_mod2.K(2).base_change_map(universal_2_cocycle()) # universal_2_cocycle() lives in Hom(CS_2, X^vee) (and isn't actually a cocycle)
        _comp = {}
        def comparison_on_gens(k):
            if not k in _comp:
                _comp[k] = [ WC_mod2.H(k).quotient_map()(WC_mod2.K(k).base_change_map(x.lift()) ) for x in WC.H(k).gens() ]
            return _comp[k]
        mod2_ker = WC_mod2.d(2).kernel()
        M = mod2_ker / mod2_ker.submodule([ WC_mod2.K(2).base_change_map(x) for x in WC.d(2).kernel().gens() ])
        return (WC, WC_mod2, comparison_on_gens, phi_u, WC_mod2.H(2).quotient_map()(phi_u), M.quotient_map()(phi_u))

def subgroups_of_finite_abelian_group(A):
    assert len(A.invariants()) == len(A.gens())
    B = AbelianGroup(A.invariants()) # A is an instance of FGP_Module, which is not a subclass of AbelianGroup
    # The Sage function AbelianGroup.subgroups is *really, really* slow (as of version 8.1)
    for BB in reversed(B.subgroups()): # I prefer to get smaller subgroups first
        yield A.submodule([ A.sum(( gen.exponents()[i]*A.gens()[i] for i in range(len(A.gens())) )) for gen in BB.gens() ])

def faster_kernel(f):
    '''Computes the kernel of a morphism between FreeModules.

    Until this is fixed in Sage, it is necessary if we want to compute
    kernels of integer matrices in our lifetime.
    '''
    if f.base_ring() == ZZ:
        K = f.matrix().change_ring(QQ).kernel().intersection(FreeModule(ZZ, f.domain().rank()))
        return f.domain().submodule([ f.domain().linear_combination_of_basis(x) for x in K.gens() ])
    else:
        return f.kernel()

# base class for cocomplexes that are dimension-wise finite free R-modules
class CocomplexOfFreeModules:
    def __init__(self, R=ZZ):
        self.R = R

    def base_ring(self):
        return self.R

    # the k-th dimensional module
    def K(self, k):
        raise NotImplementedError # override in subclass

    # the k-th dimensional differential d(k): K(k) --> K(k+1)
    def d(self, k):
        raise NotImplementedError # override in subclass

    def base_change(self, new_base):
        assert new_base.has_coerce_map_from(self.R) # in a just world, we would start from a ring morphism, and wouldn't need this code
        new_cocomplex = CocomplexOfFreeModules(new_base)
        _K = {}
        _d = {}
        def new_K(new_self, k):
            if not k in _K:
                _K[k] = base_change_module(self.K(k), new_base) # self.K(k).change_ring(new_base)
```

```
            return _K[k]
        def new_d(new_self, k):
            if not k in _d:
                _d[k] = base_change_morphism(self.d(k), new_base) # self.d(k).change_ring(new_base)
            return _d[k]
        # comparison map, given k returns a (python) function
        #      self.K(k) --> self.base_change(new_base).K(k)
        # at the moment, this is just the identity (because of Sage's weirdness)
        def comparison(k):
            def identity(x):
                return x
            return identity
        new_cocomplex.K = new_K.__get__(new_cocomplex, CocomplexOfFreeModules) # thanks to Mad Physicist! https://stackoverflow.com/q/394770/
        new_cocomplex.d = new_d.__get__(new_cocomplex, CocomplexOfFreeModules)

        return (new_cocomplex, comparison)

    # the k-th dimensional cohomology group
    def H(self, k):
        if not hasattr(self, '_H'):
            self._H = {}
        if not k in self._H:
            self._H[k] = faster_kernel(self.d(k))/self.d(k-1).image()
        return self._H[k]

# A cocomplex K that computes the cohomology of a R[W]-module M.
# More precisely, the cocomplex K(k) = Hom_R[W](C(k),M), where C(k) is
# the DeConcini-Salvetti resolution of the trivial R[W]-module R.
#
# It is assumed that R is a principal ideal domain and that M is a FreeModule over R.
class WeylCohomology(CocomplexOfFreeModules):
    def __init__(self, W, M, action, R=ZZ):
        self.DCSR = DeConciniSalvettiResolution(W, R)
        self.M = M
        self.action = action
        self.R = R
        self.modules = {}
        self.differential = {}
        self.cohomology = {}

    def _K(self, k):
        if not k in self.modules:
            self.modules[k] = hom_module(self.DCSR.C(k), self.M, self.action)
        return self.modules[k]

    def K(self, k):
        return self._K(k)[0]

    def d(self, k):
        if not k in self.differential:
            self.differential[k] = hom_module_induced_morphism(self.DCSR.delta(k+1), self.M, self.action, self._K(k+1), self._K(k))
        return self.differential[k]

def test():
    def d(cartan_type):
        CRD = CohomologyOfRootData(RootSystem(cartan_type))
        _, WC_mod2, _, _, _ = CRD.cohomology_of_cocharacter_lattice(CRD.fundamental_group.submodule([]))
        return WC_mod2.H(2).dimension()
    dims = {1: 1,
            2: 0,
            3: 2,
            4: 0,
            5: 3,
            6: 0,
            7: 3,
            8: 0}
    for ell in dims.keys():
        print "Checking that dim_\\F_2 H^2(W_0,X^\\vee \\otimes_\\Z \\F_2) = %d for A_%d" % (dims[ell], ell)
        dim = d(['A',ell])
```

```
        if dim == dims[ell]:
            print "OK"
        else:
            print "Test␣FAILED:␣dimension␣=␣%d" % dim
```

B.3 LaTeXOutput.py

```
import itertools
from sage.all import *
from CoxeterCohomology import *
from cypari2.handle_error import PariError
from os.path import isfile

def latex_rep_of_finite_abelian_group_with_invariants(invariants):
    if len(invariants) == 0:
        return '0'
    return '␣\\oplus␣'.join(map(lambda n: ('\\Z/%d\\Z' % n) if n > 0 else '\\Z', invariants))

def latex_rep_of_fundamental_group(crd):
    omega = crd.fundamental_group
    return (latex_rep_of_finite_abelian_group_with_invariants(omega.invariants()), ',␣'.join(map(lambda gen: crd.lift_to_Pvee(gen)._latex_(), omega.gens())))

def latex_rep_of_subgroup_of_fundamental_group(crd, omega_prime):
    if omega_prime == crd.fundamental_group:
        return 'P^\\vee/Q^\\vee'
    s = ',␣'.join(map(lambda x: str(crd.fundamental_group.coerce_map_from(omega_prime)(x)), omega_prime.gens()))
    if s == '':
        return '0'
    else:
        return '\\left<%s\\right>' % s

def latex_rep_of_salvetti_flag(Gamma):
    return '␣\\supseteq␣'.join(map(lambda Lambda_i: '%s' % ','.join(map(str, Lambda_i)), Gamma))

def latex_rep_of_element_of_Pvee(x):
    return x._latex_()

def latex_rep_of_element_of_MPvee(crd, x):
    return latex_rep_of_element_of_Pvee(crd.MPvee_to_Pvee(x))

def latex_rep_of_element_of_MXvee(crd, MXvee, x):
    return latex_rep_of_element_of_Pvee(crd.MPvee_to_Pvee(crd.MPvee.coerce_map_from(MXvee)(x)))

def latex_rep_of_salvetti_cochain(WC, k, phi, rep_for_M=str):
    """Returns latex representation of the element phi of WC.K(k)."""
    _, M_basis, M_basis_keys, components, _, _ = WC._K(k)
    comp = components(phi)
    def optional_parentheses(x):
        return ('\\left(%s\\right)' % x) if ('+' in x or '-' in x) else x
    s = '␣+␣'.join(map(lambda Gamma: '%s\\bm{\\text{\\mbox{$[%s]$}}}' % (optional_parentheses(rep_for_M(comp[Gamma])), latex_rep_of_salvetti_flag(Gamma)), filter(lambda Gamma: not comp[Gamma].is_zero(), M_basis_keys)))
    return s if not s == '' else '0'

def transpose_matrix(A):
    n = len(A)
    if n == 0:
        return A
    m = len(A[0])
    return [ [ A[j][i] for j in range(n) ] for i in range(m) ]

def latex_rep_of_matrix(A):
    matrix_contents = ''
    for row in A:
        matrix_contents += '␣&␣'.join(map(str, row)) + '␣\\\\␣'
    return '\\begin{pmatrix}␣%s␣\\end{pmatrix}' % (matrix_contents if matrix_contents != '' else '\\relax')

def latex_rep_of_cohomology(omega_prime, crd, WC, WC_mod2, comparison_on_gens, phi_u, class_of_phi_u, class_of_phi_u_mod, range_of_k=[0,1,2,3]):
    s = ''
```

```python
    def cochain_rep(k, x):
        return latex_rep_of_salvetti_cochain(WC, k, x, rep_for_M=lambda x: latex_rep_of_element_of_MXvee(crd, WC.M, x))
    def mod2_cochain_rep(k, x):
        x_lifted = WC.K(k).linear_combination_of_basis([ GF(2).lift(y) for y in x]) # lift to cochain in the Salvetti complex over MXvee
        return latex_rep_of_salvetti_cochain(WC, k, x_lifted, rep_for_M=lambda x: latex_rep_of_element_of_MXvee(crd, WC.M, x))
    def rep_of_comparison(comparison_on_gens):
        A = [ x.list() for x in comparison_on_gens ]
        return latex_rep_of_matrix(transpose_matrix(A))
    # The cocycle phi_u
    s += '\n\\vskip 5pt'
    s += '\n\\begin{center}'
    s += '\n\\scalebox{1.15}{\\fbox{\\begin{tabu}spread 1cm{X[-1,R,$$]X[-1,L,$$]}'
    if phi_u.is_zero():
        s += '\n \\phi_u & = \\ 0 \\\\'
    elif class_of_phi_u.is_zero():
        s += '\n \\phi_u & = \\ \\partial \\tau \\text{ with } \\tau = %s \\\\' % mod2_cochain_rep(1, WC_mod2.d(1).lift(phi_u))
    else:
        s += '\n [\\phi_u] & = \\ \\left(%s\\right) \\\\' % ', '.join(map(str, WC_mod2.H(2).coordinate_vector(class_of_phi_u)))
        if class_of_phi_u_mod.is_zero():
            s += '\n & \\textbf{ lies in the image of $\\text{comp}_2$}' # TODO: Give pre-image
        else:
            s += '\n & \\textbf{ does not lie in the image of $\\text{comp}_2$}'
    s += '\n\\end{tabu}}}'
    s += '\n\\end{center}\n'
    # Integral cohomology
    s += '\n\\begin{longtabu}{lX[-0.3,C,$$]>{\\footnotesize}X[1,L,$$]}'
    s += '\n\\toprule'
    s += '\n\\rowfont{\\bf}'
    s += '\nk & H^k(W_0, X^\\vee) & \\textbf{{\\normalsize generating cocycles}} \\\\'
    s += '\n\\midrule'
    s += '\n\\endhead'
    row_counter = 1
    n_rows = len(range_of_k)
    for k in range_of_k:
        row = '%d & ' % k
        row += '%s & ' % latex_rep_of_finite_abelian_group_with_invariants(WC.H(k).invariants() )
        row += '%s' % ' \\linebreak \\newline '.join(map(lambda x: cochain_rep(k, x.lift()), WC.H(k).gens()))
        s += '\n%s \\\\%s' % (row, '\\\\' if row_counter < n_rows else '')
        row_counter += 1
    s += '\n\\bottomrule'
    s += '\n\\end{longtabu}\n\n\\vskip 0.5cm\n'
    # Mod 2 cohomology
    s += '\n\\begin{longtabu}{lX[0.1,C,$$]>{\\footnotesize}X[1,L,$$]}'
    s += '\n\\toprule'
    s += '\n\\rowfont{\\bf}'
    s += '\nk & h^k(\\overline{X^\\vee}) & \\textbf{{\\normalsize generating cocycles}} \\\\'
    s += '\n\\midrule'
    s += '\n\\endhead'
    row_counter = 1
    for k in range_of_k:
        row = '%d & ' % k
        row += '%s & ' % str(WC_mod2.H(k).dimension())
        row += '%s' % ' \\linebreak \\newline '.join(map(lambda x: mod2_cochain_rep(k, WC_mod2.H(k).lift_map()(x)), WC_mod2.H(k).gens()))
        s += '\n%s \\\\%s' % (row, '\\\\' if row_counter < n_rows else '')
        row_counter += 1
    s += '\n\\bottomrule'
    s += '\n\\end{longtabu}\n\n\\vskip 0.5cm\n'
    # Matrices of comparison maps
    s += '\n\\begin{center}\\begin{tabu}spread 1cm {>{\\bf}X[-1,R,$$]X[-1,C,$$]X[-1,C,$$]X[-1,C,$$]X[-1,C,$$]}'
    s += '\n\\toprule'
    s += '\n%s \\\\' % ' & '.join(('\\textbf{k}',) + tuple(map(str, range_of_k)))
    s += '\n\\midrule'
    s += '\n%s \\\\' % ' & '.join(('\\textbf{comp}_k',) + tuple(map(lambda k: rep_of_comparison(comparison_on_gens(k)), range_of_k)))
    s += '\n\\bottomrule'
```

```
    s += '\n\\end{tabu}\\end{center}\n'
    s += '\n\\vskip 1 cm \n'
    return s

def latex_rep_of_trivial_cohomology(crd, WC_triv, WC_triv_mod2, range_of_k=[0,1,2,3]):
    s = ''
    def cochain_rep(WC, k, x):
        return latex_rep_of_salvetti_cochain(WC, k, x, rep_for_M=lambda x: '' if x[0] == 1 else str(x[0]))
    # Trivial integral cohomology
    s += '\n\\begin{longtabu}{lX[0.3,C,$$]>{\\footnotesize}X[1,L,$$]}'
    s += '\n\\toprule'
    s += '\n\\rowfont{\\bf}'
    s += '\nk & H^k(W_0, \\Z) & \\text{{\\normalsize generating cocycles}} \\\\'
    s += '\n\\midrule'
    s += '\n\\endhead'
    row_counter = 1
    n_rows = len(range_of_k)
    for k in range_of_k:
        row = '%d & ' % k
        row += '%s & ' % latex_rep_of_finite_abelian_group_with_invariants(WC_triv.H(k).invariants())
        row += '%s' % ' \\linebreak \\newline '.join(map(lambda x: cochain_rep(WC_triv, k, x.lift()), WC_triv.H(k).gens()))
        s += '\n%s \\\\%s' % (row, '\\\\' if row_counter < n_rows else '')
        row_counter += 1
    s += '\n\\bottomrule'
    s += '\n\\end{longtabu}\n\n\\vskip 0.5cm\n'
    # Trivial Mod 2 cohomology
    s += '\n\\begin{longtabu}{lX[0.1,C,$$]>{\\footnotesize}X[1,L,$$]}'
    s += '\n\\toprule'
    s += '\n\\rowfont{\\bf}'
    s += '\nk & h^k(\\F_2) & \\text{{\\normalsize generating cocycles}} \\\\'
    s += '\n\\midrule'
    s += '\n\\endhead'
    row_counter = 1
    for k in range_of_k:
        row = '%d & ' % k
        row += '%s & ' % str(WC_triv_mod2.H(k).dimension())
        row += '%s' % ' \\linebreak \\newline '.join(map(lambda x: cochain_rep(WC_triv_mod2, k, WC_triv_mod2.H(k).lift_map()(x)), WC_triv_mod2.H(k).gens()))
        s += '\n%s \\\\%s' % (row, '\\\\' if row_counter < n_rows else '')
        row_counter += 1
    s += '\n\\bottomrule'
    s += '\n\\end{longtabu}\n\n\\vskip 1cm \n'
    return s

def latex_rep_of_cohomology_of_type(cartan_type):
    s = ''
    R = RootSystem(cartan_type)
    CRD = CohomologyOfRootData(R)
    cartan_type_text_repr = "%s%d" % (cartan_type[0], cartan_type[1])
    s += '\\subsection{Root system \\texorpdfstring{$%s$}{%s}}' % (R.cartan_type()._latex_(), cartan_type_text_repr)
    s += '\n\\fbox{\\begin{tabular}{rp{1cm}l}'
    s += '\n\\textbf{Dynkin diagram} & & %s \\\\ [2em]' % R.dynkin_diagram()._latex_()
    group_rep, gens_rep = latex_rep_of_fundamental_group(CRD)
    s += '\n\\textbf{Fundamental group} & & \n{$\\begin{aligned} P^\\vee/Q^\\vee & \\simeq %s \\\\' % group_rep
    if len(gens_rep) > 0:
        s += '\n & \\text{ generated by } %s \\in P^\\vee \\mod Q^\\vee' % gens_rep
    s += '\n\\end{aligned}$}'
    s += '\n\\end{tabular}}\n'
    section_counter = 0
    for omega in subgroups_of_finite_abelian_group(CRD.fundamental_group):
        WC, WC_mod2, comp_on_gens, phi_u, class_of_phi_u, class_of_phi_u_mod = CRD.cohomology_of_cocharacter_lattice(omega)
        if omega.invariants() == (): # simply connected case
            s += '\n\\subsubsection{Cohomology of coroot lattice \\texorpdfstring{$X^\\vee = Q^\\vee$}{X^v = Q^v}}'
            s += '\n\\label{subsub:cohomology_of_%s_simply_connected}' % cartan_type_text_repr
        elif omega == CRD.fundamental_group: # adjoint case
            s += '\n\\subsubsection{Cohomology of coweight lattice \\texorpdfstring{$X^\\vee = P^\\vee$}{X^v = P^v}}'
```

```
            s += '\n\\label{subsub:cohomology_of_%s_adjoint}' % cartan_type_text_repr
        else: # general case
            omega_latex_rep = latex_rep_of_subgroup_of_fundamental_group(CRD, omega)
            omega_text_rep = str(omega)
            s += '\n\\subsubsection{Cohomology of lattice \\texorpdfstring{$X^\\vee$}{X^v} corresponding to \\texorpdfstring{$\\Omega = %s$}{Omega = %s}}' % (omega_latex_rep, omega_text_rep)
            s += '\n\\label{subsub:cohomology_of_%s_%d}' % (cartan_type_text_repr, section_counter)
            section_counter += 1
        s += '\n'+latex_rep_of_cohomology(omega, CRD, WC, WC_mod2, comp_on_gens, phi_u, class_of_phi_u, class_of_phi_u_mod)
    WC_triv = WeylCohomology(CRD.W, FreeModule(ZZ, 1), lambda g,x: x, R=ZZ)
    WC_triv_mod2 = WeylCohomology(CRD.W, FreeModule(GF(2), 1), lambda g,x: x, R=GF(2))
    s += '\n\\subsubsection{Cohomology with trivial coefficients}'
    s += '\n\\label{subsub:cohomology_of_%s_with_trivial_coefficients}' % cartan_type_text_repr
    s += '\n'+latex_rep_of_trivial_cohomology(CRD, WC_triv, WC_triv_mod2)
    return s

def compute_cohomology_for_type(X, range_of_ell):
    for ell in range_of_ell:
        print "Computing cohomology of %s_%d ..." % (X, ell)
        filename = 'cohomology_of_%s_%d.tex' % (X, ell)
        if isfile(filename):
            print "File already exists, SKIPPING."
        else:
            s = latex_rep_of_cohomology_of_type([X,ell])
            with open(filename, 'w') as f:
                f.write(s)
            print "DONE."
```

B.4 main.py

```
from sage.all import *
from LaTeXOutput import *
import sys

def main():
    if len(sys.argv) > 2:
        compute_cohomology_for_type(sys.argv[1], list(map(int, sys.argv[2:])))
    else:
        print "Usage: sage main.py <type> <list-of-ranks>\nFor example: sage main.py A 2 3 4\n"

if __name__ == '__main__':
    main()
```

B.5 Makefile

```
.phony: no-default-goal sync-changes compute-A compute-B compute-C compute-D compute-F compute-E compute-G compute-all

SAGE = /Applications/SageMath/sage

A_RANGE = 1 2 3 4 5 6 7 8
B_RANGE = 2 3 4 5 6 7 8
C_RANGE = 2 3 4 5 6 7 8
D_RANGE = 3 4 5 6 7 8
E_RANGE = 6 7 8
F_RANGE = 4
G_RANGE = 2

CRD-FILES = DeConciniSalvetti.py CoxeterCohomology.py LaTeXOutput.py main.py
CRD-DIR = /Users/nico/Documents/math/crd

no-default-goal:
	@echo "No default goal set. Please choose a goal manually"

sync-changes:
	rsync -v *.py $(CRD-DIR)

commit: sync-changes
	cd $(CRD-DIR) && git commit -a && git push

```

```
compute-A:
	@echo "Computing␣cohomology␣of␣type␣A"
	$(SAGE) main.py A $(A_RANGE)

compute-B:
	@echo "Computing␣cohomology␣of␣type␣B"
	$(SAGE) main.py B $(B_RANGE)

compute-C:
	@echo "Computing␣cohomology␣of␣type␣C"
	$(SAGE) main.py C $(C_RANGE)

compute-D:
	@echo "Computing␣cohomology␣of␣type␣D"
	$(SAGE) main.py D $(D_RANGE)

compute-E:
	@echo "Computing␣cohomology␣of␣type␣E"
	$(SAGE) main.py E $(E_RANGE)

compute-F:
	@echo "Computing␣cohomology␣of␣type␣F"
	$(SAGE) main.py F $(F_RANGE)

compute-G:
	@echo "Computing␣cohomology␣of␣type␣G"
	$(SAGE) main.py G $(G_RANGE)

compute: compute-A compute-B compute-C compute-D compute-E compute-F compute-G
```

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Erklärung

Ich erkläre hiermit, dass ich die beigefügte Dissertation selbstständig und nur unter Gebrauch der in §7 Absatz 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät der Humboldt-Universität zu Berlin (Stand 2014) genannten Hilfsmittel verfasst habe.

Unterschrift: ______________________________

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