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Some Exact Results for the Two Point Function of an Integrable Quantum Field Theory

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ABSTRACT

The two point correlation function for the quantum nonlinear Schrödinger (δ-function gas) model is studied. An infinite series representation for this function is derived using the quantum inverse scattering formalism. For the case of zero temperature, the infinite coupling ($c \to \infty$) result of Jimbo, Miwa, Môri and Sato is extended to give an exact expression for the order $1/c$ correction to the two point function in terms of a Painlevé transcendent of the fifth kind.

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The theory of completely integrable quantum systems encompasses a wide class of exactly soluble models in statistical mechanics and quantum field theory. Important advances have been made in the past few years, both in the application of Bethe ansatz methods to new models and in the understanding of the algebraic structure which underlies Bethe's ansatz and its connection with the classical method of inverse scattering.[1] In spite of these developments, one problem of fundamental importance remains largely unsolved--the determination of Green's functions for general integrable systems. Progress on this problem has been mostly limited to certain special cases for which the algebraic structure is that of a free fermion theory (e.g. the 2-D Ising model,[2-4] the X-Y spin chain,[4] and the impenetrable Bose gas[4-8]). The more general problem of Green's functions for Bethe's ansatz models is not well understood. In this note we describe some new results which constitute a step toward the resolution of this problem.

The model we consider is the quantum nonlinear Schrödinger (delta-function gas) model described by the Hamiltonian $H = \int dx \left[\partial_x \phi^* \partial_x \phi + c\phi^* \phi^* \phi\phi \right]$ where $\phi(x)$ is a canonical nonrelativistic boson field. For the case $c = \infty$, the Green's functions have been extensively studied.[5-8] In the present work, we study the finite c two-point function G(x) by considering its expansion for large c:

$$G(x) \sim G^{(0)}(x) + \frac{1}{c} G^{(1)}(x) + O\left(\frac{1}{c^2}\right) \quad . \tag{1}$$

Our main result is an exact closed form expression for the O(1/c) term $G^{(1)}(x)$, which like the $c = \infty$ case is given in terms of Painlevé transcendents.

Our procedure for studying Green's functions employs the quantum inverse formalism[9-11] and particularly the operator Gel'fand-Levitan transform.[11-13] Here we will outline the main elements of the calculation. Detailed proofs will be

presented elsewhere. The quantized reflection coefficient operators $R^*(k)$ and $R(k)$ are defined via the Zakharov-Shabat eigenvalue problem. [We follow essentially the notation of Refs. 10 and 12.] The Gel'fand-Levitan transform expresses the local field as an operator functional of R^* and R, which may be written as an infinite series:

$$\phi(x) = \sum_{N=0}^{\infty} \int \left(\prod_{i=1}^{N} \frac{dp_i}{2\pi} \right) \left(\prod_{j=0}^{N} \frac{dk_i}{2\pi} \right) g_N(p_i, k_i, x) R^*(p_1) \ldots R^*(p_N) R(k_N) \ldots R(k_0) \quad (2)$$

where $g_N(p_i, k_i, x)$ is given in Refs. 12 and 13. A normal ordered expression for the operator product $\phi^*(x)\phi(y)$ is obtained by the following reordering theorem: For $x > y$, $\phi^*(x)\phi(y)$ is obtained in normal ordered form by writing the series analogous to (2) for $\phi^*(x)$ and then inserting $\phi(y)$ between the R^*'s and R's in each term (i.e. each term will contain an expression of the form $R^*(p_0) \ldots R^*(p_N)\phi(y)R(k_N) \ldots R(k_1)$). Expanding $\phi(y)$ then gives a normal ordered double series for $\phi^*(x)\phi(y)$. Such a result was conjectured by Honerkamp[15] on the basis of low order calculations. We have constructed a proof to all orders which utilizes the analytic properties of the commutator $[R(k), \phi(y)]$ discussed in Ref. 12. Thus we have an operator series of the form

$$\phi^*(x)\phi(y) = \sum_{N=0}^{\infty} \int \prod_{i=0}^{N} \frac{dp_i}{2\pi} \prod_{j=0}^{N} \frac{dk_i}{2\pi} F_N(p_i, k_i; x, y)$$

$$\times \quad R^*(p_0) \ldots R^*(p_N) R(k_N) \ldots R(k_0) \quad (3)$$

where F_N is a sum of products of the g_j's, $j \leq N$.

With the above reordering theorem the finite temperature Green's function

$$G_{\beta,\mu}(x - y) = \frac{\mathrm{Tr}\,[\phi^*(x)\phi(y)e^{-\beta(H-\mu N)}]}{\mathrm{Tr}\,e^{-\beta(H-\mu N)}} \tag{4}$$

may be computed using precisely the same technique as in our previous derivation of the thermodynamic properties of the system,[14] the latter being equivalent to computing the zero separation correlation function $G_{\beta,\mu}(0)$. In this way, it may be shown that to all orders, the β and μ dependence of $G_{\beta,\mu}$ can be isolated in terms of single particle functions $\tilde{\rho}(k)$, yielding a formula

$$G_{\beta,\mu}(x - y) = \sum_{N=0}^{\infty} \int \left[\prod_{i=0}^{N} \tilde{\rho}(k_i) \frac{dk_i}{2\pi} \right] \tilde{f}_N(k_0 \ldots k_N; x, y) \tag{5}$$

where the functions $\tilde{f}_N$ are independent of β and μ. One can actually derive several such representations for $G_{\beta,\mu}$. In the form suggested by Honerkamp,[15] $\tilde{\rho}$ is just the density function $\rho(k)$ defined by Yang and Yang.[16] We have found another representation of the form (5) for which $\tilde{\rho}(k) = [\,e^{\beta\varepsilon(k)} + 1\,]^{-1}$ where $\varepsilon(k)$ is the function of Yang and Yang which describes the particle-hole excitation spectrum. This latter representation has two main advantages. One is that each function $\tilde{f}_N$ in (5) is obtained directly from the corresponding function F_N in (3), rather than from the first $N + 1$ functions $F_0, F_1 \ldots F_N$ as is the case for the $\rho(k)$ expansion. The other advantage is that the zero temperature ($\beta \to \infty$) limit of $[\,e^{\beta\varepsilon(k)} + 1\,]^{-1}$ is a simple unit step function with support between $-k_F$ and k_F (k_F = Fermi momentum). The zero temperature Green's function thus obtains the form

$$G(x - y) = \sum_{N=0}^{\infty} \int_{-k_F}^{k_F} \left[\prod_{i=0}^{N} \frac{dk_i}{2\pi} \right] \tilde{f}_N(k_0 \ldots k_N; x, y) \quad . \tag{6}$$

The prescription for determining the function $\tilde{f}_N$ from the function F_N in Eq. (3) involves first symmetrizing the integrand and then setting $p_i = k_i + n_i q$, taking the limit $q \to 0$ and retaining only those terms which are nonsingular when any subset of the n_i's is set equal to zero (e.g. terms which involve factors like (n_i/n_j), $i \neq j$, are discarded). A detailed discussion of Eq. (5) and how it is derived for both $\tilde{\rho}(k) = \rho(k)$ and for $\tilde{\rho}(k) = [e^{\beta\epsilon(k)} + 1]^{-1}$ will be given elsewhere.

Equation (6) is the most explicit form we have been able to find for the full zero temperature correlation function. However, for the first few terms in a large c expansion, the analysis may be carried much further. Upon symmetrization of the operator expression (3), the $c \to \infty$ limit may be taken straightforwardly (c.f. Ref. 13). In this limit, Eq. (6) gives

$$G^{(0)} = \sum_{N=0}^{\infty} \frac{(-2)^N}{N!} \int_{-k_F}^{k_F} \frac{dk_0}{2\pi} \cdots \frac{dk_N}{2\pi} \int_y^x dz_1 \cdots dz_N \, \mathscr{D}_N(x, y, \vec{z}, \vec{k}) \tag{7}$$

where $\mathscr{D}_N$ is the determinant of an $(N + 1) \times (N + 1)$ matrix $\mathscr{D}_{ij} = \exp -ik_i(x_i - y_j)$, $i,j = 0,1,\ldots,N$, with $x_0 = x$, $y_0 = y$ and $x_i = y_i = z_i$, $i = 1,\ldots,N$. For the leading $1/c$ correction we obtain the new result

$$G^{(1)} = \frac{2k_F}{\pi} G^{(0)} + \sum_{N=1}^{\infty} \frac{(-2)^N}{N!} \int_{-k_F}^{k_F} \frac{dk_0}{2\pi} \cdots \frac{dk_N}{2\pi} \int_y^x dz_1 \cdots dz_N$$

$$\times \left(-i \sum_{j=1}^{N} (k_0 - k_j) \right) \mathscr{D}_N(x, y, \vec{z}, \vec{k}) \quad . \tag{8}$$

By performing the k integrations and introducing the scaled variable $t = k_F(x - y)$ it is possible to express these results in terms of an integral kernel $K(u, v) = \sin(u - v)/(u - v)$ acting on the interval $[0, t]$. Let us define quantities $R(t, \lambda)$ and

$D_1(t, \lambda)$ as the usual resolvent kernel and first Fredholm minor but with their arguments evaluated at the end points, i.e.:

$$R(t, \lambda) = \lambda K(0, t) + \lambda^2 \int_0^t dz\, K(0, z)K(z, t) + \ldots \tag{9}$$

$$D_1(t, \lambda) = \lambda K(0, t) - \lambda^2 \int_0^t dz \begin{vmatrix} K(0, t) & K(0, z) \\ K(z, t) & K(z, z) \end{vmatrix} + \ldots \tag{10}$$

With these definitions we easily recover the result of Schultz[5] and Lenard[6] for the infinite c case: $G^{(0)} = (k_F/2)D_1(t, \lambda = 2/\pi)$. After some manipulation it turns out that the 1/c correction $G^{(1)}$ may also be expressed in terms of the quantities (9) and (10)

$$\frac{G^{(1)}}{G^{(0)}} = \frac{2k_F}{\pi} \left\{ 1 + \left(\frac{\partial^2 \ln R}{\partial t\, \partial \lambda} - \frac{\partial \ln R}{\partial t} \frac{\partial \ln R}{\partial \lambda} \right) - \left(\frac{\partial^2 \ln D_1}{\partial t} - \frac{\partial \ln D_1}{\partial t} \frac{\partial \ln D_1}{\partial \lambda} \right) \right\}_{\lambda = 2/\pi} . \tag{11}$$

The work of Jimbo, Miwa, Môri and Sato[8] has shown that the quantities (9) and (10) may be written in terms of Painlevé transcendents.[17] For our purposes we may summarize their elegant results as follows. Let us define a function $\phi(t, \lambda)$ by the differential equation

$$\phi'' = [\,(\phi')^2 - 1\,]\cot\phi + (1 - \phi')/t \tag{12}$$

with boundary condition $\phi \sim t - \lambda t^2$ as $t \to 0$, where the prime in eq. (12) denotes differentiation with respect to t. [The function $\phi(t, \lambda)$ is related to $y(t, \lambda)$ of Ref. 8, Eq. (7.98) by $y = e^{-2i\phi}$; in terms of y the eq. (12) is a Painlevé equation of the fifth kind.] Then the resolvent (9) and Fredholm minor (10) may be expressed as

$$R = \frac{1 - \phi'}{2\sin\phi} \tag{13}$$

$$\frac{\partial \ln D_1}{\partial t} = \frac{t(\phi'^2 - 1)}{4\sin^2\phi} + \cot\phi - \frac{1}{t} \quad . \tag{14}$$

Since $D_1(t, \lambda) = \lambda$ at $t = 0$, these equations completely specify $R(t, \lambda)$ and $D_1(t, \lambda)$ and hence the functions $G^{(0)}$ and $G^{(1)}$ in terms of the differential equation and boundary condition (12).

Using these results it is a trivial exercise to write down the short distance behaviors of $G^{(0)}$ and $G^{(1)}$. Here we will concentrate on their long distance properties, for which we need the large t behavior of the solutions to (12). For $\lambda > 1/\pi$ one finds that this is of the form

$$\phi(t, \lambda) = t + t_0 + k \ln t + O(\frac{1}{t}) \tag{15}$$

where t_0 and k depend on λ. Using the symbolic manipulation program MACSYMA we have extended this asymptotic expansion through order $1/t^6$. Inserting this expansion into (14) and integrating we obtain

$$\ln D_1(t, \lambda) = -kt + \frac{1}{2}(k^2 - 1)\ln t + \bar{B}(\lambda) + O(\frac{1}{t}) \tag{16}$$

where $\bar{B}(\lambda)$ is an integration constant. To date we have only been able to investigate the functions $k(\lambda)$, $t_0(\lambda)$ and $\bar{B}(\lambda)$ by numerical integration of the differential equation (12), though we feel that an analytic solution should be possible using techniques similar to those of Ref. 18. In any case we have concluded that $k(\lambda)$ is given by $k(\lambda) = -\frac{1}{\pi}\ln(\pi\lambda - 1)$ and that if t_0 and $\bar{B}$ are regarded as functions of k

rather than of λ then $t_0 - \pi/2$ is an odd function of k and $\bar{B}$ is an even function of k, with $d\bar{B}/dk = k\, dt_o/dk + k$. Thus at $\lambda = 2/\pi$ we have $k = 0$ and $t_0 = \pi/2$. For $\bar{B}$ we find at $\lambda = 2/\pi$, $\bar{B} = \ln(2/\pi) + \ln \rho_\infty$, where $\rho_\infty = 0.924182203782$. Using these values we obtain the long distance behavior of the infinite c correlation function,

$$G^{(0)} = \frac{k_F}{\pi}\rho_\infty \frac{1}{\sqrt{t}}\left\{1 - \frac{1 + 4\cos 2t}{32t^2} - \frac{3\sin 2t}{16t^3} + \ldots\right\} \quad . \tag{17}$$

The long distance behavior of $G^{(0)}$ has also been obtained by a different method by Vaidya and Tracy,[7] who have shown that ρ_∞ is related to Glaisher's constant A by $\rho_\infty = \pi e^{1/2} 2^{-1/3} A^{-6}$, which is in precise agreement with our numerical value. (Note that there are sign errors in Refs. 7 and 8 which may be corrected by making the replacement $t \to t - \pi/2$ in all the trigonemetric functions appearing in their formulas.)

In order to compute the long distance behavior of the 1/c term $G^{(1)}$ we need in addition the values $dt_0/d\lambda$, $dk/d\lambda$, and $d\bar{B}/d\lambda$ at $\lambda = 2/\pi$. From $k(\lambda) = -\frac{1}{\pi}\ln(\pi\lambda - 1)$, we have $dk/d\lambda = -1$ while for $dt_0/d\lambda$ and $d\bar{B}/d\lambda$ we have obtained the numerical results $dt_0/d\lambda = -2.6566572 = -(\gamma + 3\ln 2)$ where γ is Euler's constant, and $d\bar{B}/d\lambda = \pi/2 - 1.57079633 = 0$. Combining these results, we obtain the long distance behavior of the two point function through order 1/c:

$$\frac{G(t)}{G(0)} = \pi e^{1/2} 2^{-1/3} A^{-6}\left[1 + \frac{2k_F}{\pi c}(\gamma + 3\ln 2 - \frac{3}{2})\right] t^{-\frac{1}{2} + 2k_F/\pi c}\left\{1 + \frac{k_F}{\pi c}\frac{\sin 2t}{2t} + \ldots\right\} \tag{18}$$

where, to this order, $G(0) = (k_F/\pi)(1 + 2k_F/\pi c)$. Note that we have interpreted a $\ln t$ term in $G^{(1)}$ as the first order expansion of an asymptotic power $t^{-\nu}$, where $\nu = \frac{1}{2} - 2k_F/\pi c + O(1/c^2)$.[19]

The results of this paper suggest that the connections between the operator formalism of the quantum inverse method and the treatment of Green's functions by isomonodromic deformation theory go much deeper than our present level of understanding. We have found that the first two terms in the large c expansion (1) may be expressed in terms of Painlevé transcendents. However, our calculation of the O(1/c) term made no direct use of monodromy arguments. Instead, this term was obtained by relating it to derivatives of the Fredholm minor (10) and resolvent (9) with respect to λ and t. Reduction to Painlevé transcendents then followed the original analysis of Jimbo, et al. One could imagine a direct application of monodromy arguments, e.g. to the series (6) or to the operator formalism itself. Such an approach is presently under investigation.

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