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SOME EXAMPLES CONCERNING APPLICABILITY
OF THE FREDHOLM-RADON METHOD
IN POTENTIAL THEORY

JOSEF KRÁL, WOLFGANG WENDLAND

Dedicated to Professor Jan Poláček on the occasion of his sixtieth birthday

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Summary. Simple examples of bounded domains $D \subset \mathbf{R}^3$ are considered for which the presence of peculiar corners and edges in the boundary ∂D causes that the double layer potential operator acting on the space $\mathcal{C}(\partial D)$ of all continuous functions on ∂D can for no value of the parameter α be approximated (in the sup-norm) by means of operators of the form $\alpha I + T$ (where I is the identity operator and T is a compact linear operator) with a deviation less than $|\alpha|$; on the other hand, such approximability turns out to be possible for $\alpha = \frac{1}{2}$ if a new norm is introduced in $\mathcal{C}(\partial D)$ with help of a suitable weight function.

Key words: double layer potential, Fredholm-Radon method in potential theory

Classification AMS: 31 B 20 (47 A 53)

In this paper we deal with examples of open sets $D \subset \mathbf{R}^3$ whose boundary ∂D is compact and can be covered by a finite number of planes parallel to the coordinate axes; we shall consider only those D for which ∂D is locally a surface (i.e. every point of ∂D has a neighbourhood in ∂D which is homeomorphic with \mathbf{R}^2) and we shall call these sets rectangular.

Boundary value problems for sets of this type occur frequently in applications. Besides that, rectangular sets are conveniently used for approximation of more general domains, and various procedures (like that of Wiener) often permit to reduce a (generalized) boundary value problem corresponding to a general domain to a (classical) boundary value problem for a rectangular set. Among constructive methods of solving boundary value problems the Fredholm method of integral equations is of particular interest, also from the point of view of numerical treatment (cf. [16], [14], [15]). This method exhibiting the duality between the Dirichlet and the Neumann problems is especially efficient in connection with exterior problems. Let us suppose, for example that D is bounded, ∂D connected, and denote by $\mathcal{C}^*(\partial D)$ the space of all signed finite Borel measures supported by ∂D . If $\nu \in \mathcal{C}^*(\partial D)$, then its Newtonian potential $U\nu$ represents a harmonic function in $\mathbf{R}^3 \setminus \partial D$ whose

gradient ∇Uv is integrable over all bounded Borel sets contained in $\mathbf{R}^3 \setminus \partial D$; accordingly, one may introduce the functional

$$(1) \quad NUv: \varphi \mapsto \langle \varphi, NUv \rangle$$

over the space \mathcal{D} of all compactly supported infinitely differentiable functions in \mathbf{R}^3 defining

$$\langle \varphi, NUv \rangle = \int_G \nabla \varphi(x) \cdot \nabla Uv(x) \, dx,$$

where G is the complement of $\bar{D} = D \cup \partial D$. NUv may be considered as a natural weak characterization of the normal derivative of $Uv|_G$ at $\partial G = \partial D$. There is a uniquely determined $\mu \in \mathcal{C}^*(\partial D)$ representing the functional (1) in the sense that

$$\langle \varphi, NUv \rangle = \int_{\partial D} \varphi \, d\mu, \quad \varphi \in \mathcal{D};$$

if we identify $\mu \equiv NUv$ then, moreover, the operator

$$(2) \quad NU: v \mapsto NUv$$

is bounded on $\mathcal{C}^*(\partial D)$ normed by total variation. (This follows from the fact that there is a constant $n < \infty$ such that each straight line in \mathbf{R}^3 which is not parallel to the coordinate planes intersects ∂D at most n -times; cf. Th. 1.13 in [9].)

We shall describe two examples illustrating the difficulties connected with the natural question of invertibility of the operator (2) on $\mathcal{C}^*(\partial D)$. An affirmative answer (which follows from a recent result in [17]) means that the generalized Neumann problem for G with an arbitrarily prescribed normal derivative $\mu \in \mathcal{C}^*(\partial G)$ has always a solution representable as a Newtonian potential Uv of a suitable $v \in \mathcal{C}^*(\partial G)$.

Similarly, if $\mathcal{C}(\partial G)$ is the space of all continuous functions on ∂D then for each $f \in \mathcal{C}(\partial D)$ the classical double layer potential Wf with momentum density f on ∂D can be defined and represents a harmonic function on $\mathbf{R}^3 \setminus \partial D$ admitting continuous extension from D to \bar{D} . If $W_i f$ denotes the restriction to ∂D of this extension, then the operator

$$(3) \quad W_i: f \mapsto W_i f$$

is bounded on $\mathcal{C}(\partial D)$ equipped with the usual maximum norm (and the operator (2) is dual to (3)). Invertibility of (3) on $\mathcal{C}(\partial D)$ is another natural question connected with representability of solutions of the classical Dirichlet problem by means of double layer potentials with continuous momentum densities. Let us remark that in the plane \mathbf{R}^2 such problems for suitably normalized logarithmic potentials have been satisfactorily settled because the corresponding operator W_i can be decomposed into

$$\frac{1}{2}I + T + Z = W_i,$$

where I is the identity operator, T is a compact operator and Z is a bounded operator on $\mathcal{C}(\partial D)$ whose norm is less than $\frac{1}{2}$; this is true for domains $D \subset \mathbf{R}^2$ whose boundary has bounded rotation in the sense of Radon (cf. [5], [10], [1], [11]) and also for some more general boundaries admitting infinitely many angular points with angles

exceeding a positive constant (compare [7]). The Riesz-Schauder theory then permits to apply the Fredholm theorems and obtain the desired results concerning boundary value problems. In the space \mathbf{R}^3 , however, the situation is entirely different. We are going to exhibit examples of simple rectangular domains $D \subset \mathbf{R}^3$ for which such decomposition of W_i is impossible because the distance of $W_i - \frac{1}{2}I$ from the space \mathcal{Q} of all compact linear operators acting on $\mathcal{C}(\partial D)$ (in the metric defined by the maximum norm) exceeds the critical value $\frac{1}{2}$ and also the distance of $W_i - \alpha I$ from \mathcal{Q} exceeds $|\alpha|$ for any value of the parameter α . Nevertheless, the situation can be saved if a new norm (inducing the same topology of uniform convergence) is introduced in $\mathcal{C}(\partial D)$ (cf. also [17] dealing with general rectangular sets).

Let us remark that Calderón's result on boundedness of Cauchy's integral operators in L_p -spaces opened the way to the investigation of representability of solutions of boundary value problems by layer potentials whose densities are in L_p -spaces (cf. [3]); for $1 < p < \infty$ the picture is different from that considered here and Verchota's inequality relating the L_2 -norms of normal derivatives from both sides of the boundary for potentials with L_2 -densities permits to avoid the difficulties connected with the approximation by compact operators occurring in the Radon scheme. The reader is referred to [13] (cf. also [6]) for results and references concerning the application of layer potentials with densities in L_p , $1 < p < \infty$.

Notation. We shall consider rectangular sets $D \neq \emptyset$ in \mathbf{R}^3 . If $y \in \partial D$, then $n(y)$ will denote the unit vector of the exterior normal to D at y provided this is meaningful; for definiteness we put $n(y) = 0$ (= the zero vector in \mathbf{R}^3) in the case when y is situated on an edge where the normal is not defined. The symbol σ will be used to denote the 2-dimensional surface measure. Given $z \in \mathbf{R}^3$ we define the signed measure λ_z on Borel subsets of ∂D by

$$d\lambda_z(y) = \frac{n(y) \cdot (y - z)}{4\pi|y - z|^3} d\sigma(y)$$

so that, up to the normalizing factor, $d\lambda_z$ is just the element of the oriented spatial angle under which ∂D is visible from z ; the normalization is so chosen that $\lambda_z(\partial D) = 1$ for any z in D in case D is bounded, while $\lambda_z(\partial D) = -1$ for any $z \in \mathbf{R}^3 \setminus \bar{D}$ in case D is unbounded. Note that, in case $z \in \partial D$, $y - z$ is orthogonal to $n(y)$ for all y in those faces in ∂D whose closure contains z . Consequently, for any fixed $z \in \partial D$, the function

$$y \mapsto \frac{n(y) \cdot (y - z)}{4\pi|y - z|^3}$$

is bounded on ∂D and the measure λ_z is of bounded variation. $\mathcal{C}(\partial D)$ is the space of all continuous functions on ∂D equipped with the topology of uniform convergence; $\|\dots\|$ is the usual maximum norm in $\mathcal{C}(\partial D)$. For $f \in \mathcal{C}(\partial D)$ we define the corresponding double layer potential by

$$Wf(z) = \int_{\partial D} f d\lambda_z.$$

Wf is a harmonic function on $\mathbf{R}^3 \setminus \partial D$ tending to zero at infinity. We put

$$\Omega_r(y) = \{x \in \mathbf{R}^3; |x - y| < r\}$$

and denote by

$$d(y) = \lim_{r \downarrow 0} \frac{\text{volume}(\Omega_r(y) \cap D)}{\text{volume}(\Omega_r(y))}$$

the density of D at y . Then for any $y \in \partial D$,

$$\lim_{\substack{z \rightarrow y \\ z \in D}} Wf(z) = Wf(y) + [1 - d(y)]f(y) = \bar{W}f(y) + \frac{1}{2}f(y),$$

where we have put

$$\bar{W}f(y) = Wf(y) + [\frac{1}{2} - d(y)]f(y), \quad y \in \partial D, \quad f \in \mathcal{C}(\partial D).$$

The symbol I will always denote the identity operator on $\mathcal{C}(\partial D)$. Since for any $f \in \mathcal{C}(\partial D)$

$$W_i f(y) = \lim_{\substack{z \rightarrow y \\ z \in D}} Wf(z)$$

is a continuous function of the variable $y \in \partial D$, the so-called direct double layer operator

$$\bar{W} = W_i - \frac{1}{2}I$$

is a bounded operator acting on $\mathcal{C}(\partial D)$ (cf. Th. 2.19 and Prop. 2.20 in [8]). The Dirichlet problem for D with the prescribed boundary condition $g \in \mathcal{C}(\partial D)$ can be reduced to the equation

$$(4) \quad (\frac{1}{2}I + \bar{W})f = g$$

for an unknown $f \in \mathcal{C}(\partial D)$. For a rectangular set D the corresponding operator \bar{W} can never be compact, because the smoothness of the boundary (of class $\mathcal{C}^{(1)}$) is necessary for the compactness of \bar{W} on $\mathcal{C}(\partial D)$ (cf. Th. 3.15 in [8]). \mathcal{Q} will stand for the space of all compact linear operators acting on $\mathcal{C}(\partial D)$. If S is a bounded linear operator on $\mathcal{C}(\partial D)$, we denote by

$$\omega S = \inf \{\|S - T\|; T \in \mathcal{Q}\}$$

its distance from \mathcal{Q} . The Radon scheme permits to apply the Riesz-Schauder theory to the equation (4) under the assumption that

$$(5) \quad \omega \bar{W} < \frac{1}{2}.$$

The quantity $\omega \bar{W}$ (usually called the essential norm of \bar{W}) can be simply evaluated in geometric terms connected with D . Put

$$\Gamma = \{\theta \in \mathbf{R}^3; |\theta| = 1\}.$$

Given $y \in \mathbf{R}^3$, $\theta \in \Gamma$ and $r > 0$, we denote by $n_r(\theta, y)$ the number (possibly 0 or $+\infty$) of all points in $\{y + \varrho\theta; 0 < \varrho < r\} \cap D$. The function

$$\theta \mapsto n_r(\theta, y)$$

is Borel measurable on Γ (cf. Lemma 5.3 in [9]; note also that it coincides a.e.

on Γ with the function $n_r^D(\cdot, y)$ investigated in 1.11 in [9]) so that we may define

$$v_r(y) = \frac{1}{4\pi} \int_{\Gamma} n_r(\theta, y) d\sigma(\theta).$$

Then $\omega \bar{W}$ can be evaluated by the formula

$$(6) \quad \omega \bar{W} = \limsup_{r \downarrow 0} \{v_r(y); y \in \partial D\}$$

(cf. Th. 4.1 in [8]).

We are now going to discuss an example of a rectangular set $D \subset \mathbb{R}^3$ for which $\omega \bar{W}$ exceeds the critical value $\frac{1}{2}$.

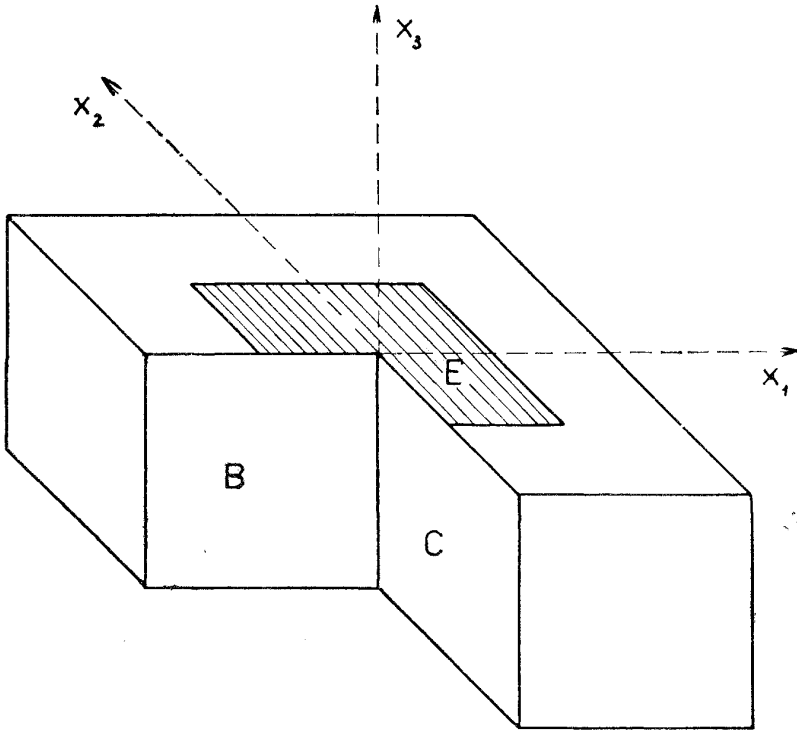


Fig. 1

Example 1.

Put

$$\begin{aligned} D &= (-1, 1) \times (0, 1) \times (-1, 0) \cup (0, 1) \times (-1, 1) \times (-1, 0), \\ B &= (-1, 0) \times \{0\} \times (-1, 0), \\ C &= \{0\} \times (-1, 0) \times (-1, 0), \\ E &= \langle -\frac{1}{2}, \frac{1}{2} \rangle \times \langle 0, \frac{1}{2} \rangle \times \{0\} \cup \langle 0, \frac{1}{2} \rangle \times \langle -\frac{1}{2}, \frac{1}{2} \rangle \times \{0\} \end{aligned}$$

(cf. Fig. 1). Consider a variable point $y = [0, y_2, y_3] \in C$ and fix $r \in (0, \frac{1}{2})$. If y approaches the origin, then the normalized spatial angle under which E is visible from y tends to $3/8$, so that

$$\lim_{\substack{y \rightarrow 0 \\ y \in C}} |\lambda_y(\Omega_r(y) \cap E)| = \frac{3}{8};$$

if, at the same time, the ratio y_2/y_3 tends to zero (so that y comes much closer to B than to E), then the normalized spatial angle under which B is visible from y approaches $\frac{1}{4}$, whence

$$|\lambda_y(\Omega_r(y) \cap B)| \rightarrow \frac{1}{4} \text{ as } \frac{y_2}{y_3} \rightarrow 0, \quad y_3 \rightarrow 0, \quad y \in C.$$

We see that

$$v_r(y) = |\lambda_y(\Omega_r(y) \cap E)| + |\lambda_y(\Omega_r(y) \cap B)| \rightarrow \frac{5}{8}$$

as

$$\frac{y_2}{y_3} \rightarrow 0, \quad y_3 \rightarrow 0, \quad y \in C.$$

Consequently, for any $r \in (0, \frac{1}{2})$,

$$\sup \{v_r(y); y \in \partial D\} = \frac{5}{8}$$

and, in view of (6), $\omega \bar{W} \geq \frac{5}{8} > \frac{1}{2}$.

Remark. One might also think of decomposing the operator W_i into

$$W_i = (W_i - \alpha I) + \alpha I$$

with another value of the parameter $\alpha \neq 0$ (for $\alpha = \frac{1}{2}$ we get $W_i - \frac{1}{2}I = \bar{W}$). It follows from the reasoning described in 4.2 in [9] that, for our set D , we have always

$$\omega(W_i - \alpha I) > |\alpha|;$$

the distance (if measured by the norm $\|\dots\|$) of the operator $1/\alpha W_i - I$ from Q can never be made smaller than 1 once we have

$$\limsup_{r \downarrow 0} \{v_r(y); y \in \partial D\} > \frac{1}{2}.$$

In order to overcome this difficulty we are now going to introduce a new norm in $\mathcal{C}(\partial D)$ inducing the same topology of uniform convergence.

Let

$$\begin{aligned} E_1 &= \langle 0, \frac{1}{2} \rangle \times \langle 0, \frac{1}{2} \rangle \times \{0\}, \\ E_2 &= \langle -\frac{1}{2}, 0 \rangle \times \langle 0, \frac{1}{2} \rangle \times \{0\}, \\ E_3 &= \langle 0, \frac{1}{2} \rangle \times \langle -\frac{1}{2}, 0 \rangle \times \{0\}. \end{aligned}$$

Fix a constant $q \in (1, \frac{4}{3})$ whose precise value will be specified later and define the function w on ∂D as follows:

$$\begin{aligned} w(y) &= \frac{1}{4}q \text{ for } y \in E_1, \quad w(y) = \frac{3}{4}q \text{ for } y \in E_2 \cup E_3, \\ w(y_1, y_2, -1) &= w(y_1, y_2, 0) \text{ if } [y_1, y_2, 0] \in E, \\ w(y) &= 1 \text{ for all remaining } y \in \partial D. \end{aligned}$$

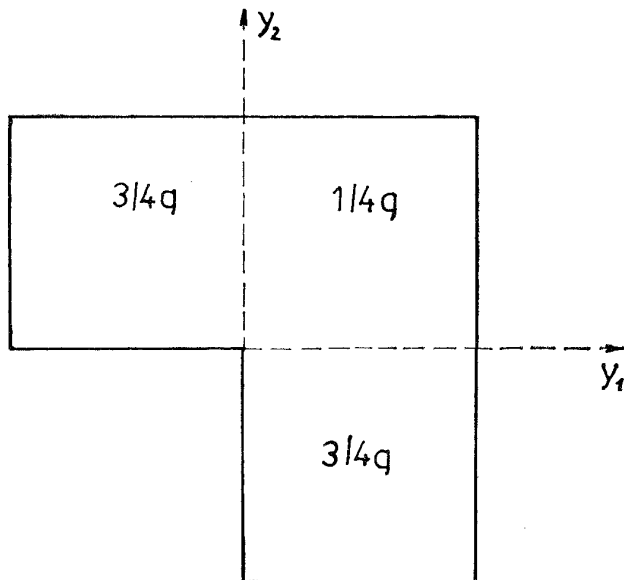


Fig. 2

The distribution of values of w in E is illustrated in Fig. 2.

Clearly, w is lower semicontinuous on ∂D and satisfies the inequalities

$$\frac{1}{4} < w \leq 1.$$

We shall now define the norm $|\dots|_{\mathcal{C}_w}$ in $\mathcal{C}(\partial D)$ by

$$|f|_{\mathcal{C}_w} = \max \left\{ \frac{|f(y)|}{w(y)}; y \in \partial D \right\}, \quad f \in \mathcal{C}(\partial D).$$

Obviously,

$$\|f\| \leq |f|_{\mathcal{C}_w} \leq 4\|f\|, \quad f \in \mathcal{C}(\partial D),$$

so that the space Q of all compact linear operators acting on the Banach space $\mathcal{C}(\partial D)$ with the new norm $|\dots|_{\mathcal{C}_w}$ remains the same. We shall prove that the distance

$$\omega_{\mathcal{C}_w}(\bar{W}) = \inf \{ |\bar{W} - T|_{\mathcal{C}_w}; T \in Q \}$$

can be made smaller than $\frac{1}{2}$ by a suitable choice of the constant q occurring in the definition of w . Let us fix $r \in (0, \frac{1}{4})$, denote by $c_{y,r}$ the characteristic function of $\partial D \setminus \Omega_r(y)$ and define

$$W_r f(y) = \int_{\partial D} c_{y,r} \cdot f \, d\lambda_y, \quad y \in \partial D, \quad f \in \mathcal{C}(\partial D).$$

It is not difficult to verify that the operator

$$W_r: f \mapsto W_r f$$

is compact on $\mathcal{C}(\partial D)$ (compare the proof of Th. 3.6 in [8]). We shall now investigate the operator

$$Z_r = \bar{W} - W_r.$$

In order to estimate $|Z_r|_{\mathcal{C}_w}$ we fix an arbitrary $f \in \mathcal{C}(\partial D)$ with $|f|_{\mathcal{C}_w} \leq 1$ and consider the ratio

$$(7) \quad |Z_r f(y)|/w(y).$$

Suppose first that $y = [y_1, y_2, y_3] \in \partial D$, $\max_{1 \leq k \leq 3} |y_k| \leq \frac{1}{4}$. We shall distinguish the cases (I)–(III) described below (where $\text{int } M$ always denotes the interior of $M \subset \partial D$ with respect to ∂D).

$$(I) \quad y \in \text{int } E_k, \quad 1 \leq k \leq 3:$$

It is easily seen that

$$|\lambda_y|(\Omega_r(y) \cap \partial D) \leq |\lambda_y(B)| + |\lambda_y(C)| \leq \begin{cases} \frac{1}{2} \cdot \frac{1}{4} & \text{if } y \in E_1, \\ \frac{1}{2} \cdot \frac{3}{4} & \text{if } y \in E_2 \cup E_3, \end{cases}$$

so that

$$|Z_r f(y)| \leq |\lambda_y(B)| + |\lambda_y(C)| \leq \frac{w(y)}{q} \cdot \frac{1}{2}$$

and the ratio (7) is bounded by $1/2q$.

$$(II) \quad y \in B:$$

Put

$$\hat{E} = E_3 \cup E_2$$

and consider the normalized spatial angle

$$(8) \quad |\lambda_y(\hat{E})| = |\lambda_y(E_3)| + |\lambda_y(E_2)|.$$

Let

$$e^1 = [-1, 0, 0], \quad e^2 = [0, 1, 0]$$

and denote by P_j ($j = 1, 2$) the half-plane determined by the straight line $\{ty; t \in \mathbf{R}^1\}$ and the half-line $\{te^j; t \geq 0\}$. Let V be that part of the unit sphere Γ which is enclosed between P_1 and P_2 . Since P_1, P_2 are orthogonal, the area of V equals

$$\sigma(V) = \pi.$$

Defining

$$\check{E}_3 = \{y - (x - y); x \in E_3\}$$

by reflection of E_3 at y , we have

$$(9) \quad |\lambda_y(E_3)| = |\lambda_y(\check{E}_3)|.$$

Observe that \check{E}_3 (as well as E_2) is contained in the wedge bounded by the half-planes P_1, P_2 and that the rays from y through the points of \check{E}_3 are different from all rays from y through the points in E_2 , because the latter are directed upwards whereas the rays from y through \check{E}_3 are directed downwards due to $y_3 < 0$. Consequently,

$$|\lambda_y(E_2)| + |\lambda_y(\check{E}_3)| \leq \frac{\sigma(V)}{4\pi} = \frac{1}{4},$$

whence we get by (9), (8)

$$|\lambda_y(\hat{E})| \leq \frac{1}{4}.$$

Further we have

$$|\lambda_y(E_1)| \leq \frac{1}{4} \cdot \frac{1}{2},$$

$$|\lambda_y(C)| \leq \frac{1}{2} \cdot \frac{1}{2}.$$

Noting that $|f| \leq w$ we summarize

$$\begin{aligned} |Z_r f(y)| &\leq \frac{1}{4}q|\lambda_y(E_1)| + \frac{3}{4}q|\lambda_y(E_2)| + \frac{3}{4}q|\lambda_y(E_3)| + |\lambda_y(C)| \leq \\ &\leq \frac{1}{2} \left(\frac{1}{16}q + \frac{3}{8}q + \frac{1}{2} \right) = \frac{1}{2} \cdot \frac{7q + 8}{16} w(y), \end{aligned}$$

so that the ratio (7) is bounded by

$$\frac{1}{2} \cdot \frac{7q + 8}{16}$$

in this case.

$$(III) \quad y \in C :$$

In view of the symmetry of D and w we again have

$$\frac{|Z_r f(y)|}{w(y)} \leq \frac{1}{2} \cdot \frac{7q + 8}{16}.$$

We have thus seen that in all cases (I)–(III) the estimate

$$(10) \quad \frac{|Z_r f(y)|}{w(y)} \leq \frac{1}{2} \max\left(\frac{1}{q}, \frac{7q + 8}{16}\right)$$

holds for any y in the intersection of the sets

$$H_1 = B \cup C \cup \bigcup_{k=1}^3 \text{int } E_k$$

and

$$H_2 = \{y = [y_1, y_2, y_3] \in \partial D; \max_{1 \leq k \leq 3} |y_k| \leq \frac{1}{2}\}.$$

It follows from the construction of w that every $y \in H_2$ is either contained in H_1 or is a limit of a sequence $y^n \in H_1 \cap H_2$ with $\lim_{n \rightarrow \infty} w(y^n) = w(y)$. Since $\lim_{n \rightarrow \infty} Z_r f(y^n) = Z_r f(y)$ because of continuity of $Z_r f$, we conclude that the inequality (10) holds for all $y \in H_2$. Since the vertex $e^3 = [0, 0, -1]$ has a similar position as the origin in ∂D , we conclude by symmetry that (10) remains valid for all $y = [y_1, y_2, y_3] \in \partial D$ satisfying

$$|y_1| \leq \frac{1}{4}, \quad |y_2| \leq \frac{1}{4}, \quad |y_3 + 1| \leq \frac{1}{4}.$$

Even simpler considerations show that for the remaining $y \in \partial D$ the estimate

$$\frac{|Z_r f(y)|}{w(y)} \leq \frac{1}{2} \max\left(\frac{2}{3q}, \frac{3}{4}\right) \leq \frac{1}{2} \max\left(\frac{1}{q}, \frac{7q + 8}{16}\right)$$

holds. Thus (10) is valid for all $y \in \partial D$. Since $f \in \mathcal{C}(\partial D)$ was an arbitrary function with $|f|_{\mathcal{G}_w} \leq 1$, we have

$$(11) \quad |Z_r|_{\mathcal{G}_w} \leq \frac{1}{2} \max \left(\frac{1}{q}, \frac{7q+8}{16} \right).$$

Let $q_0 = (\sqrt{(138) - 4})/7$, which is the positive root of the equation $1/q = (7q+8)/16$. Since $1 < q_0 < 8/7$ we may take $q = q_0$ in (11) and conclude that

$$\omega_{\mathcal{G}_w}(\bar{W}) \leq |Z_r|_{\mathcal{G}_w} \leq \frac{1}{2q_0} < \frac{1}{2}.$$

In order to exhibit another type of critical corners we shall consider the following example.

Example 2. Put

$$\begin{aligned} D &= (-2, 1) \times (-1, 0) \times (-1, 0) \cup (-1, 0) \times (0, 1) \times (-2, 1) \cup (-1, 0) \times \\ &\quad \times \{0\} \times (-1, 0), \\ C &= \{0\} \times (0, 1) \times (-2, 1), \\ C_1 &= \{[0, x_2, x_3] \in C; |x_3| \leq \sqrt{(3)} x_2\}, \\ C_2 &= C \setminus C_1, \\ E &= \langle -2, 1 \rangle \times \langle -1, 0 \rangle \times \{0\}, \\ E_1 &= \{[x_1, x_2, 0] \in E; |x_1| \leq -\sqrt{(3)} x_2\}, \\ B &= (0, 1) \times \{0\} \times (-1, 0), \\ B_1 &= \{[x_1, 0, x_3] \in B; x_3 \geq -x_1\}, \\ B_2 &= B \setminus B_1 \end{aligned}$$

(cf. Fig. 3).

In this case we have, for any $r > 0$,

$$\limsup_{\substack{y \rightarrow 0 \\ y \in \partial D}} v_r(y) = \frac{1}{2}$$

and the quantity

$$\omega \bar{W} = \limsup_{r \downarrow 0} \{v_r(y); y \in \partial D\}$$

assumes precisely the critical value $1/2$. We shall describe the construction of a lower semicontinuous function w on ∂D satisfying the inequality

$$(12) \quad 5/6 < w \leq 1$$

for which the norm $|\dots|_{\mathcal{G}_w}$ makes the distance

$$\omega_{\mathcal{G}_w}(\bar{W}) < \frac{1}{2}.$$

If $e = [e_1, e_2, e_3]$ and $\varrho > 0$, let

$$(13) \quad K_\varrho(e) = \left\{ y \in \mathbf{R}^3; \max_{1 \leq k \leq 3} |y_k - e_k| \leq \frac{1}{2}\varrho \right\};$$

if $e = 0$ is the origin, we write simply $K_\varrho = K_\varrho(0)$.

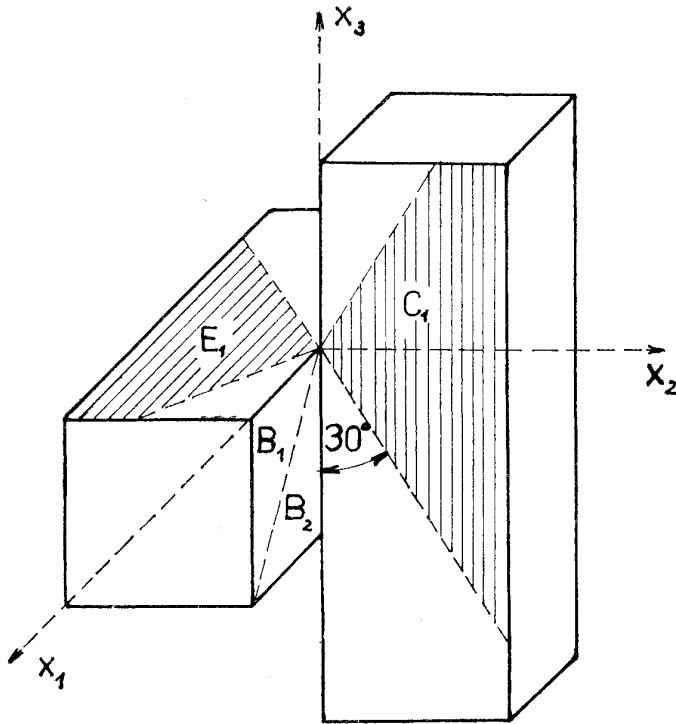


Fig. 3

Define

$$w(y) = \begin{cases} \frac{5}{6}q & \text{for } y \in K_1 \cap C_1, \\ \frac{5}{6}q & \text{for } y \in K_1 \cap E_1, \\ 1 & \text{for the remaining } y \in K_1 \cap \partial D, \end{cases}$$

where $q \in (1, \frac{6}{5})$ is a constant whose value will be specified later. If e is any of the points $e^1 = [-1, 0, 0]$, $e^3 = [0, 0, -1]$ and $e^1 + e^3 = [-1, 0, -1]$, then w is transferred into the intersection of ∂D with

$$K_1(e)$$

using the symmetry of ∂D with respect to the planes $\{x \in \mathbb{R}^3; x_3 = -\frac{1}{2}\}$ and $\{x \in \mathbb{R}^3; x_1 = -\frac{1}{2}\}$. Finally, we let $w(y) = 1$ for all $y \in \partial D$ not situated in any of the cubes $K_1, K_1(e)$, $e = e^1, e^3, e^1 + e^3$. Clearly, w is a lower semicontinuous function on ∂D satisfying (12). Let us fix an arbitrary $f \in \mathcal{C}(\partial D)$ with $|f|_{\mathcal{C}, w} \leq 1$ and $r \in (0, \frac{1}{4})$. We shall again examine the ratio (7) as in Example 1. Let us first consider $y \in K_{1/2} \cap \partial D$ and distinguish the following cases (A)–(D).

(A) $y \in B_1$:

Then the normalized spatial angle under which E is visible from y is estimated by

$$|\lambda_y(E)| \leq \frac{1}{4}.$$

Further we have

$$|\lambda_y(C_1)| \leq \frac{1}{6}.$$

Let

$$p = \{[x_1, 0, x_3]; x_3 = -x_1\},$$

denote by P_1 the half-plane determined by the straight line p and the half-line $\{te^3; t \geq 0\}$ and let P_2 be the half-plane determined by p and the half-line

$$\{[0, x_2, x_3]; 0 \geq x_3 = -\sqrt{(3)} x_2\}.$$

The angle enclosed by P_1 and P_2 equals

$$\arctan \sqrt{\left(\frac{2}{3}\right)} < \frac{2\pi}{9}.$$

Writing

$$(14) \quad \varepsilon = \frac{9}{2\pi} \arctan \sqrt{\left(\frac{2}{3}\right)} \in (0, 1)$$

we get for

$$C_2^- = \{[0, x_2, x_3] \in C; x_3 < -\sqrt{(3)} x_2, x_2 > 0\},$$

$$C_2^+ = \{-x; x \in C_2^-\}$$

the estimate

$$|\lambda_y(C_2^-)| + |\lambda_y(C_2^+)| \leq \frac{\varepsilon}{9}$$

which is based on a reasoning similar to that occurring in case (II) of Example 1 in connection with $|\lambda_y(E_3)| + |\lambda_y(E_2)|$. Clearly,

$$|\lambda_y(C \setminus C_1)| = |\lambda_y(C_2^-)| + |\lambda_y(C_2^+)| \leq \frac{\varepsilon}{9}.$$

Hence we obtain

$$\frac{|Z_r f(y)|}{w(y)} = |Z_r f(y)| \leq \frac{1}{4} + \frac{\varepsilon}{9} + \frac{5q}{6 \cdot 6} = \frac{1}{2} \cdot \frac{9 + 4\varepsilon + 5q}{18}.$$

(B)

$$y \in B_2 :$$

Now we get

$$|\lambda_y(C)| \leq \frac{1}{4},$$

$$|\lambda_y(E_1)| \leq \frac{1}{6},$$

$$|\lambda_y(E \setminus E_1)| \leq \varepsilon/9,$$

so that again

$$\frac{|Z_r f(y)|}{w(y)} = |Z_r f(y)| \leq \frac{1}{2} \cdot \frac{9 + 4\varepsilon + 5q}{18},$$

where ε is defined by (14).

(C) $y \in C_2 :$

The reasoning described in case (II) of Example 1 yields for

$$\tilde{B} = \{[x_1, x_2, x_3] \in \partial D; x_2 = 0\}$$

the estimate

$$|\lambda_y(\tilde{B})| \leq \frac{1}{4}.$$

Further we have

$$|\lambda_y(E_1)| \leq \frac{1}{2 \cdot 3},$$

$$|\lambda_y(E \setminus E_1)| \leq \frac{1}{2 \cdot 6},$$

whence

$$\frac{|Z_r f(y)|}{w(y)} = |Z_r f(y)| \leq \frac{1}{2} \left(\frac{1}{2} + \frac{5q}{3 \cdot 6} + \frac{1}{6} \right) = \frac{1}{2} \cdot \frac{12 + 5q}{18}.$$

(D) $y \in C_1 :$

Again

$$|\lambda_y(\tilde{B})| \leq \frac{1}{4}.$$

Now

$$|\lambda_y(E)| \leq \frac{1}{2 \cdot 3},$$

so that

$$|Z_r f(y)| \leq \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} \right) = \frac{1}{2} \cdot \frac{5}{6} = \frac{1}{2q} w(y),$$

$$\frac{|Z_r f(y)|}{w(y)} \leq \frac{1}{2q}.$$

Since this estimate holds also for $y \in E_1$ while for $y \in \tilde{B} \setminus B$ we get, by symmetry, from (A), (B) the estimate

$$\frac{|Z_r f(y)|}{w(y)} \leq \frac{1}{2} \cdot \frac{9 + 4\varepsilon + 5q}{18},$$

we conclude that, for any $y \in K_{1/2} \cap \partial D$,

$$\frac{|Z_r f(y)|}{w(y)} \leq \frac{1}{2} \max \left(\frac{1}{q}, \frac{12 + 5q}{18}, \frac{9 + 4\varepsilon + 5q}{18} \right) = \frac{1}{2} \max \left(\frac{1}{q}, \frac{9 + 4\varepsilon + 5q}{18} \right).$$

The same estimate holds for $y \in K_{1/2}(e) \cap \partial D$, $e = e^1, e^3, e^1 + e^3$. Since an even better estimate can be established for the remaining $y \in \partial D$ we arrive at

$$|Z_r f|_{\mathcal{G}_w} \leq \frac{1}{2} \max \left(\frac{1}{q}, \frac{9 + 4\varepsilon + 5q}{18} \right).$$

If q_1 is the positive root of the equation

$$\frac{1}{q} = \frac{9 + 4\varepsilon + 5q}{18},$$

one easily checks that $q_1 \in (1, \frac{6}{5})$. Taking $q = q_1$ we obtain

$$\omega_{\mathcal{C}_w}(\bar{W}) \leq \frac{1}{2q_1} < \frac{1}{2}.$$

The discussion of the above examples permits to establish the following results.

Let us agree for a while to denote by D_1 and D_2 the rectangular set described in Example 1 and Example 2, respectively; let w_j be the corresponding lower semicontinuous function constructed on ∂D_j ($j = 1, 2$). Consider an arbitrary rectangular set $D \subset \mathbf{R}^3$. We shall say that a point $z \in \partial D$ is critical if, for every $\varrho > 0$,

$$\limsup_{\substack{y \rightarrow z \\ y \in \partial D}} v_\varrho(y) \geq \frac{1}{2};$$

obviously, any critical $z \in \partial D$ is a vertex. D will be termed admissible if, for every critical $z \in \partial D$ there is a $\varrho \in (0, \frac{1}{4})$ and an isometric mapping of $K_\varrho(z) \cap \partial D$ onto one of the sets $K_\varrho \cap \partial D_j$ ($j = 1$ or 2).

Proposition. *If D is an admissible rectangular set in \mathbf{R}^3 with $\partial D \neq \emptyset$ and \bar{W} is the corresponding direct double layer operator on $\mathcal{C}(\partial D)$, then there is a lower semicontinuous function v on ∂D satisfying the inequalities*

$$(15) \quad \frac{1}{4} < v \leq 1$$

such that

$$(16) \quad \omega_{\mathcal{C}_w}(\bar{W}) < \frac{1}{2}.$$

Proof. Let us fix $\varrho \in (0, \frac{1}{4})$ which is less than half of the distance of any two different vertices in ∂D and satisfies

$$\sup \{v_\varrho(y); y \in K_\varrho(e) \cap \partial D\} < \frac{1}{2}$$

for each noncritical vertex $e \in \partial D$. Take ϱ small enough to guarantee that for each critical $z \in \partial D$ there is an isometric mapping τ_z of $K_\varrho(z) \cap \partial D$ onto $K_\varrho \cap \partial D_j$ ($j = 1$ or 2); in this case we define $v(x) = w_j(\tau_z(x))$, $x \in K_\varrho(z) \cap \partial D$.

Letting z run over all critical vertices in ∂D we put

$$v(y) = 1 \quad \text{for } y \in \partial D \setminus \bigcup_z K_\varrho(z).$$

In this way we obtain a lower semicontinuous function v on ∂D satisfying (15) and the discussion described in the above examples yields the inequality (16).

The following corollaries of the above theorem can be proved by arguments described in § 5 of [9].

Corollary 1. *Let D be an admissible rectangular set in \mathbf{R}^3 with $\partial D \neq \emptyset$, $G = \mathbf{R}^3 \setminus \bar{D}$, $\mu \in \mathcal{C}^*(\partial D)$.*

Then the generalized Neumann problem

$$\text{NU}v = \mu$$

for G with the boundary condition μ admits a solution $v \in \mathcal{C}^*(\partial D)$ iff $\mu(\partial H) = 0$ for each bounded component H of G ; the solution v is uniquely determined iff G is unbounded and connected.

Corollary 2. Let D be an admissible rectangular set in \mathbf{R}^3 with $\partial D \neq \emptyset$ and let G_1, \dots, G_p ($p \geq 0$) be bounded components of $G = \mathbf{R}^3 \setminus \bar{D}$; fix $x_j \in G_j$ ($j = 1, \dots, p$).

Given $g \in \mathcal{C}(\partial D)$, then there are uniquely determined constants c_1, \dots, c_p and an $f \in \mathcal{C}(\partial D)$ such that the function

$$(17) \quad x \rightarrow Wf(x) + \sum_{j=1}^p c_j |x - x_j|^{-1}$$

represents a solution of the Dirichlet problem for D and the boundary condition g ; f is uniquely determined iff D is bounded and G connected (in which case the sum

$\sum_{j=1}^p \dots$ in (17) disappears).

Remark. The word “admissible” can be dropped in the above corollaries. This follows from a recent result in [17].

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Souhrn

NĚKTERÉ PŘÍKLADY NA APLIKABILITU FREDHOLMOVY-RADONOVY METODY V TEORII POTENCIÁLU

JOSEF KRÁL, WOLFGANG WENDLAND

Jsou vyšetřeny jednoduché příklady omezených oblastí $D \subset R^3$, pro něž hrany a vrcholy na hranici ∂D způsobují, že operátor potenciálu dvojvrstvy na prostoru $\mathcal{C}(\partial D)$ všech spojitých funkcí na ∂D nelze pro žádnou hodnotu parametru α aproximovat (vzhledem k maximové normě) operátory tvaru $\alpha I + T$ (kde I je identický operátor a T je kompaktní lineární operátor) s chybou menší než $|\alpha|$; taková aproximace je však možná po zavedení nové normy na $\mathcal{C}(\partial D)$ odvozené od vhodné váhové funkce.

Резюме

НЕКОТОРЫЕ ПРИМЕРЫ ОТНОСИТЕЛЬНО ПРИМЕНИМОСТИ МЕТОДА ФРЕДГОЛМА-РАДОНА В ТЕОРИИ ПОТЕНЦИАЛА

JOSEF KRÁL, WOLFGANG WENDLAND

Рассматриваются простые примеры ограниченных областей $D \subset R^3$ с ребрами и вершинами на границе ∂D . В этих примерах оператор двойного слоя на пространстве $\mathcal{C}(\partial D)$ всех непрерывных функций на ∂D нельзя ни при каком значении параметра α аппроксимировать (относительно максимум-нормы) при помощи операторов $\alpha I + T$ (где I — тождественный оператор и T — компактный линейный оператор) с погрешностью меньше чем $|\alpha|$; с другой стороны, такая аппроксимация становится возможной для значения $\alpha = \frac{1}{2}$ после введения новой нормы на $\mathcal{C}(\partial D)$, определенной при помощи подходящей весовой функции.

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