SOME EXAMPLES OF ASYMPTOTICALLY MOST POWERFUL TESTS

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1. Introduction. In a previous paper[2] the author gave the definition of an asymptotically most powerful test and has shown that the commonly used tests, based on the maximum likelihood estimate, are asymptotically most powerful.

In this paper some further examples of asymptotically most powerful tests will be given. Let us first restate the definition of an asymptotically most powerful test. Let $f(x, \theta)$ be the probability density of a variate x involving an unknown parameter θ. For testing the hypothesis $\theta = \theta_0$ by means of n independent observations $x_1, \cdots, x_n$ on x we have to choose a region of rejection W_n in the n-dimensional sample space. Denote by $P(W_n \mid \theta)$ the probability that the sample point $E = (x_1, \cdots, x_n)$ will fall in W_n under the assumption that θ is the true value of the parameter. For any region U_n of the n-dimensional sample space denote by $g(U_n)$ the greatest lower bound of $P(U_n \mid \theta)$. For any pair of regions U_n and T_n denote by $L(U_n, T_n)$ the least upper bound of

$$P(U_n \mid \theta) - P(T_n \mid \theta).$$

In all that follows we shall denote a region of the n-dimensional sample space by a capital letter with the subscript n.

Definition 1: A sequence $\{W_n\}$ ($n = 1, 2, \cdots$, ad inf.) of regions is said to be an asymptotically most powerful test of the hypothesis $\theta = \theta_0$ on the level of significance α if $P(W_n \mid \theta_0) = \alpha$ and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n \mid \theta_0) = \alpha$ the inequality

$$\limsup_{n \to \infty} L(Z_n, W_n) \leq 0$$

holds.

Definition 2: A sequence $\{W_n\}$ ($n = 1, 2, \cdots$, ad inf.) of regions is said to be an asymptotically most powerful unbiased test of the hypothesis $\theta = \theta_0$ on the level of significance α if $P(W_n \mid \theta_0) = \lim_{n=\infty} g(W_n) = \alpha$, and if for any sequence $\{Z_n\}$ of regions for which $P(Z_n \mid \theta_0) = \lim_{n=\infty} g(Z_n) = \alpha$, the inequality

$$\limsup_{n \to \infty} L(Z_n, W_n) \leq 0$$

holds.

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[1] Research under a grant-in-aid of the Carnegie Corporation of New York.

[2] "Asymptotically most powerful tests of statistical hypotheses," *Annals of Math. Stat.* Vol. 12 (1941).

Consider the expression

$$y_n(\theta) = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} \frac{\partial}{\partial \theta} \log f(x_\alpha, \theta). \tag{1}$$

Let W'_n be the region defined by the inequality $y_n(\theta_0) \geq c'_n$, W''_n defined by the inequality $y_n(\theta_0) \leq c''_n$, and W_n defined by the inequality $|y_n(\theta_0)| \geq c_n$, where the constants c'_n, c''_n and c_n are chosen such that

$$P(W'_n \mid \theta_0) = P(W''_n \mid \theta_0) = P(W_n \mid \theta_0) = \alpha.$$

It will be shown in this paper that under certain restrictions on the probability density $f(x, \theta)$ the sequence $\{W'_n\}$ is an asymptotically most powerful test of the hypothesis $\theta = \theta_0$ if θ takes only values $\geq \theta_0$. Similarly $\{W''_n\}$ is an asymptotically most powerful test if θ takes only values $\leq \theta_0$. Finally $\{W_n\}$ is an asymptotically most powerful unbiased test if θ can take any real value.

Another example of an asymptotically most powerful unbiased test of the hypothesis $\theta = \theta_0$, as it will be shown, is the critical region of type A in the Neyman-Pearson theory of testing hypotheses. This fact gives a strong justification for the use of the critical region of type A.

2. Assumptions on the density function. Let ω be a subset of the real axis. Denote by θ^* a real variable which takes only values in ω and let θ be a variable which can take any real value. For any function $\psi(x)$ we denote by $E_\theta\psi(x)$ the expected value of $\psi(x)$ under the assumption that θ is the true value of the parameter, i.e.

$$E_\theta \psi(x) = \int_{-\infty}^{+\infty} \psi(x) f(x, \theta)\, dx.$$

For any x, for any positive δ and for any real value θ_1 denote by $\varphi_1(x, \theta_1, \delta)$ the greatest lower bound, and by $\varphi_2(x, \theta_1, \delta)$ the least upper bound of $\frac{\partial^2}{\partial \theta^2} \log f(x, \theta)$ in the interval $\theta_1 - \delta \leq \theta \leq \theta_1 + \delta$. In all that follows the symbol θ_i^*, for any integer i, will denote a value of θ^*, i.e., θ_i^* is a point of ω.

We say that a value θ lies in the ϵ-neighborhood of ω if there exists a value θ^* such that $|\theta - \theta^*| \leq \epsilon$.

Throughout the paper the following assumptions on $f(x, \theta)$ will be made:

ASSUMPTION 1: *For any pair of sequences* $\{\theta_n\}$ *and* $\{\theta_n^*\}$ $(n = 1, 2, \cdots$, ad inf.) *for which*

$$\lim_{n=\infty} E_{\theta_n} \frac{\partial}{\partial \theta} \log f(x, \theta_n^*) = 0$$

also

$$\lim_{n=\infty} (\theta_n - \theta_n^*) = 0.$$

Furthermore there exists a positive ϵ *such that* $E_\theta\left[\frac{\partial}{\partial\theta}\log f(x,\theta_1)\right]^2$ *is a bounded function of* θ *and* θ_1, $E_\theta\frac{\partial}{\partial\theta}\log f(x,\theta_1)$ *is a continuous function of* θ *and* θ_1 *and* $E_{\theta_1}\left[\frac{\partial}{\partial\theta}\log f(x,\theta_1)\right]^2 = d(\theta_1)$ *has a positive lower bound, where* θ_1 *can take any value in the* ϵ*-neighborhood of* ω.

ASSUMPTION 2: *There exists a positive* k_0 *such that* $E_{\theta_2}\varphi_1(x,\theta_1,\delta)$ *and* $E_{\theta_2}\varphi_2(x,\theta_1,\delta)$ *are uniformly continuous functions in the domain D defined as follows: the variables* θ_1 *and* θ_2 *may take any value in the* k_0*-neighborhood of* ω *and* δ *may take any value for which* $|\delta| \leq k_0$. *Furthermore it is assumed that*

$$E_{\theta_2}[\varphi_i(x,\theta_1,\delta)]^2, \qquad (i = 1, 2)$$

are bounded functions of θ_1, θ_2 *and* δ *in D.*

ASSUMPTION 3: *There exists a positive* k_0 *such that*

$$\int_{-\infty}^{+\infty}\frac{\partial}{\partial\theta}f(x,\theta)\,dx = \int_{-\infty}^{+\infty}\frac{\partial^2}{\partial\theta^2}f(x,\theta)\,dx = 0$$

for all θ *in the* k_0*-neighborhood of* ω.

Assumption 3 means simply that we may differentiate with respect to θ under the integral sign. In fact,

$$\int_{-\infty}^{+\infty} f(x,\theta)\,dx = 1,$$

identically in θ. Hence

$$\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty} f(x,\theta)\,dx = \frac{\partial^2}{\partial\theta^2}\int_{-\infty}^{+\infty} f(x,\theta)\,dx = 0.$$

Differentiating under the integral sign we obtain the relations in Assumption 3.

ASSUMPTION 4: *There exists a positive* k_0 *and a positive* η *such that*

$$E_\theta\left[\frac{\partial}{\partial\theta}\log f(x,\theta)\right]^{2+\eta}$$

is a bounded function of θ *in the* k_0*-neighborhood of* ω.

3. Some propositions. PROPOSITION 1: *To any positive* β *there exists a positive* γ *such that*

$$\lim_{n=\infty} P\left\{\frac{1}{\sqrt{n}}\,|y_n(\theta^*)| > \gamma \mid \theta\right\} = 1$$

uniformly in θ^* *and for all* θ *for which* $|\theta - \theta^*| \geq \beta$.

PROOF: From Assumption 1 it follows that $\left|E_\theta\frac{\partial}{\partial\theta}\log f(x,\theta^*)\right|$ has a positive lower bound in the domain $| \theta - \theta^* | \geq \beta$. Since according to Assumption 1 $E_\theta\left[\frac{\partial}{\partial\theta}\log f(x, \theta^*)\right]^2$ is a bounded function of θ and θ^*, Proposition 1 easily follows.

PROPOSITION 2: *There exists a positive ϵ such that*

$$\lim_{n=\infty} P[y_n(\theta) < t \mid \theta] = N(t \mid \theta)$$

uniformly in t and for all θ in the ϵ-neighborhood of ω where

$$d(\theta) = - E_\theta \frac{\partial^2}{\partial\theta^2} \log f(x, \theta) = E_\theta \left[\frac{\partial}{\partial\theta} \log f(x, \theta)\right]^2 \tag{2}$$

and

$$N(t \mid \theta) = \frac{1}{\sqrt{2\pi d(\theta)}} \int_{-\infty}^{t} e^{-\frac{1}{2}v^2/d(\theta)}\, dv. \tag{3}$$

Proposition 2 follows easily from Assumptions 3 and 4 and the general limit theorems.

PROPOSITION 3: *There exists a positive ϵ such that for any bounded sequence $\{\mu_n\}$*

$$\lim_{n=\infty} \left\{P\left[y_n(\theta) < t \mid \theta + \frac{\mu_n}{\sqrt{n}}\right] - \int_{-\infty}^{t} e^{\mu_n v - \frac{1}{2}\mu_n^2 d(\theta)}\, dN(v \mid \theta)\right\} = 0$$

uniformly in t and for all θ in the ϵ-neighborhood of ω.

PROOF: We have

$$y_n\left(\theta + \frac{\mu_n}{\sqrt{n}}\right) = y_n(\theta) + \frac{\mu_n}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n') \tag{4}$$

where θ_n' lies in the interval $\left[\theta, \theta + \frac{\mu_n}{\sqrt{n}}\right]$. From Assumption 2 and the above equation we easily obtain

$$\lim_{n=\infty} \left\{P\left[y_n\left(\theta + \frac{\mu_n}{\sqrt{n}}\right) < t \mid \theta + \frac{\mu_n}{\sqrt{n}}\right] - P\left[y_n(\theta) - \mu_n d(\theta) < t \mid \theta + \frac{\mu_n}{\sqrt{n}}\right]\right\} = 0 \tag{5}$$

uniformly in t and for all θ in the ϵ-neighborhood of ω. From Proposition 2 and (5) we get

$$\lim_{n=\infty} \left\{\int_{-\infty}^{t} dN(v \mid \theta) - P\left[y_n(\theta) < t + \mu_n d(\theta) \middle| \theta + \frac{\mu_n}{\sqrt{n}}\right]\right\} = 0$$

or

$$\lim_{n=\infty} \left\{\int_{-\infty}^{t-\mu_n d(\theta)} dN(v \mid \theta) - P\left[y_n(\theta) < t \mid \theta + \frac{\mu_n}{\sqrt{n}}\right]\right\} = 0 \tag{6}$$

uniformly in t and for all θ in the ϵ-neighbourhood of ω. This proves Proposition 3.

PROPOSITION 4: *There exists a positive* ϵ *such that for any positive* γ *and for any sequence* $\{\mu_n\}$ *for which* $\lim_{n=\infty} |\mu_n| = \infty$

$$\lim_{n=\infty} P\left\{| y_n(\theta^*) | > \gamma \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right\} = 1$$

uniformly in θ^*.

PROOF: If there exists a positive β such that $\left| \frac{\mu_n}{\sqrt{n}} \right| > \beta$ for almost all n, Proposition 4 follows from Proposition 1. Hence we have to consider only the case $\lim_{n=\infty} \frac{\mu_n}{\sqrt{n}} = 0$. Since

$$E_{\theta^*+(\mu_n/\sqrt{n})}\, y_n\left(\theta^* + \frac{\mu_n}{\sqrt{n}}\right) = 0,$$

we get from (4)

$$(7) \qquad E_{\theta^*+(\mu_n/\sqrt{n})}[y_n(\theta^*)] + \mu_n E_{\theta^*+(\mu_n/\sqrt{n})} \frac{\sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n')}{n} = 0.$$

Since $\lim \frac{\mu_n}{\sqrt{n}} = 0$, we have on account of Assumption 2

$$\lim_{n=\infty} E_{\theta^*+(\mu_n/\sqrt{n})} \frac{\sum_\alpha \frac{\partial^2}{\partial\theta^2} \log f(x_\alpha, \theta_n')}{n} = E_{\theta^*} \frac{\partial^2}{\partial\theta^2} \log f(x, \theta^*)$$

$$= -E_{\theta^*}\left[\frac{\partial}{\partial\theta} \log f(x, \theta^*)\right]^2 = -d(\theta^*)$$

uniformly in θ^*. According to Assumption 1 $d(\theta^*)$ has a positive lower bound; hence on account of $\lim |\mu_n| = \infty$ we obtain from (7)

$$(8) \qquad \lim_{n=\infty} | E_{\theta^*+(\mu_n/\sqrt{n})}\, y_n(\theta^*) | = \infty$$

uniformly in θ^*. The variance of $y_n(\theta^*)$ is equal to the variance of $\frac{\partial}{\partial\theta} \log f(x, \theta^*)$. On account of Assumption 1 the variance of $\frac{\partial}{\partial\theta} \log f(x, \theta^*)$ (under the assumption that $\theta^* + \frac{\mu_n}{\sqrt{n}}$ is the true value of the parameter) is a bounded function. Hence Proposition 4 is proved on account of (8).

PROPOSITION 5: *Let* $\{W_n(\theta^*)\}$ *be a sequence of regions of size* α, *i.e.* $P[W_n(\theta^*) \mid \theta^*] = \alpha$, *and let* $V_n(\theta^*, y)$ *be the region defined by the inequality*

$y_n(\theta^*) < y$. *Let* $U_n(\theta^*, y)$ *be the intersection of* $V_n(\theta^*, y)$ *and* $W_n(\theta^*)$ *and denote* $P[U_n(\theta^*, y) \mid \theta^*]$ *by* $F_n(y \mid \theta^*)$. *Denote furthermore* $P\left[W_n(\theta^*) \mid \theta^* + \frac{\mu}{\sqrt{n}}\right]$ *by* $G(\theta^*, \mu, n)$. *If* $\{\theta_n^*\}$ *and* $\{\mu_n\}$ *are two sequences such that* $\lim_{n=\infty} d(\theta_n^*) = d$; $\lim_{n=\infty} F_n(y \mid \theta_n^*) = F(y)$ *and* $\lim \mu_n = \mu$ *then*

$$\lim_{n=\infty} G(\theta_n^*, \mu_n, n) = \int_{-\infty}^{+\infty} e^{\mu y - \frac{1}{2}\mu^2 d} dF(y).$$

Proof: Let $\lim \mu_n = \mu$ and consider the Taylor expansion

$$\sum_\alpha \log f\left(x_\alpha, \theta^* + \frac{\mu_n}{\sqrt{n}}\right) = \sum_\alpha \log f(x_\alpha, \theta^*) + \frac{\mu_n}{\sqrt{n}} \sum_\alpha \frac{\partial}{\partial\theta} \log f(x_\alpha, \theta^*) + \frac{1}{2}\frac{\mu_n^2}{n}\frac{\partial^2}{\partial\theta^2}\sum_\alpha \log f(x_\alpha, \theta_n') \tag{9}$$

where θ_n' lies in the interval $\left[\theta^*, \theta^* + \frac{\mu_n}{\sqrt{n}}\right]$. From this we easily get on account of Assumption 2 and the fact that $\{\mu_n\}$ is bounded

$$\log \prod_{\alpha=1}^{n} \frac{f\left(x_\alpha, \theta^* + \frac{\mu_n}{\sqrt{n}}\right)}{f(x_\alpha, \theta^*)} = \mu_n y_n(\theta^*) - \tfrac{1}{2}\mu_n^2 d(\theta^*) + \epsilon(\theta^*, n) \tag{10}$$

where for arbitrary positive η

$$\lim_{n=\infty} P\left\{\mid \epsilon(\theta^*, n) \mid < \eta \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right\} = 1 \tag{11}$$

uniformly in θ^*. Denote by $R_n(\theta^*)$ the region defined by

$$\mid \epsilon(\theta^*, n) \mid < \eta > 0. \tag{12}$$

On account of (11) we have

$$\lim_{n=\infty} P\left[R_n(\theta^*) \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right] = 1, \tag{13}$$

uniformly in θ^*. Denote the intersection of $R_n(\theta^*)$ and $W_n(\theta^*)$ by $Q_n(\theta^*)$, and the intersection of $R_n(\theta^*)$ and $U_n(\theta^*, y)$ by $T_n(\theta^*, y)$. Furthermore denote $P[T_n(\theta^*, y) \mid \theta^*]$ by $\bar{F}_n(y \mid \theta^*)$. Then we have

$$e^{-\eta} \int_{-\infty}^{t} e^{\mu_n y - \frac{1}{2}\mu_n^2 d(\theta^*)} d\bar{F}_n(y \mid \theta^*) \leq P\left[T_n(\theta^*, t) \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right] \leq e^{\eta} \int_{-\infty}^{t} e^{\mu_n y - \frac{1}{2}\mu_n^2 d(\theta^*)} d\bar{F}_n(y \mid \theta^*) \tag{14}$$

for all values of t and θ^*. Furthermore we obviously have

$$\lim_{n=\infty}\left\{G(\theta^*, \mu_n, n) - P\left[Q_n(\theta^*) \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right]\right\} = 0 \tag{15}$$

uniformly in θ^*, and

$$\lim_{n=\infty}[\bar{F}_n(t \mid \theta^*) - F_n(t \mid \theta^*)] = 0 \tag{16}$$

uniformly in θ^* and t. Since η may be chosen arbitrarily small, it follows from (14) and (15) that to any $\epsilon > 0$, η may be chosen such that

$$\limsup_{n=\infty}\left|G(\theta_n^*, \mu_n, n) - \int_{-\infty}^{+\infty} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta_n^*)}\, d\bar{F}_n(t \mid \theta_n^*)\right| \leqq \frac{\epsilon}{2} \tag{17}$$

for any sequence $\{\theta_n^*\}$.

To each ϵ let L_ϵ be a positive number such that L_ϵ depends only on ϵ and

$$\int_{-\infty}^{-L_\epsilon} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta^*)}\, dN(t \mid \theta^*) + \int_{L_\epsilon}^{\infty} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta^*)}\, dN(t \mid \theta^*) \leq \frac{\epsilon}{2} \tag{18}$$

for all n and for all values of θ^*. Since $d(\theta^*)$ has a positive lower and a finite upper bound, it is easy to verify that such a L_ϵ exists. From (18) and Proposition 3 it follows

$$\begin{aligned}\limsup_{n\to\infty}\Bigg\{P\left[y_n(\theta_n^*) < -L_\epsilon \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right] \\ + P\left[y_n(\theta_n^*) > L_\epsilon \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right]\Bigg\} \leq \frac{\epsilon}{2}\end{aligned} \tag{19}$$

for any arbitrary sequence $\{\theta_n^*\}$. Since the difference $U_n(\theta^*, t_2) - U_n(\theta^*, t_1)$ is a subset of the difference $V_n(\theta^*, t_2) - V_n(\theta^*, t_1)$ and since $T_n(\theta^*, t_2) - T_n(\theta^*, t_1)$ is a subset of $U_n(\theta^*, t_2) - U_n(\theta^*, t_1)$ for $t_2 > t_1$, we get from (18) and (19)

$$\begin{aligned}\limsup_{n\to\infty}\Bigg\{P\left[U_n(\theta_n^*, -L_\epsilon) \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right] + P\left[W_n(\theta_n^*) \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right] \\ - P\left[U_n(\theta_n^*, L_\epsilon) \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right]\Bigg\} \leq \frac{\epsilon}{2}\end{aligned} \tag{20}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\Bigg\{P\left[T_n(\theta_n^*, -L_\epsilon) \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right] + P\left[Q_n(\theta_n^*) \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right] \\ - P\left[T_n(\theta_n^*, L_\epsilon) \mid \theta_n^* + \frac{\mu_n}{\sqrt{n}}\right]\Bigg\} \leq \frac{\epsilon}{2}\end{aligned} \tag{21}$$

for any sequence $\{\theta_n^*\}$. On account of (14) we get from (21)

$$e^{-\eta}\limsup_{n\to\infty}\left\{\int_{-\infty}^{-L_\epsilon} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta_n^*)}\, d\bar{F}_n(t \mid \theta_n^*) + \int_{L_\epsilon}^{\infty} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta_n^*)}\, d\bar{F}_n(t \mid \theta_n^*)\right\} \leq \frac{\epsilon}{2}. \tag{22}$$

From (17) and (22) we obtain

$$\limsup_{n\to\infty} \left| G(\theta_n^*, \mu_n, n) - \int_{-L_\epsilon}^{L_\epsilon} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta_n^*)}\, d\bar{F}_n(t \mid \theta_n^*) \right| \leq \epsilon\left(\frac{1 + e^{\eta}}{2}\right) \tag{23}$$

for any sequence $\{\theta_n^*\}$. Consider now the sequence $\{\theta_n^*\}$ which satisfies the conditions of Proposition 5. Since $F_n(t \mid \theta_n^*)$ converges to $F(t)$ uniformly in t, on account of (16) also $\bar{F}_n(t \mid \theta_n^*)$ converges to $F(t)$ uniformly in t. Hence we obtain from (23)

$$\limsup_{n\to\infty} \left| G(\theta_n^*, \mu_n, n) - \int_{-L_\epsilon}^{L_\epsilon} e^{\mu t - \frac{1}{2}\mu^2 d}\, dF(t) \right| \leq \epsilon\left(\frac{1 + e^{\eta}}{2}\right). \tag{24}$$

Since ϵ and η may be chosen arbitrarily small, Proposition 5 follows from (24).

4. Some theorems and corollaries. THEOREM 1. *Denote by $S_n(\theta^*)$ the region defined by the inequality $y_n(\theta^*) \geq A_n(\theta^*)$ where $A_n(\theta^*)$ is chosen such that $P[S_n(\theta^*) \mid \theta^*] = \alpha$. For any region $W_n(\theta^*)$ denote by $L_n[W_n(\theta^*)]$ the least upper bound of $P[W_n(\theta^*) \mid \theta] - P[S_n(\theta^*) \mid \theta]$ with respect to θ^* and θ, where θ is restricted to values $\geq \theta^*$. Then for any sequence $\{W_n(\theta^*)\}$ for which $P[W_n(\theta^*) \mid \theta^*] = \alpha$,*

$$\limsup_{n\to\infty} L_n[W_n(\theta^*)] \leq 0.$$

PROOF: Assume that Theorem 1 is not true. Then there exists a sequence of integers $\{n'\}$, a sequence $\{\theta_{n'}^*\}$ and a sequence $\{\theta_{n'}\}$ $(\theta_{n'} \geq \theta_{n'}^*)$ such that

$$\lim_{n=\infty} \{P[W_{n'}(\theta_{n'}^*) \mid \theta_{n'}] - P[S_{n'}(\theta_{n'}^*) \mid \theta_{n'}]\} = \delta > 0. \tag{25}$$

On account of Proposition 2 and Assumption 2 the sequence $\{A_{n'}(\theta_{n'}^*)\}$ is bounded. Then it follows easily from (25) and Proposition 4 (taking in account that $E_\theta \frac{\partial}{\partial\theta} \log f(x, \theta^*) > 0$ for $\theta > \theta^*$

$$(\theta_{n'} - \theta_{n'}^*)\sqrt{n'} = \mu_{n'} > 0 \tag{26}$$

must be bounded. Denote by $\{n''\}$ a subsequence of $\{n'\}$ such that

$$\lim d(\theta_{n''}^*) = d \tag{27}$$

$$\lim \mu_{n''} = \mu, \quad \text{and} \tag{28}$$

$$\lim F_{n''}(t \mid \theta_{n''}^{*}) = F(t) \tag{29}$$

uniformly in t where

$$F_n(t \mid \theta^*) = P[U_n(\theta^*, t) \mid \theta^*]$$

and $U_n(\theta^*, t)$ is the intersection of $W_n(\theta^*)$ and the region $y_n(\theta^*) < t$. The existence of a subsequence $\{n''\}$ such that (29) holds follows from the fact that

$$F_n(t_2 \mid \theta^*) - F_n(t_1 \mid \theta^*) \leq \Phi_n(t_2 \mid \theta^*) - \Phi_n(t_1 \mid \theta^*) \quad \text{for} \quad t_2 > t_1, \tag{30}$$

and

$$\lim_{n=\infty} \Phi_n(t \mid \theta_n^{*}{}'') = \frac{1}{\sqrt{2\pi d}} \int_{-\infty}^{t} e^{-\frac{1}{2}v^2/d}\, dv = N(t), \tag{31}$$

where $\Phi_n(t \mid \theta^*)$ denotes the probability $P[y_n(\theta^*) < t \mid \theta^*]$. Furthermore it can easily be shown that

$$\int_{-\infty}^{+\infty} dF(t) = \alpha. \tag{32}$$

On account of Proposition 5 we get from (25), (27), (28), (29), (30) and (31)

$$\int_{-\infty}^{+\infty} e^{\mu t - \frac{1}{2}\mu^2 d}\, dF(t) - \int_{A}^{\infty} e^{\mu t - \frac{1}{2}\mu^2 d}\, dN(t) = \delta, \tag{33}$$

where A denotes a value such that

$$\int_{A}^{\infty} dN(t) = \alpha.$$

It has been shown in a previous paper[3] that (33) leads to a contradiction. Hence Theorem 1 is proved.

THEOREM 2: *Denote by $S_n(\theta^*)$ the region defined by the inequality $y_n(\theta^*) \leq A_n(\theta^*)$ where $A_n(\theta^*)$ is chosen such that $P[S_n(\theta^*) \mid \theta^*] = \alpha$. For any region $W_n(\theta^*)$ denote by $L_n[W_n(\theta^*)]$ the least upper bound of*

$$P[W_n(\theta^*) \mid \theta] - P[S_n(\theta^*) \mid \theta]$$

with respect to θ^* and θ, where θ is restricted to values $\leq \theta^*$. Then for any sequence $\{W_n(\theta^*)\}$ for which $P[W_n(\theta^*) \mid \theta^*) = \alpha$,

$$\limsup_{n \to \infty} L_n[W_n(\theta^*)] \leq 0.$$

The proof is omitted, since it is analogous to that of Theorem 1.

THEOREM 3: *Let $\{W_n(\theta^*)\}$ be for each θ^* a sequence of regions for which $P[W_n(\theta^*) \mid \theta^*] = \alpha$ and $\lim_{n=\infty} g[W_n(\theta^*)] = \alpha$ uniformly in θ^*. Denote by $L_n[W_n(\theta^*)]$ the least upper bound of*

$$P[W_n(\theta^*) \mid \theta] - P[\mid y_n(\theta^*) \mid \geq A_n(\theta^*) \mid \theta]$$

with respect to θ and θ^*, where $A_n(\theta^*)$ is chosen such that

$$P[\mid y_n(\theta^*) \mid \geq A_n(\theta^*) \mid \theta^*] = \alpha.$$

Then

$$\limsup_{n \to \infty} L_n[W_n(\theta^*)] \leq 0.$$

[3] See p. 12 of the paper cited in [2].

PROOF: Denote $P[y_n(\theta^*) < t \mid \theta^*]$ by $\Phi_n(t \mid \theta^*)$ and denote by $F_n(t \mid \theta^*)$ the probability (under the hypothesis $\theta = \theta^*$) of the intersection of $W_n(\theta^*)$ with the region $y_n(\theta^*) < t$. Assume that Theorem 3 is not true. Then there exists a subsequence $\{n''\}$, a sequence $\{\theta^*_{n''}\}$ and a sequence $\{\theta_{n''}\}$ such that

$$\lim_{n=\infty} d(\theta^*_{n''}) = d; \qquad \lim\, (\theta_{n''} - \theta^*_{n''})\sqrt{n''} = \lim \mu_{n''} = \mu;$$

$$\lim F_{n''}(t \mid \theta^*_{n''}) = F(t)$$

uniformly in t, and

$$\int_{-\infty}^{+\infty} e^{\mu t - \frac{1}{2}\mu^2 d}\, dF(t) - \int_{-\infty}^{-A} e^{\mu t - \frac{1}{2}\mu^2 d}\, dN(t) - \int_{A}^{\infty} e^{\mu t - \frac{1}{2}\mu^2 d}\, dN(t) = \delta \tag{34}$$

where A is a positive number such that

$$\int_{-\infty}^{-A} dN(t) = \frac{\alpha}{2}, \qquad \text{and} \qquad N(t) = \frac{1}{\sqrt{2\pi d}} \int_{-\infty}^{t} e^{-\frac{1}{2}v^2/d}\, dv.$$

This can be proved in the same way as (33) has been proved. The author has shown in a previous paper[4] that (34) leads to a contradiction. Hence Theorem 3 is proved.

THEOREM 4: *Denote by $A_n(\theta^*)$ the region of type[5] A of size α for testing the hypothesis $\theta = \theta^*$. Denote by $B_n(\theta^*)$ the region $\mid y_n(\theta^*) \mid \geq C_n(\theta^*)$ where $C_n(\theta^*)$ is determined such that*

$$P[\mid y_n(\theta^*) \mid \geq C_n(\theta^*) \mid \theta^*] = \alpha.$$

Then, under the assumption that $E_\theta \left[\frac{\partial^2}{\partial \theta^2} \log f(x, \theta^*) \right]^2$ *is bounded,*

$$\lim_{n=\infty} \{P[A_n(\theta^*) \mid \theta] - P[B_n(\theta^*) \mid \theta]\} = 0$$

uniformly in θ and θ^*.

PROOF: The region $A_n(\theta^*)$ is given by the inequality[6]

$$\left[\sum_\alpha \frac{\partial}{\partial \theta} \log f(x_\alpha, \theta^*)\right]^2 + \sum_\alpha \frac{\partial^2}{\partial \theta^2} \log f(x_\alpha, \theta^*) \geq k'_n(\theta^*) \left[\sum_\alpha \frac{\partial}{\partial \theta} \log f(x_\alpha, \theta^*)\right] + k''_n(\theta^*), \tag{35}$$

where $k'_n(\theta^*)$ and $k''_n(\theta^*)$ are chosen such that $A_n(\theta^*)$ should be unbiased and of size α. The inequality (35) can be written also in the form

$$[y_n(\theta^*)]^2 + \frac{1}{n} \sum_\alpha \frac{\partial^2}{\partial \theta^2} \log f(x_\alpha, \theta^*) \geq l'_n(\theta^*) y_n(\theta^*) + l''_n(\theta^*). \tag{36}$$

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[4] See p. 14 of the paper cited in [2].

[5] Neyman, J. and Pearson, E. S., "Contributions to the theory of testing statistical hypotheses," *Stat. Res. Mem.*, Vol. 1.

[6] See the paper cited in [5].

Let $\{\mu_n\}$ be a bounded sequence. From Assumption 2 it follows that for any positive ϵ

$$(37)\qquad P\left\{\left|\frac{1}{n}\sum_{\alpha}\frac{\partial^2}{\partial\theta^2}\log f(x_\alpha, \theta^*) + d(\theta^*)\right| < \epsilon \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right\} = 1$$

uniformly in θ^*. Since (37) holds for arbitrarily small ϵ, we get easily on account of Proposition 3

$$(38)\qquad \lim_{n=\infty}\left\{P\left[A_n(\theta^*) \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right] - P\left[A_n'(\theta^*) \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right]\right\} = 0$$

uniformly in θ^*, where $A_n'(\theta^*)$ is defined by

$$(39)\qquad [y_n(\theta^*)]^2 \geq l_n'(\theta^*)y_n(\theta^*) + l_n''(\theta^*) + d(\theta^*).$$

Since $A_n(\theta^*)$ is unbiased and of size α, we have on account of (38) and (39)

$$(40)\qquad \lim l_n'(\theta^*) = 0 \quad \text{and}$$

$$(41)\qquad \lim l_n''(\theta^*) + d(\theta^*) = \lambda(\theta^*) > 0$$

uniformly in θ^*, where $\lambda(\theta^*)$ is given by the condition

$$(42)\qquad \frac{1}{\sqrt{2\pi d(\theta^*)}}\int_{-\sqrt{\lambda(\theta^*)}}^{+\sqrt{\lambda(\theta^*)}} e^{-\frac{1}{2}t^2/d(\theta^*)}\,dt = \alpha.$$

Inequality (39) is obviously equivalent to the simultaneous inequalities:

$$y_n(\theta^*) \leq c_n'(\theta^*) \qquad \text{and} \qquad y_n(\theta^*) \geq c_n''(\theta^*)$$

where $c_n'(\theta^*)$ and $c_n''(\theta^*)$ are the roots of the equation in $y_n(\theta^*)$

$$[y_n(\theta^*)]^2 = l_n'(\theta^*)y_n(\theta^*) + l_n''(\theta^*) + d(\theta^*).$$

Since

$$\lim c_n'(\theta^*) = -\sqrt{\lambda(\theta^*)} \qquad \text{and} \qquad \lim c_n''(\theta^*) = +\sqrt{\lambda(\theta^*)}$$

uniformly in θ^*, from Proposition 3 it follows that

$$(43)\qquad \lim_{n=\infty}\left\{P\left[A_n(\theta^*) \mid \theta^* + \frac{\mu_n}{\sqrt{n}}\right] - \int_{-\infty}^{-\sqrt{\lambda(\theta^*)}} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta^*)}\,dN(t \mid \theta^*) - \int_{+\sqrt{\lambda(\theta^*)}}^{\infty} e^{\mu_n t - \frac{1}{2}\mu_n^2 d(\theta^*)}\,dN(t \mid \theta^*)\right\} = 0$$

uniformly in θ^*.

Now let us consider a sequence $\{\nu_n\}$ such that $\lim |\nu_n| = \infty$ and $\lim \dfrac{\nu_n}{\sqrt{n}} = 0$.

We shall prove that

$$P\left[A_n(\theta^*) \mid \theta^* + \frac{\nu_n}{\sqrt{n}}\right] = 1 \tag{44}$$

uniformly in θ^*. Since $E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(x, \theta^*)\right]^2$ is assumed to be bounded,

$$E_{\theta^*+(\nu_n/\sqrt{n})}\left[\frac{\partial^2}{\partial\theta^2}\log f(x, \theta^*)\right] \tag{45}$$

and

$$E_{\theta^*+(\nu_n/\sqrt{n})}\left[\frac{\partial^2}{\partial\theta^2}\log f(x, \theta^*)\right]^2 \tag{46}$$

are bounded functions of θ^* and n. We get by Taylor expansion

$$\begin{aligned}\sum_\alpha \frac{\partial}{\partial\theta}\log f(x_\alpha, \theta^*) &= \sum_\alpha \frac{\partial}{\partial\theta}\log f\left(x_\alpha, \theta^* + \frac{\nu_n}{\sqrt{n}}\right) \\ &\quad - \frac{\nu_n}{\sqrt{n}}\sum_\alpha \frac{\partial^2}{\partial\theta^2}\log f(x_\alpha, \bar{\theta}_n^*)\end{aligned} \tag{47}$$

where $\bar{\theta}_n^*$ lies in $\left[\theta^*, \theta^* + \frac{\nu_n}{\sqrt{n}}\right]$. Hence

$$E_{\theta^*+(\nu_n/\sqrt{n})}[y_n(\theta^*)] = -\nu_n E_{\theta^*+(\nu_n/\sqrt{n})}\left[\frac{1}{n}\frac{\partial^2}{\partial\theta^2}\sum_\alpha \log f(x_\alpha, \bar{\theta}_n^*)\right]. \tag{48}$$

From Assumption 2 and $\lim |\nu_n| = \infty$ it follows that the absolute value of the right hand side of (48) converges to ∞. Hence

$$\lim | E_{\theta^*+\nu_n/\sqrt{n}}[y_n(\theta^*)] | = \infty.$$

Since on account of Assumption 1

$$E_{\theta^*+(\nu_n/\sqrt{n})}\left[\frac{\partial}{\partial\theta}\log f(x_\alpha, \theta^*)\right]^2$$

is a bounded function of n and θ^*, also the variance of $y_n(\theta^*)$ (under the assumption that $\theta = \theta^* + \nu_n/\sqrt{n}$ is the true value of the parameter) is a bounded function of n and θ^*. Hence for any arbitrary large constant C

$$\lim P\left[\,| y_n(\theta^*) | \geq C \mid \theta^* + \frac{\nu_n}{\sqrt{n}}\right] = 1, \tag{49}$$

uniformly in θ^*. The equation (44) follows easily from (36), (40), (41), (45), (46) and (49).

Consider a sequence $\{\rho_n\}$ such that $\left|\frac{\rho_n}{\sqrt{n}}\right| > \beta > 0$ for all n. Then it follows easily from Proposition 1 that for any arbitrary C

$$\lim P\left[\,|\,y_n(\theta^*)\,| \geq C \mid \theta^* + \frac{\rho_n}{\sqrt{n}}\right] = 1 \tag{50}$$

uniformly in θ^*. Since $E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(x_\alpha\,,\ \theta^*)\right]^2$ is assumed to be bounded, and therefore also $E_\theta \frac{\partial^2}{\partial\theta^2}\log f(x,\,\theta^*)$ is bounded, there exists a finite g such that

$$\lim P\left\{\left|\frac{1}{n}\sum_\alpha \frac{\partial^2}{\partial\theta^2}\log f(x_\alpha,\,\theta^*)\right| < g \mid \theta^* + \frac{\rho_n}{\sqrt{n}}\right\} = 1 \tag{51}$$

uniformly in θ^*. From (36), (40), (41), (50) and (51) it follows

$$\lim P\left[A_n(\theta^*) \mid \theta^* + \frac{\rho_n}{\sqrt{n}}\right] = 1 \tag{52}$$

uniformly in θ^*. Since on account of Propositions 3 and 4, the relations (43), (44) and (52) hold if we substitute $B_n(\theta^*)$ for $A_n(\theta^*)$, Theorem 4 is proved.

If Assumptions 1–4 are fulfilled for the set ω consisting of the single point $\theta = \theta_0$, then we get from Theorems 1–4 the following corollaries:

COROLLARY 1: *Let* W_n' *be the region defined by the inequality* $y_n(\theta_0) \geq c_n'$; W_n'' *defined by the inequality* $y_n(\theta_0) \leq c_n''$, *and* W_n *defined by the inequality* $|\,y_n(\theta_0)\,| \geq c_n$, *where the constants* c_n', c_n'' *and* c_n *are chosen such that*

$$P(W_n' \mid \theta_0) = P(W_n'' \mid \theta_0) = P(W_n \mid \theta_0) = \alpha.$$

Then $\{W_n'\}$ *is an asymptotically most powerful test of the hypothesis* $\theta = \theta_0$ *if* θ *takes only values* $\geq \theta_0$. *Similarly* $\{W_n''\}$ *is an asymptotically most powerful test if* θ *takes only values* $\leq \theta_0$. *Finally* $\{W_n\}$ *is an asymptotically most powerful unbiased test if* θ *can take any real value.*

COROLLARY 2: *The sequence* $\{A_n(\theta_0)\}$ *is an asymptotically most powerful unbiased test of the hypothesis* $\theta = \theta_0$, *where* $A_n(\theta_0)$ *denotes the critical region of type A for testing* $\theta = \theta_0$.