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SOME EXAMPLES OF MANIFOLDS OF NONNEGATIVE CURVATURE

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The purpose of this note is to describe some examples of manifolds of nonnegative curvature and positive Ricci curvature. Apart from homogeneous spaces, no such examples appear in the literature. Our main tool is the formula of O'Neill [8] for riemannian submersions.

Recall that the map $\pi: M^{n+k} \to N^n$ of riemannian manifolds is called a *riemannian* submersion if

1. π is a differentiable submersion, i.e., for all $m \in M$, rank $d\pi_m = n$,
2. $d\pi | H_m$ is an isometry for all $m \in M$.

Here H_m is the orthogonal complement of the kernel V_m of $d\pi$. If $\bar{X}, \bar{Y}$ are horizontal fields, then the vertical component $[\bar{X}, \bar{Y}]_m^V$ of $[\bar{X}, \bar{Y}](m)$ depends only on $\bar{X}(m), \bar{Y}(m)$. Let $x, y \in N_{\pi(m)}$ be orthonormal, $\bar{x}, \bar{y}$ their horizontal lifts at m, and $K, \bar{K}$ denote sectional curvature. Then the formula of O'Neill says

$$(*) \qquad K(x, y) = \bar{K}(\bar{x}, \bar{y}) + \tfrac{3}{4}\|[\bar{x}, \bar{y}]^V\|^2 .$$

Let $G \times M \to M$ be an action of a Lie group on M such that all orbits are closed and of the same type. Then $\pi: M \to G/M$ is a submersion, and any G-invariant riemannian structure on M induces in an obvious way a riemannian structure on $G\backslash M$ such that π becomes a riemannian submersion. If M has nonnegative curvature, then so does $G\backslash M$.[1]

If G acts on N_1, M_1 freely and properly discontinuously on N_1, then it acts freely and properly discontinuously on $N_1 \times M_1$ by the diagonal action. Hence further examples arise by taking products.

Example 1 (*Associated bundles*). Let $M = G_1 \times M_1$, where G_1 is a Lie group with bi-invariant metric, and M_1 has nonnegative curvature. Suppose $G \subset G_1$ is a closed subgroup which acts on M_1 by isometries. Then $(g_1, m) \to (g_1 \cdot g^{-1}, g_m)$ defines a free properly discontinuous action of G on M. As above, $G\backslash M$ inherits a metric of nonnegative curvature. Topologically, $G\backslash M$ is of course the bundle with fibre M_1 associated to the principal fibration $G \to G_1 \to$

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[1] Recently, Gromoll and Meyer [4] have constructed a free action of S^3 on SP(2) which preserves the bi-invariant metric. The quotient is an exotic 7-sphere.

G_1/G. G_1 acts by isometries of $G\backslash M$ via $(g_1, m) \to (gg_1, m)$. In general, however, even if M_1 is homogeneous, $G\backslash M$ will not be homogeneous if G_1 does not act transitively; e.g. the S^2 bundle over S^2 of Example 3 below.

Example 2 (*Change of metric*). It is sometimes of interest to consider the case $G = G_1$ in the preceeding example. In this situation, the projection $\pi|(M, e) \to M \times_G G = G\backslash M \times G$ is a diffeomorphism. Therefore we have actually obtained a new metric $\hat{g}$ on M. In order to describe the new metric, we proceed as follows: Let T denote the tangent space to the orbit $G(m)$ of m, and $N = T^{\perp}$. Let $E_1, \cdots, E_p$ be an orthonormal basis of left invariant fields of G, and let λE_i denote the corresponding Killing fields on M. If X is a field on G (resp. M), we denote by $\tilde{X}$ the field (X, O) (resp. (O, X)) on $G \times M$. Then the space $\bar{T}$ tangent to the orbits of G on $G \times M$ is spanned by the fields $\tilde{E}_i + \lambda\tilde{E}_i$. Set $(\langle\lambda E_i, \lambda E_j\rangle) = (k_{i,j}) = K$. Then the normal space $\bar{N} = \bar{T}^{\perp}$ is spanned by orthonormal fields $\{\tilde{N}_i\}$ (where $N_i \in N$) and $\{\lambda\tilde{E}_i - \sum_l k_{i,l}E_l\}$. In order to find a vector in $\bar{N}$ which projects down under $d\pi$ to say λE_i, we must decompose $\lambda\tilde{E}_i$ as $\lambda\tilde{E}_i = \lambda\tilde{E}_i^{\bar{T}} + \lambda\tilde{E}_i^{\bar{N}}$. Then $\lambda\tilde{E}_i^{N}$ is the required vector. Set

$$\lambda\tilde{E}_i = \sum_k x_{i,k}(\lambda\tilde{E}_h + \tilde{E}_k) + \sum_l y_{i,l}\Big(\lambda\tilde{E}_l - \sum_k k_{l,k}\tilde{E}_k\Big) .$$

If $X = (x_{i,k})$ and $Y = (y_{i,l})$, by collecting terms we have

$$X + Y = I , \qquad X - Y\cdot K = 0 ,$$

or

$$X = K(I + K)^{-1}Y = (1 + K)^{-1} .$$

In particular, Y is symmetric and commutes with K. Then

$$\begin{aligned} k_{ij} = \langle\lambda\tilde{E}_i^N, \lambda\tilde{E}_j^N\rangle &= \Big\langle \sum_l y_{i,l}\Big(\lambda\tilde{E}_l - \sum_k k_{l,k}\tilde{E}_k\Big), \sum_r y_{j,r}\Big(\lambda\tilde{E}_r - \sum_s k_{r,s}E_s\Big)\Big\rangle \\ &= \sum_{lr} y_{i,l}k_{l,r}y_{j,r} + \sum_{l,k,r,s} y_{i,l}k_{l,k}\delta_{k,s}k_{r,s}y_{j,r} . \end{aligned}$$

So

$$\hat{K} = Y^2(K + K^2) = K\cdot(I + K)^{-1} .$$

Thus the new metric $\hat{g}$ may be described by

$$\begin{gathered} \hat{g}|N = g|N , \qquad \hat{g}(T, N) = 0 , \\ (\hat{g}(\lambda E_i, \lambda E_j)) = (\langle\lambda E_i, \lambda E_k\rangle)\cdot(I + \langle\lambda E_k, \lambda E_j\rangle)^{-1} . \end{gathered}$$

$\hat{g}$ is closely related to deformations of metric which have been studied in [2]. It is a straightforward matter to compute the curvature of $\hat{g}$; we will not carry this out because we do not need it. Observe, however, that a plane section in $\bar{N}$ can have zero curvature with respect to the product metric on $M \times G$ only if its projection on M has zero curvature with respect to the original metric. On the other hand, $d\pi \colon \bar{N} \to M_m$ is curvature nondecreasing with respect to the metric $\hat{g}$ on M. Hence in general $\hat{g}$ has "fewer" sections of zero curvature than g does. Since G is often the largest group to act by isometries with respect to $\hat{g}$, this improvement may have been obtained at the expense of destroying some of the symmetry of the metric g.

Example 3 (*Connected sum of symmetric spaces of rank one*). Let S^1 act freely on S^{2n+1} so that $S^1 \to S^{2n+1} \to CP(n)$ is the Hopf fibration. S^1 also acts on $\boldsymbol{R}^2$ by rotation about the origin. The quotient of $S^{2n+1} \times \boldsymbol{R}^2$ by the diagonal action is the normal bundle η of $CP(n)$ in $CP(n+1)$. η is diffeomorphic to $CP(n+1)$ with a ball removed. If S^{2n+1} and $\boldsymbol{R}^2$ are equipped with S^1-invariant metrics, then η inherits a metric of nonnegative curvature. It is interesting to choose such metrics as follows: Let g_0 denote the metric of constant curvature 1 on S^{2n+1}, and $T \oplus N$ be the splitting of S^{2n+1}_p into the tangent space to the orbit of S^1 and its orthogonal complement. Define a new metric g_ε on S^{2n+1} by

$$g_\varepsilon | N = g_0 | N \,, \quad g_\varepsilon(N, T) = 0 \,, \quad g_\varepsilon | T = (1+\varepsilon) g | T \,.$$

Clearly S^1 still acts by isometries with respect to g_ε, and for sufficiently small positive ε, which we now fix, g_ε still has positive curvature. Now equip $\boldsymbol{R}^2$ with a metric h_ε given in polar coordinates by

$$h_\varepsilon\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1 \,, \quad h_\varepsilon\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = 0 \,, \quad h_\varepsilon\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = f_\varepsilon^2(r) \,,$$

where $f_\varepsilon(r)$ is a smooth convex function with the properties $f_\varepsilon(0) = 0$, $f_\varepsilon'(0) = 1$, and $f_\varepsilon(r) \equiv 2\pi\,(1+\varepsilon)/\sqrt{(1+\varepsilon)^2 - 1}$ for sufficiently big $r > R$.

$\boldsymbol{R}^2$ has nonnegative curvature with respect to h_ε, and hence $(g_\varepsilon, h_\varepsilon)$ gives rise to a metric of nonnegative curvature on $\eta = S^{2n+1} \times_{S^1} \boldsymbol{R}^2$. If we restrict to the disc bundle $D_{\bar{R}}(\eta)$ with $\bar{R} > R$, then an annular neighborhood of the boundary *splits isometrically* as $\partial D_{\bar{R}}(\eta) \times I$, where I denotes an interval.

In fact, $A = \{X \in \boldsymbol{R}^2 | R \le \|X\| \le \bar{R}\}$ splits isometrically as $S^1 \times I$, and S^1 acts trivially on I. Then

$$S^{2n+1} \times_{S^1} A = S^{2n+1} \times_{S^1} (S^1 \times I) = (S^{2n+1} \times_{S^1} S^1) \times I = S^{2n+1} \times I \,,$$

and the calculation of the previous example shows that $S^{2n+1} = \partial D_{\bar{R}}(\eta)$ gets back the original metric g_0 of curvature 1. It is a routine manner to check that analogous constructions work for the normal bundles of the cut loci of the other symmetric spaces of rank one. Since the metrics split as a product $S^{2n+1} \times I$ near the boundary, by gluing two such disc bundles together along their common boundary we obtain

Theorem 1. *The connected sum of a symmetric space of rank one with another symmetric space of rank one or its negative admits a metric of nonnegative curvature.*

Remark These manifolds contain a totally geodesic hypersurface, the common boundary of the disc bundles. Conversely, the arguments of [3] show that any manifold M of nonnegative curvature which contains a totally geodesic hypersurface H with trivial normal bundle is topologically the union of two disc bundles with common boundary H.

Also, these manifolds are not homogeneous in general. For example, $CP(2) + CP(2)$ has signature 2 while $CP(2) - CP(2)$ is the nontrivial S^2 bundle over S^2. By the Meyer-Victoris sequence, both spaces have the same integral homology groups as $S^2 \times S^2$. But by a result of J. Wolf (unpublished) any Riemannian homogeneous space with the integral homology groups of $S^2 \times S^2$ is diffeomorphic to $S^2 \times S^2$.

Example 4 (*Kervaire spheres*). In order to produce a metric of nonnegative curvature by the method of gluing together different disc bundles, it was necessary that the metrics g_ε had positive curvature for sufficiently small ε. While there are not many such examples known, an easier condition to fullfill is that of having positive Ricci curvature. For k odd, consider the Brieskorn variety $B_{n,k}$ defined by the equations

$$Z_0^k + Z_1^2 + \cdots + Z_n^2 = 0 \,, \qquad |Z_0|^2 + |Z_1^2| + \cdots + |Z_n|^2 = 2 \,.$$

As is well known $SO(n) \times S^1$ acts on this variety by isometries of the metric induced from the imbedding as follows: $(\xi, \Theta)(Z_0, Z) = (\xi^2 Z_0, \Theta(\xi^k Z))$, where $Z = (Z_1, \cdots, Z_n)$, $\Theta \in SO(\eta)$, and ξ is a complex number of norm 1. The principal orbits of this action are codimension 1, and are given by the level surfaces $|Z_0| = a, O < a < 1$. It is easy to check that with respect to the induced metric, the curve $[0, 1] \to (t, i, \sqrt{1 - t^n}, 0, 0, \cdots, 0)$ is orthogonal to all orbits. The isotropy groups of the points $(0, i, 1, 0, \cdots, 0)$, $(1, i, 0, 0, \cdots, 0)$ and $(a, i, \sqrt{1 - a^n}, 0, \cdots, 0)$ are easily computed to be $SO(n-2) \times \Phi$, $SO(n-1)$ and $SO(n-2)$ respectively, where Φ is the circle imbedded as

$$\left(\cos 2\theta - i \sin 2\theta, \begin{pmatrix} \begin{matrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{matrix} & 0 \\ 0 & I \end{pmatrix}\right) .$$

It follows from [7] that $D(\eta_1) = \{(Z_0, \cdots, Z_n) | 0 \le |Z_0| \le a\} \cap B_{n,k}, D(\eta_2) = \{|(Z_0, \cdots, Z_n) | a \le |Z_0| \le 1\} \cap B_{n,k}$ are disc bundles given as follows:

$$D(\eta_1) = SO(n) \underset{SO(n-2)\times\Phi}{\times} D^2 \,, \qquad D(\eta_2) = (SO(n) \underset{SO(n-1)}{\times} D^{n-1}) \times S^1 \,.$$

Now equip $D(\eta_1)$, $D(\eta_2)$ with metrics as follows: Let $so(n) = p + so(n-1)$ be the standard decomposition of the Lie algebra $so(n)$ of $SO(n)$, and let g

denote the bi-invariant metric on $so(n)$. Define a new left invariant metric g_ε on $SO(n)$ by setting

$$g_\varepsilon|p = g|p \,, \qquad g_\varepsilon(p, so(n-1)) = 0 \,.$$
$$g_\varepsilon|so(n-1) = (1+\varepsilon)g|so(n-1) \,.$$

g_ε is right invariant under $so(n-2)$, and for sufficiently small ε it has positive Ricci curvature for $n \neq 4$, and nonnegative Ricci curvature for $n = 4$, but unfortunately not nonnegative sectional curvature. Let $(r, \theta_1, \cdots, \theta_n)$ be polar coordinates on the n-disc D^n, and $f_\varepsilon(r)$ be a convex function satisfying the conditions of f_ε of Example 3. Then equipping $SO(n)$, D^n with the metrics $g_\varepsilon, h_\varepsilon$ respectively we produce as in Example 3 a metric on $D(\eta_2) = SO(n) \underset{SO(n-1)}{\times} D^{n-1} \times S^1$ with the property that near the boundary the metric is isometrically a product of an interval and the boundary $SO(n)|SO(n-2) \times S^1$ equipped with its normal metric. By the same technique we construct such a metric on $D(\eta_1)$. Then by gluing $D(\eta_1)$, $D(\eta_2)$ together we obtain a metric of nonnegative Ricci curvature on $B_{n,k}$. On $D(\eta_1)$ this metric is easily seen to have positive Ricci curvature near the orbit $|Z_0| = 0$. ($D(\eta_2)$ splits off S^1 isometrically and hence has a direction of zero Ricci curvature.) However, by a theorem of Aubin [1], the metric can be deformed to one of strictly positive Ricci curvature. For n odd, $k \equiv 3, 5 \bmod 8$, $B_{n,k}$ is the Kervaire sphere [7]. Hence

Theorem 2. *The Kervaire spheres admit metrics of positive Ricci curvature.*

Theorem 2 should be contrasted with results of Hitchen [6], which give examples of exotic spheres which do not even admit a metric of positive scalar curvature.

A modification of Aubin's arguments shows that one can actually choose the metrics of positive Ricci curvature to be invariant under $SO(n) \times S^1$; the proof will appear in the thesis of P. Ehrlich. Motivated by our examples, Hernandez [5] has constructed imbeddings of a large family of Brieskorn varieties for which the Ricci curvature is positive. In particular, he also gets all the Kervaire spheres. On the other hand, clearly various other examples arise from our method by looking at G-spaces with orbits of codimension 1.

One might ask if by careful choice of the function f_ε, it is possible to compensate for the negative curvatures of g_ε so as to make the sectional curvatures of $D(\eta_1)$, $D(\eta_2)$ come out nonnegative. This is possible for $D(\eta_1)$, but seems not to be possible for $D(\eta_2)$.

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