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SOME EXAMPLES OF SQUARE INTEGRABLE REPRESENTATIONS OF SEMISIMPLE p-ADIC GROUPS

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ABSTRACT. We construct irreducible representations of the Hecke algebra of an affine Weyl group analogous to Kilmoyer's reflection representation corresponding to finite Weyl groups, and we show that in many cases they correspond to a square integrable representation of a simple p-adic group.

1. Introduction.

1.1. Let $\mathcal{G}$ (resp. $\tilde{\mathcal{G}}$) be the group of rational points of a simple split adjoint (resp. simply connected) algebraic group over a nonarchimedian local field K, and let π: $\tilde{\mathcal{G}} \to \mathcal{G}$ be the natural map. The image under π of an Iwahori (resp. parahoric) subgroup of $\tilde{\mathcal{G}}$ is said to be an Iwahori (resp. parahoric) subgroup of $\mathcal{G}$. Let us fix an Iwahori subgroup $\mathcal{I}$ of $\mathcal{G}$.

We shall be interested in the irreducible admissible representations of $\mathcal{G}$ (over $\mathbf{C}$) which possess nonzero vectors invariant under $\mathcal{I}$. The isomorphism classes of such representations form the set $\mathcal{U}(\mathcal{G})$ of representations in the "unramified principal series" of $\mathcal{G}$. By a theorem of Bernstein, Borel [2] and Matsumoto [9], the set $\mathcal{U}(\mathcal{G})$ is naturally in 1-1 correspondence with the set $\mathcal{U}(\tilde{H})$ of irreducible (finite-dimensional) complex representations of the Hecke algebra $\tilde{H}$ (algebra of double cosets) of $\mathcal{G}$ with respect to $\mathcal{I}$. The correspondence is obtained as follows. To a representation in $\mathcal{U}(\mathcal{G})$ one associates the space of its $\mathcal{I}$-invariant vectors which is an irreducible $\tilde{H}$-module. Conversely to an irreducible H-module E one associates the space of $\tilde{H}$-linear maps from E to the space locally constant functions on $\mathcal{G}/\mathcal{I}$; one thus gets a $\mathcal{G}$-module in $\mathcal{U}(\mathcal{G})$.

This correspondence reduces the question of classifying the elements of $\mathcal{U}(\mathcal{G})$ to the question of classifying the irreducible representations of $\tilde{H}$, which is an algebra with finitely many generators and relations, explicitly known from the work of Iwahori and Matsumoto [7]. These questions are still unsolved, in general.

1.2. A remarkable conjecture of Deligne and Langlands says that there should be a natural finite-to-one correspondence between $\mathcal{U}(\mathcal{G})$ (or $\mathcal{U}(\tilde{H})$) and the set

$$(1.2.1) \quad \{(s, N) \in G \times \operatorname{Lie} G \mid s \text{ semisimple}, sNs^{-1} = q^{-1}N\} \text{ modulo action of } G,$$

where G is a simply connected group over $\mathbf{C}$ whose root system is dual to that of $\mathcal{G}$ and q is the number of elements in the residue field of K; we denote by sNs^{-1} the

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image of N under the adjoint action of s. For $\mathcal{G}$ of type A_n, such a correspondence, which turned out to be 1-1, was found by Bernstein and Zelevinski **[1]**.

Let $\mathfrak{U}_2(\mathcal{G})$ be the set of square integrable representations in $\mathfrak{U}(\mathcal{G})$ and let $\mathfrak{U}_2(\tilde{H})$ be the corresponding subset of $\mathfrak{U}(\tilde{H})$. According to Langlands, the pairs (s, N) in G corresponding to the subset $\mathfrak{U}_2(\tilde{H})$ of $\mathfrak{U}(\tilde{H})$ should be the pairs (s, N) as in (1.2.1) which are not centralized by any nontrivial torus in G. (We shall call them L_2-pairs.)

1.3. In this paper, we shall construct for any $\mathcal{G}$ of type $\neq A_n$, some representations in $\mathfrak{U}_2(\tilde{H})$ (or, equivalently, square integrable representations in $\mathfrak{U}_2(\mathcal{G})$) which should correspond to L_2-pairs (s, N) in G with N subregular. To do this we use W-graphs (as in **[8]**) arising from the left cells (see loc. cit.) of a Coxeter group, containing a simple reflection. Our construction generalizes Kilmoyer's construction **[5]** of the reflection representation of the Hecke algebra of a finite Coxeter group. To prove that our representations are in $\mathfrak{U}_2(\tilde{H})$, we apply a criterion of Casselman; to be able to apply it, we had to do some rather long case by case computations.

1.4. Our results suggest some striking connections between the structure of representations of $\tilde{H}$ on the one hand, and the geometry of the varieties

$$\mathfrak{B}(s, N) = \{B\text{: Borel subgroup in } G \mid s \in \mathfrak{B}, N \in \mathrm{Lie}(B)\}$$

((s, N) as in (1.2.1)), on the other hand.

Let E be a finite dimensional (complex) $\tilde{H}$-module. One can associate to E an element λ_E in the group ring $\mathbf{Z}[(\mathbf{C}^*)^n]$ ($n = \operatorname{rank} \mathcal{G}$) as follows. We consider n standard elements $\hat{T}_{\omega_1}, \ldots, \hat{T}_{\omega_n} \in \tilde{H}$ in 1-1 correspondence with vertices of the Coxeter graph of $\mathcal{G}$ (see 4.2, 4.3 for the definition). Since the elements $\hat{T}_{\omega_i}$ commute among themselves, there exists a filtration $E = E_0 \supset E_1 \supset \cdots \supset E_m = 0$ invariant under each $\hat{T}_{\omega_i}$, whose successive quotients are one-dimensional. Since each $\hat{T}_{\omega_i}$ is invertible in $\tilde{H}$, there exist $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jn}) \in (\mathbf{C}^*)^n$, $j = 1, \ldots, m$, such that $\hat{T}_{\omega_i}$ acts on E_j/E_{j-1} as the scalar $\lambda_{ji} \in \mathbf{C}^*$ ($1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$). We set $\lambda_E = \sum_{j=1}^m \lambda_j \in \mathbf{Z}[(\mathbf{C}^*)^n]$. This is clearly independent of the choice of filtration.

On the other hand, let (s, N) be a pair as in (1.2.1). Let $C_1, C_2, \ldots, C_r$ be the connected components of the variety $\mathfrak{B}(s, N)$, and let $\pi_0(\mathfrak{B}(s, N))$ be the set of these components. Let $Z(s, N)$ be the group of elements in G which centralize both s and N, and let $\bar{Z} = \bar{Z}(s, N) = Z(s, N)/Z^0(s, N) \cdot \mathrm{centre}(G)$. Then $\bar{Z}$ acts naturally, by permutation, on $\pi_0(\mathfrak{B}(s, N))$. Let φ be an irreducible complex representation of $\bar{Z}$ which appears in this permutation representation. We associate to the triple (s, N, φ) an element

$$\lambda'_{(s,N,\varphi)} = |\bar{Z}|^{-1} \sum_{\substack{1 \leqslant j \leqslant r \\ z \in \bar{Z} \\ z(C_j) = C_j}} \mathfrak{L}(z, C_j) \mathrm{Tr}(z, \varphi) \lambda'_j \in \mathbf{Z}\big[(\mathbf{C}^*)^n\big].$$

Here, $\mathfrak{L}(z, C_j)$ is the Lefschetz number of the transformation $z\colon C_j \to C_j$, and for each j ($1 \leqslant j \leqslant r$), we define

$$\lambda'_j = (\lambda'_{j1}, \ldots, \lambda'_{jn}) \in (\mathbf{C}^*)^n$$

by the requirement that, for some $B \in C_j$, the value of the ith fundamental weight $\omega_i\colon B \to \mathbf{C}^*$ at s is λ'_{ji}. It is clear that λ'_j depends only on C_j, and not on the choice of B, $B \in C_j$. Moreover, $\lambda'_j = \lambda'_k$ whenever $C_j = z(C_k)$ for some $z \in \bar{Z}$; from this, it follows easily that the coefficients of $\lambda'_{(s,N,\varphi)}$ are integers, rather than rational numbers. (In particular, we have

$$\lambda'_{(s,N,1)} = \sum_{C_j} \chi(C_j)\lambda'_j$$

where C_j runs over a set of representatives for the $\bar{Z}$-orbits on $\pi_0(\mathcal{B}(s, N))$ and χ denotes the Euler characteristic.)

1.5. We now state a conjecture which is a refinement of the conjecture of Deligne and Langlands. It is convenient to enlarge the set $\mathfrak{U}(\mathcal{G})$ to a set $\mathfrak{U}'(\mathcal{G}) \supset \mathfrak{U}(\mathcal{G})$, defined as follows. $\mathfrak{U}'(\mathcal{G})$ is the set of isomorphism classes of irreducible admissible representations V of $\mathcal{G}$ such that there exists a parahoric subgroup P, $\mathcal{J} \subset P \subset \mathcal{G}$ with "unipotent radical" $\mathfrak{U}_P$ and "reductive part" $\bar{P} = P/U_P$ such that the U_P-invariant part of V (a finite-dimensional $\bar{P}$-module) contains some unipotent representation of the reductive group $\bar{P}$ (over F_q). The representations in $\mathfrak{U}'(\mathcal{G})$ are said to be unipotent representations.

One could conjecture that there is a 1-1 corespondence between the set $\mathfrak{U}'(\mathcal{G})$ and the set of all triples (s, N, φ) (up to G-conjugacy) where (s, N) is a pair as in (1.2.1) and φ is an irreducible representation of the finite group $\bar{Z}(s, N)$. Under this correspondence, the elements of $\mathfrak{U}(\mathcal{G})$ should correspond precisely to the triples (s, N, φ) such that φ appears in the permutation representation of $\bar{Z}(s, N)$ on $\pi_0(\mathcal{B}(s, N))$. (Note the similarity with Springer's correspondence **[12]** between unipotent classes and Weil group representations.) Moreover, the square integrable representations in $\mathfrak{U}'(\mathcal{G})$ should correspond precisely to the triples (s, N, φ) such that there is no torus $\neq e$ in G centralizing both s and N. Let V be in $\mathfrak{U}(\mathcal{G})$, let E be the corresponding element in $\mathfrak{U}(\tilde{H})$ and (s, N, φ) the corresponding triple. If $V \in \mathfrak{U}_2(\mathcal{G})$, or more generally, if $V \in \mathfrak{U}(\mathcal{G})$ is tempered, we should have the identity

$$\lambda_E = \lambda'_{(s,N,\varphi)}. \tag{1.5.1}$$

1.6. Let $V \in \mathfrak{U}'(\mathcal{G})$. I don't know how to attach to V a triple (s, N, φ). However, at least when $V \in \mathfrak{U}(\mathcal{G})$, one can attach to V a pair (s, N), as in (1.2.1), as follows. The definition of the s-component is well known. Consider the $\tilde{H}$-module E corresponding to V, let $\lambda_1, \ldots, \lambda_m \in (\mathbf{C}^*)^n$ be defined as in 1.4. Then there is a unique semisimple $s \in G$ (up to conjugacy) such that $\lambda_1, \ldots, \lambda_m$ are terms in the sum $\lambda'_{(s,0,1)} \in \mathbf{Z}[(\mathbf{C}^*)^n]$. To define the N-component, we may assume that V is tempered. (The general case reduces to this by the p-adic version of the Langlands quotient theorem **[11]**: we require that the N-component of V is equal to the N-component attached to the tempered representation of a Levi subgroup of $\mathcal{G}$, corresponding to V). When V is tempered, let A be the set of all elements $x \in \operatorname{Lie} G$ such that $sxs^{-1} = q^{-1}x$. Then A consists of nilpotent elements and $Z(s)$ has a unique open orbit on A. We select N to be any element of this open orbit. The pair (s, N) is then well defined up to G-conjugacy.

1.7. Consider, for example, the case where $\mathcal{G}$ is of type G_2. There are precisely 8 triples (s, N, φ), up to conjugacy in $\mathcal{G}$, with N subregular nilpotent. They are in a natural 1-1 correspondence with the set $\mathfrak{M}(\mathfrak{S}_3)$ of pairs (x, φ) where x is an element in the symmetric group $\mathfrak{S}_3$, defined up to conjugacy, and φ is an irreducible representation of the centralizer of x in $\mathfrak{S}_3$. (The set $\mathfrak{M}(\mathfrak{S}_3)$ plays a role in the classification of unipotent representations of the Chevalley groups of type G_2 over F_q.) According to 1.5, these 8 elements of $\mathfrak{M}(\mathfrak{S}_3)$ should correspond to 8 square integrable representations of $\mathcal{G}$. Four of them are in $\mathfrak{U}(\mathcal{G})$; they are constructed in this paper. (Their existence was first pointed out by Matsumoto [**9**].) The other four are supercuspidal: they are induced from one of the four unipotent cuspidal representations of $G_2(F_q)$ via a maximal special parahoric subgroup of $\mathcal{G}$.

1.8. Here are some comments on earlier work on construction elements in $\mathfrak{U}_2(\mathcal{G})$. Borel [**2**] constructed the representations in $\mathfrak{U}_2(\mathcal{G})$ corresponding to one-dimensional representations of the Hecke algebra. Recently, Gustafson constructed some (but not all) representations in $\mathfrak{U}_2(\mathcal{G})$ for $\mathcal{G}$ of type C_n. Rodier [**10**] has constructed the representations in $\mathfrak{U}_2(\mathcal{G})$ such that the corresponding $s \in G$ is regular.

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2. L_2-pairs.

2.1. Let D be a diagonalizable algebraic group over $\mathbf{C}$ and let $X(D)$ be the group of (algebraic) characters $D \to \mathbf{C}^*$. Define

$$D_c = \{g \in D: |\lambda(g)| = 1, \text{ for any } \lambda \in X(D)\},$$
$$D_v = \{g \in D: \lambda(g) \in \mathbf{R}_{>0}, \text{ for any } \lambda \in X(D)\}.$$

Thus D_c is the maximal compact subgroup of D and D_v is isomorphic to $\mathbf{R}^n$ as a Lie group. Clearly, D is the direct product of the subgroups D_c, D_v.

Now let G be any algebraic group over $\mathbf{C}$ and let $s \in G$ be a semisimple element. Then s has a canonical decomposition $s = s_c \cdot s_v = s_v \cdot s_c$, where s_c, s_v are the projections of s onto the two factors of $D = D_c \times D_v$ where D is the smallest diagonalizable algebraic subgroup of G containing s. This decomposition is preserved by homomorphisms of algebraic groups.

2.2. LEMMA. *Let T be a torus/$\mathbf{C}$ and let $s \in T$ be such that $s = s_v$. Then the smallest diagonalizable algebraic subgroup D of T containing s is connected.*

PROOF. As D is closed in T, the restriction map $X(T) \to X(D)$ is onto. If it is also injective, then $T = D$ and there is nothing to prove. Thus, we may assume that there exists $\lambda \in \ker(X(T) \to X(D))$, $\lambda \not\equiv 1$. Write $\lambda = \lambda_1^m$, where λ_1 is an indivisible element of $X(T)$ and m is an integer $\geqslant 1$. We have $\lambda_1(s)^m = 1$. Since $s = s_v$, it follows that $\lambda_1(s) = 1$ hence $s \in T' = \ker\{\lambda_1: T \to \mathbf{C}^*\}$. Now T' is a torus of codimension 1 in T, and we may assume that the lemma is already proved for (T', s). The lemma follows.

2.3. DEFINITION. Let us fix a real number $q > 1$. An L_2-pair (with respect to q) in a connected semisimple algebraic group G over $\mathbf{C}$ is a pair (s, N) where $s \in G$ is semisimple, $N \in \operatorname{Lie} G$ is nilpotent and $sNs^{-1} = q^{-1}N$.

2.4. EXAMPLE. Let $N \in \operatorname{Lie} G$ be a nilpotent element which is not centralized by any nontrivial torus in G. Then $N \neq 0$. Let G_N be a 3-dimensional subgroup of type A_1 of G such that $N \in \operatorname{Lie}(G_N)$, and let S_N be a 1-dimensional torus in G_N stabilizing the line $\mathbf{C} \cdot N$. There is a unique element $s \in S_N$ such that $sNs^{-1} = q^{-1}N$ and $s = s_v$. It is clear that (s, N) is an L_2-pair. Conversely, let (s, N) be an L_2-pair in G such that $s = s_v$. We show that it is of the type just described. We have necessarily $N \neq 0$. Define S_N and G_N as before.

Let $\mathfrak{N}(N) = \{g \in G \mid gNg^{-1} \in \mathbf{C}^* \cdot N\}$. Then we have an exact sequence

$$1 \to Z(N) \to \mathfrak{N}(N) \xrightarrow{\pi} \mathbf{C}^* \to 1$$

where $Z(N)$ is the centralizer of N in G and π is defined by $gNg^{-1} = \pi(g)N$. Moreover, we have $\mathfrak{N}^0(N) = Z^0(N) \cdot S_N$, since $\pi\colon S_N \to \mathbf{C}^*$ is onto. We have $s = s_v \in \mathfrak{N}(N)$. By Lemma 2.2, the smallest diagonalizable algebraic subgroup of G containing s is connected. It is contained in $\mathfrak{N}(N)$. It follows that $s \in \mathfrak{N}^0(N)$. Let H be a maximal torus in $\mathfrak{N}^0(N)$ containing s. Then $H \cap Z^0(N)$ must be finite since (s, N) is an L_2-pair. Hence $\dim(H) = 1$. It follows that H is conjugate to S_N under $\mathfrak{N}^0(N)$. As $\mathfrak{N}^0(N) = Z^0(N) \cdot S_N$, H is also conjugate to S_N under $Z^0(N)$. Thus, we may assume that $s \in S_N$. We also see that $Z^0(N)$ contains no nontrivial torus, for if it did, it would be a maximal torus in $\mathfrak{N}^0(N)$, hence it would again be conjugate to S_N under $Z^0(N)$. But $S_N \not\subset Z^0(N)$ and we find a contradiction.

We can now prove

2.5. PROPOSITION. *Assume that G is simply connected. The map $(s, N) \to (s_c, N)$ defines a* 1-1 *correspondence between the set of L_2-pairs (s, N) in G up to conjugacy, and the set of pairs (s', N') (up to conjugacy) where $s' \in G$ is an element whose centralizer $Z(s')$ is semisimple and $N' \in \operatorname{Lie} Z(s')$ is a nilpotent element whose centralizer in $Z(s')$ contains no nontrivial torus.*

PROOF. Let (s, N) be an L_2-pair in G. Then $s \in \mathfrak{N}(N)$; hence $s_c \in \mathfrak{N}(N)$, $s_v \in \mathfrak{N}(N)$. As in the example, we have a map $\pi\colon \mathfrak{N}(N) \to \mathbf{C}^*$, $gNg^{-1} = \pi(g)N$. This map is compatible with decomposition $s = s_c \cdot s_v$. Thus $\pi(s_c) = \pi(s)_c$. But $\pi(s) = q^{-1}$ hence $\pi(s_c) = 1$, so that $s_c \in Z(N)$, and $N \in \operatorname{Lie} Z(s_c)$. (Note that $Z(s_c)$ is connected, since G is simply connected.) We also have $s \in Z(s_c)$. Since (s, N) is an L_2-pair, it follows that $Z(s_c)$ contains no central torus $\neq \{e\}$, hence $Z(s_c)$ is semisimple. Let T be a torus in $Z(s_c)$ such that $T \subset Z(s_v) \cap Z(N)$. Then $T \subset Z(s)$. Since (s, N) is an L_2-pair, T must be trivial. Hence (s_v, N) is an L_2-pair relative to $Z(s_c)$ and, by the discussion in Example 2.4, N is not centralized by any torus $\neq e$ in $Z(s_c)$.

Conversely, let $s' \in G$ be a semisimple element whose centralizer is semisimple, (hence s' is necessarily of finite order) and let N' be a nilpotent element in $\operatorname{Lie}(Z(s'))$ whose centralizer in $Z(s')$ contains no torus $\neq e$. Let $s'' \in Z(s')$ be a semisimple element such that $s'' = s''_v$ and such that (s'', N) is an L_2-pair relative to $Z(s')$. (Note that, by the discussion in Example 2.4, s'' is uniquely determined up to conjugacy by an element in $Z(N') \cap Z(s')$.) Let $s = s' \cdot s''$. We must show that

(s, N') is an L_2-pair in G. Let T be a torus in $Z(s) \cap Z(N')$. We have $Z(s) = Z(s_c) \cap Z(s_v)$, hence $T \subset Z(s') \cap Z(s'') \cap Z(N)$. As (s'', N') is an L_2-pair for $Z(s')$, it follows that T must be trivial. The proposition is proved.

2.6. COROLLARY. (a) *There are only finitely many conjugacy classes of L_2-pairs in G.*
(b) *If (s, N) is an L_2-pair of G, then $Z(s) \cap Z(N)$ is finite.*

PROOF. (a) The elements $s \in G$ such that $Z(s)$ is semisimple fall into $\Pi(n_i + 1)$ conjugacy classes, where n_i are the ranks of the simple factors of G. It remains to use the finiteness of the number of nilpotent classes in a semisimple Lie algebra.

(b) As we have seen, there exists a three-dimensional subgroup of type A_1 of $Z(s_c)$ containing s_v, and whose Lie algebra contains N. We may apply to it **[13]** which shows that $(Z(s_c) \cap Z(s_v) \cap Z(N))^0$ is reductive. Since (s, N) is an L_2-pair, the last group contains no torus $\neq e$, hence is trivial.

2.7. PROPOSITION. *Assume G simply connected. Let (s, N) be a pair with $s \in G$ semisimple, $N \in \mathrm{Lie}(G)$ nilpotent, $sNs^{-1} = q^{-1}N$. Then the variety*

$$\mathfrak{B}(s, N) = \{B = \textit{Borel subgroup of } G \mid s \in B, N \in \mathrm{Lie}\, B\}$$

is nonempty.

PROOF. A Borel subgroup B contains s if and only if it contains both s_c and s_v. Note that $s_v \in Z(s_c)$, $N \in \mathrm{Lie}\, Z(s_c)$ (see the proof of Proposition 2.5). Hence $\mathfrak{B}(s, N)$ is a disjoint union of f copies of the variety

$$\{B'\text{: Borel subgroup of } Z(s_c) \mid s_v \in B', N \in \mathrm{Lie}\, B'\}$$

where f is the number of Borel subgroups of G containing a fixed Borel subgroup of $Z(s_c)$. Let D be the smallest diagonalizable algebraic group in $Z(s_c)$ containing s_v. By Lemma 2.2, D is connected. It clearly normalizes the one parameter additive subgroup A of $Z(s_c)$ corresponding to N, hence it generates together with A a connected solvable group. It remains to note that $D \cdot A$ is contained in a Borel subgroup of $Z(s_c)$.

2.8. PROPOSITION. *Assume G simply connected. Let (s, N) be an L_2-pair in G, and let $B \in \mathfrak{B}(s, N)$. Let $\lambda\colon B \to \mathbf{C}^*$ be any dominant weight, $\not\equiv 1$. Then $\lambda(s) = \varepsilon q^{-n/2}$, where ε is a root of 1 and n is an integer $\geqslant 1$. In particular, we have $|\omega_i(s)| < 1$ for any fundamental weight $\omega_i\colon B \to \mathbf{C}^*$.*

PROOF. Write $s = s_c \cdot s_v$. By Proposition 2.5, the centralizer of s_c is semisimple, hence s_c is of finite order, hence $\lambda(s_c) = \varepsilon$ is a root of 1. Let V be a finite-dimensional irreducible complex representation of $Z(s_c)$ with highest weight $\lambda \mid B'$ with respect to the Borel subgroup $B' = B \cap Z(s_c)$ of $Z(s_c)$.

Then, there exists a nonzero vector $x_0 \in V$ such that $bx_0 = \lambda(b)x_0$ for all $b \in B'$. By Example 2.4 and Proposition 2.5, there exists an algebraic homomorphism α: $\mathrm{SL}_2(\mathbf{C}) \to Z(s_c)$ which maps $\left(\begin{smallmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{smallmatrix}\right)$ to s_v and whose differential maps $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to N.

We shall regard V as an $\mathrm{SL}_2(\mathbf{C})$-module, via α. By a general property of $\mathrm{SL}_2(\mathbf{C})$-modules of finite dimension, we can decompose $V = \bigoplus_{i \in \mathbf{Z}} V_i$, where

$$V_i = \left\{ x \in V \;\middle|\; \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} x = q^{i/2} x \right\}$$

and we have

$$\text{(2.8.1)} \qquad \ker\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : V \to V\right) \subset \bigoplus_{i \leqslant 0} V_i,$$

$$\text{(2.8.2)} \qquad \ker\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : V_0 \to V\right) = V^{\mathrm{SL}_2(\mathbf{C})} \quad \text{(invariant part)}.$$

(These statements are proved by reduction to the case of an irreducible SL_2 (**C**)-module of finite dimension.)

Consider now the vector $x_0 \in V$. Since $s_v(x_0) = \lambda(s_v)x_0$, x_0 must lie in one of the subspaces V_i. Since $N \in \mathrm{Lie}\, B$, we have $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)x_0 = Nx_0 = 0$, hence, by (2.8.1), we have $x_0 \in V_i$ with $i \leqslant 0$. Assume that $x_0 \in V_0$. Then, by (2.8.2), we have $x_0 \in V^{\mathrm{SL}_2(\mathbf{C})}$. Let $P = \{g \in Z(s_c) \mid gx_0 \in \mathbf{C}^* \cdot x_0\}$. This group contains B' hence it is a parabolic subgroup of $Z(s_c)$. We have $P = G$ since $\lambda \not\equiv 1$. Also, P contains $\alpha(\mathrm{SL}_2(\mathbf{C}))$, which is reductive. Hence $\alpha(\mathrm{SL}_2(\mathbf{C}))$ must be contained in a Levi subgroup of P. The identity component of the centre of that Levi subgroup centralizes both s_v and N, contradicting the fact that (s_v, N) is an L_2-pair of $Z(s_c)$. Thus, we must have $i < 0$, and hence $\lambda(s_v) = q^{i/2}$ with i an integer, $i < 0$. This completes the proof.

2.9. With the notations in 1.2, consider $E \in \mathfrak{U}(\tilde{H})$ and let $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jn}) \in (\mathbf{C}^*)^n$, $j = 1, \ldots, m = \dim E$ be defined as in 1.4. According to Casselman [**4**], we have $E \in \mathfrak{U}_2(\tilde{H})$ if and only if $|\lambda_{ji}| < 1$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Proposition 2.8 shows that the conjecture 1.5 is compatible with Casselman's criterion. Casselman shows also that $E \in \mathfrak{U}(\tilde{H})$ corresponds to a tempered representation of $\mathcal{G}$ if and only if $|\lambda_{ij}| \leqslant 1$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. This suggests that, in the conjecture 1.5, the tempered representations in $\mathfrak{U}'(\mathcal{G})$ correspond precisely to the triples (s, N, φ) in G such that there exists a 3-dimensional subgroup of type A_1 contained in $Z(s_c)$, which contains s_v and whose Lie algebra contains N. Indeed, for such a triple, the argument in the proof of Proposition 2.8 is still applicable, but yields the weaker conclusion $|\omega_i(s)| \leqslant 1$ for all i.

2.10. We shall now describe the L_2-pairs (s, N) in G (assumed to be simple, simply connected) with N subregular. We shall also describe the varieties $\mathfrak{B}(s, N)$ in these cases. The results in this section are easily proved using Steinberg's discussion of subregular elements in [**14**].

If G is of type A_n, there is no such L_2-pair (s, N). If G is of type B_n $(n \geqslant 2)$ there is a unique such L_2-pair (s, N): take $s' \in G$ to be the unique element (up to conjugacy) such that $Z(s')$ is of type D_n, take (s'', N) to be an L_2-pair in $Z(s')$ with N regular nilpotent in $\mathrm{Lie}(Z(s'))$ and set $s = s's''$. Then (s, N) is the required L_2-pair in G. The variety $\mathfrak{B}(s, N)$ consists of two points (the two Borel subgroups in G containing the unique Borel subgroup of $Z(s')$ which has N in its Lie algebra). The group $\bar{Z}(s, N)$ has order 2; it acts trivially on $\pi_0(\mathfrak{B}(s, N))$. In the remaining cases, s_v is uniquely determined by N, up to conjugacy by an element in $Z^0(N)$. We shall regard s_v as fixed. We set $s' = s_c$.

We shall now discuss the possibilities for s'. If G is of type D_n $(n \geqslant 4)$ or E_n $(n = 6, 7 \text{ or } 8)$, then s' runs through the elements of the centre of G. The variety $\mathfrak{B}(s's_v, N)$ is independent of s'; it consists of $(n - 1)$ isolated points and a single projective line (which is the irreducible component of the "Dynkin curve" $\mathfrak{B}_N = \{B$: Borel subgroups in $G \mid N \in \mathrm{Lie}(B)\}$ corresponding to the branch point of the Coxeter graph). In these cases, $\bar{Z}(s's_v, N)$ is trivial.

Assume now that G is of type C_n ($n \geqslant 3$). Then there are four possibilities for $s' \in Z(s_v, N)$. If s' is in the center of G (of order 2), then the variety $\mathfrak{B}(s's_v, N)$ consists of n isolated points and a single projective line. The group $\bar{Z}(s's_v, N)$ has two elements; it permutes two of the points and acts as identity on the other components. The other two elements s' in $Z(s_v, N)$ are such that $Z(s')$ is of type $C_{n-1} \times C_1$. Then N is regular in Lie $Z(s')$, hence $\mathfrak{B}(s's_v, N)$ consists of n isolated points. The group $\bar{Z}(s's_v, N)$ has order 2; it acts trivially on $\pi_0(\mathfrak{B}(s_v s', N))$.

Assume next that G is of type F_4. Then there are two possibilities for $s' \in Z(s_v, N)$. If $s' = e$, then $\mathfrak{B}(s's_v, N)$ consists of five isolated points and a single projective line. The group $\bar{Z}(s's_v, N)$ has two elements; it permutes two pairs of points and leaves the fifth point and the line invariant. The second possibility is $s' \in Z(s_v, N)$ such that $Z(s')$ is of type B_4. Then N is regular in Lie $Z(s')$, hence $\mathfrak{B}(s's_v, N)$ consists of 3 isolated points. The group $\bar{Z}(s's_v, N)$ has order 2; it acts trivially on $\pi_0(\mathfrak{B}(s_v s', N))$.

Finally, assume that G is of type G_2. Then $Z(s, N)$ is isomorphic to the symmetric group $\mathfrak{S}_3$. If $s' = e$, then $\mathfrak{B}(s's_v, N)$ consists of three isolated points and a single projective line. The group $\bar{Z}(s's_v, N) \approx \mathfrak{S}_3$ acts on the three isolated points by permuting them in all possible ways, and it acts as identity on the line. If s' is an element of order 2 (resp. 3) then $Z(s')$ is of type $A_1 \times A_1$ (resp. A_2) and N is regular in Lie $Z(s')$. Hence $\mathfrak{B}(s's_v, N)$ consists of 3 (resp. 2) isolated points. The group $\bar{Z}(s's_v, N)$ has order 2 (resp. 3); it acts trivially on $\pi_0(\mathfrak{B}(s's_v, N))$.

2.11. It would be interesting to understand the structure of the variety $\mathfrak{B}(s, N)$ where (s, N) is an arbitrary L_2-pair in G. Let us assume that $s = s_v$. (From the proof of Proposition 2.7, we see that we can reduce ourselves to this case.) It seems that this variety is nonsingular with a number of connected components equal to the number of irreducible components of the variety $\mathfrak{B}_N$.

3. Construction of some representations of Hecke algebras.

3.1. Let (W, S), (W', S') be two Coxeter groups. A map β: $S' \to S$ is said to be admissible if:

(a) For any $s \in S$, the set $\beta^{-1}(s)$ consists of commuting involutions (it is possibly empty).

(b) Let $s \neq t \in S$ be such that st has order $m_{s,t} < \infty$ and let Γ_0 be any connected component of the full subgraph $\beta^{-1}\{s, t\}$ of the Coxeter graph of (W', S'). Then we require that Γ_0 is the Coxeter graph of a finite Coxeter group with Coxeter number dividing $m_{s,t}$.

Assume that β is admissible. Let Γ' be the underlying graph of the Coxeter graph of (W', S'). For each oriented edge $\{s', t'\}$ in Γ' we choose $\mu(s', t') \in \mathbf{N}$ such that $\mu(s', t')\mu(t', s') = 4\cos^2(\pi/m_{s',t'})$. (We assume, for simplicity, that $m_{s',t'}$ takes only the values 2, 3, 4 or 6.) Let E' be the free $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-module with basis $\{e_{s'}\}$, $s' \in S'$. (Here $q^{1/2}$ is an indeterminate.) For each $s \in S$, we define an endomorphism T_s: $E' \to E'$ by

$$T_s(e_{s'}) = \begin{cases} -e_{s'} & \text{if } \beta(s') = s, \\ qe_{s'} + q^{1/2} \sum\limits_{\substack{t' \in S' \\ \beta(t') = s \\ m_{s',t'} \geqslant 3}} \mu(t', s')e_{t'} & \text{if } \beta(s') \neq s. \end{cases}$$

(Condition (b) assures that the last sum is finite.) It is clear that $(T_s - q)(T_s + 1) = 0$ $(s \in S)$. We have

3.2. LEMMA. *Assume that β is admissible. If $s \neq t \in S$ are such that $m_{s,t} < \infty$ then $T_sT_tT_s \cdots = T_tT_sT_t \cdots : E' \to E'$ (both sides of the equality contain $m_{s,t}$ factors). In other words, the T_s define on E' a left module structure over the Hecke algebra H of (W, S) (over $\mathbf{Z}[q^{1/2}, q^{-1/2}]$). Equivalently, the graph Γ' together with the assignment $s' \in S' \to \{\beta(s')\}$ and the function $\mu(s', t')$ is a W-graph in the sense of* [**8**] *(except that μ is not assumed to be symmetric).*

PROOF. In the case where $(W', S') = (W, S)$ and β is the identity map, the lemma is due to Kilmoyer, see [**5**, 9.8] (he assumes that W is finite, but his proof can be easily adapted to infinite W). We shall refer to this case as "the special case". We now turn to the general case; for each $s_1' \in S'$, we define an endomorphism $T'_{s_1'}$: $E' \to E'$ by

$$T'_{s_1'}(e_{s'}) = \begin{cases} -e_{s'}, & \text{if } s' = s_1', \\ qe_{s'}, & \text{if } s' \neq s_1' \text{ and } m_{s',s_1'} = 2, \\ qe_{s'} + \mu(s_1', s')q^{1/2}e_{s_1'}, & \text{if } s' \neq s_1' \text{ and } m_{s',s_1'} \geqslant 3. \end{cases}$$

By "the special case" the endomorphisms $T_{s_1'}$ define on E' a structure of left module over H', the Hecke algebra of (W', S'). It is clear that for any $s \in S$, we have

$$q^{-1}T_s = \prod_{s_1' \in \beta^{-1}(s)} \left(q^{-1}T'_{s_1'}\right): E' \to E'. \tag{3.2.1}$$

(This is an infinite product of commuting linear maps such that any element in E' is kept fixed by all but finitely many of these maps. Hence the product has a meaning.) In particular $q^{-1}T_s$ is identity, if $\beta^{-1}(s)$ is empty.

Now let $s \neq t$ in S be such that $m_{s,t} = 2$. We must show that T_s commutes with T_t. For this it is enough to show that $T'_{s_1'}$ commutes with $T'_{t_1'}$ for any $s_1' \in \beta^{-1}(s)$, $t_1' \in \beta^{-1}(t)$. By assumption 3.1(b), we must have $m_{s_1',t_1'} = 2$; hence $s_1't_1' = t_1's_1'$. Hence by "the special case" we have $T'_{s_1'}T'_{t_1'} = T'_{t_1'}T'_{s_1'}$. (Note that $s_1't_1'$ is a reduced expression.) Next, we assume that $s \neq t$ in S are such that $3 \leqslant m_{s,t} < \infty$. By assumption 3.1(b), we are reduced to the case where W is the dihedral group generated by s and t and (W', S') is a finite irreducible Coxeter group with Coxeter number h dividing $m_{s,t}$. Then $\beta^{-1}(s)$ and $\beta^{-1}(t)$ consist of commuting involutions and they form a partition of S'. According to Bourbaki [**3**, Chapter V, §6, Exercise 2], the longest element of

W' has two reduced expressions of the form

$$\Big(\prod_{s_1'\in\beta^{-1}(s)} s_1'\Big)\Big(\prod_{t_1'\in\beta^{-1}(t)} t_1'\Big)\Big(\prod_{s_1'\in\beta^{-1}(s)} s_1'\Big)\cdots,$$

$$\Big(\prod_{t_1'\in\beta^{-1}(t)} t_1'\Big)\Big(\prod_{s_1'\in\beta^{-1}(s)} s_1'\Big)\Big(\prod_{t_1'\in\beta^{-1}(t)} t_1'\Big)\cdots$$

(in both cases the number of $\prod$ signs is equal to h). It follows then from "the special case" that

$$\Big(\prod T'_{s_1'}\Big)\Big(\prod T'_{t_1'}\Big)\Big(\prod T'_{s_1'}\Big)\cdots=\Big(\prod T'_{t_1'}\Big)\Big(\prod T'_{s_1'}\Big)\Big(\prod T'_{t_1'}\Big)\cdots$$

where s_1' runs through $\beta^{-1}(s)$, t_1' runs through $\beta^{-1}(t)$, and both sides contain a number of $\prod$ signs equal to h. It follows then from (3.2.1) that

$$T_sT_tT_s\cdots=T_tT_sT_t\cdots : E'\to E'$$

(with h factors on both sides). Since h divides $m_{s,t}$, we have also

$$T_sT_tT_s\cdots=T_tT_sT_t\cdots : E'\to E'$$

(with $m_{s,t}$ factors on both sides). This completes the proof.

3.3. Corollary of the Proof. *If S, S' are finite and $\beta\colon S'\to S$ is admissible, then the map $s\in S\to\prod_{s'\in\beta^{-1}(s)}s'$ defines a homomorphism of W into W' and also a homomorphism between the corresponding braid monoids.*

3.4. Let W be a Coxeter group and let S be the corresponding set of simple reflections. We denote by l the length function on W and we set for each $w\in W$: $\mathfrak{L}(W)=\{s\in S\mid l(sw)<l(w)\}$, $\mathfrak{R}(W)=\{s\in S\mid l(ws)<l(w)\}$.

In [8] a function μ from a subset of $W\times W$ to the nonzero integers was defined. We shall not repeat the definition here, but we recall that if $\mu(y,w)$ is defined, then y,w are comparable for the Bruhat order, their lengths have different parities and $\mu(w,y)=\mu(y,w)$.

Following [8], we say that $w,w'\in W$ satisfy $w\leqslant_L w'$ if there exists a sequence of elements $w=w_0,w_1,\ldots,w_n=w'$ in W such that for each i, $1\leqslant i\leqslant n$, $\mu(w_{i-1},w_i)$ is defined and $\mathfrak{L}(w_{i-1})\not\subset\mathfrak{L}(w_i)$. We say that $w\leqslant_R w'$ if $w^{-1}\leqslant_L w'^{-1}$; we say that $w\leqslant_{LR} w'$ if there exists a sequence of elements $w=w_0, w_1,\ldots,w_n=w'$ in W such that for each i, $1\leqslant i\leqslant n$, we have $w_{i-1}\leqslant_L w_i$ or $w_{i-1}\leqslant_R w_i$. We say that $w\sim_L w'$ if $w\leqslant_L w'\leqslant_L w$. The relations $w\sim_R w'$, $w\sim_{LR} w'$ are defined similarly, replacing L by R, LR. The equivalence classes for $\sim_L$, $\sim_R$, $\sim_{LR}$ have been called in [8] left cells, right cells, 2-sided cells respectively. One reason for introducing these concepts is the following result in [8].

(3.4.1) *Let L be a left cell of W and let E_L be the free $\mathbf{Z}[q^{1/2},q^{-1/2}]$-module with basis $\{e_w\}_{w\in L}$ ($q^{1/2}$ is an indeterminate). Define, for each $s\in S$, an endomorphism T_s: $E_L\to E_L$ by*

$$T_s(e_w)=\begin{cases}-e_w & \text{if } s\in\mathfrak{L}(w),\\ qe_w+q^{1/2}\displaystyle\sum_{\substack{y\in L\\ s\in\mathfrak{L}(y)\\ \mu(y,w)\text{ defined}}}\mu(y,w)e_y & \text{if } s\notin\mathfrak{L}(w).\end{cases}$$

Then, for any $s \neq t$ in S such that st has finite order m, we have

$$\underbrace{T_sT_tT_s\cdots}_{m\ \text{factors}} = \underbrace{T_tT_sT_t\cdots}_{m\ \text{factors}} : E_L \to E_L.$$

In other words, each left cell L gives rise to a left module for the Hecke algebra H associated with (W, S). Similarly, each right cell gives rise to a right H-module and each 2-sided cell gives rise to an H-bimodule.

3.5. I conjecture that, each two-sided cell of W should have a nonempty intersection with some *finite* parabolic subgroup $W_I \subset W$ $(I \subset S)$. In particular, if S is finite, then W should have only finitely many 2-sided cells; moreover each 2-sided cell should be a union of finitely many left cells.

3.6. Assume, for example, that (W, S) is an affine Weyl group of type $\tilde{A}_{n-1}$ $(n \geq 2)$. We may regard W as the group of all permutations σ: $\mathbf{Z} \to \mathbf{Z}$ such that $\sigma(i + n) = \sigma(i) + n$ for all $i \in \mathbf{Z}$ and such that $\sum_{i=1}^{n}(\sigma(i) - i) = 0$. The simple reflections are $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ where $\sigma_i(j) = j + 1$ for $j \equiv i \pmod n$, $\sigma_i(j) = j - 1$ for $j \equiv i + 1 \pmod n$, $\sigma_i(j) = j$ for $j \not\equiv i, i + 1 \pmod n$, $i = 0, 1, 2, \ldots, n - 1$. With each $\sigma \in W$, we associate a sequence of integers $d_1 \leq d_2 \leq \cdots \leq d_n = n$ as follows: d_k is the maximum cardinal of a subset of $\mathbf{Z}$ whose elements are noncongruent to each other mod n and which is a disjoint union of k subsets each of which has its natural order reversed by σ. This definition is suggested by the work of C. Greene [**6**]; I am indebted to C. Greene for showing me how Theorem 1.5 of [**6**] implies that $d_1 \geq d_2 - d_1 \geq d_3 - d_2 \geq \cdots \geq d_n - d_{n-1}$ (a partition of n). I conjecture that two elements $\sigma, \sigma' \in W$ are in the same 2-sided cell if and only if they give rise to the same partition of n. (Notice that the identity element of W is the unique element of W giving rise to the partition $1 \geq 1 \geq 1 \geq \cdots$.)

I also conjecture that the number of left cells contained in the 2-sided cell giving rise to the partition $\lambda_1 \geq \lambda_2 \geq \cdots$ of n is equal to $n!(\lambda'_1!)^{-1}(\lambda'_2!)^{-1}\cdots$, where $\lambda'_1 \geq \lambda'_2 \geq \cdots$ is the dual partition. More generally, if (W, S) is the affine Weyl group associated to the group $\mathcal{G}$ (see 1.1) then there should be a 1-1 correspondence between the two-sided cells of W and the set of nilpotent classes in Lie(G) (see 1.2) such that the number of left sided cells contained in the two-sided cell corresponding to the nilpotent $N \in \mathrm{Lie}(G)$ is equal to $\Sigma(-1)^i \dim H^i(\mathfrak{B}_N)^{Z(N)}$ where $\mathfrak{B}_N$ is the variety of Borel subgroups of G containing N and $Z(N)$ is the centralizer of N in G.

3.7. We return to a general Coxeter group (W, S), assumed to be irreducible. Let $\mathcal{C}$ be the set of elements $w \in W$, $w \neq e$, such that w has a unique reduced expression. For each $s \in S$, let $\mathcal{C}_s = \{w \in \mathcal{C} \mid \mathfrak{R}(w) = \{s\}\}$. Then clearly, $\mathcal{C} = \coprod_{s \in S} \mathcal{C}_s$. For $s \in S$, let Γ_s be the graph with set of vertices $\mathcal{C}_s$ and edges $\{y, w\} \subset \mathcal{C}_s$ such that $yw^{-1} \in S$. The graph Γ_s is isomorphic to the graph Γ'_s whose vertices are the sequences $s_1, s_2, \ldots, s_p$ of reflections in S such that $s_p = s$, $s_i \neq s_{i+1}$ $(1 \leq i \leq p - 1)$ and such that whenever $s, t \in S$ have product st of order m $(2 \leq m < \infty)$, there are no m consecutive terms of $s_1, s_2, \ldots, s_p$ of form $s, t, s, t, \ldots$; the edges of Γ'_s are of form $\{(s_1, s_2, \ldots, s_p), (s_2, \ldots, s_p)\}$. (The correspondence between Γ_s and Γ'_s is defined by $(s_1, s_2, \ldots, s_p) \to s_1 s_2 \cdots s_p$.)

Let π: $\Gamma_s \to S$ be defined by $\{\pi(w)\} = \mathfrak{R}(w)$. This corresponds to the map $(s_1, s_2, \ldots, s_p) \to s_1$ of Γ'_s into S. Let $\hat{W}_s$ be the Coxeter group with Coxeter graph Γ_s

(simply laced) and let $s_w \in \hat{W}_s$ be the simple reflection of $\hat{W}_s$ corresponding to $w \in \mathcal{C}_s$. We have

3.8. Proposition. (a) *If $y \in \mathcal{C}_s$, $w \in W$ $(w \neq e)$ are such that $\mu(y, w)$ is defined and $\mathcal{L}(y) \not\subset \mathcal{L}(w)$, then $w \in \mathcal{C}_s$.*

(b) *$\mathcal{C}_s$ is a left cell, for any $s \in S$.*

(c) *$\mathcal{C}$ is a 2-sided cell.*

(d) *Let y, $w \in \mathcal{C}_s$. Then $\{y, w\}$ is an edge of Γ_s if and only if $\mathcal{L}(y) \neq \mathcal{L}(w)$ and $\mu(y, w)$ is defined. We then have $\mu(y, w) = 1$.*

(e) *The graph Γ_s is a tree. The map π: $\Gamma_s \to S$ defines an isomorphism of Γ_s onto the Coxeter graph of (W, S) if and only if the latter is a tree and is simply laced. If $s, t \in S$ satisfy $(st)^3 = e$, then there exists an isomorphism of graphs $\Gamma_s \approx \Gamma_t$, compatible with the map π.*

(f) *Let E_s be the free $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-module with basis (e_w), $w \in \mathcal{C}_s$. For each $t \in S$, let T_t: $E_s \to E_s$ be the endomorphism defined by*

$$T_t(e_w) = \begin{cases} -e_w & \text{if } \pi(w) = t, \\ qe_w + q^{1/2} \displaystyle\sum_{\substack{y \in \mathcal{C}_s \\ \pi(y) = t \\ y,\, w \text{ edge of } \Gamma_s}} e_y & \text{if } \pi(w) \neq t. \end{cases}$$

These endomorphisms define a left H-module structure on E_s.

(g) *Consider the Coxeter groups $(\hat{W}, S)$, (W_s, Γ_s). Then the map π: $\Gamma_s \to S$ is admissible.*

(h) *Assume that S is finite. Then $\mathcal{C}_s$ is finite for some $s \in S$ if and only if $\mathcal{C}_s$ is finite for any $s \in S$ and if and only if the Coxeter graph of (W, S) is a tree and there is at most one pair $s \neq t$ of elements in S such that st has order* > 4.

Proof. (a) Since $w \neq e$, $\mathcal{L}(w)$ has at least one element. It cannot be contained in $\mathcal{L}(y)$ since then $\mathcal{L}(y)$ would have at least two elements contradicting the assumption that y has a unique reduced expression. Thus $\mathcal{L}(y) \not\subset \mathcal{L}(w) \not\subset \mathcal{L}(y)$. It follows then from [**8**, (2.3.e)] that $w = ty$ $(t \in S)$ and that $\mu(y, w) = 1$. If $\mathcal{L}(y) = \{t\}$, then $w = ty$ clearly has a unique reduced expression (obtained by omitting t at the front of the reduced expression for y). If $t \notin \mathcal{L}(y)$, and if $w = ty$ has more than one reduced expression then we must have $y = s_1 s_2 \cdots s_p$ (reduced) and $s_1 s_2 \cdots s_{m-1} = s_1 t s_1 t \cdots$ $(m - 1$ factors), where m is the order of $s_1 t$ (necessarily finite). But then

$$w = (t s_1 t s_1 \cdots) s_m s_{m+1} \cdots s_p$$

(where there are m factors in the parenthesis) is a reduced expression for w and it follows that $s_1 \in \mathcal{L}(W)$. But this contradicts $\mathcal{L}(y) \not\subset \mathcal{L}(w)$. Thus, we have proved that $w \in \mathcal{C}_s$.

(b) Let $w \in \mathcal{C}_s$, $w \neq s$, and let $w = s_1 s_2 \cdots s_p$ be the unique reduced expression for w. We have $s_p = s$. Moreover, by [**8**, (2.3.e)], we have $\mu(w', w) = 1$, where $w' = s_2 \cdots s_p$. Note that $\mathcal{L}(w) = \{s_1\} \neq \{s_2\} = \mathcal{L}(w')$ hence $w' \sim_L w$. Repeating this, we find $s_1 s_2 \cdots s_p \sim_L s_2 \cdots s_p \sim_L \cdots \sim_L s_p = s$. Hence w is in the left cell containing s.

Conversely, let w' be an element in the left cell containing s. Then there exists a sequence of elements $s = w_0, w_1, \ldots, w_n = w'$ such that for each i, $1 \leq i \leq n$, $\mu(w_{i-1}, w_i)$ is defined and $\mathcal{L}(w_{i-1}) \not\subset \mathcal{L}(w_i)$. It follows that $w_i \neq e$ for $i = 0, 1, \ldots, n - 1$. We have also $w_n \neq e$, since e is not in the left cell containing s ($\mathcal{R}(e) \neq \mathcal{R}(s)$). Applying (a) repeatedly, we find then that $w_i \in \mathcal{C}_s$ for $i = 1, \ldots, n$. In particular, $w' \in \mathcal{C}_s$ and (b) is proved.

(c) To show that $\mathcal{C}$ is contained in a 2-sided cell it is enough (because of (b)) to show that any two elements s, t of S satisfy $s \sim_{LR} t$. Moreover, since (W, S) is irreducible, we can reduce ourselves to the case where s, t are not commuting. But then $s \cdot s' \sim_L s'$ and $s's \sim_L s$ by (b). Hence $(s's)^{-1} \sim_R s^{-1}$. In particular, we have $ss' \sim_{LR} s'$, $ss' \sim_{LR} s$ hence $s' \sim_{LR} s$ as required.

Now let w be any element in the 2-sided cell containing $\mathcal{C}$. We have necessarily $w \neq e$ since $\{e\}$ is a 2-sided cell by itself. We want to show that $w \in \mathcal{C}$. We may assume that there exists $s \in S$ and $w' \in \mathcal{C}_s$ such that $\mu(w', w)$ is defined and either $\mathcal{L}(w') \not\subset \mathcal{L}(w)$ or $\mathcal{R}(w') \not\subset \mathcal{R}(w)$. In the first case, we have (by (a)) $w \in \mathcal{C}_s$, hence $w \in \mathcal{C}$. In the second case, we have $\mathcal{L}(w'^{-1}) \not\subset \mathcal{L}(w^{-1})$, and if $w'^{-1} \in \mathcal{C}_{s'}$, we have (by (a)) $w^{-1} \in \mathcal{C}_{s'}$ hence $w^{-1} \in \mathcal{C}$, hence $w \in \mathcal{C}$. Thus, (c) is proved. The proof of (d) is contained in the proof of (a).

(e) Any element $w \in \mathcal{C}_s$ of length $n \geq 2$ is joined in the graph Γ_s with a unique element $y \in \mathcal{C}_s$ of length $n - 1$: y is obtained from w by removing the first simple reflection in a reduced expression of w. This gives a contraction of the graph Γ_s onto its vertex s, and shows that Γ_s is a tree. The second statement of (e) follows from the isomorphism $\Gamma_s \approx \Gamma'_s$. (When the Coxeter graph of (W, S) is a tree and is simply laced, the vertices of Γ'_s are the sequences $s_1, s_2, \ldots, s_p$ in S which define a geodesic on the Coxeter graph from $s_p = s$ to s_1.) If s, $t \in S$ satisfy $(st)^3 = 1$, we define $\Gamma'_s \to \Gamma'_t$ by

$$(s_1, s_2, \ldots, s_{p-1}, s_p) \to \begin{cases} (s_1, s_2, \ldots, s_{p-1}), & \text{if } s_{p-1} = t,\ s_p = s, \\ (s_1, s_2, \ldots, s_{p-1}, s_p, t), & \text{if } s_{p-1} \neq t,\ s_p = s. \end{cases}$$

This is a graph isomorphism $\Gamma'_s \approx \Gamma'_t$.

(f) follows from (3.4.1) and (d).

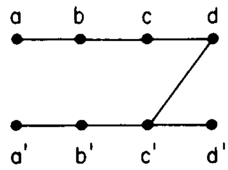
(g) follows from the isomorphism $\Gamma_s \approx \Gamma'_s$. For example, if $s_1 \neq t_1$ are elements of S such that the product $s_1 t_1$ has finite order $m \geq 3$, then the inverse image of $\{s_1, t_1\}$ under $\Gamma'_s \to S$, regarded as a full subgraph of Γ'_s, has as connected components only graphs of type A_{m-1} (whose Coxeter number is m).

(h) This also follows easily from the isomorphism $\Gamma_s \approx \Gamma'_s$.

3.9. Remarks. (a) If S is finite but C_s is infinite, Corollary 3.3 is not applicable, but the same method gives a homomorphism of W into a "completion" of $\hat{W}_s$. For example, if (W, S) is an affine Weyl group of type $\tilde{A}_n$ ($n \geq 1$) we get in this way an imbedding of W into an infinite symmetric group. This is essentially the description of W used in 3.6.

(b) If (W, S) is of type H_4, then combining Proposition 3.8(g) and Corollary 3.3, we get an imbedding of W into a Coxeter group of type E_8; if we represent the

Coxeter diagram of E_8 in the form shown in the diagram ($a, b, c, d, a', b', c', d'$ are the simple reflections in E_8) then the subgroup generated by aa', bb', cc', dd' is the Coxeter group of type H_4. The imbedding of W in E_8 doubles the length of any element.



3.10. We return to the setup in 3.1, and assume that $\beta: S' \to S$ is admissible and that S, S' are finite sets. Following Kilmoyer **[5]**, we consider the $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-bilinear form $(\ ,\ ): E' \times E' \to \mathbf{Z}[q^{1/2}, q^{-1/2}]$ defined by

$$(e_{s'}, e_{s'}) = q^{1/2} + q^{-1/2}, \quad (e_{s'}, e_{t'}) = -\mu(s', t'), \qquad s' \neq t',$$

where we agree to set $\mu(s', t') = 0$, if $s' \neq t'$, $m_{s',t'} = 2$. We then have

$$T'_{s'}(e) = qe - q^{1/2}(e_{s'}, e)e_{s'}$$

for all $e \in E'$. Let φ be a homomorphism of $\mathbf{Z}[q^{1/2}, q^{-1/2}]$ into a field K. According to **[5**, (9.13)**]**, $E' \otimes_\varphi K$ is an irreducible $H \otimes_\varphi K$-module in the case where $W = W'$ are finite, $\beta =$ identity, and φ is the imbedding of $\mathbf{Z}[q^{1/2}, q^{-1/2}]$ into its quotient field. We shall prove here a more general result.

3.11. PROPOSITION. *Assume that* $\varphi(\mu(s', t')) \neq 0$ *for all* $s' \neq t'$ *in* S' *such that* $m_{s',t'} \geq 3$, *that* $\beta: S' \to S$ *is admissible and injective and that* S *has finitely many (and at least two) elements. Then the following two conditions are equivalent.*

(a) $E \otimes_\varphi K$ *is an irreducible* $H \otimes_\varphi K$*-module.*

(b) $\varphi(\det(e_{s'}, e_{t'})_{(s',t') \in S' \times S'}) \neq 0$ *in* K.

PROOF. Using (3.2.1) and the injectivity of β we are clearly reduced to the case where $W = W'$ and β is the identity. We shall assume this in the rest of the proof. Then $T_s = T'_s$ for $s \in S$.

Assume first that (b) is satisfied. Let M be a nonzero $H \otimes_\varphi K$-submodule of $E' \otimes_\varphi K$. If $m \in M$, $m \neq 0$ satisfies $T_s(m) = \varphi(q)m$ for all $s \in S$, then $(e_s, m)_\varphi = 0$ for all $s \in S$, where $(\ ,\ )_\varphi$ denotes the K-bilinear form on $E \otimes_\varphi K$ deduced from $(\ ,\ )$ by extension of scalars via φ. But such m cannot exist if (b) holds. Hence, if $m \in M$, $m \neq 0$, there exists $s \in S$ such that $T_s(m) \neq \varphi(q)m$. Then $M \supset T_s(m) - \varphi(q)m = -(e_s, m)e_s$ and $(e_s, m) \neq 0$. It follows that $e_s \in M$. If $t \in S$ is joined with s in the Coxeter graph of W, then $M \ni T_t(e_s) - \varphi(q)e_s = -\varphi(\mu(t, s))e_t$ and $\varphi(\mu(t, s)) \neq 0$, hence $e_t \in M$. Since (W, S) is irreducible, it follows then that $e_t \in M$ for all $t \in S$ so that $M = E \otimes_\varphi K$ and hence $E \otimes_\varphi K$ is irreducible.

Conversely, let us assume that $E \otimes_\varphi K$ is irreducible and that (b) is not satisfied. Then there exists $m \neq 0$ in $E \otimes_\varphi K$ such that $(e_s, m)_\varphi = 0$ for all $s \in S$. Hence

$$\{m \in E \otimes_\varphi K \mid T_s(m) = \varphi(q)m, \forall s \in S\}$$

is nonzero. But this is clearly an $H \otimes_\varphi K$-submodule, hence it must coincide with $E \otimes_\varphi K$. Thus, $T_s(e_t) = \varphi(q)e_t$ for all $s, t \in S$, hence

$$\varphi(\mu(s, t)) = 0 \quad \text{for all } s \neq t \text{ in } S.$$

By assumption, S has at least two elements and we find a contradiction.

3.12. REMARK. The polynomial (in $q^{1/2}$)

$$\Delta_{W'} = q^{|S|/2} \det\big((e_{s'}, e_{t'})_{(s',t') \in S \times S}\big)$$

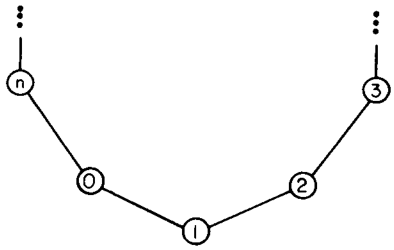
has been determined for finite W' by Kostant who showed that it is equal to the characteristic polynomial in q of a Coxeter element of W' in the standard reflection representation. One can also compute directly $\Delta_{W'}$ for (W', S') an irreducible affine Weyl group (see the tables in Examples 3.13).

3.13. EXAMPLES. We now assume that (W, S) is an irreducible affine Weyl group. We shall describe in each case (omitting the trivial case $\tilde{A}_1$) certain W-graphs obtained by the method of 3.2. Each of these W-graphs Γ gives rise to a representation E_Γ of the Hecke algebra H of (W, S), and when Γ is finite, to a homomorphism of W into the Coxeter group with Coxeter graph Γ.

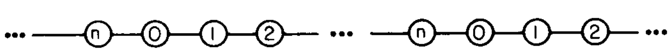
For each such graph, we shall specify the function β from the vertices of Γ to S by writing i inside a small circle at each vertex mapped by β to $s_i \in S$. We also give (when appropriate) conditions for $E_\Gamma \otimes_\varphi K$ to be an irreducible $H \otimes_\varphi K$-module, where φ is a homomorphism of $\mathbf{Z}[q^{1/2}, q^{1/2}]$ into a field K. The symbol "$\Gamma \sim \mathcal{G}_1 + \mathcal{G}_2$" will mean that under the specified conditions, the $H \otimes_\varphi K$-module $E_\Gamma \otimes_\varphi K$ is isomorphic to the direct sum of the $H \otimes_\varphi K$-modules $E_{\mathcal{G}_1} \otimes_\varphi K$, $E_{\mathcal{G}_2} \otimes_\varphi K$. Our examples will include all graphs Γ_s associated in 3.7 to the simple reflections $s \in S$. The Coxeter graph will be denoted $\mathcal{G}$.

Type $\tilde{A}_n$ $(n \geqslant 2)$:

$\mathcal{G}$:

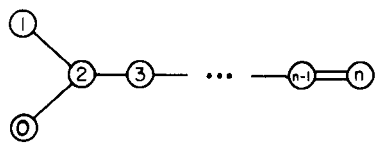


Γ_{s_0}:

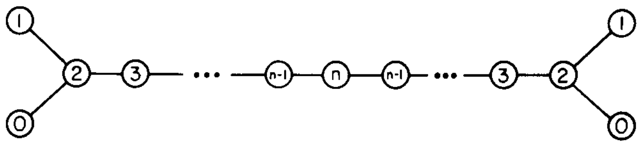


Type $\tilde{B}_n$ $(n \geqslant 3)$:

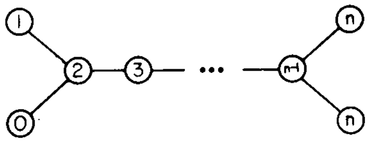
$\mathcal{G}$:



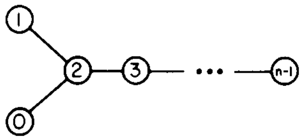
Γ_{s_0}:



Γ_{s_n}:



$\mathcal{G}'$:



$\mathcal{G}''$:

Assume $2 \neq 0$ in K. Then

$$\Gamma_{s_0} \sim \mathcal{G} \oplus \mathcal{G}', \quad \Gamma_{s_n} \sim \mathcal{G} \oplus \mathcal{G}''.$$

(Consider the nontrivial involution of the graph Γ_{s_i}, $i = 0, \ldots, n$, compatible with the labels and take the two eigenspaces of the corresponding $E_{\Gamma_{s_i}} \otimes K$. This gives the required decomposition.)

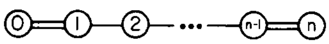
$E_{\mathcal{G}} \otimes K$ irreducible $\Leftrightarrow (q^2 - 1)(q^{n-1} - 1) \neq 0$ in K,

$E_{\mathcal{G}'} \otimes K$ irreducible $\Leftrightarrow (q + 1)(q^{n-1} + 1) \neq 0$ in K,

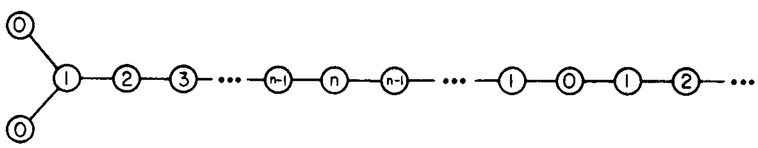
$E_{\mathcal{G}''} \otimes K$ irreducible always.

Type $\tilde{C}_n$ $(n \geqslant 2)$:

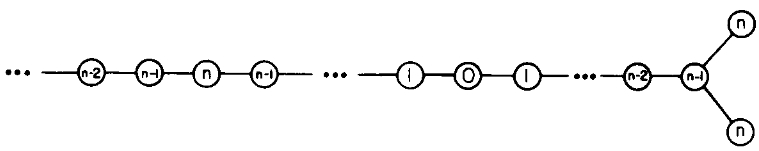
$\mathcal{G}$:

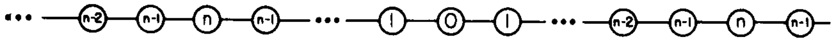


Γ_{s_0}:



Γ_{s_n}:

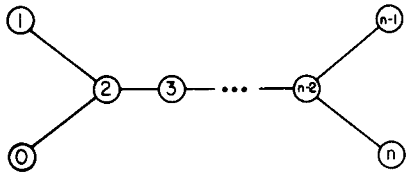


Γ_{S_1}: 

$\mathcal{G}'$: 

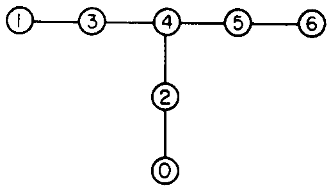
$\mathcal{G}''$: 

Type $\tilde{D}_n$ $(n \geqslant 4)$:

$\mathcal{G} = \Gamma_{S_0}$ 

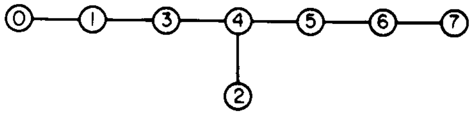
$$E_{\mathcal{G}} \otimes K \quad \text{irreducible} \Leftrightarrow (q+1)(q^2-1)(q^{n-2}-1) \neq 0 \text{ in } K.$$

Type $\tilde{E}_6$:

$\mathcal{G} = \Gamma_{S_0}$: 

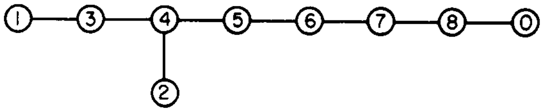
$$E_{\mathcal{G}} \otimes K \quad \text{irreducible} \Leftrightarrow (q+1)(q^3-1)^2 \neq 0 \text{ in } K.$$

Type $\tilde{E}_7$:

$\mathcal{G} = \Gamma_{S_0}$: 

$$E_{\mathcal{G}} \otimes K \quad \text{irreducible} \Leftrightarrow (q+1)(q^3-1)(q^4-1) \neq 0 \text{ in } K.$$

Type $\tilde{E}_8$:

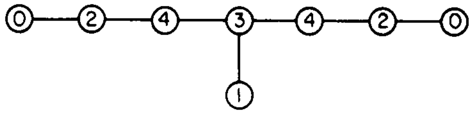
$\mathcal{G} = \Gamma_{S_0}$: 

$$E_{\mathcal{G}} \quad \text{irreducible} \Leftrightarrow (q+1)(q^3-1)(q^5-1) \neq 0 \text{ in } K.$$

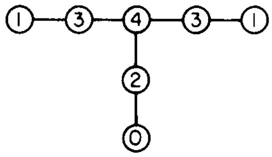
Type $\tilde{F}_4$:

$\mathcal{G}$:

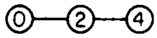
Γ_{s_0}:



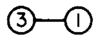
Γ_{s_1}:



$\mathcal{G}'$:



$\mathcal{G}''$:



Assume $2 \neq 0$ in K. Then

$$\Gamma_{s_0} \sim \mathcal{G} \oplus \mathcal{G}', \quad \Gamma_{s_1} \sim \mathcal{G} \oplus \mathcal{G}''.$$

(These are obtained as for type B_n.)

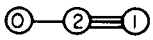
$E_{\mathcal{G}} \otimes K$ irreducible $\Leftrightarrow (q^2-1)(q^3-1) \neq 0$ in K,

$E_{\mathcal{G}'} \otimes K$ irreducible $\Leftrightarrow (q+1)(q^2+1) \neq 0$ in K,

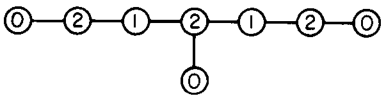
$E_{\mathcal{G}''} \otimes K$ irreducible $\Leftrightarrow q^2+q+1 \neq 0$ in K.

Type $\tilde{G}_2$:

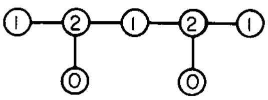
$\mathcal{G}$:



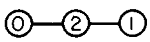
Γ_{s_0}:

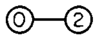


Γ_{s_1}:



$\mathcal{G}'$:



$\mathcal{G}''$: 

$\mathcal{G}'''$: 

Assume that $6 \neq 0$ in K. Then

(3.13.1)
$$\Gamma_{s_0} \sim \mathcal{G} \oplus \mathcal{G}' \oplus \mathcal{G}'',$$
$$\Gamma_{s_1} \sim \mathcal{G} \oplus \mathcal{G}' \oplus \mathcal{G}''',$$
$$E_{\mathcal{G}} \otimes K \quad \text{irreducible} \Leftrightarrow (q-1)(q^2-1) \neq 0 \text{ in } K,$$
$$E_{\mathcal{G}'} \otimes K \quad \text{irreducible} \Leftrightarrow (q+1)(q^2+1) \neq 0 \text{ in } K,$$
$$E_{\mathcal{G}''} \otimes K \quad \text{irreducible} \Leftrightarrow q^2+q+1 \neq 0 \text{ in } K,$$
$$E_{\mathcal{G}'''} \otimes K \quad \text{irreducible always.}$$

The decomposition (3.13.1) of the space $E_{\Gamma_{s_0}} \otimes K$ is achieved as follows. We denote the canonical basis of this space as

$$\begin{matrix} e_0, e_2, e_1, e_2', e_1', e_2'', e_0' \\ e_0'' \end{matrix}$$

(in correspondence with the vertices of Γ_{s_0}). Then the three subspaces

$$\langle e_1 + e_1', e_2 + 2e_2' + e_2'', e_0 + 2e_0'' + e_0' \rangle,$$
$$\langle e_1 - e_1', e_2 - e_2'', e_0 - e_0' \rangle, \quad \langle e_2 - e_2' + e_2'', e_0 - e_0'' + e_0' \rangle$$

are H-submodules and give the required decomposition.

The decomposition (3.13.1) of $E_{\Gamma_{s_1}} \otimes K$ is achieved as follows. We denote the canonical basis of this space as

$$\begin{matrix} e_1, e_2, e_1', e_2', e_1''. \\ e_0 \qquad e_0' \end{matrix}$$

Then the three subspaces

$$\langle e_1 + 2e_1' + e_1'', e_2 + e_2', e_0 + e_0' \rangle,$$
$$\langle e_1 - e_1'', e_2 - e_2', e_0 - e_0' \rangle, \quad \langle e_1 - e_1' + e_1'' \rangle$$

are H-submodules and give the required decomposition.

4. Computations with affine Weyl groups.

4.1. Let (W, S) be an irreducible affine Weyl group regarded as a Coxeter group. (For information on affine Weyl groups see [3].)

W has a normal free abelian finitely generated subgroup Q, of finite index. It consists of all elements in W which have only finitely many conjugates.

Let Ω be the group of all automorphisms of (W, S) which have the property that their restriction to W is the same as conjugation by some element of W. This is a finite group.

Let $\tilde{W}$ be the group of all pairs (τ, W), $\tau \in \Omega$, $w \in W$ with multiplication $(\tau, w)(\tau', w') = (\tau\tau', \tau'^{-1}(w)w')$; in other words, $\tilde{W}$ is the semidirect product of Ω and W. We write $(\tau, w) = \tau w$ and we then have $\tau(w) = \tau w \tau^{-1}$ in $\tilde{W}$. Let P be the set of elements in $\tilde{W}$ which have only finitely many conjugates in $\tilde{W}$. Then P is a normal free abelian subgroup of $\tilde{W}$ of finite index, containing Q as a subgroup of finite index. We have $P/Q \stackrel{\approx}{\to} \tilde{W}/W \stackrel{\sim}{\to} \Omega$.

The length function l: $W \to N$ extends uniquely to a function l: $\tilde{W} \to \mathbf{N}$ by $l(\tau w) = l(w)$ $(\tau \in \Omega, w \in W)$. Let us fix $s_0 \in S$ in such a way that W is generated by Q and by the finite group W_0 generated by $S - s_0$. We shall denote the elements of $S - \{s_0\}$ by $s_1, \ldots, s_n$ in a way compatible with the numbering of the Coxeter graph of (W, S) in 1.13.

4.2. For each i, $1 \leqslant i \leqslant n$, there is a unique element $\alpha_i \in Q$ and a unique homomorphism $\check{\alpha}_i$: $P \to \mathbf{Z}$ such that $s_i x s_i^{-1} x^{-1} = \alpha_i^{-\check{\alpha}_i(x)}$ for all $x \in X$ and such that $l(s_i\alpha_i) > l(\alpha_i)$. Then $\alpha_i (1 \leqslant i \leqslant n)$ form a basis for Q. For any i, $1 \leqslant i \leqslant n$, there is a unique element $\omega_i \in P$ such that

$$\check{\alpha}_j(\omega_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (1 \leqslant j \leqslant n).$$

In other words, we have

$$s_j\omega_i = \omega_i s_j, \quad s_i\omega_i s_i^{-1}\omega_i^{-1} = \alpha_i^{-1}, \qquad 1 \leqslant j \leqslant n, j \neq i.$$

The ω_i $(1 \leqslant i \leqslant n)$ form a basis for P. Let P^{++} be the semigroup in P generated by $\omega_1, \ldots, \omega_n$. We have

$$P^{++} = \{x \in P \mid l(s_i x) > l(x), i = 1, \ldots, n\}.$$

Let ht: $Q \to \mathbf{Z}$ be the homomorphism defined by $\mathrm{ht}(\alpha_i) = 1$ $(1 \leqslant i \leqslant n)$. This extends uniquely to a homomorphism ht: $P \to (1/2)\mathbf{Z}$. If $x \in P^{++}$, we have $l(x) = 2\,\mathrm{ht}(x)$. In particular, we have

(4.2.1) $$l(xy) = l(x) + l(y) \quad \text{for } x, y \in P^{++}.$$

4.3. Besides the Hecke algebra H of (W, S), we shall also consider the Hecke algebra $\tilde{H}$ defined as follows. $\tilde{H}$ is the free $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-module with basis T_w $(w \in \tilde{W})$ and multiplication defined by

$$(T_s + 1)(T_s - q) = 0 \quad \text{if } s \in S,$$
$$T_w T_{w'} = T_{ww'} \quad \text{if } l(ww') = l(w) + l(w'), w, w' \in W.$$

The algebra $\tilde{H}$ considered in the introduction is the algebra we have just defined tensored with $\mathbf{C}$, via the ring homomorphism $\mathbf{Z}[q^{1/2}, q^{-1/2}] \to \mathbf{C}$ taking q to a prime power and $q^{1/2}$ to its positive square root. I hope that this ambiguity in the notation will not create confusion. Then H is a subalgebra of $\tilde{H}$. We shall denote T_τ also by τ $(\tau \in \Omega)$. We shall also set $T_i = T_{s_i}$, $i \in [0, n]$.

It follows from (4.2.1) that in $\tilde{H}$, we have

(4.3.1) $$T_{xy} = T_x T_y, \quad \text{for } x, y \in P^{++}.$$

Following J. Bernstein, we define, for each $x \in P$, an element $\hat{T}_x \in \tilde{H}$ by

$$\hat{T}_x = q^{(-l(x_1)+l(x_2))/2} T_{x_1} T_{x_2}^{-1}$$

where x_1, x_2 are elements of P^{++} such that $x = x_1 x_2^{-1}$. It follows from (4.2.1), (4.3.1) that $\hat{T}_x$ is well defined. We then have $\hat{T}_{xy} = \hat{T}_x \hat{T}_y$ for all $x, y \in P$ and $\hat{T}_x = q^{-l(x)/2} T_x$ for $x \in P^{++}$. We shall also set $\hat{T}_i = q^{-1/2} T_i$ $(0 \leq i \leq n)$. We have $\hat{T}_i^{-1} = \hat{T}_i - (q^{1/2} - q^{-1/2})$. The following lemma is a special case of a result of J. Bernstein (unpublished).

4.4. LEMMA. *Let $i \in [1, n]$ and let $x \in P$.*
(a) *If $s_i x = s x_i$, then $\hat{T}_i \hat{T}_x = \hat{T}_x \hat{T}_i$.*
(b) *If $s_i x s_i^{-1} x^{-1} = \alpha_i^{-1}$, then $\hat{T}_i^{-1} \hat{T}_x \hat{T}_i^{-1} = \hat{T}_{s_i x s_i^{-1}}$.*

PROOF. In case (a), we may write $x = x_1 x_2^{-1}$ with $x_1, x_2 \in P^{++}$, $s_i x_1 = x_1 s_i$, $s_i x_2 = x_2 s_i$, and we are reduced to the case where $x \in P^{++}$. In that case, $l(sx) = l(xs) = l(x) + 1$ hence $T_s T_x = T_x T_s = T_{sx} = T_{xs}$ and (a) follows.

We now consider the case (b). Write $x = x_1 x_2^{-1}$ with

$$x_1, x_2 \in P^{++}, \quad s_i x_1 s_i^{-1} x_1^{-1} = \alpha_i^{-1}, \quad s_i x_2 = x_2 s_i.$$

Using (a), we are reduced to the case where $x = x_1$. Thus, we may assume that $x \in P^{++}$. Let $l = l(x)$. Then $l(s_i) = l + 1$ (since $x \in P^{++}$) and $l(x s_i) = l - 1$ (since $s_i x s_i \neq x$). It is easy to see that $s_i x s_i x \in P^{++}$ hence

$$l(s_i x s_i x) = 2\,\mathrm{ht}(s_i x s_i x) = 2\,\mathrm{ht}(s_i x s_i x^{-1}) + 4\,\mathrm{ht}(x) = 2\,\mathrm{ht}(\alpha_i^{-1}) + 2l = 2l - 2.$$

Since $s_i x s_i x \in P^{++}$, we have $l(x s_i x) = l(s_i x s_i x) + 1 = 2l - 1$. We have

$$\begin{aligned} T_i^{-1} T_x T_i^{-1} T_x &= T_i^{-1} T_{x s_i} T_x, \quad \text{since } l(x s_i) = l(x) - 1, \\ &= T_i^{-1} T_{x s_i x}, \quad \text{since } l(x s_i x) = l(x s_i) + l(x), \\ &= T_{s_i x s_i x}, \quad \text{since } l(s_i x s_i x) = l(x s_i x) - 1. \end{aligned}$$

Hence $\hat{T}_i^{-1} \hat{T}_x \hat{T}_i^{-1} = \hat{T}_{s_i x s_i x} \hat{T}_x^{-1} = \hat{T}_{s_i x s_i}$ and (b) is proved.

4.5. We shall now give some formulas describing the elements $\hat{T}_{\omega_i} \in \tilde{H}$ $(1 \leq i \leq n)$ for each irreducible affine Weyl group.

Type $\tilde{A}_n$: We have

$$\hat{T}_{\omega_n} = \tau \hat{T}_1 \hat{T}_2 \cdots \hat{T}_n \tag{4.5.1}$$

where $\tau \in \Omega$ is defined by $s_0 \tau = \tau s_1, s_1 \tau = \tau s_2, \ldots, s_{n-1} \tau = \tau s_n, s_n \tau = \tau s_0$.

Indeed $\tau s_1 s_2 \cdots s_n$ is in P^{++}, commutes with $s_1, s_2, \ldots, s_{n-1}$ and has length n hence it is equal to ω_n. Using now Lemma 4.4(b), we have

$$\hat{T}_{\omega_{n-1}} \hat{T}_{\omega_n}^{-1} = \hat{T}_{\omega_{n-1}\omega_n^{-1}} = \hat{T}_n^{-1} \hat{T}_{\omega_n} \hat{T}_n^{-1} = \tau \hat{T}_0^{-1} \hat{T}_1 \hat{T}_2 \cdots \hat{T}_{n-1},$$

$$\hat{T}_{\omega_{n-2}} \hat{T}_{\omega_{n-1}}^{-1} = \hat{T}_{\omega_{n-2}\omega_{n-1}^{-1}} = \hat{T}_{n-1}^{-1} \hat{T}_n^{-1} \hat{T}_{\omega_n}^{-1} \hat{T}_n^{-1} \hat{T}_{n-1}^{-1} = \tau \hat{T}_n^{-1} \hat{T}_0^{-1} \hat{T}_1 \hat{T}_2 \cdots \hat{T}_{n-2},$$

$$\hat{T}_{\omega_2} \hat{T}_{\omega_3}^{-1} = \hat{T}_{\omega_2\omega_3^{-1}} = \hat{T}_3^{-1} \hat{T}_4^{-1} \cdots \hat{T}_n^{-1} \hat{T}_{\omega_n} \hat{T}_n^{-1} \cdots \hat{T}_4^{-1} \hat{T}_3^{-1} = \tau \hat{T}_4^{-1} \hat{T}_5^{-1} \cdots \hat{T}_n^{-1} \hat{T}_0^{-1} \hat{T}_1 \hat{T}_2,$$

$$\hat{T}_{\omega_1} \hat{T}_{\omega_2}^{-1} = \hat{T}_{\omega_1\omega_2^{-1}} = \hat{T}_2^{-1} \hat{T}_3^{-1} \cdots \hat{T}_n^{-1} \hat{T}_{\omega_n} \hat{T}_n^{-1} \cdots \hat{T}_3^{-1} \hat{T}_2^{-1} = \tau \hat{T}_3^{-1} \hat{T}_4^{-1} \cdots \hat{T}_n^{-1} \hat{T}_0^{-1} \hat{T}_1.$$

It follows that

$$\hat{T}_{\omega_{n-1}} = \tau^2(\hat{T}_2\hat{T}_1)(\hat{T}_3\hat{T}_2)\cdots(\hat{T}_i\hat{T}_{i-1})\cdots(\hat{T}_n\hat{T}_{n-1}),$$
$$\hat{T}_{\omega_{n-2}} = \tau^3(\hat{T}_3\hat{T}_2\hat{T}_1)(\hat{T}_4\hat{T}_3\hat{T}_2)\cdots(\hat{T}_i\hat{T}_{i-1}\hat{T}_{i-2})\cdots(\hat{T}_n\hat{T}_{n-1}\hat{T}_{n-2}),$$
$$\cdots$$
$$\hat{T}_{\omega_1} = \tau^n\hat{T}_n\hat{T}_{n-1}\cdots\hat{T}_1.$$

Type $\tilde{B}_n$: We have

$$\hat{T}_{\omega_1} = \tau\hat{T}_1\hat{T}_2\hat{T}_3\cdots\hat{T}_{n-1}\hat{T}_n\hat{T}_{n-1}\cdots\hat{T}_3\hat{T}_2\hat{T}_1$$

where $\tau \in \Omega$ is defined by $s_1\tau = \tau s_0$, $s_0\tau = \tau s_1$, $\tau s_i = s_i\tau$ for $i = 2,3,\ldots,n$. Indeed, $\tau s_1 s_2 s_3 \cdots s_{n-1}s_n s_{n-1}\cdots s_3 s_2 s_1$ is in P^{++}, commutes with $s_2,s_3,\ldots,s_n$ and has length $2n-1$ hence it is equal to ω_1.

Using now Lemma 4.4(b), we have

$$\hat{T}_{\omega_2}\hat{T}_{\omega_1}^{-1} = \hat{T}_{\omega_2\omega_1^{-1}} = \hat{T}_1^{-1}\hat{T}_{\omega_1}\hat{T}_1^{-1},$$
$$\hat{T}_{\omega_3}\hat{T}_{\omega_2}^{-1} = \hat{T}_{\omega_3\omega_2^{-1}} = \hat{T}_2^{-1}\,\hat{T}_{\omega_2\omega_1^{-1}}\hat{T}_2^{-1},$$
$$\cdots$$
$$\hat{T}_{\omega_{n-1}}\hat{T}_{\omega_{n-2}}^{-1} = \hat{T}_{\omega_{n-1}\omega_{n-2}^{-1}} = \hat{T}_{n-2}^{-1}\hat{T}_{\omega_{n-2}\omega_{n-3}^{-1}}\hat{T}_{n-2}^{-1},$$
$$\hat{T}_{\omega_n}\hat{T}_{\omega_{n-1}}^{-1} = \hat{T}_{\omega_n\omega_{n-1}^{-1}} = \hat{T}_{n-1}^{-1}\hat{T}_{\omega_{n-1}\omega_{n-2}^{-1}}\hat{T}_{n-1}^{-1}$$

which determines also $\hat{T}_{\omega_2}, \hat{T}_{\omega_3}, \ldots, \hat{T}_{\omega_n}$.

Type $\tilde{C}_n$ $(n \geqslant 2)$: We have

$$\hat{T}_{\omega_1} = \hat{T}_0\hat{T}_1\hat{T}_2\cdots\hat{T}_{n-1}\hat{T}_n\hat{T}_{n-1}\cdots\hat{T}_2\hat{T}_1.$$

Using Lemma 4.4(b), we have

$$\hat{T}_{\omega_2}\hat{T}_{\omega_1}^{-1} = \hat{T}_{\omega_2\omega_1^{-1}} = \hat{T}_1^{-1}\hat{T}_{\omega_1}\hat{T}_1^{-1},$$
$$\hat{T}_{\omega_3}\hat{T}_{\omega_2}^{-1} = \hat{T}_{\omega_3\omega_2^{-1}} = \hat{T}_2^{-1}\hat{T}_{\omega_2\omega_1^{-1}}\hat{T}_2^{-1},$$
$$\cdots$$
$$\hat{T}_{\omega_{n-1}}\hat{T}_{\omega_{n-2}}^{-1} = \hat{T}_{\omega_{n-1}\omega_{n-2}^{-1}} = \hat{T}_{n-2}^{-1}\hat{T}_{\omega_{n-2}\omega_{n-3}^{-1}}\hat{T}_{n-2}^{-1},$$
$$\hat{T}_{\omega_n}^2\hat{T}_{\omega_{n-1}} = \hat{T}_{\omega_n^2\omega_{n-1}^{-1}} = \hat{T}_{n-1}^{-1}\hat{T}_{\omega_{n-1}\omega_{n-2}^{-1}}\hat{T}_{n-1}^{-1}$$

which determines also $\hat{T}_{\omega_2}, \hat{T}_{\omega_3}, \ldots, \hat{T}_{\omega_{n-1}}, \hat{T}_{\omega_n}^2$. We can compute separately

$$\hat{T}_{\omega_n} = \tau(\hat{T}_n\hat{T}_{n-1}\cdots\hat{T}_1)(\hat{T}_n\hat{T}_{n-1}\cdots\hat{T}_2)\cdots(\hat{T}_n\hat{T}_{n-1})(\hat{T}_n)$$

where τ is the unique nontrivial element of Ω. Indeed, $\tau(s_n s_{n-1}\cdots s_1)(s_n s_{n-1}\cdots s_2)\cdots(s_n s_{n-1})(s_n)$ is in P^{++}, commutes with $s_1,s_2,\ldots,s_{n-1}$ and has length $n+(n-1)+\cdots+2+1$, hence it is equal to ω_n.

Type $\tilde{D}_n$ $(n \geqslant 4)$: We have

$$\hat{T}_{\omega_1} = \hat{T}_1\hat{T}_2\cdots\hat{T}_{n-2}\hat{T}_{n-1}\hat{T}_n\hat{T}_{n-2}\hat{T}_{n-3}\cdots\hat{T}_2\hat{T}_1$$

where $\tau \in \Omega$ is defined by $s_n\tau = \tau s_{n-1}$, $s_{n-1}\tau = \tau s_n$, $s_1\tau = \tau s_0$, $s_0\tau = \tau s_1$, $\tau s_i = s_i\tau$ $(i = 2,3,\ldots,n-2)$.

Using Lemma 4.4(b), we have

$$\hat{T}_{\omega_2}\hat{T}_{\omega_1}^{-1} = \hat{T}_{\omega_2\omega_1^{-1}} = \hat{T}_1^{-1}\hat{T}_{\omega_1}\hat{T}_1^{-1},$$
$$\hat{T}_{\omega_3}\hat{T}_{\omega_2}^{-1} = \hat{T}_{\omega_3\omega_2^{-1}} = \hat{T}_2^{-1}\hat{T}_{\omega_2\omega_1^{-1}}\hat{T}_2^{-1},$$
$$\cdots$$
$$\hat{T}_{\omega_{n-2}}\hat{T}_{\omega_{n-3}}^{-1} = \hat{T}_{\omega_{n-2}\omega_{n-3}^{-1}} = \hat{T}_{n-3}^{-1}\hat{T}_{\omega_{n-3}\omega_{n-4}^{-1}}\hat{T}_{n-3}^{-1},$$
$$\hat{T}_{\omega_n}\hat{T}_{\omega_{n-1}}\hat{T}_{\omega_{n-2}}^{-1} = \hat{T}_{\omega_n\omega_{n-1}\omega_{n-2}^{-1}} = \hat{T}_{n-2}^{-1}\hat{T}_{\omega_{n-2}\omega_{n-3}^{-1}}\hat{T}_{n-2}^{-1},$$
$$\hat{T}_{\omega_n}^{-1}\hat{T}_{\omega_{n-1}} = \hat{T}_{\omega_n^{-1}\omega_{n-1}} = \hat{T}_n^{-1}\hat{T}_{\omega_n\omega_{n-1}\omega_{n-2}^{-1}}\hat{T}_n^{-1}$$

which determines also $\hat{T}_{\omega_2}, \hat{T}_{\omega_3}, \ldots, \hat{T}_{\omega_{n-2}}, \hat{T}_{\omega_{n-1}}^2, \hat{T}_{\omega_n}^2$ (and also $\hat{T}_{\omega_{n-1}}\hat{T}_{\omega_n}$). We can check separately that the expression

$$(4.5.2)\quad \tau'(\hat{T}_{n-1}\hat{T}_{n-2}\hat{T}_{n-3}\cdots\hat{T}_1)(\hat{T}_n\hat{T}_{n-2}\hat{T}_{n-3}\cdots\hat{T}_2)(\hat{T}_{n-1}\hat{T}_{n-2}\hat{T}_{n-3}\cdots\hat{T}_3)\cdots$$

(the last three factors are $(\hat{T}_{n-1}\hat{T}_{n-2}\hat{T}_{n-3})(\hat{T}_n\hat{T}_{n-2})(\hat{T}_{n-1})$ if n is even and $(\hat{T}_n\hat{T}_{n-2}\hat{T}_{n-3})(\hat{T}_{n-1}\hat{T}_{n-2})(\hat{T}_n)$ if n is odd) is equal to $\hat{T}_{\omega_{n-1}}$ if n is even and to $\hat{T}_{\omega_n}$ if n is odd.

Here τ' is the unique element of Ω such that $\tau' s_{n-1} = s_0\tau'$, $\tau' s_n = s_1\tau'$. (We then have $\tau' s_0 = s_{n-1}\tau'$, $\tau' s_1 = s_n\tau'$ if n is even and $\tau' s_0 = s_n\tau'$, $\tau' s_1 = s_{n-1}\tau'$ if n is odd). To get $\hat{T}_{\omega_{n-1}}$ (for n odd) and $\hat{T}_{\omega_n}$ (for n even) replace $\hat{T}_{n-1}$ by $\hat{T}_n$ and $\hat{T}_n$ by $\hat{T}_{n-1}$ in the expression (4.5.2); we also replace τ' by $\tau\tau'$ in that expression.

Type $\tilde{E}_6$: We have

$$\hat{T}_{\omega_2} = \hat{T}_0\hat{T}_2\hat{T}_4\hat{T}_3\hat{T}_5\hat{T}_4\hat{T}_1\hat{T}_3\hat{T}_2\hat{T}_4\hat{T}_5\hat{T}_6\hat{T}_5\hat{T}_4\hat{T}_2\hat{T}_3\hat{T}_1\hat{T}_4\hat{T}_5\hat{T}_3\hat{T}_4\hat{T}_2.$$

Using Lemma 4.4(b), we have

$$\hat{T}_{\omega_4}\hat{T}_{\omega_2} = \hat{T}_{\omega_4\omega_2^{-1}} = \hat{T}_2^{-1}T_{\omega_2}\hat{T}_2^{-1},$$
$$\hat{T}_{\omega_3}\hat{T}_{\omega_5}\hat{T}_{\omega_4}^{-1} = \hat{T}_{\omega_3\omega_5\omega_4^{-1}} = \hat{T}_4^{-1}T_{\omega_4\omega_2^{-1}}^{-1}\hat{T}_4^{-1},$$
$$\hat{T}_{\omega_3}\hat{T}_{\omega_5}^{-1}\hat{T}_{\omega_6} = \hat{T}_{\omega_3\omega_5^{-1}\omega_6} = \hat{T}_5^{-1}\hat{T}_{\omega_3\omega_5\omega_4^{-1}}\hat{T}_5^{-1},$$
$$\hat{T}_{\omega_3}\hat{T}_{\omega_6}^{-1} = \hat{T}_{\omega_3\omega_6^{-1}} = \hat{T}_6^{-1}\hat{T}_{\omega_3\omega_5^{-1}\omega_6}\hat{T}_6^{-1},$$
$$\hat{T}_{\omega_3}^{-1}\hat{T}_{\omega_5}\hat{T}_{\omega_1} = \hat{T}_{\omega_3^{-1}\omega_5\omega_1} = \hat{T}_3^{-1}T_{\omega_3\omega_5\omega_4^{-1}}\hat{T}_3^{-1}$$

which determines also $\hat{T}_{\omega_4}, \hat{T}_{\omega_3}^3\hat{T}_{\omega_5}^3, \hat{T}_{\omega_6}^3, \hat{T}_{\omega_1}^3$.

Type $\tilde{E}_7$: We have

$$\hat{T}_{\omega_1} = \hat{T}_0\hat{T}_1\hat{T}_3\hat{T}_4\hat{T}_5\hat{T}_6\hat{T}_7\hat{T}_2\hat{T}_4\hat{T}_3\hat{T}_1\hat{T}_5\hat{T}_6\hat{T}_4\hat{T}_3\hat{T}_2\hat{T}_5\hat{T}_4\hat{T}_5\hat{T}_2\hat{T}_3\hat{T}_4\hat{T}_6\hat{T}_5\hat{T}_1\hat{T}_3\hat{T}_4\hat{T}_2\hat{T}_7\hat{T}_6\hat{T}_5\hat{T}_4\hat{T}_3\hat{T}_1$$

(34 factors).

Using now Lemma 4.4(b), we have

$$\hat{T}_{\omega_3}\hat{T}_{\omega_1}^{-1} = \hat{T}_{\omega_3\omega_1^{-1}} = \hat{T}_1^{-1}\hat{T}_{\omega_1}\hat{T}_1^{-1},$$
$$\hat{T}_{\omega_4}\hat{T}_{\omega_3}^{-1} = \hat{T}_{\omega_4\omega_3^{-1}} = \hat{T}_3^{-1}\hat{T}_{\omega_3\omega_1^{-1}}\hat{T}_3^{-1},$$
$$\hat{T}_{\omega_2}\hat{T}_{\omega_5}\hat{T}_{\omega_4}^{-1} = \hat{T}_{\omega_2\omega_5\omega_4^{-1}} = \hat{T}_4^{-1}\hat{T}_{\omega_4\omega_3^{-1}}\hat{T}_4^{-1},$$
$$\hat{T}_{\omega_2}\hat{T}_{\omega_5}^{-1}\hat{T}_{\omega_6} = \hat{T}_{\omega_2\omega_5^{-1}\omega_6} = \hat{T}_5^{-1}\hat{T}_{\omega_2\omega_5\omega_4^{-1}}\hat{T}_5^{-1},$$

$$\hat T_{\omega_2}\hat T_{\omega_6}^{-1}\hat T_{\omega_7} = \hat T_{\omega_2\omega_6^{-1}\omega_7} = \hat T_6^{-1}\hat T_{\omega_2\omega_5^{-1}\omega_6}\hat T_6^{-1},$$

$$\hat T_{\omega_2}\hat T_{\omega_5}^{-1} = \hat T_{\omega_2\omega_5^{-1}} = \hat T_2^{-1}\hat T_{\omega_2\omega_5\omega_4^{-1}}\hat T_2^{-1}$$

which determines also $\hat T_{\omega_3}$, $\hat T_{\omega_4}$, $\hat T_{\omega_6}$, $\hat T_{\omega_2}^2$, $\hat T_{\omega_5}^2$, $\hat T_{\omega_7}^2$.

Type $\tilde E_8$: We have

$$\text{(4.5.3)}\quad \hat T_{\omega_8} = \hat T_0\hat T_8\hat T_7\hat T_6\hat T_5\hat T_4\hat T_2\hat T_3\hat T_1\hat T_4\hat T_5\hat T_6\hat T_3\hat T_4\hat T_5\hat T_7\hat T_6\hat T_2\hat T_4\hat T_5\hat T_8\hat T_7\hat T_6\hat T_3\hat T_4\hat T_5\hat T_2\hat T_4\hat T_3\hat T_1\hat T_3$$
$$\times \hat T_4\hat T_2\hat T_5\hat T_4\hat T_3\hat T_6\hat T_7\hat T_8\hat T_5\hat T_4\hat T_2\hat T_6\hat T_7\hat T_5\hat T_4\hat T_3\hat T_6\hat T_5\hat T_4\hat T_1\hat T_3\hat T_2\hat T_4\hat T_5\hat T_6\hat T_7\hat T_8$$

(58 factors). Note that if we omit the factor $\hat T_0$ and replace each $\hat T_i$ by s_i, we get a reduced expression, in the Weyl group of type E_8, for the reflection with respect to the highest root (of length 57). Using now Lemma 4.4(b), we have

(4.5.4)
$$\hat T_{\omega_7}\hat T_{\omega_8}^{-1} = \hat T_{\omega_7\omega_8^{-1}} = \hat T_8^{-1}\hat T_{\omega_8}\hat T_8^{-1},$$

$$\hat T_{\omega_6}\hat T_{\omega_7}^{-1} = \hat T_{\omega_6\omega_7^{-1}} = \hat T_7^{-1}\hat T_{\omega_7\omega_8^{-1}}\hat T_7^{-1},$$

$$\hat T_{\omega_5}\hat T_{\omega_6}^{-1} = \hat T_{\omega_5\omega_6^{-1}} = \hat T_6^{-1}\hat T_{\omega_6\omega_7^{-1}}\hat T_6^{-1},$$

$$\hat T_{\omega_4}\hat T_{\omega_5}^{-1} = \hat T_{\omega_4\omega_5^{-1}} = \hat T_5^{-1}\hat T_{\omega_5\omega_6^{-1}}\hat T_5^{-1},$$

$$\hat T_{\omega_2}\hat T_{\omega_3}\hat T_{\omega_4}^{-1} = \hat T_{\omega_2\omega_3\omega_4^{-1}} = \hat T_4^{-1}\hat T_{\omega_4\omega_5^{-1}}\hat T_4^{-1},$$

$$\hat T_{\omega_1}\hat T_{\omega_2}\hat T_{\omega_3}^{-1} = \hat T_{\omega_1\omega_2\omega_3^{-1}} = \hat T_3^{-1}\hat T_{\omega_2\omega_3\omega_4^{-1}}\hat T_3^{-1},$$

$$\hat T_{\omega_3}\hat T_{\omega_2}^{-1} = \hat T_{\omega_3\omega_2^{-1}} = \hat T_2^{-1}\hat T_{\omega_2\omega_3\omega_4^{-1}}\hat T_2^{-1}$$

which determines also $\hat T_{\omega_7}$, $\hat T_{\omega_6}$, $\hat T_{\omega_5}$, $\hat T_{\omega_4}$, $\hat T_{\omega_2}^2$, $\hat T_{\omega_3}^2$, $\hat T_{\omega_1}^2$.

Type $\tilde F_4$: We have

$$\hat T_{\omega_2} = \hat T_0\hat T_2\hat T_4\hat T_3\hat T_1\hat T_4\hat T_2\hat T_3\hat T_4\hat T_3\hat T_2\hat T_4\hat T_1\hat T_3\hat T_4\hat T_2$$

(16 factors).

Using Lemma 4.4(b), we have

$$\hat T_{\omega_4}\hat T_{\omega_2}^{-1} = \hat T_{\omega_4\omega_2^{-1}} = \hat T_2^{-1}\hat T_{\omega_2}\hat T_2^{-1},$$

$$\hat T_{\omega_3}\hat T_{\omega_4}^{-1} = \hat T_{\omega_3\omega_4^{-1}} = \hat T_4^{-1}\hat T_{\omega_4\omega_2^{-1}}\hat T_4^{-1},$$

$$\hat T_{\omega_1}\hat T_{\omega_3}^{-1}\hat T_{\omega_4} = \hat T_{\omega_1\omega_3^{-1}\omega_4} = \hat T_3^{-1}\hat T_{\omega_3\omega_4^{-1}}\hat T_3^{-1}$$

which determines also $\hat T_{\omega_4}$, $\hat T_{\omega_3}$, $\hat T_{\omega_1}$.

Type $\tilde G_2$: We have

$$\hat T_{\omega_2} = \hat T_0\hat T_2\hat T_1\hat T_2\hat T_1\hat T_2,$$

$$\hat T_{\omega_1} = \hat T_0\hat T_2\hat T_1\hat T_2\hat T_1\hat T_0\hat T_2\hat T_1\hat T_2\hat T_1.$$

4.6. We now consider the Coxeter graph $\mathcal{G}$ of the affine Weyl group (W, S) as in 3.13 and the corresponding H-module $E_{\mathcal{G}}$. The definition of $E_{\mathcal{G}}$ depends on the choice of the function $\mu(s, t)$. When $\mathcal{G}$ is not simply laced this choice is not unique; to make it unique we shall require that $\mu(s, t) = 1$ whenever st has order $\geqslant 4$ and t is nearer to an extremal point of $\mathcal{G}$, than s.

By definition, $E_{\mathcal{G}}$ has given basis elements e_s, one for each $s \in S$. We shall write e_i instead of e_{s_i} $(0 \leqslant i \leqslant n)$. The H-module $E_{\mathcal{G}}$ can be made naturally into an $\tilde{H}$-module as follows. For each $\tau \in \Omega$, we define a $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-linear map τ: $E_{\mathcal{G}} \to E_{\mathcal{G}}$ by $\tau(e_s) = e_{\tau(s)}$ $(s \in S)$. It is clear that $\tau(T_t(e_s)) = T_{\tau(t)}(\tau(e_s))$ for all $\tau \in \Omega$, and $s, t \in S$. Hence the endomorphisms τ $(\tau \in \Omega)$ and T_w $(w \in W)$ make $E_{\mathcal{G}}$ into an $\tilde{H}$-module.

We wish to describe the action of the elements $\hat{T}_{\omega_i} \in \tilde{H}$ $(1 \leqslant i \leqslant n)$ on the $\tilde{H}$-module $E_{\mathcal{G}}$, assuming that (W, S) is of type $\neq \tilde{A}_n, \tilde{C}_n$.

We define an integer b, $1 \leqslant b \leqslant n$, as follows: $b = n - 1$ (for type $\tilde{B}_n$), $b = n - 2$ (for type $\tilde{D}_n$), $b = 4$ (for type $\tilde{E}_n$ and $\tilde{F}_4$) and $b = 2$ (for type $\tilde{G}_2$). Thus, for types $\tilde{D}_n$ and $\tilde{E}_n$, s_b corresponds to a branch point of $\mathcal{G}$. For any i, $0 \leqslant i \leqslant n$, we denote by d_i the distance on $\mathcal{G}$ between the vertices corresponding to i and b. Thus, $d_b = 0$. Let m be the exponent with which the simple root α_b appears in the highest root, expressed as a product of simple roots. (For example, for $\tilde{E}_8$, we have $m = 6$.)

There is a unique vector $v_b \in E_{\mathcal{G}}$ such that

$$\begin{aligned}
\hat{T}_i(v_b) &= q^{1/2} v_b \qquad (1 \leqslant i \leqslant n, i \neq b), \\
\hat{T}_b(v_b) &= q^{1/2} v_b - q^{(d_0 - 3)/2 + m}(q - 1) e_b, \\
\hat{T}_0(v_b) &= q^{1/2} v_b + q^{-1/2}(q^m - 1) e_0.
\end{aligned}$$

The vector v_b is given explicitly as follows:

For type $\tilde{B}_n$ $(n \geqslant 3)$, we have

$$v_b = e_0 + q e_1 + \sum_{i=2}^{n-1} \left(q^{(i-1)/2} + q^{(i+1)/2}\right) e_i + 2 q^{(n-1)/2} e_n.$$

For type $\tilde{D}_n$ $(n \geqslant 4)$, we have

$$v_b = e_0 + q e_1 + \sum_{i=2}^{n-2} \left(q^{(i-1)/2} + q^{(i+1)/2}\right) e_i + q^{(n-2)/2} e_{n-1} + q^{(n-2)/2} e_n.$$

For type $\tilde{E}_6$, we have

$$\begin{aligned}
v_b = e_0 &+ \left(q^{1/2} + q^{5/2}\right) e_2 + \left(q + q^2 + q^3\right) e_4 + \left(q^{3/2} + q^{5/2}\right) e_3 \\
&+ \left(q^{3/2} + q^{5/2}\right) e_5 + q^2 e_1 + q^2 e_6.
\end{aligned}$$

For type $\tilde{E}_7$, we have

$$\begin{aligned}
v_b = e_0 &+ \left(q^{1/2} + q^{7/2}\right) e_1 + \left(q + q^3 + q^4\right) e_3 + \left(q^{3/2} + q^{5/2} + q^{7/2} + q^{9/2}\right) e_4 \\
&+ \left(q^2 + q^3 + q^4\right) e_5 + \left(q^{5/2} + q^{7/2}\right) e_6 + q^3 e_7 + \left(q^2 + q^4\right) e_2.
\end{aligned}$$

For type $\tilde{E}_8$, we have

$$\begin{aligned}
v_b = e_0 &+ \left(q^{1/2} + q^{11/2}\right) e_8 + \left(q + q^5 + q^6\right) e_7 \\
&+ \left(q^{3/2} + q^{9/2} + q^{11/2} + q^{13/2}\right) e_6 + \left(q^2 + q^4 + q^5 + q^6 + q^7\right) e_5 \\
&+ \left(q^{5/2} + q^{7/2} + q^{9/2} + q^{11/2} + q^{13/2} + q^{15/2}\right) e_4 \\
&+ \left(q^3 + q^4 + q^6 + q^7\right) e_3 + \left(q^{7/2} + q^{13/2}\right) e_1 + \left(q^3 + q^5 + q^7\right) e_2.
\end{aligned}$$

For type $\tilde{F}_4$, we have

$$v_b = e_0 + \left(q^{1/2} + q^{5/2}\right)e_2 + \left(q + q^2 + q^3\right)e_4 + \left(2q^{3/2} + 2q^{5/2}\right)e_2 + 2q^2 e_1.$$

For type $\tilde{G}_2$, we have

$$v_b = e_0 + \left(q^{1/2} + q^{3/2}\right)e_2 + 3qe_1.$$

Now let i be an integer, $1 \leqslant i \leqslant n$, $i \neq b$. We define a vector $v_i \in E_{\mathcal{G}}$ by

$$v_i = v_b + \sum_{a=1}^{d_i} \left(q^{(d_0+a)/2+m-1} - q^{(d_0-a)/2+m-1}\right)e_{i_a}$$

where $b = i_0, i_1, i_2, \ldots, i_{d_i} = i$ is the geodesic (of length d_i) on $\mathcal{G}$ from b to i. (We identify the vertices of $\mathcal{G}$ with their labels.) In particular, i_a, i_{a+1} is joined in $\mathcal{G}$ for $a = 0, 1, \ldots, i_{d_i - 1}$.

We also set $v_0 = e_b$.

Let λ_b: $Q \to \mathbf{Z}$ be the homomorphism defined by

$$\lambda_b(\alpha_j) = 1 \qquad (1 \leqslant j \leqslant n, j \neq b), \qquad \lambda_b(\alpha_b) = 0.$$

This extends uniquely to a homomorphism λ_b: $P \to (1/2)\mathbf{Z}$ (case by case verification).

For any integer i $(1 \leqslant i \leqslant n,\ i \neq b)$ we define a homomorphism λ_i: $P \to (1/2)\mathbf{Z}$ by

$$\lambda_i(x) = \lambda_b\left(s_{i_1}s_{i_2}\cdots s_{i_{d_i}} x s_{i_{d_i}}^{-1}\cdots s_{i_2}^{-1}s_{i_1}^{-1}\right)$$

where $b = i_0, i_1, \ldots, i_{d_i} = i$ is the geodesic in $\mathcal{G}$ from b to i.

We have, for any j $(1 \leqslant j \leqslant n)$ and any i $(1 \leqslant j \leqslant n,\ i \neq b)$

$$\lambda_i(\omega_j) = \begin{cases} \lambda_b(\omega_j) - d_j & \text{if } j \text{ belongs to the geodesic from } b \text{ to } i, \\ \lambda_b(\omega_j) & \text{otherwise.} \end{cases}$$

With these notations, we can state

4.7. THEOREM. *Assume that* (W, S) *is an irreducible affine Weyl group of type* $\neq \tilde{A}_n$, $\tilde{C}_n$.

We have

$$\hat{T}_{\omega_j}(v_i) = q^{\lambda_i(\omega_j)}v_i \qquad (1 \leqslant i, j \leqslant n)$$

and

$$\hat{T}_j(v_0) = \begin{cases} q^{\lambda_b(\omega_j)}\left(v_0 - q^{-d_0/2-m+1}v_b\right), & j = b, \\ q^{\lambda_b(\omega_j)}v_0, & 1 \leqslant j \leqslant n, j \neq b. \end{cases}$$

4.8. The proof of 4.7 consists of rather long, case by case computations. We shall indicate what computations are necessary in the case of $\tilde{E}_8$ (which is the most complicated one).

For each simple reflection s_i $(0 \leqslant i \leqslant 8)$, the action of $\hat{T}_i$ on $E_{\mathcal{G}}$ (in the basis e_i, $0 \leqslant i \leqslant 8$) is given by a very simple formula (see 3.1). We multiply directly the 58 matrices corresponding to the factors in (4.5.3) and we find

$$\hat{T}_{\omega_8}(e_i) = q^{23}e_i \qquad (1 \leqslant i \leqslant 7),$$
$$\hat{T}_{\omega_8}(e_8) = q^{23}e_8 - q^{27/2}v_8,$$
$$\hat{T}_{\omega_8}(e_0) = q^{23}e_0 + (q^{14} + q^{19})v_8$$

from which it follows that $\hat{T}_{\omega_8}$ acts in the basis v_i $(0 \leqslant i \leqslant 8)$, as stated in Theorem 4.7. Next, we use the formulas (4.5.4) and we check that $\hat{T}_{\omega_7}$, $\hat{T}_{\omega_6}$, $\hat{T}_{\omega_5}$, $\hat{T}_{\omega_4}$, $\hat{T}^2_{\omega_2}$, $\hat{T}^2_{\omega_3}$, $\hat{T}^2_{\omega_1}$ act as stated in Theorem 4.7. Then, $\hat{T}_{\omega_j}$ $(1 \leqslant j \leqslant 3)$ whose action is still unknown will necessarily map v_i to $\pm q^{\lambda_i(\omega_j)}v_i$ $(1 \leqslant i \leqslant 8)$ and v_0 to $\pm q^{\lambda_4(\omega_j)}v_0$ + a multiple of v_4 (since they commute with the previous transformations and their square is known). If we tensor $\tilde{H}$ and E_g with $\mathbf{C}$ via the ring homomorphism $\mathbf{Z}[q^{1/2}, q^{-1/2}] \to \mathbf{C}$ taking $q^{1/2}$ to 1, $\tilde{H}$ becomes the group algebra of $\tilde{W}$ and E_g becomes the 9-dimensional standard reflection representation of $\tilde{W}$; the vectors v_i $(1 \leqslant i \leqslant 8)$ all reduce to the same vector: the unique vector (up to a scalar) fixed by $\tilde{W}$. The action of T_{ω_j} becomes the usual action of a translation in $\tilde{W}$ on the reflection representation, hence it is a unipotent transformation.

It follows that all the $\pm$ signs above are necessarily $+1$. Thus, for $j = 1, 2, 3$,

$$\hat{T}_{\omega_j}(v_i) = q^{\lambda_i(\omega_j)}v_i \qquad (1 \leqslant i \leqslant 8),$$
$$\hat{T}_{\omega_j}(v_0) = q^{\lambda_4(\omega_j)}(v_0 + xv_4).$$

But the action of the square of $\hat{T}_{\omega_j}$ is known. We have

$$\hat{T}^2_{\omega_j}(v_0) = q^{2\lambda_4(\omega_j)}v_0 = q^{2\lambda_4(\omega_j)}(v_0 + 2xv_4);$$

hence $x = 0$, and hence $\hat{T}_{\omega_j}$ $(1 \leqslant j \leqslant 3)$ act as stated in Theorem 4.7. The proofs for the cases $\neq \tilde{E}_8$ are similar.

4.9. We now consider the graph $\mathcal{G}'$ associated to the affine Weyl group of type $\tilde{B}_n$ $(n \geqslant 3)$ (see 3.13) and let $E_{\mathcal{G}'}$ be the corresponding H-module. It has a basis $e_0, e_1, \ldots, e_{n-1}$ (in 1-1 correspondence with the vertices of $\mathcal{G}'$). We extend $E_{\mathcal{G}'}$ to an $\tilde{H}$-module by letting the nontrivial element $\tau \in \Omega$ act by

$$\tau(e_0) = e_1, \quad \tau(e_1) = e_0, \quad \tau(e_i) = e_i \qquad (2 \leqslant i \leqslant n-1).$$

We consider the vectors $v_1, v_2, \ldots, v_{n-1}, \bar{v}_{n-1}$ in $E_{\mathcal{G}'}$, defined by

$$\bar{v}_{n-1} = e_0 + qe_1 + \sum_{i=2}^{n-1} \left(q^{(i-1)/2} + q^{(i+1)/2}\right)e_i,$$

$$v_j = e_0 + qe_1 + \sum_{i=2}^{j-1} \left(q^{(i-1)/2} - q^{(i+1)/2}\right)e_i + \sum_{i=j}^{n-1} \left(q^{(i-1)/2} - q^{n-(i+1)/2}\right)e_i \qquad (2 \leqslant j \leqslant n-1),$$

$$v_1 = e_0 - q^{n-1}e_1 + \sum_{i=2}^{n-1} \left(q^{(i-1)/2} - q^{n-(i+1)/2}\right)e_i.$$

With these notations, we can state

4.10. PROPOSITION. *We have*

$$\hat{T}_{\omega_k}(v_j) = \varepsilon_{k,j} q^{\lambda_j(\omega_k)} v_j \qquad (1 \leqslant k \leqslant n, 1 \leqslant j \leqslant n-1)$$

where

$$\varepsilon_{k,j} = \begin{cases} 1 & \text{if } k < j, \\ -1 & \text{if } k \geqslant j \end{cases}$$

and

$$\hat{T}_{\omega_k}(\bar{v}_{n-1}) = \varepsilon_k q^{\lambda_{n-1}(\omega_k)} \bar{v}_{n-1} \qquad (1 \leqslant k \leqslant n)$$

where

$$\varepsilon_k = \begin{cases} 1 & \text{if } k < n, \\ -1 & \text{if } k = n. \end{cases}$$

The proof of this proposition (as well as that of 4.12, 4.14, 4.16, 4.18, 4.19, 4.20, 4.22) is of the same nature (direct computation) as that of Theorem 4.7.

4.11. We now consider the graph $\mathcal{G}''$ for (W, S) as in 4.9 and the corresponding H-module $E_{\mathcal{G}''}$ (of rank 1). We extend $E_{\mathcal{G}''}$ to an $\tilde{H}$-module with Ω acting trivially.

4.12. PROPOSITION. *If* $1 \leqslant k \leqslant n$, $\hat{T}_{\omega_k}$ *acts on* $E_{\mathcal{G}''}$ *as the scalar* $q^{\lambda_n(\omega_k)}$.

4.13. We now consider the graphs $\mathcal{G}'$, $\mathcal{G}''$ for (W, S) of type $\tilde{C}_n$ $(n \geqslant 2)$ (see 3.13) and the corresponding H-modules $E_{\mathcal{G}'}$, $E_{\mathcal{G}''}$ of rank 1. They do not extend to $\tilde{H}$-modules, but their direct sum $E_{\mathcal{G}'} + E_{\mathcal{G}''}$ does: we set $\tau e' = e''$, $\tau e'' = e'$ where e' is a basis element for $E_{\mathcal{G}'}$, e'' is a basis element for $E_{\mathcal{G}''}$ and τ is the nontrivial element of Ω. We then have

4.14. PROPOSITION. *Let* φ: $\mathbf{Z}[q^{1/2}, q^{1/2}] \to K$ *be a ring homomorphism with* K *a field, such that* $2 \neq 0$ *in* K *and* $(-1)^{n/2} \in K$. *Then* $e = e' + (-1)^{n/2} q^{n/2} e''$, $\bar{e} = e' - (-1)^{n/2} q^{n/2} e''$ *form a* K*-basis of* $(E_{\mathcal{G}'} \oplus E_{\mathcal{G}''}) \otimes_\varphi K$ *and*

$$\hat{T}_{\omega_i}(e) = (-1)^i q^{in-(i^2+i)/2} e \qquad (1 \leqslant i \leqslant n-1),$$

$$T_{\omega_i}(\bar{e}) = (-1)^i q^{in-(i^2+i)/2} \bar{e} \qquad (1 \leqslant i \leqslant n-1),$$

$$\hat{T}_{\omega_n}(e) = (-1)^{n/2} q^{n(n-1)/2} e,$$

$$\hat{T}_{\omega_n}(\bar{e}) = -(-1)^{n/2} q^{n(n-1)/2} \bar{e}.$$

4.15. We now consider the graph $\mathcal{G}'$ for (W, S) of type $\tilde{F}_4$ (see 3.13) and the corresponding H-module $E_{\mathcal{G}'}$. It has basis e_0, e_2, e_4 (in 1-1 correspondence with the vertices of $\mathcal{G}'$). We consider the vector v_2, v_4, $\bar{v}_4$ in $E_{\mathcal{G}'}$ defined by

$$v_4 = e_0 + (q^{1/2} - q^{5/2}) e_2 + (q - q^2 + q^3) e_4,$$

$$\bar{v}_4 = e_0 + (q^{1/2} - q^{5/2}) e_2 + (q - q^2 - q^3) e_4,$$

$$v_2 = e_0 + (q^{1/2} + q^{7/2}) e_2 + (q - q^2 + q^3) e_4.$$

We can now state

4.16. PROPOSITION. *We have*

$$\hat{T}_{\omega_i}(v_2) = \begin{cases} q^{\lambda_2(\omega_i)}v_2 & \text{for } i = 1,2,3, \\ -q^{\lambda_2(\omega_i)}v_2 & \text{for } i = 4, \end{cases}$$

$$\hat{T}_{\omega_i}(v_4) = \begin{cases} q^{\lambda_4(\omega_i)}v_4 & \text{for } i = 1,3, \\ -q^{\lambda_4(\omega_i)}v_4 & \text{for } i = 2,4, \end{cases}$$

$$\hat{T}_{\omega_i}(\bar{v}_4) = \begin{cases} q^{\lambda_4(\omega_i)}\bar{v}_4 & \text{for } i = 1,3,4, \\ -q^{\lambda_4(\omega_i)}\bar{v}_4 & \text{for } i = 2. \end{cases}$$

4.17. Next, we consider the graph $\mathcal{G}''$ for (W, S) of type $\tilde{F}_4$ (see 3.13), and the corresponding H-module $E_{\mathcal{G}''}$. It has a basis e_1, e_3 (in 1-1 correspondence with the vertices of $\mathcal{G}''$). We consider the vector $e_1' = e_3 + (q^{1/2} + q^{-1/2})e_1$ in $E_{\mathcal{G}''}$. We can state

4.18. PROPOSITION. *We have*

$$\hat{T}_{\omega_i}(e_3) = q^{\lambda_3(\omega_i)} \qquad (1 \leqslant i \leqslant 4),$$
$$\hat{T}_{\omega_i}(e_1') = q^{\lambda_1(\omega_i)} \qquad (1 \leqslant i \leqslant 4).$$

4.19. We now consider the graph $\mathcal{G}'$ for (W, S) of type $\tilde{G}_2$ (see 3.13) and the corresponding H-module $E_{\mathcal{G}'}$. It has a basis e_0, e_2, e_1 (in 1-1 correspondence with the vertices of $\mathcal{G}'$). We consider the vectors

$$v_2 = e_0 + (q^{1/2} + q^{3/2})e_2 + qe_1,$$
$$\bar{v}_2 = e_0 + (q^{1/2} - q^{3/2})e_2 + qe_1,$$
$$v_1 = e_0 + (q^{1/2} - q^{3/2})e_2 - q^2e_1.$$

We then have

$$T_{\omega_2}(v_2) = qv_2, \quad T_{\omega_1}(v_2) = -q^2v_2,$$
$$T_{\omega_2}(\bar{v}_2) = -q\bar{v}_2, \quad T_{\omega_1}(\bar{v}_2) = -q^2\bar{v}_2,$$
$$T_{\omega_2}(v_1) = -qv_1, \quad T_{\omega_1}(v_1) = qv_1.$$

We now consider the graph $\mathcal{G}''$ for (W, S) of type $\tilde{G}_2$ (see 3.13) and the corresponding H-module $E_{\mathcal{G}''}$. It has a basis e_0, e_2 (in 1-1 correspondence with the vertices of $\mathcal{G}''$). We can state

4.20. PROPOSITION. *Let* φ: $\mathbf{Z}[q^{1/2}, q^{-1/2}] \to K$ *be a ring homomorphism, with* K *a field such that the equation* $x^2 + x + 1 = 0$ *has two distinct solutions* θ, $\bar{\theta}$ *in* K. *Then the vectors*

$$v = e_0 + (q^{1/2} + \bar{\theta}q^{3/2})e_2, \quad \bar{v} = e_0 + (q^{1/2} + \bar{\theta}q^{3/2})e_2$$

form a K-basis of $E_{\mathcal{G}''} \otimes_\varphi K$ *and*

$$\hat{T}_{\omega_2}(v) = \theta q v, \quad \hat{T}_{\omega_2}(\bar{v}) = \theta q \bar{v},$$

$$\hat{T}_{\omega_1}(v) = q^2 v, \quad \hat{T}_{\omega_1}(\bar{v}) = q^2 \bar{v}.$$

4.21. Finally, we consider the graph $\mathcal{G}'''$ for (W, S) of type $\tilde{G}_2$ (see 3.13) and the corresponding H-module $E_{\mathcal{G}'''}$ (of rank 1). We can state

4.22. Proposition. *We have*

$$\hat{T}_{\omega_i} = q \quad \text{on } E_{\mathcal{G}'''}\ (i = 1, 2).$$

4.23. Following Iwahori and Matsumoto [7], to each H-module E one can associate a new $\tilde{H}$-module E^* as follows. E^* has the same underlying space as E, and the action of T_w $(w \in \tilde{W})$ on E^* is the same as the action of $q^{l(w)} \varepsilon_w T_{w^{-1}}^{-1}$ on E, where ε: $\tilde{W} \to \{\pm 1\}$ is the homomorphism defined by $\varepsilon(s_i) = -1$ $(1 \leq i \leq n)$, $\varepsilon \mid P = 1$. (This differs slightly from the definition in [7].) When H, E have scalars extended to $\mathbf{C}$ (via a ring homomorphism $\mathbf{Z}[q^{1/2}, q^{-1/2}] \to \mathbf{C}$ taking q to a prime power, and $q^{1/2}$ to a number > 1) then E^* is irreducible if E is, $(E^*)^* = E$ and $\lambda_{E^*} = \bar{\lambda}_E$ (with the notation in 1.4), where the bar is the involution of $\mathbf{Z}[(\mathbf{C}^*)^n]$ defined by

$$(x_1, x_2, \ldots, x_n) \to (x_{\bar{1}}^{-1}, x_{\bar{2}}^{-1}, \ldots, x_{\bar{n}}^{-1})$$

and $i \to \bar{i}$ represents the opposition involution of the Coxeter graph of $\mathcal{G}$. (On the level of representations of $\mathcal{G}$ the involution $*$ was defined by J. Bernstein.) Using 2.9, it now follows that *for each of the representations* E_Γ of $\tilde{H}$ described in 4.7, 4.10, 4.12, 4.14, 4.16, 4.18, 4.19, 4.20, 4.22, the $\tilde{H}$-module E_Γ^* is in $\mathcal{U}_2(\tilde{H})$.

Given a one-dimensional Ω-module X, and an $\tilde{H}$-module E, we can define a new $\tilde{H}$-module $E \otimes X$. The action of T_w $(w \in \tilde{W})$ on $E \otimes X$ is the action of T_w on E tensored with the action of τ on X, where $w \in \tau \cdot W$. In particular, from E^* above, we get new $\tilde{H}$-modules $E_\Gamma^* \otimes X \in \mathcal{U}_2(\tilde{H})$. We see then that *for each L_2-pair (s, N) in G with N*-subregular we have *constructed a representation E in $\mathcal{U}_2(\tilde{H})$, such that the identity* (1.5.1) *is satisfied.*

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