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SOME EXISTENCE THEOREMS FOR FUNCTIONAL EQUATIONS ARISING IN DYNAMIC PROGRAMMING

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Abstract. The existence, uniqueness and iterative approximation of solutions for a few classes of functional equations arising in dynamic programming of multistage decision processes are discussed. The results presented in this paper extend, improve and unify the results due to Bellman [2, 3], Bhakta-Choudhury [6], Bhakta-Mitra [7], and Liu [12].

1. Introduction and preliminaries

Bellman [2, 3] first studied the existence of solutions for several classes of functional equations arising in dynamic programming. Bellman and Roosta [5] constructed an approximation solution for a class of the infinite-stage equation arising in dynamic programming. Bellman and Lee [4] pointed out that the basic form of the functional equations of dynamic programming is

$$f(x) = \sup_{y \in D} H(x, y, f(T(x, y))), \ x \in S, \tag{1.1}$$

where x and y represent the state and decision vectors, respectively, T represents the transformation of the process, and $f(x)$ represents the optimal return function with initial state x. Baskaran and Subrahmanyam [1], Bhakta and Choudhury [6], Bhakta and Mitra [7], Chang [8], Chang and Ma [9], Liu [10]-[12] and others extended the results of [2]-[5] in

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various directions. Bhakta and Mitra [7] established the existence and uniqueness of solutions for the functional equation:

$$f(x) = \sup_{y \in D}\{u(x,y) + f(T(x,y))\}, \ x \in S. \tag{1.2}$$

Under suitable conditions, Bellman [2], Bhakta and Choudhury [6] and Liu [12] obtained the existence or uniqueness of solutions for the functional equations:

$$f(x) = \inf_{y \in D} \max\{u(x,y), f(T(x,y))\}, \ x \in S. \tag{1.3}$$

$$\begin{aligned} f(x) = \inf_{y \geq x}\{u(x,y) + v(x,y)[\int_y^{+\infty} p(s-y)q(s)ds \\ + f(0)\int_y^{+\infty} q(s)ds + \int_0^y f(y-s)q(s)ds]\}. \end{aligned} \tag{1.4}$$

Inspired and motivated by the work in [1]-[12], in this paper, we prove the existence, uniqueness and iterative approximation of solutions for the functional equations (1.4)-(1.6):

$$f(x) = \mathrm{opt}_{y \in D}\{u(x,y) + f(T(x,y))\}, \ x \in S, \tag{1.5}$$

$$f(x) = \mathrm{opt}_{y \in D} \max\{u(x,y), f(T(x,y))\}, \ x \in S, \tag{1.6}$$

where the opt denotes the sup or inf. The results presented in this paper extend, improve and unify the corresponding results of Bellman [2, 3], Bhakta-Choudhury [6], Bhakta-Mitra [7], and Liu [12].

Throughout this paper, N denotes the set of all positive integers, $R = (-\infty, +\infty)$ and $R^+ = [0, +\infty)$. Define

$$\Phi_1 = \{\varphi : \varphi : R^+ \to R^+ \text{ is nondecreasing}\},$$

$$\Phi_2 = \{\varphi : \varphi \in \Phi_1 \text{ and } \lim_{n \to \infty} \varphi^n(t) = 0 \text{ for } t > 0\},$$

$$\Phi_3 = \{\varphi : \varphi \in \Phi_1, \varphi(0) = 0 \text{ and } \varphi \text{ is right continuous at } 0\}.$$

REMARK 1.1. It is easy to see that $\varphi \in \Phi_2$ implies $\varphi(t) < t$ for any $t > 0$.

Let us recall the following concept. Let X be a nonempty set and let $\{d_n\}_{n\in N}$ be a countable family of pseudometrics on X such that for any distinct $x, y \in X$, $d_k(x,y) \neq 0$ for some $k \in N$. Define

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x,y)}{1+d_k(x,y)} \quad \text{for all } x, y \in X.$$

It is clear that d is a metric on X. A sequence $\{x_n\}_{n\in N}$ in X converges to a point $x \in X$ if and only if $d_k(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}_{n\in N}$ is a Cauchy sequence if and only if $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $k \in N$.

LEMMA 1.1. ([12]) *Let a, b, c be in R. Then*

$$|\text{opt}\ \{a,c\} - \text{opt}\ \{b,c\}| \leq |a-b|.$$

2. Existence and uniqueness theorems

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ be real Banach spaces, $S \subseteq X$ be the state space, and $D \subseteq Y$ be the decision space. Denote by $BB(S)$ the set of all real-valued mappings on S that are bounded on bounded subsets of S. It is easy to verify that $BB(S)$ is a linear space over R under usual definitions of addition and multiplication by scalars. For any $k \in N$ and $a, b \in BB(S)$, let

$$d_k(a,b) = \sup\{|a(x) - b(x)| : x \in \overline{B}(0,k)\},$$
$$d(a,b) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a,b)}{1+d_k(a,b)},$$

where $\overline{B}(0,k) = \{x : x \in S \text{ and } \|x\| \leq k\}$. Clearly, $\{d_k\}_{k\in N}$ is a countable family of pseudometrics on $BB(S)$ and $(BB(S), d)$ is a complete metric space.

THEOREM 2.1. *Let $u : S \times D \to R$, $T : S \times D \to S$ be mappings and*

$$\text{(2.1)} \qquad \begin{aligned} a_0(x) &= \sup_{y\in D} u(x,y), \\ a_n(x) &= \sup_{y\in D} \{u(x,y) + a_{n-1}(T(x,y))\}, \ x \in S, \ n \in N. \end{aligned}$$

If there exist $\varphi \in \Phi_2$ *and* $\psi \in \Phi_1$ *such that*

$$\|T(x,y)\| \le \varphi(\|x\|) \text{ for all } (x,y) \in S \times D, \tag{2.2}$$

$$|u(x,y)| \le \psi(\|x\|) \text{ for all } (x,y) \in S \times D, \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty \text{ for all } t > 0, \tag{2.4}$$

then the functional equation (1.2) *possesses a solution* $w \in BB(S)$ *such that*

$$\begin{gathered} \lim_{n\to\infty} w(x_n) = 0 \text{ for any } x_0 \in S, \\ \{y_n\}_{n\in N} \subseteq D, \ x_n = T(x_{n-1}, y_n), \ n \in N. \end{gathered} \tag{2.5}$$

Moreover, the solution w *of the functional equation* (1.2) *is also unique with respect to* (2.5).

Proof. Put

$$\begin{aligned} H(x,y,a) &= u(x,y) + a(T(x,y)) \text{ for all } (x,y,a) \in S \times D \times BB(S), \\ fa(x) &= \sup_{y\in D} H(x,y,a) \text{ for all } (x,a) \in S \times BB(S). \end{aligned}$$

For any $k \in N$, $y \in D$ and $x \in \overline{B}(0,k)$, by (2.2), (2.3), and Remark 1.1, we have

$$|u(x,y)| \le \psi(\|x\|) \le \psi(k), \|T(x,y)\| \le \varphi(\|x\|) \le \varphi(k). \tag{2.6}$$

Using (2.6) and the definition of f, we infer that $fa \in BB(S)$ for any $a \in BB(S)$. That is, f maps $BB(S)$ into $BB(S)$.

Now we prove that f is a nonexpansive mapping in $BB(S)$. For any $a, b \in BB(S)$, $\varepsilon > 0$, $k \in N$ and $x \in \overline{B}(0,k)$, there exist $y, z \in D$ such that

$$\begin{gathered} fa(x) - \varepsilon < H(x,y,a), \ fb(x) - \varepsilon < H(x,z,b), \\ fa(x) \ge H(x,z,a), \ fb(x) \ge H(x,y,b). \end{gathered} \tag{2.7}$$

It follows from (2.7) that

$$\begin{aligned}
&|fa(x) - fb(x)| \\
&< \max\{|H(x,z,a) - H(x,z,b)|, |H(x,y,a) - H(x,y,b)|\} + \varepsilon \\
&= \max\{|a(T(x,z) - b(T(x,z))|, |a(T(x,y)) - b(T(x,y))|\} + \varepsilon \\
&\le d_k(a,b) + \varepsilon,
\end{aligned}$$

which implies that $d_k(fa, fb) \le d_k(a,b) + \varepsilon$. Letting $\varepsilon \to 0$, we have

$$d_k(fa, fb) \le d_k(a,b) \text{ for all } \ a, b \in BB(S), \ k \in N,$$

which yields that

$$d(fa, fb) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(fa, fb)}{1 + d_k(fa, fb)} \le \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(a,b)}{1 + d_k(a,b)} = d(a,b).$$

That is,

$$d(fa, fb) \le d(a,b) \ \text{ for all } a, b \in BB(S). \tag{2.8}$$

We claim that for any $n \ge 0$,

$$|a_n(x)| \le \sum_{i=0}^{n} \psi(\varphi^i(\|x\|)) \text{ for all } x \in S. \tag{2.9}$$

In fact, by (2.3) we conclude that

$$-\psi(\|x\|) \le u(x,y) \le \psi(\|x\|) \text{ for all } (x,y) \in S \times D,$$

which means that

$$|a_0(x)| = |\sup_{y \in D} u(x,y)| \le \psi(\|x\|) \text{ for all } x \in S.$$

Assume that (2.9) holds for some $n \ge 0$. It follows from (2.2) that

$$|a_n(T(x,y))| \le \sum_{i=0}^{n} \psi(\varphi^i(\|T(x,y)\|)) \le \sum_{i=0}^{n} \psi(\varphi^{i+1}(\|x\|)) \tag{2.10}$$

for all $(x, y) \in S \times D$. From (2.3) and (2.10) we know that

$$-\sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \leq u(x,y) + a_n(T(x,y)) \leq \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \text{ for all } (x,y) \in S \times D.$$

This yields that

$$|a_{n+1}(x)| = |\sup_{y \in D}\{u(x,y) + a_n(T(x,y))\}| \leq \sum_{i=0}^{n+1} \psi(\varphi^i(\|x\|)) \text{ for all } x \in S.$$

That is, (2.9) holds for all $n \geq 0$.

Next we prove that $\{a_n\}_{n\geq 0}$ is a Cauchy sequence in $BB(S)$. Let $k \in N$, $\varepsilon > 0$ and $x_0 \in \overline{B}(0,k)$ be given. (2.4) ensures that there exists some $m \in N$ such that

$$\sum_{i=n}^{n+p} \psi(\varphi^i(k)) < \varepsilon \text{ for all } n \geq m \text{ and } p \in N. \tag{2.11}$$

For any $n \geq m$ and $p \in N$, by (2.1) we know that there exist v_1, $w_1 \in D$ and $y_1 = T(x_0, v_1)$, $z_1 = T(x_0, w_1)$ satisfying

$$\begin{gathered} a_{n+p}(x_0) < H(x_0, v_1, a_{n+p-1}) + 2^{-1}\varepsilon, \ a_n(x_0) \geq H(x_0, v_1, a_{n-1}), \\ a_n(x_0) < H(x_0, w_1, a_{n-1}) + 2^{-1}\varepsilon, \ a_{n+p}(x_0) \geq H(x_0, w_1, a_{n+p-1}). \end{gathered} \tag{2.12}$$

From (2.12) we have

$$\begin{aligned} &|a_{n+p}(x_0) - a_n(x_0)| \\ &< \max\{|H(x_0, v_1, a_{n+p-1}) - H(x_0, v_1, a_{n-1})|, \\ &\qquad |H(x_0, w_1, a_{n+p-1}) - H(x_0, w_1, a_{n-1})|\} + 2^{-1}\varepsilon \\ &= \max\{|a_{n-1}(y_1) - a_{n+p-1}(y_1)|, |a_{n-1}(z_1) - a_{n+p-1}(z_1)|\} + 2^{-1}\varepsilon \\ &= |a_{n-1}(x_1) - a_{n+p-1}(x_1)| + 2^{-1}\varepsilon, \end{aligned} \tag{2.13}$$

where $x_1 = y_1$ or z_1 and

$$\begin{aligned}&|a_{n-1}(x_1) - a_{n+p-1}(x_1)|\\ =\ &\max\{|a_{n-1}(y_1) - a_{n+p-1}(y_1)|, |a_{n-1}(z_1) - a_{n+p-1}(z_1)|\}.\end{aligned}$$

Similarly, we conclude that there exist $v_i, w_i \in D$, $y_i = T(x_{i-1}, v_i)$, $z_i = T(x_{i-1}, w_i)$, $x_i = y_i$ or z_i for $2 \le i \le n$ satisfying

$$\begin{aligned}&|a_{n+p-1}(x_1) - a_{n-1}(x_1)| < |a_{n+p-2}(x_2) - a_{n-2}(x_2)| + 2^{-2}\varepsilon,\\ &|a_{n+p-2}(x_2) - a_{n-2}(x_2)| < |a_{n+p-3}(x_3) - a_{n-3}(x_3)| + 2^{-3}\varepsilon,\\ &\qquad\vdots\\ &|a_{p+1}(x_{n-1}) - a_1(x_{n-1})| < |a_p(x_n) - a_0(x_n)| + 2^{-n}\varepsilon.\end{aligned} \tag{2.14}$$

It follows from $\varphi \in \Phi_2$, (2.2) and Remark 1.1 that

$$\|x_n\| \le \varphi(\|x_{n-1}\|) \le \varphi^2(\|x_{n-2}\|) \le \cdots \le \varphi^n(\|x_0\|) \le \|x_0\| \le k \tag{2.15}$$

for any $n \in N$. In the light of (2.9), (2.11), and (2.13)-(2.15), we obtain that

$$\begin{aligned}|a_{n+p}(x_0) - a_n(x_0)| &< |a_p(x_n) - a_0(x_n)| + \varepsilon,\\ &\le |a_p(x_n)| + |a_0(x_n)| + \varepsilon\\ &\le \psi(\|x_n\|) + \sum_{i=0}^{p} \psi(\varphi^i(\|x_n\|)) + \varepsilon\\ &\le \psi(\varphi^n(\|x_0\|)) + \sum_{i=0}^{p} \psi(\varphi^{i+n}(\|x_0\|)) + \varepsilon\\ &\le 2\sum_{i=n}^{n+p} \psi(\varphi^i(\|k\|)) + \varepsilon\\ &< 3\varepsilon\end{aligned} \tag{2.16}$$

for any $n \ge m$ and $p \in N$. Thus (2.15) and (2.16) yield that $d_k(a_n, a_{n+p}) \le 3\varepsilon$. That is, $\{a_n\}_{n\ge 0}$ is a Cauchy sequence in $(BB(S), d)$ and hence it converges to some $w \in BB(S)$. By virtue of (2.8), we get that

$$d(fw, w) \le d(fw, fa_n) + d(a_{n+1}, w) \le d(w, a_n) + d(a_{n+1}, w) \to 0$$

as $n \to \infty$. That is, $w = fw$ is a fixed point of f and hence the functional equation (1.2) possesses a solution $w \in BB(S)$.

We prove that (2.5) holds. For any $x_0 \in S$, $\{y_n\}_{n\in N} \subseteq D$, $x_n = T(x_{n-1}, y_n)$, $n \in N$, we have by (2.2)

(2.17)
$$\|x_n\| = \|T(x_{n-1}, y_n)\| \le \varphi(\|x_{n-1}\|) \le \cdots \le \varphi^n(\|x_0\|) \to 0, \text{ as } n \to \infty.$$

Put $k = [\|x_0\|] + 1$, where $[t]$ denotes the largest integer not exceeding t. From Remark 1.1 and (2.17) we conclude that $\{x_n\}_{n\ge 0} \subseteq \overline{B}(0, k)$. Let k be in N. Note that $\lim_{m\to\infty} d_k(w, a_m) = 0$. For given $\varepsilon > 0$, by (2.4) and (2.17) we know that there exists some $m \in N$ such that

$$\max\left\{d_k(w, a_m), \sum_{i=n}^{m+n} \psi(\varphi^i(\|x_0\|))\right\} < \varepsilon \text{ for any } n \ge m. \tag{2.18}$$

By virtue of (2.9) and (2.18), we infer that

$$\begin{aligned}
|w(x_n)| &\le |w(x_n) - a_m(x_n)| + |a_m(x_n)| \\
&\le d_k(w, a_m) + \sum_{i=0}^{m} \psi(\varphi^i(\|x_n\|)) \\
&\le d_k(w, a_m) + \sum_{i=n}^{m+n} \psi(\varphi^i(\|x_0\|)) \\
&< 2\varepsilon
\end{aligned}$$

for all $n \ge m$. Therefore $\lim_{m\to\infty} w(x_n) = 0$.

Finally we prove that w is a unique solution of the functional equation (1.2) in $BB(S)$ satisfying (2.5). Suppose that v is also a solution of the functional equation (1.2) in $BB(S)$ satisfying (2.5). Let $x_0 = t_0 \in S$ and $\varepsilon > 0$ be given. By the definition of w and v, we conclude that there exist $\{y_n\}_{n\ge 1} \subseteq D$, $\{z_n\}_{n\ge 1} \subseteq D$, $\{x_n\}_{n\ge 1} \subseteq S$ and $\{t_n\}_{n\ge 1} \subseteq S$ with $x_n = T(x_{n-1}, y_n)$, $t_n = T(t_{n-1}, z_n)$ for all $n \in N$, such that

$$\begin{aligned}
w(x_i) &< u(x_i, y_{i+1}) + w(x_{i+1}) + 2^{-i-1}\varepsilon, \\
v(t_i) &< u(t_i, z_{i+1}) + v(t_{i+1}) + 2^{-i-1}\varepsilon, \\
w(t_i) &\ge u(t_i, z_{i+1}) + w(t_{i+1}), \ \ v(x_i) \ge u(x_i, y_{i+1}) + v(x_{i+1})
\end{aligned} \tag{2.19}$$

for all $i \geq 0$. By (2.19) we easily deduce that

$$\begin{aligned} w(x_0) &< u(x_0, y_1) + u(x_1, y_2) + \cdots + u(x_{n-1}, y_n) \\ &\quad + w(x_n) + (1 - 2^{-n})\varepsilon, \\ v(t_0) &< u(t_0, z_1) + u(t_1, z_2) + \cdots + u(t_{n-1}, z_n) \\ &\quad + v(t_n) + (1 - 2^{-n})\varepsilon, \\ w(t_0) &\geq u(t_0, z_1) + u(t_1, z_2) + \cdots + u(t_{n-1}, z_n) + w(t_n), \\ v(x_0) &\geq u(x_0, y_1) + u(x_1, y_2) + \cdots + u(x_{n-1}, y_n) + v(x_n) \end{aligned} \tag{2.20}$$

for any $n \in N$. Using (2.20) and $x_0 = t_0$, we have

$$|w(x_0) - v(x_0)| < |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + (1 - 2^{-n})\varepsilon.$$

Letting $n \to \infty$ in the above inequality, we obtain that $|w(x_0) - v(x_0)| \leq \varepsilon$, which implies that $w(x_0) = v(x_0)$ by letting $\varepsilon \to 0$. This completes the proof. □

REMARK 2.1. Theorem 2.4 in [7] is a special case of Theorem 2.1 with $\psi(t) = Mt$ for all $t \in R^+$, where M is a positive constant. The following example reveals that Theorem 2.1 generalizes properly Theorem 2.4 in [7].

EXAMPLE 2.1. Let $X = Y = R$, $S = D = R^+$. Define $u : S \times D \to R$, $T : S \times D \to S$ by

$$u(x, y) = \frac{x^2(1 - xy)}{1 + xy}, \quad T(x, y) = \frac{x \sin^2(x + y)}{2 + y^2} \quad \text{for all } (x, y) \in S \times D.$$

Choose $\varphi(t) = 2^{-1}t$ and $\psi(t) = t^2$ for all $t \in R^+$. It is easy to verify that the conditions of Theorem 2.1 are satisfied. Hence the functional equation (1.2) possesses a solution in $BB(S)$. But Theorem 2.4 in [7] is not applicable since

$$|u(x, y)| = |u(M + 1, 0)| = (M + 1)^2 > M|x|$$

for any $M > 0$ with $(x, y) = (M + 1, 0) \in S \times D$.

A proof similar to that of Theorem 2.1 gives the following result and is thus omitted.

THEOREM 2.2. *Let $u : S \times D \to R$, $T : S \times D \to S$ be mappings and*

$$a_0(x) = \inf_{y\in D} u(x,y), a_n(x) = \inf_{y\in D}\{u(x,y) + a_{n-1}(T(x,y))\},\ x \in S,\ n \in N. \tag{2.21}$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_1$ satisfying (2.2)-(2.4). *Then the functional equation*

$$f(x) = \inf_{y\in D}\{u(x,y) + f(T(x,y))\},\ x \in S, \tag{2.22}$$

possesses a solution $w \in BB(S)$ such that (2.5) *holds. Moreover, the solution w of the functional equation* (2.22) *is also unique with respect to* (2.5).

THEOREM 2.3. *Let $u : S \times D \to R$, $T : S \times D \to S$ be mappings and*

$$a_0(x) = \sup_{y\in D} u(x,y), a_n(x) = \sup_{y\in D} \max\{u(x,y), a_{n-1}(T(x,y))\},\ x \in S,\ n \in N. \tag{2.23}$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_3$ satisfying (2.2) *and* (2.3). *Then the functional equation*

$$f(x) = \sup_{y\in D} \max\{u(x,y), f(T(x,y))\},\ x \in S, \tag{2.24}$$

possesses a solution $w \in BB(S)$ such that (2.5) *holds and*

$$w(x) \ge 0 \quad \text{for all } x \in S. \tag{2.25}$$

Moreover, the solution w of the functional equation (2.24) *is also unique with respect to* (2.5).

Proof. Set

$$H(x,y,a) = \max\{u(x,y), a(T(x,y))\} \text{ for all } (x,y,a) \in S \times D \times BB(S),$$
$$fa(x) = \sup_{y\in D} H(x,y,a) \text{ for all } (x,a) \in S \times BB(S).$$

As in the proof of Theorem 2.1, we can conclude that f maps $BB(S)$ into $BB(S)$ and (2.8) holds. Now we claim that for all $n \ge 0$,

$$|a_n(x)| \le \psi(\|x\|) \text{ for all } x \in S. \tag{2.26}$$

It is easy to verify that (2.3) implies that (2.26) holds for $n = 0$. Suppose that (2.26) holds for some $n \geq 0$. From (2.2), $\varphi \in \Phi_2$ and Remark 1.1, we infer that

(2.27)
$$\begin{aligned} |a_n(T(x,y))| &\leq \psi(\|T(x,y)\|) \\ &\leq \psi(\varphi(\|x\|)) \leq \psi(\|x\|) \text{ for all } (x,y) \in S \times D. \end{aligned}$$

Using (2.3) and (2.27), we have

$$-\psi(\|x\|) \leq \max\{u(x,y), a_n(T(x,y))\} \leq \psi(\|x\|) \text{ for all } (x,y) \in S \times D,$$

which implies that

$$|a_{n+1}(x)| = |\sup_{y \in D} \max\{u(x,y), a_n(T(x,y))\} \leq \psi(\|x\|) \text{ for all } \ x \in S.$$

Hence (2.26) holds for all $n \geq 0$. On the other hand, (2.23) ensures that

$$(2.28) \qquad a_0(x) \leq a_1(x) \leq \cdots \leq a_n(x) \leq a_{n+1}(x) \leq \cdots \text{ for all } x \in S.$$

Next we show that $\{a_n\}_{n \geq 0}$ is a Cauchy sequence in $BB(S)$. Let $k \in N$, $\varepsilon > 0$ and $x_0 \in \overline{B}(0,k)$ be given. Since $\varphi \in \Phi_2$ and $\psi \in \Phi_3$, it follows that there exists some $m \in N$ such that

$$(2.29) \qquad \psi(\varphi^n(k)) < \varepsilon \text{ for all } n \geq m.$$

For any $n \geq m$ and $p \in N$, by (2.23) we easily conclude that there exist $y_1 \in D$ and $x_1 = T(x_0, y_1)$ satisfying

$$(2.30) \quad a_{n+p}(x_0) < H(x_0, y_1, a_{n+p-1}) + 2^{-1}\varepsilon, \ a_n(x_0) \geq H(x_0, y_1, a_{n-1}).$$

By virtue of (2.28), (2.30), and Lemma 1.1, we have

(2.31)
$$\begin{aligned} 0 &\leq a_{n+p}(x_0) - a_n(x_0) \\ &< H(x_0, y_1, a_{n+p-1}) - H(x_0, y_1, a_{n-1}) + 2^{-1}\varepsilon \\ &= \max\{u(x_0, y_1), a_{n+p-1}(x_1)\} - \max\{u(x_0, y_1), a_{n-1}(x_1)\} + 2^{-1}\varepsilon \\ &\leq a_{n+p-1}(x_1) - a_{n-1}(x_1) + 2^{-1}\varepsilon. \end{aligned}$$

Similarly, we conclude that there exist $y_i \in D$, $x_i = T(x_{i-1}, y_i) \in S$, $2 \le i \le n$ satisfying

$$(2.32)\qquad \begin{aligned} &0 \le a_{n+p-1}(x_1) - a_{n-1}(x_1) < a_{n+p-2}(x_2) - a_{n-2}(x_2) + 2^{-2}\varepsilon, \\ &0 \le a_{n+p-2}(x_2) - a_{n-2}(x_2) < a_{n+p-3}(x_3) - a_{n-3}(x_3) + 2^{-3}\varepsilon, \\ &\qquad\vdots \\ &0 \le a_{p+1}(x_{n-1}) - a_1(x_{n-1}) < a_p(x_n) - a_0(x_n) + 2^{-n}\varepsilon. \end{aligned}$$

It follows from (2.2), (2.3), (2.26), (2.29), (2.31), and (2.32) that

$$\begin{aligned} 0 &\le a_{n+p}(x_0) - a_n(x_0) < a_p(x_n) - a_0(x_n) + \varepsilon \\ &\le |a_p(x_n)| + |a_0(x_n)| + \varepsilon \le 2\psi(\|x_n\|) + \varepsilon \\ &= 2\psi(\|T(x_{n-1}, y_n)\|) + \varepsilon \le 2\psi(\varphi(\|x_{n-1}\|)) + \varepsilon \\ &\le 2\psi(\varphi^n(\|x_0\|)) + \varepsilon \le 2\psi(\varphi^n(k)) + \varepsilon < 3\varepsilon \end{aligned}$$

for any $n \ge m$ and $p \in N$. This gives that $d_k(a_n, a_{n+p}) \le 3\varepsilon$ for any $n \ge m$ and $p \in N$. Consequently, $\{a_n\}_{n \ge 0}$ is a Cauchy sequence in $(BB(S), d)$ and it converges to some $w \in BB(S)$. From (2.8), we deduce that $w = fw$. That is, the functional equation (2.24) possesses a solution $w \in BB(S)$.

We prove that (2.5) holds. For any $x_0 \in S$, $\{y_n\}_{n \in N} \subseteq D$, $x_n = T(x_{n-1}, y_n)$, $n \in N$, (2.2) yields that (2.17) holds. Note that $\psi(0) = 0$ and ψ is right continuous at 0. Thus (2.17) means that

$$(2.33)\qquad \lim_{n \to \infty} \psi(\|x_n\|) = \psi(0) = 0.$$

Put $k = [\|x_0\|] + 1$. It is easy to verify that $\{x_n\}_{n \in 0} \subseteq \overline{B}(0, k)$. Let k be in N and $\varepsilon > 0$. Since $\{a_n\}_{n \in N}$ converges to w, by (2.33) we know that there exists some $m \in N$ such that

$$(2.34)\qquad \max\{d_k(w, a_m), \psi(\|x_n\|)\} < \varepsilon \text{ for any } n \ge m.$$

By virtue of (2.26) and (2.34), we have

$$|w(x_n)| \le |w(x_n) - a_m(x_n)| + |a_m(x_n)| \le d_k(w, a_m) + \psi(\|x_n\|) < 2\varepsilon$$

for all $n \ge m$. That is, $\lim_{n \to \infty} w(x_n) = 0$.

Given $x_0 \in S$ and $\{y_n\}_{n \in N} \subset D$, take $x_n = T(x_{n-1}, y_n)$ for all $n \in N$. Since w is a solution of the functional equation (2.24), by (2.5) we immediately infer that

$$w(x_0) \geq \max\{u(x_0, y_1), w(T(x_0, y_1))\} \geq w(x_1) \geq \cdots \geq w(x_n) \to 0$$

as $n \to \infty$. That is, $w(x_0) \geq 0$ for all $x_0 \in S$.

Finally we prove that w is a unique solution of the functional equation (2.24) in $BB(S)$ satisfying (2.5). Suppose that v is also a solution of the functional equation (2.24) in $BB(S)$ satisfying (2.5). Let $x_0 = t_0 \in S$ and $\varepsilon > 0$ be given. By the definition of w and v, we conclude that there exist $\{y_n\}_{n\geq 1} \subseteq D$, $\{z_n\}_{n\geq 1} \subseteq D$, $\{x_n\}_{n\geq 1} \subseteq S$ and $\{t_n\}_{n\geq 1} \subseteq S$ with $x_n = T(x_{n-1}, y_n)$, $t_n = T(t_{n-1}, z_n)$ for all $n \in N$, such that
(2.35)

$$\begin{aligned} w(x_i) &< \max\{u(x_i, y_{i+1}), w(x_{i+1})\} + 2^{-i-1}\varepsilon, \\ v(t_i) &< \max\{u(t_i, z_{i+1}), v(t_{i+1})\} + 2^{-i-1}\varepsilon, \\ w(t_i) &\geq \max\{u(t_i, z_{i+1}), w(t_{i+1})\}, \ v(x_i) \geq \max\{u(x_i, y_{i+1}), v(x_{i+1})\} \end{aligned}$$

for all $i \geq 0$. By (2.35) we easily deduce that
(2.36)

$$\begin{aligned} w(x_0) &< \max\{u(x_0, y_1), u(x_1, y_2), \cdots, u(x_{n-1}, y_n), w(x_n)\} + (1 - 2^{-n})\varepsilon, \\ v(t_0) &< \max\{u(t_0, z_1), u(t_1, z_2), \cdots, u(t_{n-1}, z_n), v(t_n)\} + (1 - 2^{-n})\varepsilon, \\ w(t_0) &\geq \max\{u(t_0, z_1), u(t_1, z_2), \cdots, u(t_{n-1}, z_n), w(t_n)\}, \\ v(x_0) &\geq \max\{u(x_0, y_1), u(x_1, y_2), \cdots u(x_{n-1}, y_n), v(x_n)\} \end{aligned}$$

for any $n \in N$. Using (2.36), Lemma 1.1 and $x_0 = t_0$, we have

$$|w(x_0) - v(x_0)| < |w(x_n) - v(x_n)| + |w(t_n) - v(t_n)| + (1 - 2^{-n})\varepsilon.$$

Letting $n \to \infty$ in the above inequality, we obtain that $|w(x_0) - v(x_0)| \leq \varepsilon$, which implies that $w(x_0) = v(x_0)$ by letting $\varepsilon \to 0$. This completes the proof. □

Following a similar argument as in the proof of Theorem 2.3, we have the following.

THEOREM 2.4. *Let $u : S \times D \to R$, $T : S \times D \to S$ be mappings and*

$$a_0(x) = \inf_{y \in D} u(x, y), a_n(x) = \inf_{y \in D} \max\{u(x, y), a_{n-1}(T(x, y))\}, \ x \in S, \ n \in N. \tag{2.37}$$

Suppose that there exist $\varphi \in \Phi_2$ and $\psi \in \Phi_3$ satisfying (2.2) *and* (2.3). *Then the functional equation*

$$f(x) = \inf_{y \in D} \max\{u(x, y), f(T(x, y))\}, \ x \in S, \tag{2.38}$$

possesses a solution $w \in BB(S)$ such that (2.5) *and* (2.25) *hold. Moreover, the solution w of the functional equation* (2.38) *is also unique with respect to* (2.5).

REMARK 2.2. Theorem 2.4 extends, improves and unifies Theorem 3.5 of Bhakta and Choudhury [6], Theorem 3.5 of Liu [12] and a result of Bullman [2, p.135]. The example below shows that Theorem 2.4 is indeed a generalization of the results due to Bhakta and Choudhury [6], Liu [12], and Bullman [2].

EXAMPLE 2.2. Let X, Y, S, D be as in Example 2.1. Define $u : S \times D \to R$, $T : S \times D \to S$ by

$$u(x, y) = \frac{x^4(1 + xy)}{1 + x^2 + y^2}, \quad T(x, y) = \frac{x|\sin(x + y)|}{1 + x} \quad \text{for all } (x, y) \in S \times D.$$

Put $\varphi(t) = \frac{t}{1+t}$, $\psi(t) = t^4$ for all $t \in R^+$. Then the assumptions of Theorem 2.4 are fulfilled. However, we cannot invoke the results of Bhakta and Choudhury [6], Liu [12], and Bullman [2] to establish that the functional equation (2.38) possesses a solution in $BB(S)$ because

$$|u(x, y)| = \left|u\Big(\frac{3}{2}(M + 1), \frac{3}{2}(M + 1)\Big)\right| \geq \frac{2}{3}\Big[\frac{3}{2}(M + 1)\Big]^2 > M|x|$$

for any $M > 0$ with $(x, y) = \Big(\frac{3}{2}(M + 1), \frac{3}{2}(M + 1)\Big) \in S \times D$.

Let $BC(R^+)$ denote the set of all bounded continuous real-valued functions on R^+. Put $d(a, b) = \sup\{|a(x) - b(x)| : x \in R^+\}$ for all $a, b \in BC(R^+)$. It is easily seen that $(BC(R^+), d)$ is a complete metric space.

THEOREM 2.5. *Let* $X = Y = R$, $S = D = R^+$. *Let* $u, v : S \times D \to R^+$ *be continuous,* $u(x,x)$ *be bounded on* S, $\lim_{y\to+\infty} u(x,y) = +\infty$, $u(x,\cdot)$ *and* $v(x,\cdot)$ *be nondecreasing with respect to the second argument on* $[x,+\infty)$ *for every* $x \in S$, *and*

$$0 \le v(x,y) \le r \text{ for all } (x,y) \in S \times D, \tag{2.39}$$

where r *is a positive constant. Let* $p, q : S \to R^+$ *satisfy that* p *is* continuous, nondecreasing, $\int_0^{+\infty} p(s)q(s)ds < +\infty$ and

$$\int_0^{+\infty} q(s)ds = t > 0. \tag{2.40}$$

Assume that

$$a_0(x) = \inf_{y\ge x} u(x,y),\ x \in S,$$

(2.41)

$$a_{n+1}(x) = \inf_{y\ge x} \Big\{ u(x,y) + v(x,y)\Big[\int_y^{+\infty} p(s-y)q(s)ds + a_n(0)\int_y^{+\infty} q(s)ds + \int_0^y a_n(y-s)q(s)ds\Big]\Big\},\ x \in S, n \ge 0.$$

If $rt < 1$, *then the functional equation* (1.4) *possesses a unique solution* $w \in BC(R^+)$ *and*

$$d(a_{n+1}, w) \le (rt)^{n+1}(1-rt)^{-1} d(a_0, a_1) \text{ for all } n \ge 0. \tag{2.42}$$

Proof. For all $(x,y,b) \in S \times D \times BC(R^+)$, set

$$H(x,y,b) = u(x,y) + v(x,y)\Big[\int_y^{+\infty} p(s-y)q(s)ds + b(0)\int_y^{+\infty} q(s)ds + \int_0^y b(y-s)q(s)ds\Big].$$

Let

$$fb(x) = \inf_{y\ge x} H(x,y,b) \text{ for all } (x,b) \in S \times BC(R^+). \tag{2.43}$$

It is easy to see that f maps $BC(R^+)$ into itself. Let $\varepsilon > 0$, $x \in S$ and $b, c \in BC(R^+)$ be given. It follows from (2.43) that there exist $y_1 \geq x$ and $y_2 \geq x$ such that

$$\begin{aligned} &fb(x) > H(x, y_1, b) - \varepsilon, \ fc(x) > H(x, y_2, c) - \varepsilon, \\ &fb(x) \leq H(x, y_2, b), \ fc(x) \leq H(x, y_1, c). \end{aligned} \tag{2.44}$$

By virtue of (2.39), (2.40), and (2.44), we deduced that

$$\begin{aligned} &|fb(x) - fc(x)| \\ &< \max\Big\{|H(x, y_i, b) - H(x, y_i, c)| : i = 1, 2\Big\} + \varepsilon \\ &\leq \max\Big\{v(x, y_i)\Big[|b(0) - c(0)| \int_{y_i}^{+\infty} q(s)ds \\ &\quad + \int_0^{y_i} |b(y_i - s) - c(y_i - s)| q(s)ds\Big] : i = 1, 2\Big\} + \varepsilon \\ &\leq \max\Big\{rd(b, c)\Big[\int_{y_i}^{+\infty} q(s)ds + \int_0^{y_i} q(s)ds\Big] : i = 1, 2\Big\} + \varepsilon \\ &= rtd(b, c) + \varepsilon, \end{aligned}$$

which implies that

$$d(fb, fc) = \sup\{|fb(x) - fc(x)| : x \in S\} \leq rtd(b, c) + \varepsilon.$$

Letting $\varepsilon \to 0$, we easily conclude that

$$d(fb, fc) \leq rtd(b, c) \text{ for all } b, c \in BC(R^+).$$

It follows from Banach fixed-point theorem that f has a unique fixed point $w \in BC(R^+)$ and (2.42) holds. Obviously, w is a unique solution of the functional equation (1.4). This completes the proof. □

Remark 2.3. Theorem 2.5 generalizes Theorem 3.6 of Bhakta and Choudhury [6] and a result of Bellman [2, p.129].

Problem 2.1. If $rt < 1$ is replaced by $rt = 1$ in Theorem 2.5, does the functional equation (1.4) possess a solution in $BC(R^+)$?

Problem 2.2. If the answer to Problem 2.1 is no, then what additional hypotheses on u, v, p, q are needed to guarantee the existence of a solution of the functional equation (1.4)?

References

[1] R. Baskaran and P. V. Subrahmanyam, *A note on the solution of a class of functional equations*, Appl. Anal. **22** (1986), no. 3-4, 235–241.

[2] R. Bellman, *Dynamic programming*, Princeton University Press, Princeton, N. J., 1957.

[3] ———, *Methods of Non-linear analysis, Vol. II*, Mathematics in Science and Engineering, Vol. 61-II, Academic Press, New York-London, 1973.

[4] R. Bellman and E. S. Lee, *Functional equations in dynamic programming*, Aequationes Math. **17** (1978), no. 1, 1–18.

[5] R. Bellman and M. Roosta, *A technique for the reduction of dimensionality in dynamic programming*, J. Math. Anal. Appl. **88** (1982), no. 2, 543–546.

[6] P. C. Bhakta and S. R. Choudhury, *Some existence theorems for functional equations arising in dynamic programming II*, J. Math. Anal. Appl. **131** (1988), no. 1, 217–231.

[7] P. C. Bhakta and S. Mitra, *Some existence theorems for functional equations arising in dynamic programming*, J. Math. Anal. Appl. **98** (1984), no. 2, 348–362.

[8] S. S. Chang, *Some existence theorems of common and coincidence solutions for a class of functional equations arising in dynamic programming*, Appl. Math. Mech.(English Ed.) **12** (1991), no. 1, 33–39.

[9] S. S. Chang and Y. H. Ma, *Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming*, J. Math. Anal. Appl. **160** (1991), no. 2, 468–479.

[10] Z. Liu, *Coincidence theorems for expansive mappings with applications to the solutions of functional equations arising in dynamic programming*, Acta Sci. Math. (Szeged) **65** (1999), no. 1-2, 359–369.

[11] ———, *Compatible mappings and fixed points*, Acta Sci. Math. (Szeged) **65** (1999), no. 1-2, 371–383.

[12] ———, *Existence theorems of solutions for certain classes of functional equations arising in dynamic programming*, J. Math. Anal. Appl. **262** (2001), no. 2, 529–553.

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