The Korean Communications in Statistics **Vol. 14** No. 1, 2007, pp. 93–109

Some Exponentiated Distributions

M. Masoom Ali[1] Manisha Pal[2] and Jungsoo Woo[3]

Abstract

In this paper we study a number of new exponentiated distributions. The survival function, failure rate and moments of the distributions have been derived using certain special functions. The behavior of the failure rate has also been studied.

Keywords: Inverse Weibull distribution; logistic distribution; Pareto distribution; general exponential distribution; double exponential distribution; double Weibull distribution; inverse double Weibull distribution; survival function; failure rate; moments.

1. Introduction

In this paper we study the properties of some exponentiated distributions. The idea of exponentiated distribution was introduced by Gupta *et al.* (1998), who discussed a new family of distributions termed as exponentiated exponential distribution. The family has two parameters (scale and shape) similar to the Weibull or gamma family. Some properties of the distribution was studied by Gupta and Kundu (2001a). They observed that many properties of the new family are similar to those of the Weibull or gamma family. Hence the distribution can be used as an alternative to a Weibull or gamma distribution. They (2001b, 2002) also examined the estimation and inference aspects of the distribution. The distribution has been further studied by Nadarajah and Kotz (2003). A class of goodness-of-fit tests for the distribution with estimated parameter has been proposed by Hassan (2005). Pal *et al.* (2006) studied the exponentiated Weibull family as an extension of the Weibull and exponentiated exponential families. The exponentiated Frechet distribution was considered by Kotz and Nadarajah

1) Professor, Department of Mathematical Sciences, Ball State University, Muncie, Indiana, U.S.A.
2) Professor, Department of Statistics, University of Calcutta, India.
Correspondence : manishapal2@gmail.com
3) Professor, Department of Statistics, Yeungnam University, Gyongsan, South Korea.

(2003). Shirke *et al.* (2005) provided tolerance intervals for exponentiated scale family of distributions.

In this paper some further exponentiated distributions have been considered. The survival function, failure rate and moments of the distributions have been derived using certain special functions. The behavior of the failure rate has also been studied. Since survival function and failure rate are associated with a lifetime distribution and lifetime is non-negative, for exponentiated distribution defined over the range $(-\infty, \infty)$ we study the same for the distribution truncated at zero.

2. Exponentiated Distributions

Let X be a random variable with probability density function (*pdf*) $f(x)$ and the cumulative distribution function (*cdf*) $F(x), x \in R^1$. Consider a random variable Z with *cdf* given by

$$G_\alpha(z) = [F(z)]^\alpha, z \in R^1, \alpha > 0. \tag{2.1}$$

Then Z is said to have an exponentiated distribution.

The *pdf* of Z is given by

$$g_\alpha(z) = \alpha\,[F(z)]^{\alpha-1} f(z). \tag{2.2}$$

The survival function $S_\alpha(z)$ and the failure rate $r_\alpha(z)$ of the distribution of Z are defined by

$$\begin{aligned} S_\alpha(z) &= 1 - G_\alpha(z), \\ r_\alpha(z) &= \frac{g_\alpha(z)}{1 - G_\alpha(z)}. \end{aligned}$$

For a system with lifetime distribution having the *cdf* $G_\alpha(z), S_\alpha(z)$ defines the probability that the system will survive at least z units of time, while $r_\alpha(z)dz$ defines the probability that the system will fail in the interval $(z, z+dz)$, given that it has survived upto z units of time. The distribution is said to be IFR (increasing failure rate) if $r_\alpha(z)$ is a non-decreasing function of z, and it is said to be DFR (decreasing failure rate) if $r_\alpha(z)$ in non-increasing in z.

Some properties of an exponentiated distribution are observed in the following lemmas.

Lemma 2.1 *If α be an integer, then*

$$E(Z^k) = E\left(X^k_{(\alpha;\alpha)}\right),$$

where $X_{(\alpha;\alpha)}$ is the largest order statistic in a random sample of size α drawn from the distribution of X.

Lemma 2.2 *If there are n components in a parallel system and the lifetimes of the components are independently and identically distributed with common cdf $G_\alpha(.)$ given by (2.1), then the system lifetime has cdf $G_{\alpha n}(.)$, that is, the system lifetime is also an exponentiated distribution of the form (2.1), but with exponentiating parameter αn.*

3. Exponentiated Inverse Weibull Distribution

The inverse Weibull distribution has the *cdf*

$$F(x) = \exp\left(-\left(\frac{1}{\beta x}\right)^\gamma\right), x > 0,\ \beta, \gamma > 0.$$

The exponentiated inverse Weibull distribution is, therefore, defined by the *pdf*

$$g_\alpha(z) = \frac{\alpha\gamma}{\beta^\gamma}\exp\left(-\alpha\left(\frac{1}{\beta z}\right)^\gamma\right) z^{-\gamma-1}, z > 0,\ \alpha, \beta, \gamma > 0$$

and its *cdf* is given by

$$G_\alpha(z) = \exp\left(-\alpha\left(\frac{1}{\beta z}\right)^\gamma\right),\ z > 0.$$

Figure 3.1 shows that the peakedness of the distribution increases as α decreases, *i.e.* the distribution becomes more and more flattened as α increases.

The survival function and the failure rate of the distribution are given by

$$S_\alpha(z) = 1 - \exp\left(-\alpha\left(\frac{1}{\beta z}\right)^\gamma\right),$$

$$r_\alpha(z) = \frac{\frac{\alpha\gamma}{\beta^\gamma}\exp\left(-\alpha\left(\frac{1}{\beta z}\right)^\gamma\right) z^{-\gamma-1}}{1 - \exp\left(-\alpha\left(\frac{1}{\beta z}\right)^\gamma\right)}.$$

Clearly, the failure rate $r_\alpha(z)$ is increasing in z for $z \le 1/\beta z_0$ and decreasing in z for $z > 1/\beta z_0$, where

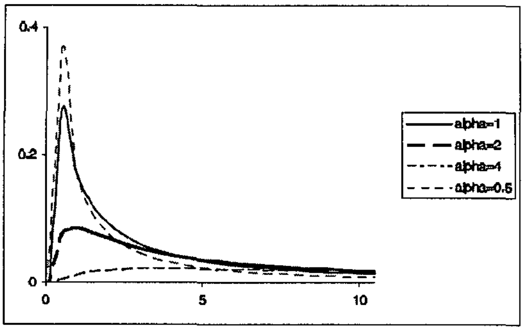

Figure 3.1: Showing the *pdf* of exponentiated inverse Weibull distribution with $\beta = 0.5, \gamma = 0.5$, when $\alpha = 1, 2, 4$ and 0.5.

$$z_0 = \left[\frac{1}{\alpha}\log_e\left(\frac{\gamma+1}{\gamma}\right)\right]^{\frac{1}{\gamma}},$$

i.e., $z = 1/\beta z_0$ is the change point of the failure rate function

From formula 3.15 in Oberhettinger (1974), the k^{th} moment of the distribution is given by

$$E(Z^k) = \alpha^{k/\gamma}\beta^{-k}\Gamma\left(1-\frac{k}{\gamma}\right), \quad \text{provided} \quad \gamma > k.$$

Hence, for $\gamma > 2$,

$$\begin{aligned} E(Z) &= \frac{\alpha^{1/\gamma}}{\beta}\Gamma\left(1-\frac{1}{\gamma}\right), \\ \mathrm{Var}(Z) &= \frac{\alpha^{2/\gamma}}{\beta^2}\left[\Gamma\left(1-\frac{2}{\gamma}\right) - \Gamma^2\left(1-\frac{1}{\gamma}\right)\right]. \end{aligned}$$

Clearly, both the mean and variance increase as α increases, when $\gamma > 2$.

4. Exponentiated Logistic Distribution

Noting that the *cdf* of a logistic distribution is given by

$$F(x) = \frac{1}{1+e^{-\frac{x}{\beta}}}, x \in R^1, \beta > 0,$$

the exponentiated logistic distribution is obtained in the following way.

Its *pdf* is given by

$$g_\alpha(z) = \frac{\alpha}{\beta}\left(1 + e^{-z/\beta}\right)^{-\alpha-1} e^{-z/\beta}, z \in R^1, \alpha, \beta > 0$$

and its *cdf* is

$$G_\alpha(z) = \left(1 + e^{-z/\beta}\right)^{-\alpha}, z \in R^1.$$

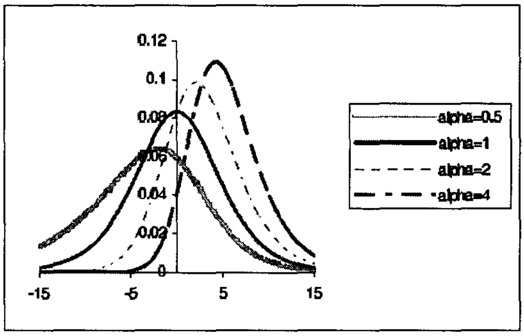

Figure 4.1: Showing the *pdf* of exponentiated logistic distribution with $\beta = 0.5$, when $\alpha = 1, 2, 4$ and 0.5.

Figure 4.1 shows that the distribution shifts to the right as α increases. The peakedness of the distribution also increases with increase in α.

The moments of the distribution, obtained using the formula 3.381(4) in Gradshteyn and Ryzhik (1965) and the binomial expansion, are as follows :

$$E(Z^k) = \alpha\beta^k k! \left[\sum_{i=0}^{\infty} \frac{(-\alpha-1)P_i}{i!(i+1)^{k+1}} + (-1)^k \sum_{i=0}^{\infty} \frac{(-\alpha-1)P_i}{i!(i+\alpha)^{k+1}}\right].$$

To study the survival function and failure rate, we consider the lower truncated distribution, truncated at $z = 0$.

The survival function of the truncated distribution is

$$S_\alpha(z) = \frac{\left[1 - \left(1 + e^{-z/\beta}\right)^{-\alpha}\right]}{1 - 2^{-\alpha}}$$

and the failure rate is

$$r_\alpha(z) = \frac{\frac{\alpha}{\beta}\left(1 + e^{-z/\beta}\right)^{-\alpha-1} e^{-z/\beta}}{1 - \left(1 + e^{-z/\beta}\right)^{-\alpha}}, z > 0.$$

It is easily seen that the failure rate $\gamma_\alpha(z)$ is an increasing function of z for all $\alpha, \beta > 0$, *i.e.* the distribution is IFR.

5. Exponentiated Pareto Distribution

The exponentiated Pareto distribution has *pdf* given by

$$g_\alpha(z) = \alpha\gamma\beta^\gamma \left[1 - \left(\frac{\beta}{z}\right)^\gamma\right]^{\alpha-1} z^{-\gamma-1}, z \geq \beta > 0, \alpha, \gamma > 0$$

and its *edf* is

$$G_\alpha(z) = [1 - (\beta/z)^\gamma]^\alpha, z \geq \beta.$$

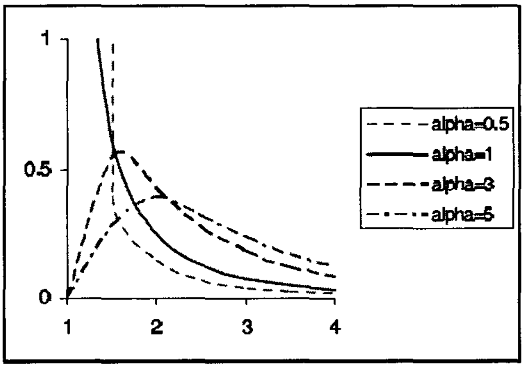

Figure 5.1: Showing the *pdf* of exponentiated Pareto distribution with $\beta = 1, \gamma = 2$, when $\alpha = 1, 3$ and 5.

Figure 5.1 shows that the distribution becomes more and more flattened as α increases.

The survival function and the failure rate of the distribution are

$$S_\alpha(z) = 1 - \left[1 - \left(\frac{\beta}{z}\right)^\gamma\right]^\alpha,$$

$$r_\alpha(z) = \frac{\alpha\gamma\beta^\gamma \left[1 - \left(\frac{1}{\beta z}\right)^\gamma\right]^{\alpha-1} z^{-\gamma-1}}{1 - [1 - (\beta/z)^\gamma]^\alpha}.$$

So, for $\alpha, \gamma > 1, r_\alpha(z)$ is a decreasing function of z, *i.e.* for $\alpha, \gamma > 1$ the distribution is DFR.

We obtain the k^{th} moment of the distribution as

$$E(Z^k) = \begin{cases} \alpha\beta^k\gamma\sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}\frac{(-1)^i}{\gamma(i+1)-k}, & \text{if } \gamma > k, \alpha \in N \\ \alpha\beta^k\gamma\sum_{i=0}^{\infty}\frac{(\alpha-1)P_i}{i!}\frac{(-1)^i}{\gamma(i+1)-k}, & \text{if } \gamma > k, \alpha \notin N. \end{cases}$$

It is possible to express the moment generating function *(mgf)* in terms of incomplete gamma functions, which in turn may be used to find the different moments. From formula 3.6 in Oberhettinger (1974), the *mgf* can be written as

$$E(e^{tZ}) = \begin{cases} \alpha\gamma\sum_{i=0}^{\alpha-1}\binom{\alpha-1}{i}(-1)^i(-\beta t)^{\gamma(i+1)}\Gamma(-\gamma(i+1),-\beta t), & \text{if } t < 0, \alpha \in N \\ \alpha\gamma\sum_{i=0}^{\infty}\frac{(\alpha-1)P_i}{i!}(-1)^i(-\beta t)^{\gamma(i+1)}\Gamma(-\gamma(i+1),-\beta t), & \text{if } t < 0, \alpha \notin N, \end{cases}$$

where

$$\Gamma(\alpha, x) = \int_x^{\infty} e^{-y}y^{\alpha-1}dy,\ a > 0.$$

6. Exponentiated Generalized Uniform Distribution

The *cdf* of a generalized uniform distribution is of the form

$$F(x) = \left(\frac{x}{\beta}\right)^{\gamma+1}, 0 < x < \beta, \gamma > -1,$$

Hence the *pdf* and *cdf* of an exponentiated generalized uniform distribution will be defined by

$$\begin{aligned} g_\alpha(z) &= \frac{\alpha(\gamma+1)}{\beta}\left(\frac{z}{\beta}\right)^{\alpha(\gamma+1)-1}, 0 < z < \beta, \gamma > -1, \alpha > 0, \\ G_\alpha(z) &= \left(\frac{z}{\beta}\right)^{\alpha(\gamma+1)}. \end{aligned}$$

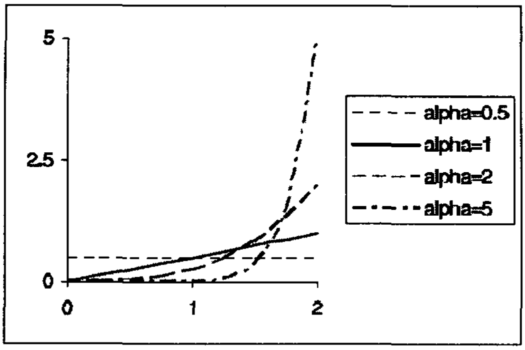

Figure 6.1: Showing the *pdf* of exponentiated generalized uniform distribution with $\beta = 2, \gamma = 1$, when $\alpha = 0.5, 1, 2$ and .5.

Figure 6.1 shows that the distribution shifts to the right as α increases and the right tail of the distribution becomes steeper with increase in α.

The survival function and failure rate of the distribution are given by

$$S_\alpha(z) = 1 - \left(\frac{z}{\beta}\right)^{\alpha(\gamma+1)},$$

$$r_\alpha(z) = \frac{\frac{\alpha(\gamma+1)}{\beta}\left(\frac{z}{\beta}\right)^{\alpha(\gamma+1)-1}}{1-\left(\frac{z}{\beta}\right)^{\alpha(\gamma+1)}}, 0 < z < \beta.$$

Substituting $t = z/\beta$, it can be easily seen that the distribution is IFR for $\gamma > 0$ and $\alpha > 1$.

Using formula 3.381(1) in Gradshteyn and Ryzhik (1965), the expression for the mgf is obtained as

$$E(e^{tZ}) = \alpha(\gamma+1)(-t\beta)^{-\alpha(\gamma+1)}[1 - \Gamma(\alpha(\gamma+1), -t\beta)], t \neq 0.$$

The k^{th} moment of the distribution is, therefore, given by

$$E(Z^k) = \frac{\alpha(\gamma+1)}{\alpha(\gamma+1)+k}\beta^k.$$

Thus, the mean and variance come out to be

$$E(Z) = \frac{\alpha(\gamma+1)}{\alpha(\gamma+1)+1}\beta$$

$$\mathrm{Var}(Z) = \alpha(\gamma+1)\left[\frac{1}{\alpha(\gamma+1)+2} - \frac{\alpha(\gamma+1)}{\{\alpha(\gamma+1)+1\}^2}\right]\beta^2.$$

7. Exponentiated General Exponential Distribution

With $F(x) = 1 - e^{-\frac{x-\theta}{\beta}}, x > \theta, \beta > 0, \theta \in R^1$, we define the exponentiated general exponential distribution as having the density function

$$g_\alpha(z) = \frac{\alpha}{\beta}\left(1 - e^{-\frac{z-\theta}{\beta}}\right)^{\alpha-1} e^{-\frac{z-\theta}{\beta}}, z > \theta, \alpha, \beta > 0, \theta \in R^1.$$

The *cdf* of the distribution is

$$G_\alpha(z) = \left(1 - e^{-\frac{z-\theta}{\beta}}\right)^\alpha.$$

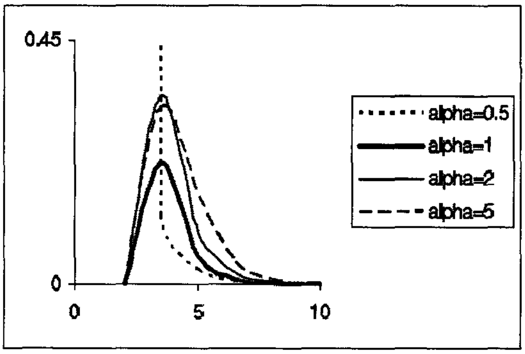

Figure 7.1: Showing the *pdf* of exponentiated general exponential distribution with $\theta = 2, \beta = 1$, when $\alpha = 0.5, 1, 2$ and $.5$.

Figure 7.1 shows that the distribution is unimodal with mode more or less at the sam point, whatever be α.

We study the survival probability and failure rate of the distribution for $\theta \geq 0$. The survival function and failure rate come out as

$$S_\alpha(z) = 1 - \left(1 - e^{-\frac{z-\theta}{\beta}}\right)^\alpha,$$

$$r_\alpha(z) = \frac{\frac{\alpha}{\beta}\left(1 - e^{-\frac{z-\theta}{\beta}}\right)^{\alpha-1} e^{-\frac{z-\theta}{\beta}}}{1 - \left(1 - e^{-\frac{z-\theta}{\beta}}\right)^\alpha}.$$

Here α is the shape parameter, β the scale parameter and θ the location parameter. For $\theta = 0$, it becomes the exponentiated exponential distribution studied by Gupta and Kundu (2001). For any β and $\theta > 0$, the failure rate is an increasing function of z if $\alpha > 1$, and decreases as z increases for $\alpha < 1$. For $\alpha = 1$, it is constant.

It is possible to express the *mgf* of the distribution in terms of the gamma function. From formula 2.20 in Oberhettinger (1974), the *mgf* is

$$E(e^{tZ}) = \alpha e^{\theta t}\frac{\Gamma(1-\beta t)\Gamma(\alpha)}{\Gamma(\alpha+1-\beta t)}, \text{ if } t < \frac{1}{\beta}.$$

Defining $m(t) = \ln E(e^{tZ})$ and evaluating $\frac{\partial}{\partial t}m(t)|_{t=0}$ and $\frac{\partial^2}{\partial t^2}m(t)|_{t=0}$ we obtain the mean and variance of the distribution as

$$\begin{aligned} E(Z) &= \theta + \beta[\Psi(\alpha+1) - \Psi(1)], \\ \text{Var}(Z) &= \beta^2[\Psi'(1) - \Psi'(\alpha+1)], \end{aligned}$$

where $\Psi(.)$ is the digamma function and $\Psi'(.)$ is its first order derivative. $\Psi(1) = \gamma$, the Euler's constant, and $\Psi(1) = \xi(2)$, where $\xi(r) = \sum_{j=1}^{\infty} j^{-r}$ is Riemann zeta function.

In particular, for $\alpha = 2$, using formulas 6.3.5 and 6.4.2 in Abramowitz and Stegun (1972), we get

$$E(Z) = \theta + \frac{3}{2}\beta \quad \text{and} \quad \text{Var}(Z) = \frac{5}{4}\beta^2.$$

8. Exponentiated Double Exponential Distribution

The *cdf* of a double exponential random variable X has the form

$$F(x) = \frac{1}{2}[1 + \text{ sgn }(x-\theta)(1-e^{-\frac{|x-\theta|}{\beta}})], x \in R^1, \theta \in R^1, \beta > 0,$$

where

$$\text{sgn }(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Hence an exponentiated double exponential distribution is defined by the density function

$$g_\alpha(z) = \frac{\alpha}{2^\alpha \beta}\left[1 + \text{ sgn }(z-\theta)\left(1-e^{-\frac{|z-\theta|}{\beta}}\right)\right]^{\alpha-1} e^{-\frac{|z-\theta|}{\beta}}, z \in R^1, \theta \in R^1, \beta, \alpha > 0.$$

Therefore, the *cdf* of the distribution is

$$G_\alpha(z) = \frac{1}{2^\alpha}\left[1 + \text{ sgn } (z-\theta)\left(1 - e^{-\frac{|z-\theta|}{\beta}}\right)\right]^\alpha, z \in R^1.$$

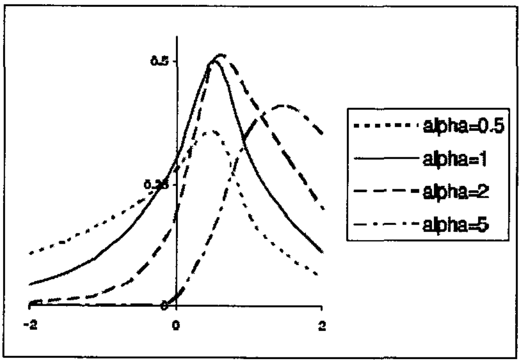

Figure 8.1: Showing the *pdf* of exponentiated double exponential distribution with $\theta = 0.5, \beta = 1$, when $\alpha = 0.5, 1, 2$ and .5.

Figure 8.1 shows that the distribution shifts to the right with increase in α.

We may obtain the moments of the distribution with the help of formula 3.381(3) in Gradshteyn and Ryzhik (1965) :

$$\begin{aligned} E(Z^k) &= 2^{-\alpha}(-\alpha/\beta)^{-k}e^{-\frac{\alpha}{\beta}\theta}\Gamma(k+1, \alpha\theta/\beta) + \alpha\beta^k\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i} \\ &\quad \times\Gamma(k+1,(i+1)\theta/\beta)\frac{e^{(i+1)\theta/\beta}}{2^{i+1}(i+1)^{k+1}}, \text{ if } \alpha \in N \\ &= 2^{-\alpha}(-\alpha/\beta)^{-k}e^{-\frac{\alpha}{\beta}\theta}\Gamma(k+1, \alpha\theta/\beta) + \alpha\beta^k\sum_{i=0}^{\infty}(-1)^{k+i}\frac{{}_{(\alpha-1)}P_i}{i!} \\ &\quad \times\Gamma(k+1,(i+1)\theta/\beta)\frac{e^{(i+1)\theta/\beta}}{2^{i+1}(i+1)^{k+1}}, \text{ if } \alpha \notin N. \end{aligned}$$

In particular, for $\theta = 0$ and $\beta = 1$, and α a natural number, we get

$$E(Z^k) = (-1)^k 2^{-\alpha}\alpha^{-k}k! + \alpha\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\frac{k!}{2^{i+1}(i+1)^{k+1}}.$$

A study of the skewness measure $\gamma_1 = \frac{E[Z-E(Z)]^3}{Var^{3/2}(Z)}$ shows that the distribution is skewed to the right and the skewness increases with increase in α.

Consider $\theta \geq 0$. For the lower truncated distribution, truncated at $z = 0$, the survival function is

$$S_\alpha(z) = \frac{1 - \frac{1}{2^\alpha}[1 + \text{ sgn } (z-\theta)(1 - e^{-\frac{|z-\theta|}{\beta}})]^\alpha}{1 - \frac{e^{-\frac{\alpha\theta}{\beta}}}{2^\alpha}}$$

and the failure rate is

$$r_\alpha(z) = \frac{\frac{1}{2^\alpha \beta}[1 + \text{ sgn } (z-\theta)(1 - e^{-\frac{|z-\theta|}{\beta}})]^{\alpha-1} e^{-\frac{|z-\theta|}{\beta}}}{1 - \frac{1}{2^\alpha}[1 + \text{ sgn } (z-\theta)(1 - e^{-\frac{|z-\theta|}{\beta}})]^\alpha}.$$

Clearly, the failure rate function is increasing for $0 < z < \theta$, whatever $\alpha > 0$, and is decreasing in z for $z > \theta$ if $\alpha \leq 2$. Thus, for $0 < \alpha \leq 2$, $r_\alpha(z)$ is a concave function with change point at $z = \theta$.

9. Exponentiated Double Weibull Distribution

The *cdf* of a double Weibull random variable X has the form

$$F(x) = \frac{1}{2}\left[1 + \text{ sgn } (x)\left\{1 - \exp\left(-\left(\frac{|x|}{\beta}\right)^\gamma\right)\right\}\right], x \in R^1,\ \beta, \gamma > 0.$$

The exponentiated double Weibull random variable Z, therefore, has the *pdf*

$$g_\alpha(z) = \frac{\alpha\gamma}{2^\alpha \beta^\gamma}\left[1 + \text{ sgn } (z)\left\{1 - \exp\left(-\left(\frac{|z|}{\beta}\right)^\gamma\right)\right\}\right]^{\alpha-1}$$
$$\times \exp\left(-\left(\frac{|z|}{\beta}\right)^\gamma\right) |z|^{\gamma-1}, \qquad z \in R^1, \alpha, \beta, \gamma > 0.$$

Cdf of the distribution is given by

$$G_\alpha(z) = \frac{1}{2^\alpha}\left[1 + \text{ sgn } (z)\left\{1 - \exp\left(-\left(\frac{|z|}{\beta}\right)^\gamma\right)\right\}\right]^\alpha, z \in R^1.$$

For $\gamma = 1$, we get the exponentiated double exponential distribution.

Figure 9.1 shows that as α increases, the distribution shifts to the right and the part of the distribution in the positive range of z becomes steeper.

We obtain the moments of the distribution by utilizing formula 3.478(1) in Gradshteyn and Ryzhik (1965). The k^{th} moment is given by

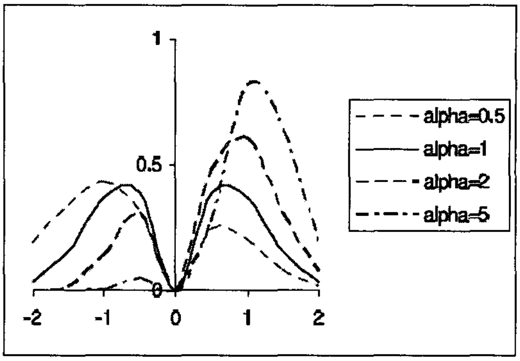

Figure 9.1: Showing the *pdf* of exponentiated double Weibull distribution with $\beta = 1, \gamma = 2$, when $\alpha = 0.5, 1, 2$ and 5.

$$
\begin{aligned}
E(Z^k) &= \beta^k \Gamma(1+\frac{k}{\gamma})[(-1)^k 2^{-\alpha} \alpha^{-k/\gamma} + \frac{\alpha}{2}\sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} \frac{(-1)^i}{2^i (i+1)^{1+k/\gamma}}], \text{ if } \alpha \in N \\
&= \beta^k \Gamma(1+\frac{k}{\gamma})[(-1)^k 2^{-\alpha} \alpha^{-k/\gamma} + \frac{\alpha}{2}\sum_{i=0}^{\infty} \frac{{}^{(\alpha-1)}P_i}{i!} \frac{(-1)^i}{2^i (i+1)^{1+k/\gamma}}], \text{ if } \alpha \notin N.
\end{aligned}
$$

Putting $\gamma = 1$, we get the k^{th} moment of the exponentiated double exponential distribution.

The survival function $S_a(z)$ and the failure rate $r_a(z)$ of the exponentiated double Weibull distribution truncated at $z = 0$ come out to be

$$
S_\alpha(z) = \frac{1 - \frac{1}{2^\alpha}\left[1 + \text{ sgn } (z)\left\{1 - \exp\left(-\left(\frac{|z|}{\beta}\right)^\gamma\right)\right\}\right]^\alpha}{1 - \frac{1}{2^\alpha}}
$$

$$
r_\alpha(z) = \frac{\frac{\alpha\gamma}{2^\alpha \beta^\gamma}\left[2 - \exp\left(-\left(\frac{z}{\beta}\right)^\gamma\right)\right]^{\alpha-1} \exp\left(-\left(\frac{z}{\beta}\right)^\gamma\right) z^{\gamma-1}}{1 - \frac{1}{2^\alpha}\left[2 - \exp\left(-\left(\frac{z}{\beta}\right)^\gamma\right)\right]^\alpha}.
$$

Clearly, $r_\alpha(z)$ is a decreasing function of z if $\alpha, \gamma \leq 1$.

10. Exponentiated Double Inverse Weibull Distribution

The *cdf* of a double inverse Weibull distribution is given by

$$F(x) = \frac{1}{2}\left[1 + \text{ sgn }(x)\exp\left(-\left(\frac{1}{\beta \mid x \mid}\right)^{\gamma}\right)\right], \, x \in R^1, \beta, \gamma > 0.$$

We therefore define an exponentiated double inverse Weibull distribution as having *pdf* and *cdf* given by $g_a(z)$ and $G_a(z)$, respectively, where

$$\begin{aligned} g_\alpha(z) &= \frac{\alpha\gamma}{2^\alpha \beta^\gamma}\left[1 + \text{ sgn }(z)\exp\left(-\left(\frac{1}{\beta \mid z \mid}\right)^{\gamma}\right)\right]^{\alpha-1} \exp\left(-\left(\frac{1}{\beta \mid z \mid}\right)^{\gamma}\right) \mid z \mid^{-\gamma-1}, \\ G_\alpha(z) &= \frac{1}{2^\alpha}\left[1 + \text{ sgn }(z)\exp\left(-\left(\frac{1}{\beta \mid z \mid}\right)^{\gamma}\right)\right]^{\alpha}, z \in R^1, \alpha, \beta, \gamma > 0. \end{aligned}$$

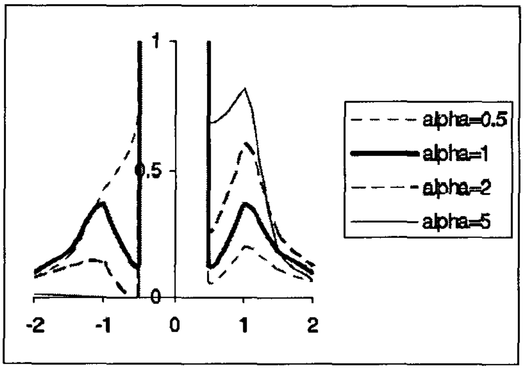

Figure 10.1: Showing the *pdf* of exponentiated double inverse Weibull distribution with $\beta = 1, \gamma = 2$, when $\alpha = 0.5, 1, 2$ and 5.

We obtain the moments of the distribution by utilizing formula 3.478(1) in Gradshteyn and Ryzhik (1965). The k^{th} moment is given by

$$E(Z^k) = \begin{cases} 2^{-\alpha}\alpha\beta^{-k}\Gamma(1-\frac{k}{\gamma})\sum_{i=0}^{\alpha-1}\{1+(-1)^{k+i}\}\binom{\alpha-1}{i}(i+1)^{\frac{k-\gamma}{\gamma}}, \text{ if } \alpha \in N \\ 2^{-\alpha}\alpha\beta^{-k}\Gamma(1-\frac{k}{\gamma})\sum_{i=0}^{\infty}\{1+(-1)^{k+i}\}\frac{(\alpha-1)P_i}{i!}(i+1)^{\frac{k-\gamma}{\gamma}}, \text{ if } \alpha \notin N. \end{cases}$$

The survival function $S_\alpha(z)$ and the failure rate $r_\alpha(z)$ of the distribution truncated at $z = 0$ are given by

$$S_\alpha(z) = \frac{1 - \frac{1}{2^\alpha}[1 + \exp(-(\frac{1}{\beta|z|})^\gamma)]^\alpha}{1 - \frac{1}{2^\alpha}},$$

$$r_\alpha(z) = \frac{\frac{\alpha\gamma}{2^\alpha \beta^\gamma}[1 + \exp(-(\frac{1}{\beta|z|})^\gamma)]^{\alpha-1} \exp(-(\frac{1}{\beta|z|})^\gamma \mid z \mid^{-\gamma-1}}{1 - \frac{1}{2^\alpha}[1 + \exp(-(\frac{1}{\beta|z|})^\gamma)]^\alpha}.$$

11. Conclusions

The paper discusses some properties of a number of exponentiated distributions. The distributions are skewed and some of them will be useful in analyzing many lifetime skewed data. For example, the exponentiated general exponential distribution will fit lifetime data, where items are known to survive for at least a minimum age. The exponentiated Inverse Weibull would be a suitable model for describing degradation phenomenon of mechanical components like dynamic components of diesel engines. The exponentiated Pareto distribution could be effectively used for modelling of financial data.

We now illustrate two applications of the exponentiated distributions.

Data Set 1 : (Lawless, 1986, page 228). The data relate to tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.40.

Here we fitted the exponentiated general exponential distribution with $\theta = 0$.

The MLEs of the parameters β and α came out as

$$\hat{\beta} = 30.96455, \ \hat{\alpha} = 5.2829.$$

The observations were grouped into 4 intervals in order that the expected frequency of each interval is at least 5, which is essential for good χ^2 approximation.

Intervals	Observed	Expected
0 - 45.5	5	5.780
45.5 - 65.5	6	5.885
65.5 - 85.5	6	6.292
85.5 -	6	5.043
	23	23

The value of the frequency Chi-square (χ^2) for the fit was obtained as $\chi^2 = 0.3027$, which shows the fit to be very good.

Data Set 2 : (NASDAQ data - 1985 to 2005). The data is not reproduced here because of its volume. We first transformed the data by taking logarithm of the daily returns and then computing the first differences. An exponentiated Pareto distribution was fitted to the extremities of the transformed data, where the model showed extremely good fit. The parameters of the distribution have been estimated using the maximum likelihood method.

(The dashed curve is the *cdf* based on exponentiated Pareto fit, while the solid curve is the empirical *cdf.*)

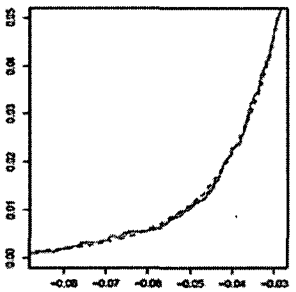

Figure 11.1

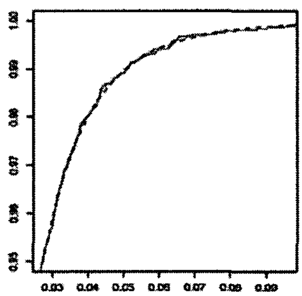

Figure 11.2

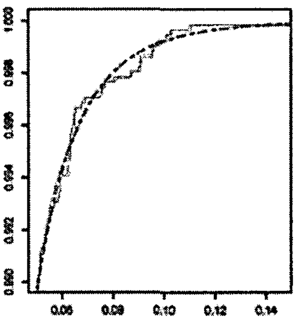

Figure 11.3

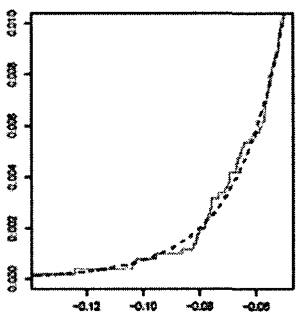

Figure 11.4

Figure 11.1 shows an exponentiated Pareto fitted to the excess gains above the 95% quantile $((\gamma, \beta, \alpha) = (6.55, 0.088, 0.989))$.

Figure 11.2 shows an exponentiated Pareto fitted to the excess losses below the 5% quantile $(\gamma, \beta, \alpha) = (7.02, 0.120, 1.046)$.

Figure 11.3 shows an exponentiated Pareto fitted to the excess gains above the 99% quantile $((\gamma, \beta, \alpha) = (8.36, 0.103, 1.10))$.

Figure 11.4 shows an exponentiated Pareto fitted to the excess losses below the 1% quantile ($(\gamma, \beta, \alpha) = (11.98, 0.214, 1.098)$).

Clearly, the exponentiated Pareto is found to fit the empirical *cdf* curve very well over the corresponding ranges.

Acknowledgments

The authors thank the referees for their constructive suggestions which immensely improved the presentation of the paper.

References

Abramowitz, M. and Stegun, I. A. (1972). *Handbook of Mathematical Functions.* Dover Publications, New York.

Bain, L. J. (1976). *Statistical Analysis of Reliability and Life-Testing Models.* Marcel Dekker, New York.

Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Tables of Integrals, Series and Products.* Academic Press, New York.

Gupta, R. C., Gupta, R. D. and Gupta, P. L. (1998). Modelling failure time data by Lehman alternatives. *Communications in Statistics-Theory and Methods*, **27**, 887–904.

Gupta, R. D. and Kundu, D. (2001a). Exponentiated exponential family: an alternative to gamma and Weibull distributions, *Biometrical Journal,* **43**, 117–130.

Gupta, R. D. and Kundu, D. (2001b). Generalized exponential distribution: different method of estimations. *Journal of Statistical Computation and Simulation,* **69**, 315–337.

Gupta, R. D. and Kundu, D. (2002). Generalized exponential distribution: statistical inferences. *Journal of Statistical Theory and Applications*, **1**, 101–118.

Hassan, A. S. (2005). Goodness-of-fit for the generalized exponential distribution. *InterStat,* July, No. 1.

Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data.* John Wiley & Sons, New York.

Linhart, H. and Zucchini, W. (1986). *Model Selection.* Wiley, New York.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions,* Vol. 1, 2nd ed, John Wiley & Sons, New York.

Nadarajah, S. and Kots, S. (2003). The exponentiated Frechet distribution. *InterStat,* December, No. 1.

Nadarajah, S. and Kotz, S. (2003). On the exponentiated exponential distribution, To appear in *Statistica.*

Pal, M., Ali, M. Masoom and Woo, J. (2006). Exponentiated Weibull distribution, communicated.

Shirke, D. T., Kumbhar, R. R. and Kundu, D. (2005). Tolerance intervals for exponentiated scale family of distributions. *Journal of Applied Statistics*, **32**, 1067–1074.

[Received July 2006, Accepted December 2006]