SOME EXTENSIONS OF THE WISHART DISTRIBUTION[1]

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1. Introduction. The well-known Wishart distribution is the distribution of the variances and covariances of a sample drawn from a multivariate normal population assuming that the expected value of each variate remains the same from observation to observation. For problems such as testing collinearity [1], comparing scales of measurement [2], and multiple regression in times series analysis [3], it is desirable to have the distribution of sample variances and covariances for observations, the expected values of which are not all identical. Such a distribution could be considered as a generalization to several variates of the χ'^2 (non-central χ^2) distribution, as well as a generalization of the Wishart distribution to the non-central case. In this paper we shall discuss the general problem of finding the distribution in question and shall derive this distribution for two particular cases. We shall start out with the problem in its most general form and as a result of linear transformations express the distribution as a certain multiple integral.

We can think of the expected values of the observations as defining points in a space of dimension equal to the number of variates. If these points lie on a line, the non-central Wishart distribution is essentially the Wishart distribution multiplied by a Bessel function; if the points lie in a plane, it is a Wishart distribution multiplied by an infinite series of Bessel functions. For higher dimensionality the integration of the multiple integral becomes extremely troublesome; it has not been possible yet to express the general integration in a concise form. These results are summarized precisely at the end of the paper.

2. Reduction to canonical form. Consider a set of N multivariate normal populations each of p variates. Let the ith $(i = 1, 2, \cdots, p)$ variate of the αth $(\alpha = 1, 2, \cdots, N)$ population be $x_{i\alpha}$; let the mean of this variate be

$$E(x_{i\alpha}) = \mu_{i\alpha} \quad (i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N); \tag{1}$$

and let the variance-covariance matrix (of rank p) common to all N distributions be

$$\| E[(x_{i\alpha} - \mu_{i\alpha})(x_{j\alpha} - \mu_{j\alpha})] \| = \| \sigma_{ij} \| \quad (\alpha = 1, 2, \cdots, N).$$

Now consider a sample of observations $\{x_{i\alpha}\}$ one from each population.

The purpose of this paper is to find the joint distribution of the quantities

$$a_{ij} = \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j), \tag{2}$$

[1] The results given below were arrived at independently by the two authors. Some preliminary results were given before the Institute of Mathematical Statistics at Washington, D. C., May 6, 1944, by Girshick.

where

$$\bar{x}_i = \frac{1}{N} \sum_{\alpha=1}^{N} x_{i\alpha} .$$

To simplify the notation in the subsequent work we treat the quantities a_{ij} instead of the sample variances and covariances which are simply multiples (by $1/(N-1)$) of the a_{ij}.

The a_{ij} may be considered as sums of squares and cross products, for there exists a linear transformation[2].

$$x'_{i\alpha} = \sum_{\beta=1}^{N} \theta_{\alpha\beta} x_{i\beta} \qquad (i = 1, 2, \cdots, p),$$

where the matrix $\| \theta_{\alpha\beta} \|$ is orthogonal (and $\theta_{N1} = \theta_{N2} = \cdots = \theta_{NN} = 1/\sqrt{N}$), such that

$$a_{ij} = \sum_{\alpha=1}^{n} x'_{i\alpha} x'_{j\alpha} ,$$

where $n = N - 1$ and

$$N\bar{x}_i\bar{x}_j = x'_{iN}x'_{jN} .$$

For a given α the $x'_{i\alpha}$ have a multivariate normal distribution with the same variances and covariances as the x's and with expected values

$$E(x'_{i\alpha}) = \sum_{\beta=1}^{N} \theta_{\alpha\beta} \mu_{i\beta} = \mu'_{i\alpha} , \quad \text{say}.$$

Let

$$\tau_{ij} = \sum_{\alpha=1}^{n} \mu'_{i\alpha} \mu'_{j\alpha} .$$

Then it is clear that the τ_{ij} are the same functions of the μ's that the a_{ij} are of the x's, namely,

$$\tau_{ij} = \sum_{\alpha=1}^{N} (\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j), \tag{3}$$

where

$$\bar{\mu}_i = \frac{1}{N} \sum_{\alpha=1}^{N} \mu_{i\alpha} .$$

Now consider the two p by p matrices

$$\Sigma = \| \sigma_{ij} \|$$

and

$$\mathrm{T} = \| \tau_{ij} \|.$$

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[2] See, for example, [4].

Let $\kappa_1^2, \kappa_2^2, \cdots, \kappa_p^2$ be the real, non-negative roots of the determinantal equation

$$|\mathrm{T} - \lambda\Sigma| = 0. \tag{4}$$

There exists a non-singular p by p matrix

$$\Psi = \| \psi_{ij} \| \tag{5}$$

such that[3]

$$\Psi\Sigma\Psi' = \mathrm{I} \tag{6}$$

and

$$\Psi\mathrm{T}\Psi' = \begin{Vmatrix} \kappa_1^2 & 0 & \cdots & 0 \\ 0 & \kappa_2^2 & \cdots & 0 \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 0 & 0 & \cdots & \kappa_p^2 \end{Vmatrix}, \tag{7}$$

where I is the identity matrix and Ψ' is the transpose of Ψ. Suppose the rank of T is t; then t of the roots are non-zero and $p - t$ are zero. For the sake of convenience we shall choose $\kappa_1^2, \kappa_2^2, \cdots, \kappa_t^2$ to be the non-zero roots. If T is of rank t, then the means $\mu_{i\alpha}$ lie in a t dimensional sub-space of the original p dimensional space. Let us make the transformation

$$z_{i\alpha} = \sum_{j=1}^{p} \psi_{ij} x'_{j\alpha} . \tag{8}$$

The $z_{i\alpha}$ are normally and, because of relationship (6), independently distributed with unit variances. The mean value of $z_{i\alpha}$ is

$$E(z_{i\alpha}) = \sum_{j=1}^{p} \psi_{ij} \mu'_{j\alpha} = \nu_{i\alpha} ,$$

say. As a result of (7)

$$\sum_{\alpha=1}^{n} \nu_{i\alpha}^2 = \kappa_i^2 \qquad (i = 1, 2, \cdots, p), \tag{9}$$

$$\sum_{\alpha=1}^{n} \nu_{i\alpha} \nu_{j\alpha} = 0 \qquad (i \neq j). \tag{10}$$

Let the new sum of squares of cross-products be

$$b_{ij} = \sum_{\alpha=1}^{n} z_{i\alpha} z_{j\alpha} . \tag{11}$$

We shall first find the joint distribution of the b_{ij} and then obtain the distribution of the a_{ij} by using the fact that the b_{ij} can be considered simply as a linear

[3] See, for example, [5].

transformation of the a_{ij} in a $\frac{1}{2}p(p + 1)$ dimensional space. For we can write b_{ij} as

$$(12) \qquad b_{ij} = \sum_{\alpha=1}^{n} \sum_{h,k=1}^{p} \psi_{ih}\psi_{jk} x'_{h\alpha} x'_{k\alpha} = \sum_{h,k=1}^{p} \psi_{ih}\psi_{jk} a_{hk} .$$

The transformation (8) is performed on all variates of each observation. The next transformation, which is the one that results in the canonical form of the problem, is performed on all observations of each variate. We wish to construct the n by n matrix of this transformation

$$\Phi = || \phi_{\alpha\beta} ||$$

in the following manner: Let

$$\phi_{\eta\alpha} = \frac{\nu_{\eta\alpha}}{\kappa_\eta} \qquad (\alpha = 1, 2, \cdots, n; \eta = 1, 2, \cdots, t).$$

In view of (9) and (10)

$$\sum_{\alpha=1}^{n} \phi_{\xi\alpha}\phi_{\eta\alpha} = \delta_{\xi\eta}, \qquad (\xi, \eta = 1, 2, \cdots, t)$$

where $\delta_{\xi\eta}$ is the Kronecker delta. The remaining elements in Φ are chosen in any way to make Φ orthogonal.

Now make the transformation

$$y_{i\alpha} = \sum_{\beta=1}^{n} \phi_{\alpha\beta} z_{i\beta} \qquad (i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, n).$$

Because Φ is orthogonal,

$$(13) \qquad b_{ij} = \sum_{\alpha=1}^{n} y_{i\alpha} y_{j\alpha},$$

and the y's are independently normally distributed. By virtue of the construction of Φ and the properties of $|| \nu_{i\alpha} ||$ the expected value of each $y_{i\alpha}$ is zero except for t of the variates, namely,

$$E(y_{\eta\eta}) = \kappa_\eta \qquad (\eta = 1, 2, \cdots, t).$$

Now the problem can be put in this form: Find the distribution of b_{ij} (given by (13)) when the distribution of the y's (in the canonical form) is

$$\frac{1}{(2\pi)^{pn/2}} e^{-\frac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=1}^{n}(y_{i\alpha}-\kappa_i\delta_{i\alpha})^2},$$

where $\kappa_1, \kappa_2, \cdots, \kappa_t$ are different from zero.

The solution of our problem can be expressed as a certain multiple integral of tp variables. Let

$$b'_{ij} = \sum_{\alpha=t+1}^{n} y_{i\alpha} y_{j\alpha} .$$

Since the b'_{ij} have the Wishart distribution with $n - t$ degrees of freedom (we assume $n \geq t + p$) we can write the joint distribution of the b'_{ij} and $y_{i\eta}$ ($i = 1, 2, \cdots, p$; $\eta = 1, 2, \cdots, t$) as

$$\frac{1}{2^{\frac{1}{2}p(n-t)}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n - t + 1 - i])} | b'_{ij} |^{\frac{1}{2}(n-p-t-1)} e^{-\frac{1}{2}\sum_{i,j=1}^{p} b'_{ii}}$$

$$\times \frac{1}{(2\pi)^{pt/2}} e^{-\frac{1}{2}\sum_{i=1}^{p}\sum_{\eta=1}^{t}(y_{i\eta}-\kappa_\eta\delta_{i\eta})^2}$$

Considering the equations

$$b_{ij} = b'_{ij} + \sum_{\eta=1}^{t} y_{i\eta}y_{j\eta}$$

as a transformation of the b'_{ij} we immediately obtain the joint distribution of the b's and the $y_{i\eta}$($i = 1, 2, \cdots, p$; $\eta = 1, 2, \cdots, t$) as

$$(14) \quad \frac{e^{-\frac{1}{2}\sum_{\eta=1}^{t}\kappa_\eta^2}}{2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)+\frac{1}{2}pt}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n - t + 1 - i])} | b_{ij} - \sum_{\eta=1}^{t} y_{i\eta}y_{j\eta} |^{\frac{1}{2}(n-p-t-1)}$$

$$\times e^{-\frac{1}{2}\sum_{i=1}^{p} b_{ii} + \sum_{\eta=1}^{t}\kappa_\eta y_{\eta\eta}}$$

To find the distribution of the b_{ij} we must integrate out the $y_{i\eta}$, where the range of integration is such that the matrix

$$|| b_{ij} - \sum_{\eta=1}^{t} y_{i\eta}y_{j\eta} ||$$

is positive. For $t = 1$ or 2 we can integrate (14) and express the results in a convenient form. However, for higher values of t the integration affords considerable difficulty and has not been done for the general case. In terms of geometry the case $t = 1$ is the case in which the expected values of the observations lie on a line in the p dimensional space. In the case of $t = 2$, similarly the expected values lie in a plane in this space. Hence, we shall call these two cases the linear and planar cases, respectively.

3. The linear case. In the linear case there is one root of the equation (4) which is not equal to zero, that is, there is simply one κ in the distribution (14) and one set of y's, namely y_{i1}($i = 1, 2, \cdots, p$). The problem is to integrate the y_{i1} over the range for which the matrix

$$|| b_{ij} - y_{i1}y_{j1} ||$$

is positive; the integrand we are interested in is (dropping the subscript "1" from the κ_1 and y_{i1} and neglecting the part not involving the y's)

$$|b_{ij} - y_i y_j|^{\frac{1}{2}(n-p-2)} e^{\kappa y_1} \prod_{i=1}^{p} dy_i . \tag{15}$$

The determinant in expression (15) can be expressed as

$$|b_{ij} - y_i y_j| = |b_{ij}| \left(1 - \sum_{i,j=1}^{p} b^{ij} y_i y_j\right),$$

where

$$\| b^{ij} \| = \| b_{ij} \|^{-1}.$$

The inverse exists because the probability is zero that $\| b_{ij} \|$ is singular. There is a linear transformation

$$y_i = \sum_{j=1}^{p} q_{ij} u_j ,$$

such that

$$\sum_{i,j=1}^{p} b^{ij} y_i y_j = \sum_{j=1}^{p} u_j^2$$

and

$$\kappa y_1 = l u_1 ,$$

where l^2 is the one non-zero root of the equation:

$$| \lambda B^{-1} - \kappa^2 E_{11} | = 0, \tag{16}$$

where $B^{-1} = \| b^{ij} \|$ and E_{11} is the matrix with unity in the upper left hand corner and zeros elsewhere. This fact is a result of the well-known theorem concerning diagonalization of pairs of quadratic forms.[4] The Jacobian of this transformation is

$$| q_{ij} | = | b_{ij} |^{\frac{1}{2}},$$

and the range of integration is $\sum_{i=1}^{p} u_i^2 \leq 1$. The integrand is transformed into

$$| b_{ij} |^{\frac{1}{2}(n-p-1)} \left(1 - \sum_{i=1}^{p} u_i^2\right)^{\frac{1}{2}(n-p-2)} e^{l u_1} \prod_{i=1}^{p} du_i .$$

Now let

$$u_1 = \sin w,$$

$$u_i = \cos w \, v_{i-1} \qquad (i = 2, 3, \cdots, p)$$

[4] The transformation is the so-called "regression transformation." See Madow [6].

The Jacobian of this transformation is $\cos^p w$ and

$$1 - \sum_{i=1}^{p} u_i^2 = \cos^2 w \left(1 - \sum_{i=1}^{p-1} v_i^2\right).$$

The integration is over the ranges of $\frac{\pi}{2} \leq w \leq \frac{\pi}{2}$ and

$$\sum_{i=1}^{p-1} v_i^2 \leq 1.$$

We integrate the following expression

$$| b_{ij} |^{\frac{1}{2}(n-p-1)} \left[\left(1 - \sum_{i=1}^{p-1} v_i^2\right)^{\frac{1}{2}(n-p-2)} \prod_{i=1}^{p-1} dv_i\right] \{\cos^{n-2} w \, e^{l \sin w} \, dw\}.$$

The integral of the quantity within the brackets is simply a Dirichlet integral [7] and its value is

$$\frac{\Gamma(\frac{1}{2}[n - p])\pi^{\frac{1}{2}(p-1)}}{\Gamma(\frac{1}{2}[n - 1])}.$$

The integral of the expression within the braces is a multiple of a Bessel function of purely imaginary argument [8, p. 79]; that is,

$$\frac{\Gamma(\frac{1}{2}[n - 1])\sqrt{\pi}}{(l/2)^{\frac{1}{2}(n-2)}} I_{\frac{1}{2}(n-2)}(l).$$

Hence, the integral of (15) is

$$(17) \qquad | b_{ij} |^{\frac{1}{2}(n-p-1)}\Gamma(\tfrac{1}{2}[n - p])\pi^{\frac{1}{2}p}(l/2)^{-\frac{1}{2}(n-2)}I_{\frac{1}{2}(n-2)}(l).$$

Multiplying equation (16) by the determinant $|B|$ one can easily show that the non-zero root, l^2, is simply $\kappa^2 b_{11}$. The distribution of the b_{ij} in the linear case then is

$$\frac{e^{-\frac{1}{2}\kappa^2} e^{-\frac{1}{2}\sum_{i=1}^{p} b_{ii}}}{2^{\frac{1}{2}pn-\frac{1}{2}(n-2)}\pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n - i])} | b_{ij} |^{\frac{1}{2}(n-p-1)} (\kappa^2 b_{11})^{-\frac{1}{2}(n-2)} I_{\frac{1}{2}(n-2)}(\kappa\sqrt{b_{11}}),$$

In §5 we shall give the distribution in terms of the original variables, namely, the a_{ij}.

4. The planar case. The case of two non-zero roots of equation (4) can be handled by continuing the process of integration of §3 another step. The essential problem is the integration of

$$(18) \qquad | b_{ij} - \sum_{\eta=1}^{2} y_{i\eta} y_{j\eta} |^{\frac{1}{2}(n-p-3)} e^{\kappa_1 y_{11} + \kappa_2 y_{22}} \prod_{i=1}^{p} \prod_{\eta=1}^{2} dy_{i\eta}$$

over the range of y's for which the matrix

$$\| b_{ij} - \sum_{\eta=1}^{2} y_{i\eta} y_{j\eta} \|$$

is positive. The integration is done in two stages, first with respect to the y_{i1}, then with respect to the y_{i2}. Letting

$$\begin{aligned} \tilde{b}_{ij} &= b_{ij} - y_{2i} y_{2j}, \\ \tilde{n} &= n - 1, \\ \tilde{\kappa} &= \kappa_1, \\ \tilde{y}_i &= y_{i1}, \end{aligned}$$

and omitting for the time being

$$e^{\kappa_2 y_{22}} \prod_{i=1}^{p} dy_{i2}, \tag{19}$$

we can write the first stage of the integration of (18) as

$$\int | \tilde{b}_{ij} - \tilde{y}_i \tilde{y}_j |^{\frac{1}{2}(\tilde{n}-p-2)} e^{\tilde{\kappa}\tilde{y}_1} \prod_{i=1}^{p} d\tilde{y}_i$$

over the range $\| \tilde{b}_{ij} - \tilde{y}_i \tilde{y}_j \|$ positive. But, the only difference between this and the integration of (15) which has been shown to be (17), is that we are now writing all variables with "$\sim$" signs. Keeping this in mind, changing back again to our other variables of Section 4 and inserting again (19) we can write the first stage of the integration of (18) as

$$\begin{aligned} &\Gamma(\tfrac{1}{2}[n - p - 1])\pi^{\frac{1}{2}p} \, | b_{ij} - y_{i2} y_{j2} |^{\frac{1}{2}(n-p-2)} e^{\kappa_2 y_{22}} \left[\frac{\kappa_1^2 (b_{11} - y_{12}^2)}{4} \right]^{-\frac{1}{4}(n-3)} \\ &\qquad \cdot I_{\frac{1}{2}(n-3)}\left(\sqrt{\kappa_1^2 b_{11} - \kappa_1^2 y_{12}^2}\right) \prod_{i=1}^{p} dy_{i2}. \end{aligned} \tag{20}$$

Now we must integrate (20) with respect to the y_{i2} over the range $\| b_{ij} - y_{i2} y_{j2} \|$ positive. The determinant in (20) can be written as

$$| b_{ij} | \left(1 - \sum_{i,j=1}^{p} b^{ij} y_{i2} y_{j2} \right).$$

There is a transformation

$$y_{i2} = \sum_{j=1}^{p} g_{ij} s_j, \tag{21}$$

such that

$$\begin{aligned} \sum_{i,j=1}^{p} b^{ij} y_{i2} y_{j2} &= \sum_{j=1}^{p} s_j^2, \\ \kappa_1^2 y_{12}^2 &= f^2 s_1^2, \\ \kappa_2 y_{22} &= d_1 s_1 + d_2 s_2, \end{aligned} \tag{22}$$

where f^2 is the one non-zero root of the equation

$$| \lambda B^{-1} - \kappa_1^2 E_{11} | = 0,$$

where B^{-1} and E_{11} are used as in equation (16). Since this equation is similar to (14), then $f^2 = \kappa_1^2 b_{11}$. The values of d_1 and d_2 will be considered later. This result is deduced from an extension of the theorem concerning the diagonalization of pairs of quadratic forms.[5] The Jacobian is

$$| g_{ij} | = | b_{ij} |^{\frac{1}{2}},$$

and the range of integration is $\sum_{i=1}^{p} s_i^2 \leq 1$. The integrand (20) is now changed to

$$\Gamma(\tfrac{1}{2}[n - p - 1])\pi^{\frac{1}{2}p} | b_{ij} |^{\frac{1}{2}(n-p-1)} \left(1 - \sum_{i=1}^{p} s_i^2\right)^{\frac{1}{2}(n-p-2)} e^{d_1 s_1 + d_2 s_2}$$
$$\cdot \left[\frac{f^2(1 - s_1^2)}{4}\right]^{-\frac{1}{4}(n-3)} I_{\frac{1}{2}(n-3)} (\sqrt{f^2(1 - s_1^2)}) \prod_{i=1}^{p} ds_i .$$

Next the following transformation is made:

$$s_1 = \sin w_1 ,$$
$$s_2 = \cos w_1 \sin w_2 ,$$
$$s_i = \cos w_1 \cos w_2 v_{i-2} \qquad (i = 3, 4, \cdots , p).$$

The Jacobian is $\cos^p w_1 \cos^{p-1} w_2$, and

$$1 - \sum s_i^2 = \cos^2 w_1 \cos^2 w_2 \left(1 - \sum_{i=1}^{p-2} v_i^2\right).$$

We now integrate

$$\Gamma(\tfrac{1}{2}[n - p - 1])\pi^{\frac{1}{2}p} | b_{ij} |^{\frac{1}{2}(n-p-1)} \left[\left(1 - \sum_{i=1}^{p-2} v_i^2\right)^{\frac{1}{2}(n-p-2)} \prod_{i=1}^{p-2} dv_i\right]$$
$$(23) \qquad \cdot\left\{\cos^{n-2} w_1 \cos^{n-3} w_2 e^{d_1 \sin w_1 + d_2 \cos w_1 \sin w_2} \left(\frac{f}{2} \cos w_1\right)^{-\frac{1}{2}(n-3)}\right.$$
$$\left. \cdot I_{\frac{1}{2}(n-3)}(f \cos w_1)\, dw_2\, dw_3\right\}.$$

The integral of the expression within the square brackets is another Dirichlet integral; its value is

$$\frac{\Gamma(\tfrac{1}{2}[n - p])\pi^{\frac{1}{2}(p-2)}}{\Gamma(\tfrac{1}{2}[n - 2])} .$$

[5] Again the "regression transformation" is used. See footnote for Section 3.

Similar to Section 3 the integration of the quantity within the braces with respect to w_2 is

$$\Gamma(\tfrac{1}{2}[n-2])\sqrt{\pi}\left(\frac{d_2 \cos w_1}{2}\right)^{-\frac{1}{2}(n-3)} I_{\frac{1}{2}(n-3)}(d_2 \cos w_1) \cdot \cos^{n-2} w_1 e^{d_1 \sin w_1}\left(\frac{f}{2}\cos w_1\right)^{-\frac{1}{2}(n-3)} I_{\frac{1}{2}(n-3)}(f \cos w_1).$$

Since the range of integration of w_1 is $-\pi/2 \leq w_1 \leq \pi/2$ and since $\sin w_1$ is an odd function and $\cos w_1$ is an even function, the integral of the above expression can be transformed into an integration over the range $0 \leq w_1 \leq \pi/2$ by replacing $e^{d_1 \sin w_1}$ by $2 \sinh (d_1 \sin w_1)$. In view of the relationship between the Bessel functions of purely imaginary argument and $\sinh (d_1 \sin w_1)$ [8, p. 54] we can write the integral of the above expression as

$$\Gamma(\tfrac{1}{2}[n-2])\sqrt{\pi}\left(\frac{d_2}{2}\right)^{-\frac{1}{2}(n-3)} \sqrt{d_1}\sqrt{2\pi} \times \int_0^{\pi/2} I_{-\frac{1}{2}}(d_1 \sin w_1) I_{\frac{1}{2}(n-3)}(d_2 \cos w_1) I_{\frac{1}{2}(n-3)}(f \cos w_1) \sin^{\frac{1}{2}} w_1 \cos w_1 \, dw_1 .$$

This integral can be expressed in another form by virtue of a formula in Watson's *Bessel Functions* [8, p. 377] as

$$\sqrt{\pi}\, 2^{\frac{1}{2}(n-2)} \int_0^{\pi} \frac{I_{\frac{1}{2}(n-2)}\left(\sqrt{d_1^2 + d_2^2 + f^2 - 2d_2 f \cos u}\right)}{(d_1^2 + d_2^2 + f^2 - 2d_2 f \cos u)^{\frac{1}{2}(n-2)}} \sin^{n-3} u \, du. \tag{24}$$

Letting $d_1^2 + d_2^2 + f^2 = x$ and $d_2 f = y$ and using an expresion formula for Bessel functions [8, p. 140] we can write (24) as

$$\sqrt{\pi}\, 2^{\frac{1}{2}(n-2)} \int_0^{\pi} \sum_{\gamma=0}^{\infty} \frac{(-y)^{\gamma} \cos^{\gamma} u}{\gamma!} x^{-\frac{1}{2}(n-2)+\gamma} I_{\frac{1}{2}(n-2)+\gamma}(\sqrt{x}) \sin^{n-3} u \, du.$$

Since the integral of $\cos^{\gamma} u \sin^{n-3} u$ where γ is odd is zero, the result of the integration (using the "duplication formula" for Γ functions and letting $\gamma = 2\omega$) is

$$\Gamma(\tfrac{1}{2}[n-2])\pi 2^{\frac{1}{2}(n-2)} \sum_{\omega=0}^{\infty} \frac{(y^2)^{\omega} x^{-\frac{1}{2}(n-2)+2\omega}}{2^{2\omega}\omega!\Gamma\left(\frac{n-1}{2}+\omega\right)} I_{\frac{1}{2}(n-2)+2\omega}(\sqrt{x}).$$

From the relationship (22) it is clear that the equation

$$| \kappa_i^2 \delta_{ij} - \lambda b^{ij} | = 0 \quad (\kappa_i^2 = 0 \text{ for } i = 3, 4, \cdots, p) \tag{25}$$

is transformed by (21) into

$$\begin{vmatrix} f^2 + d_1^2 - \lambda & d_1 d_2 & 0 & \cdots & 0 \\ d_1 d_2 & d_2^2 - \lambda & 0 & \cdots & 0 \\ 0 & 0 & -\lambda & \cdots & 0 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \cdots & -\lambda \end{vmatrix} = 0. \tag{26}$$

Expression (25) is equivalent to

$$(27) \qquad | k_i k_j b_{ij} - \lambda \delta_{ij} | = 0.$$

Hence, x and y^2, which are the sum and product, respectively, of the non-zero roots of the two equivalent equations (26) and (27), are given by

$$x = f^2 + d_1^2 + d_2^2 = \kappa_1^2 b_{11} + \kappa_2^2 b_{22},$$

$$y^2 = f^2 d_2^2 = \kappa_1^2 \kappa_2^2 (b_{11} b_{22} - b_{12}^2).$$

In view of this result we can now write the integration of (23) as

$$\Gamma(\tfrac{1}{2}[n-p-1])\Gamma(\tfrac{1}{2}[n-p])\pi^p 2^{\frac{1}{2}(n-2)} | b_{ij} |^{\frac{1}{2}(n-p-1)} \times \sum_{\omega=0}^{\infty} \frac{[\kappa_1^2 \kappa_2^2 (b_{11} b_{22} - b_{12}^2)]^\omega}{2^{2\omega} \omega! \Gamma\left(\frac{n-1}{2} + \omega\right)} \cdot (\kappa_1^2 b_{11} + \kappa_2^2 b_{22})^{-\frac{1}{2}(n-2/2+2\omega)} I_{\frac{1}{2}(n-2)+2\omega}\left(\sqrt{\kappa_1^2 b_{11} + \kappa_2^2 b_{22}}\right).$$

Finally, by multiplying in what was left out of (18) we obtain the integral of (14) which is the solution to the problem as stated in the canonical form:

$$(28) \qquad \frac{e^{-\frac{1}{2}(\kappa_1^2+\kappa_2^2)} | b_{ij} |^{\frac{1}{2}(n-p-1)} e^{-\frac{1}{2}\sum_{i=1}^{p} b_{ii}}}{2^{\frac{1}{2}pn - \frac{1}{2}(n-2)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p-2} \Gamma(\frac{1}{2}[n-1-i])} \times \sum_{\omega=0}^{\infty} \frac{[\kappa_1^2 \kappa_2^2 (b_{11} b_{22} - b_{12}^2)]^\omega}{2^{2\omega} \omega! \Gamma\left(\frac{n-1}{2} + \omega\right)} (\kappa_1^2 b_{11} + \kappa_2^2 b_{22})^{-\frac{1}{2}(\frac{1}{2}(n-2)+2\omega)} \times I_{\frac{1}{2}(n-2)+2\omega}\left(\sqrt{\kappa_1^2 b_{11} + \kappa_2^2 b_{22}}\right).$$

5. Final form. To answer the problem as stated originally it is necessary to make the transformation (12) and obtain the distribution in terms of the a_{ij} for the linear and planar cases.

It is clear that equation (25) is equivalent to

$$(29) \qquad | T - \lambda A^{-1} | = 0,$$

where $A^{-1} = || a_{ij} ||^{-1}$, for T is the transform (by (5)) of $|| \kappa_i^2 \delta_{ij} ||$ and A^{-1} is the transform of $|| b_{ij} ||$. The sum and product of the non-zero roots, which are the arguments of the infinite series in (24), remain unchanged.

Since the quantity $\sum_{i=1}^{p} \kappa_i^2$ is the sum of the roots of (4) it can be expressed as

$$\sum_{i=1}^{p} \kappa_i^2 = \sum_{i,j=1}^{p} \sum_{\alpha=1}^{N} \sigma^{ij} (\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j),$$

where $|| \sigma^{ij} || = || \sigma_{ij} ||^{-1}$. Furthermore we have

$$| b_{ij} | = | \Sigma | \cdot | a_{ij} |$$

and

$$\sum_{i=1} b_{ii} = \operatorname{tr} \| b_{ij} \| = \operatorname{tr} (\Psi \| a_{ij} \| \Psi') = \sum_{i,j=1}^{p} \sigma^{ij} a_{ij}.$$

Moreover, the Jacobian of the transformation (12) is[6]

$$| J | = | \Psi |^{p+1} = | \Sigma |^{-\frac{1}{2}(p+1)}. \tag{30}$$

Hence, we have the following results:

Given N multivariate normal populations each of p variates with identical variance-covariance matrices $\| \sigma_{ij} \|$ *and with expected values of the pN variates* $x_{i\alpha}$ *given by* (1). *Let* a_{ij} *be defined by equation* (2); *let the rank of the matrix* $\| \tau_{ij} \|$ *defined by* (3) *be t.*

(i) *When* $t = 0$, *the joint distribution of the* a_{ij} *is given by the Wishart distribution.*

(ii) *When* $t = 1$, *the joint distribution of the* a_{ij} *is*

$$\begin{aligned} &\frac{e^{-\frac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}\sigma^{ij}(\mu_{i\alpha}-\bar{\mu}_i)(\mu_{j\alpha}-\bar{\mu}_j)} | \sigma^{ij} |^{\frac{1}{2}(N-1)} | a_{ij} |^{\frac{1}{2}(N-p-2)}}{2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)} \pi^{\frac{1}{4}p(p-1)} \sum_{i=1}^{p-1} \Gamma(\frac{1}{2}[N-1-i])} \\ &\qquad \times e^{\frac{1}{2}\sum_{i,j=1}^{k}\sigma^{ij}a_{ij}} \left[\sum_{i,j=1}^{p} \sum_{\alpha=1}^{N} a_{ij}(\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j) \right]^{-\frac{1}{4}(N-3)} \\ &\qquad \times I_{\frac{1}{2}(N-3)}\left(\sqrt{\sum_{i,j=1}^{p} \sum_{\alpha=1}^{N} a_{ij}(\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j)} \right). \end{aligned} \tag{31}$$

(iii) *When* $t = 2$, *the joint distribution of the* a_{ij} *is given by*

$$\begin{aligned} &\frac{e^{-\frac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}\sigma^{ij}(\mu_{i\alpha}-\bar{\mu}_i)(\mu_{j\alpha}-\bar{\mu}_j)} | \sigma^{ij} |^{\frac{1}{2}(N-1)} | a_{ij} |^{\frac{1}{2}(N-p-2)}}{2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p-2} \Gamma(\frac{1}{2}[N-2-i])} \\ &\qquad \times e^{-\frac{1}{2}\sum_{i,j=1}^{p}\sigma^{ij}a_{ij}} \times \sum_{\omega=0}^{\infty} \frac{(u_1 u_2)^{\omega}}{2^{2\omega} \omega! \Gamma\left(\frac{N-2}{2} + \omega\right)} (u_1 + u_2)^{-\frac{1}{2}(\frac{1}{2}(N-3)+2\omega)} \\ &\qquad \times I_{\frac{1}{2}(N-3)+2\omega}(\sqrt{u_1 + u_2}), \end{aligned} \tag{32}$$

where u_1 *and* u_2 *are the two non-zero roots of* (29).

(iv) *When* $t > 2$, *the joint distribution of the* a_{ij} *can be written by means of expression* (14) *as a multiple integral. The explicit form of the distribution has not yet been obtained.*

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[6] One method of demonstrating this fact is to apply (8) to centrally distributed variates and compare the Wishart distribution of the transformed variates with the Wishart distribution of the original variates.

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