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SOME FAMILIES OF RAPIDLY CONVERGENT SERIES REPRESENTATIONS FOR THE ZETA FUNCTIONS

H. M. Srivastava

Abstract. Many interesting families of rapidly convergent series representations for the Riemann Zeta function $\zeta(2n+1)$ $(n \in \mathbb{N})$ were considered recently by various authors. In this survey-cum-expository paper, the author presents a systematic (and historical) investigation of these series representations. Relevant connections of the results presented here with several other known series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ are also pointed out. In one of many computationally useful special cases presented here, it is observed that $\zeta(3)$ can be represented by means of a series which converges much faster than that in Euler's celebrated formula as well as the series used recently by Apéry in his proof of the irrationality of $\zeta(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux show, among other things, that only 50 terms of this series are capable of producing an accuracy of seven decimal places.

1. Introduction and Historical Background

Let $\mathcal{S}$ denote the set of all nontrivial integer kth powers, that is,

$$\begin{aligned} \mathcal{S} &:= \left\{ n^k : n,\ k \in \mathbb{N} \setminus \{1\}\ (\mathbb{N} := \{1,2,3,\dots\}) \right\} \\ &= \{4,\ 8,\ 9,\ 16,\ 25,\ 27,\ 32,\ 36,\ \dots\}. \end{aligned} \tag{1.1}$$

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An over two centuries old theorem of Christian Goldbach (1690 - 1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700 - 1782), was revived, not too long ago, as the following problem [19]:

$$\sum_{\omega \in \mathcal{S}} (\omega - 1)^{-1} = 1, \tag{1.2}$$

the sum being extended over all members ω of the set $\mathcal{S}$.

In terms of the Riemann Zeta function $\zeta(s)$, Goldbach's theorem (1.2) can easily be restated as

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} = 1 \tag{1.3}$$

or, equivalently, as

$$\sum_{k=2}^{\infty} \mathcal{F}(\zeta(k)) = 1, \tag{1.4}$$

where, for convenience, $\mathcal{F}(x) := x - [x]$ denotes the fractional part of $x \in \mathbb{R}$. In fact, it is fairly easy to show also that

$$\sum_{k=2}^{\infty} (-1)^k \ \mathcal{F}(\zeta(k)) = \frac{1}{2}, \tag{1.5}$$

$$\sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k)) = \frac{3}{4}, \tag{1.6}$$

and

$$\sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k+1)) = \frac{1}{4}. \tag{1.7}$$

Here, as usual, the Riemann Zeta function $\zeta(s)$ and the (Hurwitz's) generalized Zeta function $\zeta(s, a)$ are defined (for $\Re(s) > 1$) by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ \displaystyle\frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0;\ s \neq 1) \end{cases} \tag{1.8}$$

and

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad \left(\Re(s) > 1;\ a \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right), \tag{1.9}$$

and (for $\Re(s) \leqq 1;\ s \neq 1$) by their meromorphic continuations (see, for details, Titchmarsh [28]), so that (obviously)

$$\zeta(s,1) = \zeta(s) = (2^s - 1)^{-1}\ \zeta\left(s, \frac{1}{2}\right) \quad \text{and} \quad \zeta(s,2) = \zeta(s) - 1. \tag{1.10}$$

Another remarkable result involving Riemann's ζ-function is the following series representation for $\zeta(3)$:

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}, \tag{1.11}$$

which was contained in a 1772 paper, entitled "*Exercitationes Analyticae*", by Leonhard Euler (1707 - 1783) (cf., e.g., Ayoub [2, pp. 1084 - 1085]). In fact, this result of Euler was rediscovered (among others) by Ramaswami [18] (see also Srivastava [20, p. 7, Equation (2.23)]) and (more recently) by Ewell [8]. And, as pointed out by (for example) Chen and Srivastava [4, pp. 180 - 181], another series representation:

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}, \tag{1.12}$$

which played a key rôle in Apéry's *celebrated* proof [1] of the irrationality of $\zeta(3)$, was proven independently by (among others) Hjortnaes [13], Gosper [11], and Apéry [1].

Clearly, Euler's series in (1.11) converges faster than the defining series for $\zeta(3)$, but obviously not as fast as the series in (1.12). Such Zeta values as $\zeta(3)$, $\zeta(5)$, *et cetera* are known to arise naturally in a wide variety of applications (see, for example, Tricomi [29], Witten [32], and Nash and O'Connor [16, 17]). On the other hand, in the case of *even* integer arguments, we have the computationally useful relationship:

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!}\ B_{2n} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \tag{1.13}$$

with the well-tabulated Bernoulli numbers defined by the generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi), \tag{1.14}$$

as well as the familiar recursion formula:

$$\zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{k=1}^{n-1} \zeta(2k)\ \zeta(2n-2k) \quad (n \in \mathbb{N} \setminus \{1\}). \tag{1.15}$$

Thus there is a need for expressing $\zeta(2n+1)$ as a rapidly converging series for all $n \in \mathbb{N}$. With this objective in view, we propose to develop here a rather systematic investigation of the various families of rapidly convergent series representations for the Riemann $\zeta(2n+1)$ $(n \in \mathbb{N})$. We also consider relevant connections of the results presented here with many other known series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$. In one of many computationally useful special cases considered here, it is observed that $\zeta(3)$ can be represented by means of a series which converges much more rapidly than that in Euler's celebrated formula (1.11) as well as the series (1.12) used recently by Apéry [1] in his proof of the irrationality of $\zeta(3)$. Symbolic and numerical computations using *Mathematica* (Version 4.0) for Linux show, among other things, that only 50 terms of this series are capable of producing an accuracy of seven decimal places.

2. The First Set of Series Representations

The various series identities considered in the preceding section, including (for example) Goldbach's theorem (1.2), are known to be derivable also from the following simple consequence of the binomial theorem *and* the definition (1.9):

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!} \zeta(s+k, a) t^k = \zeta(s, a-t) \quad (|t| < |a|), \tag{2.1}$$

which, for $a = 1$ and $t = \pm 1/m$, readily yields the series identity:

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{m^{2k}} \\ &= \begin{cases} (2^s - 1)\,\zeta(s) - 2^{s-1} & (m = 2) \\ \dfrac{1}{2}\left[(m^s - 1)\,\zeta(s) - m^s - \displaystyle\sum_{j=2}^{m-2} \zeta\left(s, \frac{j}{m}\right)\right] & (m \in \mathbb{N} \setminus \{1, 2\}), \end{cases} \end{aligned} \tag{2.2}$$

$(\lambda)_n := \Gamma(\lambda+n)/\Gamma(\lambda)$ being the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$).

In terms of the familiar harmonic numbers

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}), \tag{2.3}$$

the following series representations for $\zeta(2n+1)$ were proven recently by appealing appropriately to the series identity (2.2) in its special cases when $m = 2, 3, 4,$ and 6 (see Srivastava [24]):

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2^{2n+1}-1}\Bigg[\frac{H_{2n}-\log\pi}{(2n)!} \\ +\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\pi^{2k}} \\ +2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{2^{2k}}\Bigg] \quad (n\in\mathbb{N});\end{aligned} \tag{2.4}$$

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n+1}-1}\Bigg[\frac{H_{2n}-\log\left(\frac{2}{3}\pi\right)}{(2n)!} \\ +\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} \\ +2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{3^{2k}}\Bigg] \quad (n\in\mathbb{N});\end{aligned} \tag{2.5}$$

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{2^{4n+1}+2^{2n}-1}\Bigg[\frac{H_{2n}-\log\left(\frac{1}{2}\pi\right)}{(2n)!} \\ +\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} \\ +2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{4^{2k}}\Bigg] \quad (n\in\mathbb{N});\end{aligned} \tag{2.6}$$

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n}(2^{2n}+1)+2^{2n}-1}\Bigg[\frac{H_{2n}-\log\left(\frac{1}{3}\pi\right)}{(2n)!} \\ +\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{3}\pi\right)^{2k}} \\ +2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{6^{2k}}\Bigg] \quad (n\in\mathbb{N}).\end{aligned} \tag{2.7}$$

Here (*and elsewhere in this work*) an empty sum is to be interpreted (as usual) to be nil.

We choose to recall the proof of (2.4) detailed by Srivastava [24]. Each of the other results (2.5), (2.6), and (2.7) can be proven *mutatis mutandis*. The following properties of the Riemann ζ-function will also be required in these derivations:

$$\zeta(0) = -\frac{1}{2}; \quad \zeta(-2n) = 0 \ (n \in \mathbb{N}); \quad \zeta'(0) = -\frac{1}{2}\log(2\pi), \tag{2.8}$$

and (in general)

$$\begin{aligned} \zeta'(-2n) &= \lim_{\epsilon \to 0} \frac{\zeta(-2n+\epsilon)}{\epsilon} \\ &= \frac{(-1)^n}{2(2\pi)^{2n}}(2n)!\,\zeta(2n+1) \ (n \in \mathbb{N}), \end{aligned} \tag{2.9}$$

where use is made of the familiar functional equation:

$$2^s\,\Gamma(1-s)\,\zeta(1-s)\ \sin\left(\frac{1}{2}\pi s\right) = \pi^{1-s}\,\zeta(s). \tag{2.10}$$

Furthermore, by l'Hôpital's rule, it is easily seen that

$$\lim_{s \to -2n}\left\{\frac{\sin\left(\frac{1}{2}\pi s\right)}{s+2n}\right\} = (-1)^n \frac{\pi}{2} \quad (n \in \mathbb{N}) \tag{2.11}$$

and

$$\begin{aligned} \lim_{s \to -2n}\left\{\frac{\zeta(s+2k)}{s+2n}\right\} &= \frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}}(2n-2k)!\,\zeta(2n-2k+1) \\ &\qquad (k = 1, \cdots, n-1; n \in \mathbb{N} \setminus \{1\}). \end{aligned} \tag{2.12}$$

First of all, upon separating the first $n+1$ terms of the series occurring on the left-hand side of the case $m = 2$ of the general result (2.2), if we transpose the terms for $k = 0$ and $k = n$ to the right-hand side, we readily obtain the identity:

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!}\,2^{2(n-k)}\,\zeta(s+2k) + \sum_{k=1}^{\infty} \frac{(s)_{2n+2k}}{(2n+2k)!}\,\frac{\zeta(s+2n+2k)}{2^{2k}} \\ &= 2^{2n}(2^s-2)\zeta(s) - 2^{s+2n-1} - \frac{(s)_{2n}}{(2n)!}\zeta(s+2n) \quad (n \in \mathbb{N}), \end{aligned} \tag{2.13}$$

it being understood, as mentioned before, that an empty sum is to be interpreted as nil.

Now we apply the functional equation (2.10) in the first term on the right-hand side of (2.13) and divide both sides by $s + 2n$. We thus find that

$$\begin{aligned}
&\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!}\, 2^{2(n-k)} \left\{ \frac{\zeta(s+2k)}{s+2n} \right\} \\
&+ \sum_{k=1}^{\infty} \frac{(s)_{2n}(s+2n+1)_{2k-1}}{(2n+2k)!} \frac{\zeta(s+2n+2k)}{2^{2k}} \\
&= 2^{s+2n}(2^s-2)\pi^{s-1}\Gamma(1-s)\zeta(1-s) \left\{ \frac{\sin\left(\frac{1}{2}\pi s\right)}{s+2n} \right\} \\
&\quad - \left\{ \frac{2^{s+2n-1} + \dfrac{(s)_{2n}}{(2n)!}\zeta(s+2n)}{s+2n} \right\} \quad (s \neq -2n;\ n \in \mathbb{N}).
\end{aligned} \tag{2.14}$$

Since

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} \quad (k = 0, 1, \cdots, n;\ n \in \mathbb{N}),$$

so that, obviously,

$$(-n)_n = (-1)^n\ n! \quad (n \in \mathbb{N}), \tag{2.15}$$

it is easily seen by logarithmic differentiation that

$$\frac{d}{ds}\{(s)_n\} = (s)_n \sum_{j=0}^{n-1} \frac{1}{s+j} \quad (n \in \mathbb{N}), \tag{2.16}$$

so that

$$\frac{d}{ds}\{(s)_{2n}\}\bigg|_{s=-2n} = -(2n)!\, H_{2n} \quad (n \in \mathbb{N}), \tag{2.17}$$

where H_n denotes the harmonic numbers defined by (2.3). We observe also that the limit formula (2.12) is needed in the first sum on the left-hand side of (2.14) only when this sum is nonzero (that is, only when $n \in \mathbb{N} \setminus \{1\}$).

Furthermore, by l'Hôpital's rule once again, we have

$$\begin{aligned}
&\lim_{s\to -2n}\left\{\frac{2^{s+2n-1}+\dfrac{(s)_{2n}}{(2n)!}\zeta(s+2n)}{s+2n}\right\}\\
&=\left[2^{s+2n-1}\log 2+\frac{d}{ds}\{(s)_{2n}\}\cdot\frac{\zeta(s+2n)}{(2n)!}+\frac{(s)_{2n}}{(2n)!}\zeta'(s+2n)\right]\Bigg|_{s=-2n}\\
&=\frac{1}{2}(H_{2n}-\log\pi)\quad (n\in\mathbb{N}).
\end{aligned}\tag{2.18}$$

Finally, letting $s\to -2n$ in (2.14), and making use of the limit relationships (2.12) and (2.18), we obtain the first series representation for $\zeta(2n+1)$ asserted by (2.4).

The series representation (2.4) is markedly different from each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [33, p. 1590, Equation (3.13)] and (subsequently) by Cvijović and Klinowski [5, p. 1265, Theorem A]. Since $\zeta(2k)\to 1$ as $k\to\infty$, the general term in the series representation (2.4) has the order estimate:

$$O\left(2^{-2k}\cdot k^{-2n-1}\right)\quad (k\to\infty;\ n\in\mathbb{N}),$$

whereas the general term in each of these earlier series representations has the order estimate:

$$O\left(2^{-2k}\cdot k^{-2n}\right)\quad (k\to\infty;\ n\in\mathbb{N}).$$

By suitably combining (2.4) and (2.6), it is fairly straightforward to obtain the series representation:

$$\begin{aligned}
\zeta(2n+1)=(-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)}&\left[\frac{\log 2}{(2n)!}\right.\\
&+\sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k}-1)}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\pi^{2k}}\\
&\left.-2\sum_{k=1}^{\infty}\frac{(2k-1)!(2^{2k}-1)}{(2n+2k)!}\,\frac{\zeta(2k)}{2^{4k}}\right]\quad (n\in\mathbb{N}).
\end{aligned}\tag{2.19}$$

Now, in terms of the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ defined by the generating functions (1.14) and

$$\frac{2e^{xz}}{e^z+1}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}\quad (|z|<\pi),\tag{2.20}$$

respectively, it is known that (cf., e.g., Magnus *et al.* [15, p. 29])

$$E_n(0) = (-1)^n E_n(1) = \frac{2(1-2^{n+1})}{n+1} B_{n+1} \quad (n \in \mathbb{N}), \tag{2.21}$$

which, together with the identity (1.13), implies that

$$E_{2n-1}(0) = \frac{4(-1)^n}{(2\pi)^{2n}} (2n-1)!(2^{2n}-1)\,\zeta(2n) \quad (n \in \mathbb{N}). \tag{2.22}$$

By appealing to the relationship (2.22), the series representation (2.19) can immediately be put in the form:

$$\begin{aligned} \zeta(2n+1) = &(-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \left[\frac{\log 2}{(2n)!} \right. \\ &+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ &\left. + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2n+2k)!} \left(\frac{\pi}{2}\right)^{2k} E_{2k-1}(0) \right] \quad (n \in \mathbb{N}), \end{aligned} \tag{2.23}$$

which is a slightly modified (and corrected) version of a result proven in a significantly different way by Tsumura [30, p. 383, Theorem B].

Another interesting combination of our series representations (2.4) and (2.6) leads us to the following variant of Tsumura's result (2.19) or (2.23):

$$\begin{aligned} \zeta(2n+1) = &(-1)^{n-1} \frac{\pi^{2n}}{2^{2n+1}-1} \left[\frac{H_{2n} - \log\left(\frac{1}{4}\pi\right)}{(2n)!} \right. \\ &+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ &\left. - 4 \sum_{k=1}^{\infty} \frac{(2k-1)!(2^{2k-1}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \quad (n \in \mathbb{N}), \end{aligned} \tag{2.24}$$

which is essentially the same as the *determinantal* expression for $\zeta(2n+1)$ derived recently by Ewell [9, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Other similar combinations of the series representations (2.4) to (2.7) would yield the following interesting companions of Ewell's result (2.24):

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n+1}-1)(3^{2n}+1)}\left[\frac{H_{2n}-\log\left(\frac{1}{6}\pi\right)}{(2n)!}\right. \\ & +\sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k+1}-1)}{(2n-2k)!}\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} \\ & \left. -4\sum_{k=1}^{\infty}\frac{(2k-1)!(2^{2k-1}-1)}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right] \quad (n\in\mathbb{N}),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n}+1)(3^{2n+1}-1)}\left[\frac{2H_{2n}-\log\left(\frac{\pi^2}{27}\right)}{(2n)!}\right. \\ & +\sum_{k=1}^{n-1}\frac{(-1)^k(3^{2k+1}-1)}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}} \\ & \left. -6\sum_{k=1}^{\infty}\frac{(2k-1)!(3^{2k-1}-1)}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right] \quad (n\in\mathbb{N}),\end{aligned} \tag{2.26}$$

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n+2}-2^{2n+3}+1}\left[\frac{H_{2n}-\log\left(\frac{8\pi}{27}\right)}{(2n)!}\right. \\ & +\sum_{k=1}^{n-1}\frac{(-1)^k(3^{2k+1}-2^{2k+1})}{(2n-2k)!}\frac{\zeta(2k+1)}{(2\pi)^{2k}} \\ & \left. -12\sum_{k=1}^{\infty}\frac{(2k-1)!(3^{2k-1}-2^{2k-1})}{(2n+2k)!}\frac{\zeta(2k)}{6^{2k}}\right] \quad (n\in\mathbb{N}),\end{aligned} \tag{2.27}$$

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\frac{2(2\pi)^{2n}}{2^{4n+3}+2^{2n+2}-3^{2n+2}-1}\left[\frac{H_{2n}-\log\left(\frac{27\pi}{128}\right)}{(2n)!}\right. \\ & +\sum_{k=1}^{n-1}\frac{(-1)^k(4^{2k+1}-3^{2k+1})}{(2n-2k)!}\frac{\zeta(2k+1)}{(2\pi)^{2k}} \\ & \left. -24\sum_{k=1}^{\infty}\frac{(2k-1)!(4^{2k-1}-3^{2k-1})}{(2n+2k)!}\frac{\zeta(2k)}{12^{2k}}\right] \quad (n\in\mathbb{N}),\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\zeta(2n+1) = &(-1)^{n-1}\frac{2(2\pi)^{2n}}{3^{2n+1}(2^{2n}+1)-2^{4n+2}+2^{2n}-1}\left[\frac{H_{2n}-\log\left(\frac{4\pi}{27}\right)}{(2n)!}\right. \\ &+\sum_{k=1}^{n-1}\frac{(-1)^k(3^{2k+1}-2^{2k+1})}{(2n-2k)!}\frac{\zeta(2k+1)}{\pi^{2k}} \\ &\left.-12\sum_{k=1}^{\infty}\frac{(2k-1)!(3^{2k-1}-2^{2k-1})}{(2n+2k)!}\frac{\zeta(2k)}{12^{2k}}\right] \quad (n\in\mathbb{N}).\end{aligned} \tag{2.29}$$

Next we turn to the following obvious consequence of the series identity (2.1):

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(s)_{2k+1}}{(2k+1)!}\zeta(s+2k+1,a)\ t^{2k+1} \\ &=\frac{1}{2}[\zeta(s,a-t)-\zeta(s,a+t)] \quad (|t|<|a|).\end{aligned} \tag{2.30}$$

By setting $t=1/m$ and differentiating both sides with respect to s, we find from (2.30) that

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(s)_{2k+1}}{(2k+1)!m^{2k}}\left[\zeta'(s+2k+1,a)+\zeta(s+2k+1,a)\sum_{j=0}^{2k}\frac{1}{s+j}\right] \\ &=\frac{m}{2}\frac{\partial}{\partial s}\left\{\zeta\left(s,a-\frac{1}{m}\right)-\zeta\left(s,a+\frac{1}{m}\right)\right\} \quad (m\in\mathbb{N}\setminus\{1\}),\end{aligned} \tag{2.31}$$

where we have made use of the derivative formula (2.16). In particular, when $m=2$, (2.31) immediately yields

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(s)_{2k+1}}{(2k+1)!2^{2k}}\left[\zeta'(s+2k+1,a)+\zeta(s+2k+1,a)\sum_{j=0}^{2k}\frac{1}{s+j}\right] \\ &=-\left(a-\frac{1}{2}\right)^{-s}\log\left(a-\tfrac{1}{2}\right).\end{aligned} \tag{2.32}$$

By letting $s\to-2n-1$ ($n\in\mathbb{N}$) in the *further* special of this last identity (2.32) when $a=1$, Wilton [31, p. 92] obtained the following series representation for

$\zeta(2n+1)$ (see also Hansen [12, p. 357, Entry (54.6.9)]):

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1}\, \pi^{2n} \Bigg[& \frac{H_{2n+1} - \log \pi}{(2n+1)!} \\ & + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ & + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \Bigg] \quad (n \in \mathbb{N}),\end{aligned} \tag{2.33}$$

which, in view of the identity:

$$\frac{(2k)!}{(2n+2k)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2n \frac{(2k-1)!}{(2n+2k)!} \quad (n \in \mathbb{N}), \tag{2.34}$$

would combine with the result (2.4) to yield the series representation:

$$\begin{aligned}\zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{n(2^{2n+1}-1)} \Bigg[& \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ & + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \Bigg] \quad (n \in \mathbb{N}).\end{aligned} \tag{2.35}$$

The series representation (2.35) is precisely the aforementioned *main* result of Cvijović and Klinowski [5, p. 1265, Theorem A]. In fact, in view of the derivative formula (2.19), the series representation (2.35) is essentially the same as a result given earlier by Zhang and Williams [33, p. 1590, Equation (3.13)] (see also Zhang and Williams [33, p. 1591, Equation (3.16)], where an obviously more complicated (asymptotic) version of (2.35) was proven similarly).

Observing also that

$$\frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k)!} - (2n+1) \frac{(2k-1)!}{(2n+2k+1)!} \quad (n, k \in \mathbb{N}), \tag{2.36}$$

we obtain yet another series representation for $\zeta(2n+1)$ by applying (2.4) and (2.33):

$$\begin{aligned}\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n-1)2^{2n}+1} \Bigg[& \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ & + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \Bigg] \quad (n \in \mathbb{N}),\end{aligned} \tag{2.37}$$

which provides a significantly simpler (and much more rapidly convergent) version of the other *main* result of Cvijović and Klinowski [5, p. 1265, Theorem B]:

$$\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \sum_{k=0}^{\infty} \Omega_{n,k} \frac{\zeta(2k)}{2^{2k}} \quad (n \in \mathbb{N}), \tag{2.38}$$

where the coefficients $\Omega_{n,k}$ are given explicitly as a finite sum of Bernoulli numbers [5, p. 1265, Theorem B(i)] (see, for details, Srivastava [24, pp. 393 - 394]):

$$\Omega_{n,k} := \sum_{j=0}^{2n} \binom{2n}{j} \frac{B_{2n-j}}{(j+2k+1)(j+1)2^j} \quad (n \in \mathbb{N};\ k \in \mathbb{N}_0). \tag{2.39}$$

3. Another Family of Series Representations

Starting once again from the identity (2.1) with (of course) $a = 1$, $t = \pm 1/m$, and s replaced by $s+1$, and applying (2.2), we find yet another class of series identities including, for example,

$$\sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} = (2^s - 2)\zeta(s) \tag{3.1}$$

and

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{m^{2k}} \\ &= \frac{1}{2m} \Bigg[m(m^s - 3)\zeta(s) + (m^{s+1} - 1)\zeta(s+1) - 2\zeta\left(s+1, \frac{1}{m}\right) \\ &\quad - \sum_{j=2}^{m-2} \left\{ m\zeta\left(s, \frac{j}{m}\right) + \zeta\left(s+1, \frac{j}{m}\right) \right\} \Bigg] \quad (m \in \mathbb{N} \setminus \{1, 2\}). \end{aligned} \tag{3.2}$$

It is the series identity (3.1) which was first applied by Zhang and Williams [33] (and, subsequently, by Cvijović and Klinowski [5]) in order to prove two (only seemingly different) versions of the series representation (2.35). Indeed, by appealing to (3.2) *with* $m = 4$, we can derive the following much more rapidly convergent series representation for $\zeta(2n+1)$ (see Srivastava [23, p. 9, Equation (41)]):

$$
\begin{aligned}
\zeta(2n+1) = {}& (-1)^n \frac{2(2\pi)^{2n}}{n(2^{4n+1}+2^{2n}-1)} \left[\frac{4^{n-1}-1}{(2n)!} B_{2n} \log 2 \right. \\
& - \frac{2^{2n-1}-1}{2(2n-1)!} \zeta'(1-2n) - \frac{4^{2n-1}}{(2n-1)!} \zeta'\left(1-2n, \frac{1}{4}\right) \\
& \left. + \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N}),
\end{aligned} \tag{3.3}
$$

where (*and in what follows*) a prime denotes the derivative of $\zeta(s)$ or $\zeta(s,a)$ with respect to s.

In view of the identities (2.34) and (2.36), the results (2.6) and (3.3) would lead us eventually to the following *additional* series representations for $\zeta(2n+1)$ (see Srivastava [23, p. 10, Equations (42) and (43)]):

$$
\begin{aligned}
\zeta(2n+1) = {}& (-1)^{n-1} \left(\frac{\pi}{2}\right)^{2n} \left[\frac{H_{2n+1} - \log\left(\frac{1}{2}\pi\right)}{(2n+1)!} \right. \\
& + \frac{2(4^n-1)}{(2n+2)!} B_{2n+2} \log 2 - \frac{2^{2n+1}-1}{(2n+1)!} \zeta'(-2n-1) \\
& - \frac{2^{4n+3}}{(2n+1)!} \zeta'\left(-2n-1, \frac{1}{4}\right) + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} \\
& \left. + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N});
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
\zeta(2n+1) = {}& (-1)^n \frac{4(2\pi)^{2n}}{n \cdot 4^{2n+1} - 2^{2n} + 1} \left[\frac{2^{2n+1}-1}{2 \cdot (2n)!} \zeta'(-2n-1) \right. \\
& + \frac{4^{2n+1}}{(2n)!} \zeta'\left(-2n-1, \frac{1}{4}\right) - \frac{(2n+1)(4^n-1)}{(2n+2)!} B_{2n+2} \log 2 \\
& + \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} \\
& \left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N}).
\end{aligned} \tag{3.5}
$$

Explicit expressions for the derivatives $\zeta'(-2n \pm 1)$ and $\zeta'\left(-2n \pm 1, \frac{1}{4}\right)$, occurring in the series representations (3.3), (3.4), and (3.5), can be found and substituted into these results in order to represent $\zeta(2n+1)$ in terms of Bernoulli numbers and polynomials and various rapidly convergent series of ζ-functions (see, for details, Srivastava [23, Section 3]).

Of the four seemingly analogous results (2.6), (3.3), (3.4), and (3.5), the infinite series in (3.4) would obviously converge most rapidly, with its general term having the order estimate:

$$O\left(k^{-2n-2} \cdot 4^{-2k}\right) \quad (k \to \infty;\ n \in \mathbb{N}).$$

We now turn to the work of Srivastava and Tsumura [27], who derived the following three new members of the class of the series representations (2.6) and (3.4):

(3.6)

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\left(\frac{2\pi}{3}\right)^{2n}\left[\frac{H_{2n+1}-\log\left(\frac{2}{3}\pi\right)}{(2n+1)!}+\frac{(3^{2n+2}-1)\pi}{2\sqrt{3}(2n+2)!}B_{2n+2}\right.\\ & +\frac{(-1)^{n-1}}{\sqrt{3}(2\pi)^{2n+1}}\zeta\left(2n+2,\frac{1}{3}\right)+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}}\\ & \left.+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{3^{2k}}\right] \quad (n \in \mathbb{N}),\end{aligned}$$

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\left(\frac{\pi}{2}\right)^{2n}\left[\frac{H_{2n+1}-\log(\frac{1}{2}\pi)}{(2n+1)!}+\frac{2^{2n}(2^{2n+2}-1)\pi}{(2n+2)!}B_{2n+2}\right.\\ & +\frac{(-1)^{n-1}}{2(2\pi)^{2n+1}}\zeta\left(2n+2,\frac{1}{4}\right)+\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{(\frac{1}{2}\pi)^{2k}}\\ & \left.+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{4^{2k}}\right] \quad (n \in \mathbb{N}),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\left(\frac{\pi}{3}\right)^{2n}\left[\frac{H_{2n+1}-\log(\frac{1}{3}\pi)}{(2n+1)!}\right.\\ & +\frac{2^{2n}(3^{2n+2}-1)\pi}{\sqrt{3}(2n+2)!}B_{2n+2}+\frac{(-1)^{n-1}}{2\sqrt{3}(2\pi)^{2n+1}}\\ & \cdot\left\{\zeta\left(2n+2,\frac{1}{3}\right)+\zeta\left(2n+2,\frac{1}{6}\right)\right\}\\ & +\sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\frac{\zeta(2k+1)}{(\frac{1}{3}\pi)^{2k}}\\ & \left.+2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\frac{\zeta(2k)}{6^{2k}}\right] \quad (n \in \mathbb{N}).\end{aligned} \tag{3.8}$$

Indeed the general terms of the infinite series occurring in these three members [(3.6), (3.7), and (3.8)] have the order estimates:

$$O\left(k^{-2n-2} \cdot m^{-2k}\right) \quad (k \to \infty;\ n \in \mathbb{N};\ m = 3, 4, 6), \tag{3.9}$$

which exhibit the fact that *each* of the three series representations (3.6), (3.7), and (3.8) converges more rapidly than Wilton's result (2.33) and two of them (cf. Equations (3.7) and (3.8)) at least as rapidly as Srivastava's result (3.4).

4. Further Series Representations

In their aforecited work on the Ray-Singer torsion and topological field theories, Nash and O'Connor ([16] and [17]) obtained a number of remarkable integral expressions for $\zeta(3)$, including (for example) the following result [17, p. 1489 *et seq.*]:

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 - \frac{8}{7} \int_0^{\pi/2} z^2 \cot z \, dz. \tag{4.1}$$

Since [7, p. 51, Equation 1.20(3)]

$$z \cot z = -2 \sum_{k=0}^{\infty} \zeta(2k) \left(\frac{z}{\pi}\right)^{2k} \qquad (|z| < \pi), \tag{4.2}$$

the result (4.1) is obviously equivalent to the series representation (cf. Dąbrowski [6, p. 202]; see also Chen and Srivastava [4, p. 191, Equation (3.19)]):

$$\zeta(3) = \frac{2\pi^2}{7} \left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} \right). \tag{4.3}$$

Moreover, by integrating by parts, it is easily seen that

$$\int_0^{\pi/2} z^2 \cot z \, dz = -2 \int_0^{\pi/2} z \log \sin z \, dz, \tag{4.4}$$

so that the result (4.1) is equivalent also to the integral representation:

$$\zeta(3) = \frac{2\pi^2}{7} \log 2 + \frac{16}{7} \int_0^{\pi/2} z \log \sin z \, dz, \tag{4.5}$$

which was proven in the aforementioned 1772 paper by Euler (cf., e.g., Ayoub [2, p. 1084]).

Next, since

$$i \cot iz = \coth z = \frac{2}{e^{2z}-1} + 1 \qquad (i := \sqrt{-1}), \tag{4.6}$$

by replacing z in the known expansion (4.2) by $\frac{1}{2} i\pi z$, it is easily seen that (cf., e.g., Koblitz [14, p. 25]; see also Erdélyi *et al.* [7, p. 51, Equation 1.20(1)])

$$\frac{\pi z}{e^{\pi z}-1} + \frac{\pi z}{2} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\zeta(2k)}{2^{2k-1}} z^{2k} \qquad (|z| < 2). \tag{4.7}$$

By setting $z = it$ in (4.7), multiplying both sides by t^{m-1} $(m \in \mathbb{N})$, and then integrating the resulting equation from $t = 0$ to $t = \tau$ $(0 < \tau < 2)$, Srivastava [25] derived the following series representations for $\zeta(2n+1)$ (see also Srivastava *et al.* [26]):

$$\begin{aligned} \zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!(2^{2n+1}-1)} \\ & \cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n}{2j} \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \right. \\ & \left. + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+n)2^{2k}} \right] \qquad (n \in \mathbb{N}) \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n+1)!(2^{2n}-1)} \\ & \cdot \left[\log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n+1}{2j} \frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \right. \\ & \left. + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{\left(k+n+\frac{1}{2}\right) 2^{2k}} \right] \qquad (n \in \mathbb{N}). \end{aligned} \tag{4.9}$$

For $n = 1$, (4.9) immediately reduces to the following series representation for $\zeta(3)$:

$$\zeta(3) = \frac{2\pi^2}{9} \left(\log 2 + 2 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+3)2^{2k}} \right), \tag{4.10}$$

which was proven *independently* by (among others) Glasser [10, p. 446, Equation (12)], Zhang and Williams [33, p. 1585, Equation (2.13)], and Dąbrowski

[6, p. 206] (see also Chen and Srivastava [4, p. 183, Equation (2.15)]). And a special case of (4.8) when $n = 1$ yields (cf. Dąbrowski [6, p. 202]; see also Chen and Srivastava [4, p. 191, Equation (3.19)])

$$\zeta(3) = \frac{2\pi^2}{7}\left(\log 2 + \sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+1)2^{2k}}\right). \tag{4.11}$$

In view of the familiar sum:

$$\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+1)2^{2k}} = -\frac{1}{2}\log 2, \tag{4.12}$$

Euler's formula (1.11) is indeed a simple consequence of (4.11).

We remark in passing that an integral representation for $\zeta(2n+1)$, which is easily seen to be equivalent to the series representation (4.8), was given by Dąbrowski [6, p. 203, Equation (16)], who [6, p. 206] mentioned the existence of (but did not fully state) the series representation (4.9) as well. The series representation (4.8) is derived also in a forthcoming paper by Borwein *et al.* (cf. [3, Equation (57)]).

By suitably combining the series occurring in (4.3), (4.10), and (4.12), it is not difficult to derive several other series representations for $\zeta(3)$, which are analogous to Euler's formula (1.11). More generally, since

$$\begin{aligned}&\frac{\lambda k^2 + \mu k + \nu}{(2k+2n-1)(2k+2n)(2k+2n+1)}\\&= \frac{\mathcal{A}}{2k+2n-1} + \frac{\mathcal{B}}{2k+2n} + \frac{\mathcal{C}}{2k+2n+1},\end{aligned} \tag{4.13}$$

where, for convenience,

$$\mathcal{A} = \mathcal{A}_n(\lambda,\mu,\nu) := \frac{1}{2}\left[\lambda n^2 - (\lambda+\mu)n + \frac{1}{4}(\lambda + 2\mu + 4\nu)\right], \tag{4.14}$$

$$\mathcal{B} = \mathcal{B}_n(\lambda,\mu,\nu) := -(\lambda n^2 - \mu n + \nu), \tag{4.15}$$

and

$$\mathcal{C} = \mathcal{C}_n(\lambda,\mu,\nu) := \frac{1}{2}\left[\lambda n^2 + (\lambda-\mu)n + \frac{1}{4}(\lambda - 2\mu + 4\nu)\right], \tag{4.16}$$

by applying (4.8), (4.9), and another result (proven by Srivastava [25, p. 341, Equation (3.17)]):

$$\begin{aligned}&\sum_{j=1}^{n}(-1)^{j-1}\binom{2n+1}{2j}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\\&= \log 2 + \sum_{k=0}^{\infty}\frac{\zeta(2k)}{\left(k+n+\frac{1}{2}\right)2^{2k}} \qquad (n \in \mathbb{N}_0),\end{aligned} \tag{4.17}$$

with n replaced by $n-1$, Srivastava [25] derived the following unification of a large number of known (or new) series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$), including (for example) Euler's formula (1.11):

$$
\begin{aligned}
\zeta(2n+1) = {} & \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\{(2^{2n+1}-1)\mathcal{B}+(2n+1)(2^{2n}-1)\mathcal{C}\}} \\
& \cdot \left[\frac{1}{4}\lambda \log 2 + \sum_{j=1}^{n-1} (-1)^j \binom{2n-1}{2j-2} \right. \\
& \cdot \left\{ 2j(2j-1)\mathcal{A} + [\lambda(4n-1)-2\mu]nj + \lambda n \left(n + \frac{1}{2} \right) \right\} \\
& \cdot \frac{(2j-2)!(2^{2j}-1)}{(2\pi)^{2j}} \zeta(2j+1) \\
& \left. + \sum_{k=0}^{\infty} \frac{(\lambda k^2 + \mu k + \nu)\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}} \right] \\
& (n \in \mathbb{N};\ \lambda, \mu, \nu \in \mathbb{C}),
\end{aligned} \tag{4.18}
$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by (4.14), (4.15), and (4.16), respectively.

Numerous other interesting series representations for $\zeta(2n+1)$, which are analogous to (4.8) and (4.9), were also given by Srivastava *et al.* [26]. For the sake of completeness, we choose to recall their results as follows:

$$
\begin{aligned}
\zeta(2n+1) = {} & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n+1)!(3^{2n}-1)} \left[\log 3 + 4 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n+1)3^{2k}} \right. \\
& + (2n+1)! \sum_{j=1}^{n-1} \frac{(-1)^j}{(2n-2j+1)!} \left(\frac{3^{2j}-1}{(2\pi)^{2j}} \right) \zeta(2j+1) \\
& - \frac{(2n+1)!}{\sqrt{3}} \sum_{j=1}^{n+1} \frac{(-1)^j}{(2n-2j+2)!} \\
& \left. \cdot \frac{2\zeta(2j, \frac{1}{3}) - (3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}} \right] \quad (n \in \mathbb{N}),
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
\zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!(3^{2n+1}-1)} \left[\log 3 + 2 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+n)3^{2k}} \right. \\
& + (2n)! \sum_{j=1}^{n-1} \frac{(-1)^j}{(2n-2j)!} \left(\frac{3^{2j}-1}{(2\pi)^{2j}} \right) \zeta(2j+1) \\
& - \frac{(2n)!}{\sqrt{3}} \sum_{j=1}^{n} \frac{(-1)^j}{(2n-2j+1)!} \\
& \left. \cdot \frac{2\zeta(2j, \frac{1}{3}) - (3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}} \right] \quad (n \in \mathbb{N}),
\end{aligned} \tag{4.20}
$$

$$
\begin{aligned}
\zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n+1)!(2^{2n}-1)} \\
& \cdot \left[\log 2 + 4 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+2n+1)4^{2k}} \right. \\
& + (2n+1)! \sum_{j=1}^{n-1} \frac{(-1)^j}{(2n-2j+1)!} \left(\frac{2^{2j}-1}{(2\pi)^{2j}} \right) \zeta(2j+1) \\
& - (2n+1)! \sum_{j=1}^{n+1} \frac{(-1)^j}{(2n-2j+2)!} \\
& \left. \cdot \frac{\zeta(2j, \frac{1}{4}) - 2^{2j-1}(2^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}} \right] \quad (n \in \mathbb{N}),
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
\zeta(2n+1) = & (-1)^{n-1} \frac{(2\pi)^{2n}}{(2n)!(2^{4n+1}+2^{2n}-1)} \\
& \cdot \left[\log 2 + 2 \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+n)4^{2k}} \right. \\
& + (2n)! \sum_{j=1}^{n-1} \frac{(-1)^j}{(2n-2j)!} \left(\frac{2^{2j}-1}{(2\pi)^{2j}} \right) \zeta(2j+1) \\
& - (2n)! \sum_{j=1}^{n} \frac{(-1)^j}{(2n-2j+1)!} \\
& \left. \cdot \frac{\zeta(2j, \frac{1}{4}) - 2^{2j-1}(2^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}} \right] \quad (n \in \mathbb{N}),
\end{aligned} \tag{4.22}
$$

$$
\begin{aligned}
\zeta(2n+1) = {} & (-1)^{n-1}\frac{(2\pi)^{2n}}{(2^{2n}-1)(3^{2n}-1)} \\
& \cdot\Bigg[-\frac{4}{(2n+1)!}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n+1)6^{2k}} \\
& + \sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j+1)!}\left(\frac{(2^{2j}-1)(3^{2j}-1)}{(2\pi)^{2j}}\right)\zeta(2j+1) \\
& + \frac{1}{\sqrt{3}}\sum_{j=1}^{n+1}\frac{(-1)^j}{(2n-2j+2)!} \\
& \cdot\frac{\zeta(2j,\frac{1}{3})+\zeta(2j,\frac{1}{6})-2^{2j-1}(3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\Bigg] \quad (n\in\mathbb{N}),
\end{aligned}
\tag{4.23}
$$

and

$$
\begin{aligned}
\zeta(2n+1) = {} & (-1)^{n-1}\frac{(2\pi)^{2n}}{2^{2n}+3^{2n}+6^{2n}-1}\Bigg[\frac{2}{(2n)!}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+n)6^{2k}} \\
& - \sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j)!}\left(\frac{(2^{2j}-1)(3^{2j}-1)}{(2\pi)^{2j}}\right)\zeta(2j+1) \\
& - \frac{1}{\sqrt{3}}\sum_{j=1}^{n}\frac{(-1)^j}{(2n-2j+1)!} \\
& \cdot\frac{\zeta(2j,\frac{1}{3})+\zeta(2j,\frac{1}{6})-2^{2j-1}(3^{2j}-1)\zeta(2j)}{(2\pi)^{2j-1}}\Bigg] \quad (n\in\mathbb{N}).
\end{aligned}
\tag{4.24}
$$

It is not difficult to derive further series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) by appropriately combining two or more of the results (4.8), (4.9), (4.17), and (4.19) to (4.24). Thus we can arrive at several general results analogous (for example) to (4.18).

5. Some Interesting Deductions

For $\lambda = 0$, the series representation (4.18) simplifies to the form:

$$
\begin{aligned}
&\zeta(2n+1)\\
&= \frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\{(2^{2n+1}-1)(\mu n-\nu)-(2^{2n}-1)(n+\frac{1}{2})[\mu(n+\frac{1}{2})-\nu]\}}\\
&\quad\cdot\left[\sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-2}\left\{j(2j-1)\left[\nu-\mu\left(n-\frac{1}{2}\right)\right]-2\mu nj\right\}\right.\\
&\quad\cdot\frac{(2j-2)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\\
&\quad\left.+\sum_{k=0}^{\infty}\frac{(\mu k+\nu)\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}}\right]\ (n\in\mathbb{N};\mu,\nu\in\mathbb{C}).
\end{aligned}\tag{5.1}
$$

Furthermore, by setting

$$\lambda=\mu=0 \quad \text{and} \quad \nu=1$$

in (4.18) or (alternatively) by setting

$$\mu=0 \quad \text{and} \quad \nu=1$$

in (5.1), we immediately obtain the series representation:

$$
\begin{aligned}
\zeta(2n+1) = &\frac{(-1)^{n-1}(2\pi)^{2n}}{(2n)!\{2^{2n}(2n-3)-2n+1\}}\\
&\cdot\left[\sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-2}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\right.\\
&\left.+2\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(k+n)(2k+2n+1)2^{2k}}\right]\quad(n\in\mathbb{N}),
\end{aligned}\tag{5.2}
$$

which, in the special case when $n=1$, was given by Chen and Srivastava [4, p. 189, Equation (2.45)].

Of the three representations (4.18), (5.1), and (5.2) for $\zeta(2n+1)$ $(n\in\mathbb{N})$, the infinite series in (5.2) converges most rapidly.

For various other suitable special values of the parameters $\lambda,\mu,$ and ν, we can easily deduce from (4.18) and (5.1) several known (or new) series representations for $\zeta(2n+1)$ $(n\in\mathbb{N})$. For example, if we set

$$\mu=2 \quad \text{and} \quad \nu=2n+1$$

in the series representation (5.1), we shall obtain

$$\begin{aligned}\zeta(2n+1) = &(-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!(2^{2n+1}-1)}\left[\sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-1}\right.\\ &\cdot\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\\ &\left.-\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(k+n)2^{2k}}\right] \quad (n\in\mathbb{N}),\end{aligned} \tag{5.3}$$

which, in the special case when $n = 1$, immediately yields Euler's formula (1.11).

The following additional series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$, which are analogous to (5.3), can also be deduced similarly from (5.1):

$$\begin{aligned}\zeta(2n+1) = &(-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!\{(2n-1)2^{2n}-2n\}}\\ &\cdot\left[2n\sum_{j=1}^{n-1}(-1)^j\binom{2n-1}{2j-2}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\right.\\ &\left.-\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(k+n)(2k+2n+1)2^{2k}}\right] \quad (n\in\mathbb{N})\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\zeta(2n+1) = &(-1)^{n-1}\frac{(2\pi)^{2n}}{(2n+1)!(2^{2n}-1)}\\ &\cdot\left[\sum_{j=1}^{n-1}(-1)^j\left(\frac{4nj-2j+1}{2j-1}\right)\binom{2n-1}{2j-2}\right.\\ &\cdot\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\\ &\left.-4\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(2k+2n+1)2^{2k}}\right] \quad (n\in\mathbb{N}).\end{aligned} \tag{5.5}$$

The special case of each of the last two series representations (5.4) and (5.5) when $n = 1$ was given by Zhang and Williams [33, p. 1586].

Next, with a view to *further* improving the rate of convergence in the reasonably rapidly convergent series representation (5.2), we observe that

$$
\begin{aligned}
&\frac{1}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)} \\
&= \frac{1}{6}\left(\frac{1}{2k+2n-1} - \frac{1}{2k+2n+2}\right) - \frac{1}{2}\,\frac{1}{(2k+2n)(2k+2n+1)}.
\end{aligned}
\tag{5.6}
$$

Thus, by applying the series representations (4.17) *with* n replaced by $n-1$, (4.8) *with* n replaced by $n+1$, and (5.4), we obtain

$$
\begin{aligned}
&\zeta(2n+3) \\
&= \frac{2\pi^2\{2^{2n+2}+n(2n-3)(2^{2n}-1)-1\}}{(n+1)(2n+1)(2^{2n+3}-1)}\zeta(2n+1) \\
&\quad +(-1)^{n-1}\frac{(2\pi)^{2n+2}}{(2n+2)!(2^{2n+3}-1)} \\
&\quad \cdot\left[\sum_{j=1}^{n-1}(-1)^j\left\{\binom{2n-1}{2j}-\binom{2n+2}{2j}\right.\right. \\
&\quad \left.+6n\binom{2n-1}{2j-2}\right\}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\ \zeta(2j+1) \\
&\quad \left.+12\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)2^{2k}}\right] \\
&\quad (n\in\mathbb{N}),
\end{aligned}
\tag{5.7}
$$

where the series converges faster than that in (5.2).

In its special case when $n=1$, (5.7) readily yields the following improved version of the series representation derivable from (5.2) for $n=2$ (cf. [33, p. 1590, Equation (3.14)]):

$$
\zeta(5) = \frac{4\pi^2}{31}\zeta(3) + \frac{8\pi^4}{31}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)2^{2k}},
\tag{5.8}
$$

in which $\zeta(3)$ can be replaced by its known value $-4\pi^2\zeta'(-2)$ given by (2.9) for $n=1$.

Yet another rapidly convergent series representation for $\zeta(2n+3)$ ($n\in\mathbb{N}$), analogous to (5.7), can be derived by means of the identity:

$$
\begin{aligned}
&\frac{1}{(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)} \\
&= \frac{1}{6}\left(\frac{1}{2k+2n} - \frac{1}{2k+2n+3}\right) - \frac{1}{2}\,\frac{1}{(2k+2n+1)(2k+2n+2)},
\end{aligned}
\tag{5.9}
$$

together with our series representations (4.8), (4.9) *with* n replaced by $n+1$, and (5.3) *with* n replaced by $n+1$. We thus obtain the series representation:

$$\begin{aligned}
&\zeta(2n+3)\\
&= \frac{2\pi^2\{\frac{1}{3}(2n+1)(2n^2-4n+3)(2^{2n}-1)-2^{2n+1}+1\}}{(n+1)(2n+1)\{(2n-3)2^{2n+2}-2n\}}\zeta(2n+1)\\
&\quad +(-1)^{n-1}\frac{(2\pi)^{2n+2}}{(2n+2)!\{(2n-3)2^{2n+2}-2n\}}\left[\sum_{j=1}^{n-1}(-1)^j\left\{\binom{2n}{2j}\right.\right.\\
&\quad \left.-\binom{2n+3}{2j}+3\binom{2n+1}{2j-1}\right\}\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\\
&\quad \left.+12\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)2^{2k}}\right]\\
&\quad (n\in\mathbb{N}),
\end{aligned} \tag{5.10}$$

which, in the special case when $n=1$, yields

$$\zeta(5)=\frac{2\pi^2}{27}\zeta(3)-\frac{4\pi^4}{9}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+2)(2k+3)(2k+4)(2k+5)2^{2k}}, \tag{5.11}$$

where the series obviously converges faster than that derivable from (5.2) for $n=2$.

Lastly, by applying the identity:

$$\begin{aligned}
&\frac{1}{2k(2k+2n-1)(2k+2n)(2k+2n+1)}\\
&= \frac{1}{2n(2n-1)(2n+1)}\frac{1}{2k}-\frac{1}{2(2n-1)}\frac{1}{2k+2n-1}\\
&\quad +\frac{1}{2n}\frac{1}{2k+2n}-\frac{1}{2(2n+1)}\frac{1}{2k+2n+1}
\end{aligned} \tag{5.12}$$

in conjunction with the series representations (4.17) *with* n replaced by $n-1$, (4.8), (4.9), and the known result (cf., e.g., [12, p. 356, Entry (54.5.3)]):

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}t^{2k}=\log[\pi t\csc(\pi t)] \tag{5.13}$$

with $t=\frac{1}{2}$, we arrive at the following series representation for $\zeta(2n+1)$

$(n \in \mathbb{N})$:

$$\begin{aligned}\zeta(2n+1) = {} & (-1)^{n-1}\frac{(2\pi)^{2n}}{2\cdot(2n-1)!\{n-1-(n-2)2^{2n}\}} \\ & \cdot\left[\frac{12n^2-1}{2n^2(4n^2-1)^2} - \frac{\log\pi}{n(4n^2-1)}\right. \\ & - \sum_{j=2}^{n-1}(-1)^j\binom{2n-2}{2j-3}\frac{(2j-1)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1) \\ & \left. + \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+n)(2k+2n-1)(2k+2n+1)2^{2k}}\right] \quad (n \in \mathbb{N}),\end{aligned} \tag{5.14}$$

where we have also applied the fact that $\zeta(0) = -\frac{1}{2}$.

For $n = 1$, (5.14) reduces immediately to Wilton's formula (cf. Wilton [31, p. 92] and Hansen [12, p. 357, Entry (54.5.9)]; see also Chen and Srivastava [4, p. 181, Equation (2.1)]):

$$\zeta(3) = \frac{\pi^2}{2}\left(\frac{11}{18} - \frac{1}{3}\log\pi + \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+1)(2k+1)(2k+3)2^{2k}}\right). \tag{5.15}$$

Furthermore, in its special case when $n = 2$, (5.14) would yield the following interesting companion of the series representations (5.8) *and* (5.11):

$$\zeta(5) = \frac{2\pi^4}{45}\left(\log\pi - \frac{47}{60} - 30\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+2)(2k+3)(2k+5)2^{2k}}\right), \tag{5.16}$$

which does *not* contain a term involving $\zeta(3)$ on the right-hand side.

By eliminating $\zeta(2n+3)$ between the results (5.7) and (5.10), we can obtain a series representation for $\zeta(2n+1)$ $(n \in \mathbb{N})$, which would converge as rapidly as the series in (5.14). We thus find that (cf. Srivastava [25, pp. 548-549, Equation (3.47)])

$$
\begin{aligned}
&(5.17)\\
&\zeta(2n+1)\\
&= (-1)^{n-1}\frac{(2\pi)^{2n}}{(2n)!\Delta_n}\left[\sum_{j=1}^{n-1}(-1)^j\right.\\
&\quad\cdot\left(\{(2n-3)2^{2n+2}-2n\}\left\{\binom{2n-1}{2j}-\binom{2n+2}{2j}\right.\right.\\
&\quad\left.+6n\binom{2n-1}{2j-2}\right\}-(2^{2n+3}-1)\left\{\binom{2n}{2j}-\binom{2n+3}{2j}\right.\\
&\quad\left.\left.+3\binom{2n+1}{2j-1}\right\}\right)\frac{(2j)!(2^{2j}-1)}{(2\pi)^{2j}}\zeta(2j+1)\\
&\quad\left.+12\sum_{k=0}^{\infty}\frac{(\xi_n k+\eta_n)\,\zeta(2k)}{(2k+2n-1)(2k+2n)(2k+2n+1)(2k+2n+2)(2k+2n+3)2^{2k}}\right]\\
&\quad(n\in\mathbb{N}),
\end{aligned}
$$

where, for convenience,

$$
\begin{aligned}
\Delta_n :=\,&(2^{2n+3}-1)\left\{\frac{1}{3}(2n+1)(2n^2-4n+3)(2^{2n}-1)-2^{2n+1}+1\right\}\\
&-\{(2n-3)2^{2n+2}-2n\}\{2^{2n+2}+n(2n-3)(2^{2n}-1)-1\},
\end{aligned}
\tag{5.18}
$$

$$
\xi_n := 2\left\{(2n-5)2^{2n+2}-2n+1\right\}, \tag{5.19}
$$

and

$$
\eta_n := (4n^2-4n-7)2^{2n+2}-(2n+1)^2. \tag{5.20}
$$

In its special case when $n=1$, (5.17) yields the following (rather curious) series representation:

$$
\zeta(3) = -\frac{6\pi^2}{23}\sum_{k=0}^{\infty}\frac{(98k+121)\zeta(2k)}{(2k+1)(2k+2)(2k+3)(2k+4)(2k+5)2^{2k}}, \tag{5.21}
$$

where the series obviously converges much more rapidly than that in *each* of the celebrated results (1.11) and (1.12).

6. Symbolic and Numerical Computations

In this concluding section, we choose to summarize below the results of our symbolic and numerical computations with the series in (5.21) using *Mathematica* (Version 4.0) for Linux:

$$\begin{aligned}
\text{In}[1] &:= (98k+121)\text{Zeta}\,[2k]/\Big((2k+1)(2k+2)(2k+3) \\
&\quad \cdot(2k+4)(2k+5)2^{\urcorner}(2k)\Big) \\
\text{Out}[1] &= \frac{(121+98k)\ \text{Zeta}\ [2k]}{2^{2k}(1+2k)(2+2k)(3+2k)(4+2k)(5+2k)} \\
\text{In}[2] &:= \text{Sum}[\%,\ \{k,1,\ \text{Infinity}\}]\ //\ \text{Simplify} \\
\text{Out}[2] &= \frac{121}{240} - \frac{23\ \text{Zeta}[3]}{6\text{Pi}^2} \\
\text{In}[3] &:= \text{N}[\%] \\
\text{Out}[3] &= 0.0372903 \\
\text{In}[4] &:= \text{Sum}\ [\text{N}[\%1]\ //\ \text{Evaluate},\ \{k,1,50\}] \\
\text{Out}[4] &= 0.0372903 \\
\text{In}[5] &:= \text{N Sum}\ [\%1\ //\ \text{Evaluate},\ \{k,1,\ \text{Infinity}\}] \\
\text{Out}[5] &= 0.0372903
\end{aligned}$$

Since $\zeta(0) = -\frac{1}{2}$, Out[2] evidently validates the series representation (5.21) symbolically. Furthermore, our numerical computations in Out[3], Out[4], and Out[5], together, exhibit the fact that only 50 terms ($k=1$ to $k=50$) of the series in (5.21) can produce an accuracy of seven decimal places.

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References

1. R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, in: *Journées Arithmétiques de Luminy* (Colloq. Internat. CNRS, Centre Univ. Luminy, Luminy, 1978), *Astérisque* **61**, Soc. Math. France, Paris, 1979, pp. 11-13.

2. R. Ayoub, Euler and the Zeta function, *Amer. Math. Monthly* **81** (1974), 1067-1086.

3. J. M. Borwein, D. M. Bradley and R. E. Crandall, Computational strategies for the Riemann Zeta function, *J. Comput. Appl. Math.*, to appear.

4. M.-P. Chen and H. M. Srivastava, Some families of series representations for the Riemann $\zeta(3)$, *Results Math.* **33** (1998), 179-197.

5. D. Cvijović and J. Klinowski, New rapidly convergent series representations for $\zeta(2n+1)$, *Proc. Amer. Math. Soc.* **125** (1997), 1263-1271.

6. A. Dąbrowski, A note on the values of the Riemann Zeta function at positive odd integers, *Nieuw Arch. Wisk.* (4) **14** (1996), 199-207.

7. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill, New York, 1953.

8. J. A. Ewell, A new series representation for $\zeta(3)$, *Amer. Math. Monthly* **97** (1990), 219-220.

9. J. A. Ewell, On the Zeta function values $\zeta(2k+1)$, $k = 1, 2, ...$, *Rocky Mountain J. Math.* **25** (1995), 1003-1012.

10. M. L. Glasser, Some integrals of the arctangent function, *Math. Comput.* **22** (1968), 445-447.

11. R. W. Gosper, Jr., A calculus of series rearrangements, in: *Algorithms and Complexity: New Directions and Recent Results*, J. F. Traub, ed., Academic Press, New York, 1976, pp. 121-151.

12. E. R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.

13. M. M. Hjortnaes, Overføring av rekken $\sum_{k=1}^{\infty}(1/k^3)$ til et bestemt integral, in: *Proceedings of the Twelfth Scandinavian Mathematical Congress* (Lund; August 10-15, 1953), Scandinavian Mathematical Society, Lund, 1954, pp. 211-213.

14. N. Koblitz, *p-Adic Numbers, p-Adic Analysis, and Zeta-Functions*, Graduate Texts in Mathematics, Vol. **58**, Springer-Verlag, New York, 1977.

15. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd enlarged ed., Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtingung der Anwendungsgebiete, Bd. **52**, Springer-Verlag, New York, 1966.

16. C. Nash and D. O'Connor, Ray-Singer torsion, topological field theories and the Riemann Zeta function at $s = 3$, in: *Low-Dimensional Topology and Quantum Field Theory* (Proceedings of a NATO Advanced Research Workshop held at the Isaac Newton Institute at Cambridge, U.K.; September 6-12, 1992), H. Osborn, ed., Plenum Press, New York and London, 1993, pp. 279-288.

17. C. Nash and D. O'Connor, Determinants of Laplacians, the Ray-Singer torsion on lens spaces and the Riemann Zeta function, *J. Math. Phys.* **36** (1995), 1462-1505.

18. V. Ramaswami, Notes on Riemann's ζ-function, *J. London Math. Soc.* **9** (1934), 165-169.

19. J. D. Shallit and K. Zikan, A theorem of Goldbach, *Amer. Math. Monthly* **93** (1986), 402-403.

20. H. M. Srivastava, A unified presentation of certain classes of series of the Riemann Zeta function, *Riv. Mat. Univ. Parma* (4) **14** (1988), 1-23.

21. H. M. Srivastava, Sums of certain series of the Riemann Zeta function, *J. Math. Anal. Appl.* **134** (1988), 129-140.

22. H. M. Srivastava, Certain families of rapidly convergent series representations for $\zeta(2n+1)$, *Math. Sci. Res. Hot-Line* **1** (6) (1997), 1-6.

23. H. M. Srivastava, Further series representations for $\zeta(2n+1)$, *Appl. Math. Comput.* **97** (1998), 1-15.

24. H. M. Srivastava, Some rapidly converging series for $\zeta(2n+1)$, *Proc. Amer. Math. Soc.* **127** (1999), 385-396.

25. H. M. Srivastava, Some simple algorithms for the evaluations and representations of the Riemann Zeta function at positive integer arguments, *J. Math. Anal. Appl.* **246** (2000), 331-351.

26. H. M. Srivastava, M. L. Glasser and V. S. Adamchik, Some definite integrals associated with the Riemann Zeta function, *Z. Anal. Anwendungen* **19** (2000), 831-846.

27. H. M. Srivastava and H. Tsumura, A certain class of rapidly convergent series representations for $\zeta(2n+1)$, *J. Comput. Appl. Math.* **118** (2000), 323-335.

28. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University (Clarendon) Press, Oxford and London, 1951; 2nd ed. (revised by D.R. Heath-Brown), 1986.

29. F. G. Tricomi, Sulla somma delle inverse delle terze e quinte potenze dei numeri naturali, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **47** (1969), 16-18.

30. H. Tsumura, On evaluation of the Dirichlet series at positive integers by q-calculation, *J. Number Theory* **48** (1994), 383-391.

31. J. R. Wilton, A proof of Burnside's formula for $\log\Gamma(x+1)$ and certain allied properties of Riemann's ζ-function, *Messenger Math.* **52** (1922-1923), 90-93.

32. E. Witten, On quantum gauge theories in two dimensions, *Comm. Math. Phys.* **141** (1991), 153-209.

33. N.-Y. Zhang and K. S. Williams, Some series representations of $\zeta(2n+1)$, *Rocky Mountain J. Math.* **23** (1993), 1581-1592.

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
E-mail: harimsri@math.uvic.ca