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Some fixed point results for multi-valued mappings in b-metric spaces

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Abstract
The aim of this paper is to establish some fixed point theorems for set-valued mappings in the context of b-metric spaces. The proposed theorems expand and generalize several well-known comparable results in the literature. An example is also given to support our main result.
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1 Introduction and preliminaries

The notion of metric space, introduced by Fréchet in 1906, is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many different ways. An incomplete list of the results of such an attempt is the following: quasi-metric space, symmetric space, partial metric space, cone metric space, G-metric space, probabilistic metric space, fuzzy metric space and so on.

In this paper, we pay attention to the concept of b-metric space. The notion of b-metric space was introduced by Czerwik [1] in 1993 to extend the notion of metric space. In this interesting paper, Czerwik [1] observed a characterization of the celebrated Banach fixed point theorem [2] in the context of complete b-metric spaces. Following this pioneer paper, several authors have devoted their attention to research the properties of a b-metric space and have reported the existence and uniqueness of fixed points of various operators in the setting of b-metric spaces (see, *e.g.*, [3–12] and some reference therein).

The aim of this paper is to generalize various known results proved by Kikkawa and Suzuki [13], Mot and Petrusel [14], Dhompongsa and Yingtaweesittikul [15] to the case of b-metric spaces and give an example to illustrate our main results.

Definition 1 Let X be any nonempty set. An element x in X is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X.

Let (X, d) be a metric space. Let $CB(X)$ be the collection of all nonempty, closed and bounded subsets of X. In the sequel, we use the following notations:

$$d(a, A) = \inf\{d(a, x) : x \in A\},$$

$$\delta(A,B) = \sup\{d(a,B) : a \in A\},$$
$$\delta(B,A) = \sup\{d(b,A) : b \in B\}$$

and

$$H(A,B) = \max\{\delta(A,B), \delta(B,A)\}$$

for any $A, B \in CB(X)$.

Notice that H is called the Hausdorff metric induced by the metric d.

We start with recalling some basic definitions and lemmas on b-metric spaces. The definition of a b-metric space is given by Czerwik [1] (see also [4, 5]) as follows.

Definition 2 Let X be a nonempty set X and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is called a b-metric provided that, for all $x, y, z \in X$,

(bms_1) $d(x,x) = 0$,
(bms_2) $d(x,y) = d(y,x)$,
(bms_3) $d(x,z) \leq s(d(x,y) + d(y,z))$.

Note that a (usual) metric space is evidently a b-metric space. However, Czerwik [1, 4] showed that a b-metric on X need not be a metric on X (see also [5, 16, 17]). The following example shows that a b-metric on X need not be a metric on X.

Example 1 (*cf.* [18]) Let $X = \{a, b, c\}$ and $d(a,c) = d(2,c) = m \geq 2$, $d(c,b) = d(b,a) = d(b,c) = d(c,b) = 1$, and $d(a,a) = d(b,b) = d(c,c) = 0$. Then $d(x,y) \leq \frac{m}{2}[d(x,z) + d(z,y)]$ for all $x, y, z \in X$. If $m > 2$, then the ordinary triangle inequality does not hold.

Let (X, d) be a b-metric space. We cite the following lemmas from Czerwik [1, 4, 5] and Singh *et al.* [18].

Lemma 1 *Let (X, d) be a b-metric space. For any $A, B \in CB(X)$ and any $x, y \in X$, we have the following*:
(1) $d(x,B) \leq d(x,b)$ *for any* $b \in B$,
(2) $d(x,B) \leq H(A,B)$,
(3) $d(x,A) \leq s(d(x,y) + d(y,B))$.

Remark 1 Let (X, d) be a b-metric space and A be a nonempty set in (X, d) and $x \in A$, then we have

$$d(x,A) = 0 \quad \Leftrightarrow \quad x \in \overline{A} = A,$$

where $\overline{A}$ denotes the closure of A with respect to the induced metric d. Note that A is closed in (X, d) if and only if $\overline{A} = A$.

Remark 2 The mapping d in a b-metric space (X, d) need not be jointly continuous (see, *e.g.*, [19, 20]).

Lemma 2 *Let A and B be nonempty closed and bounded subsets of a b-metric space (X,d) and $q>1$. Then, for all $a\in A$, there exists $b\in B$ such that $d(a,b)\leq qH(A,B)$.*

Lemma 3 *Let (X,d) be a b-metric space. Let A and B be in $CB(X)$. Then, for each $\alpha>0$ and for all $b\in B$, there exists $a\in A$ such that $d(a,b)\leq H(A,B)+\alpha$.*

The following result was proved by Aydi *et al.* in [21].

Theorem 1 *Let (X,d) be a complete b-metric space and let $F:X\to CB(X)$ be a multi-valued mapping such that for all $x,y\in X$,*

$$H(Fx,Fy)\leq rM(x,y), \tag{1.1}$$

where $0\leq r<\frac{1}{s^2+s}<1$ and

$$M(x,y)=\max\bigl\{d(x,y),d(x,Fx),d(y,Fy),d(x,Fy),d(y,Fx)\bigr\}.$$

Then F has a fixed point in X, that is, there exists $u\in X$ such that $u\in Fu$.

The following preliminary lemma will play a crucial role in the sequel.

Lemma 4 [22] *Let (X,d) be a complete b-metric space and let $\{x_n\}$ be a sequence in X such that $d(x_{n+1},x_{n+2})\leq\beta d(x_n,x_{n+1})$ for all $n=0,1,2,\ldots$, where $0\leq\beta<1$. Then $\{x_n\}$ is a Cauchy sequence in X provided that $s\beta<1$.*

2 Main results

In this section we state and prove our main results. Inspired the results of Aydi *et al.* [21], we establish a Kikkawa and Suzuki type fixed point theorem in the framework of *b*-metric spaces as follows.

Theorem 2 *Let (X,d) be a complete b-metric space and let $F:X\to CB(X)$ be a multi-valued mapping. Then, for $s\geq1$, define a strictly decreasing function σ from $[0,1)$ onto $(\frac{1}{2},1]$ by $\sigma(r)=\frac{1}{(1+sr)}$, where $r<\frac{1}{s^2+s}<1$, such that*

$$\sigma(r)d(x,Fx)\leq sd(x,y)\quad\Longrightarrow\quad H(Fx,Fy)\leq rd(x,y) \tag{2.1}$$

for all $x,y\in X$. Then there exists $u\in X$ such that $u\in Fu$.

Proof If $d(x,y)=0$, then by (2.1) we deduce that $x=y$ is a fixed point of F. Hence the proof is completed. Thus, throughout the proof, we assume that $d(x,y)>0$ for all $x,y\in X$. Take

$$\alpha=\frac{1}{2}\left(\frac{1}{s^2+s}-r\right)$$

and

$$\beta=r+\alpha=\frac{1}{2}\left(\frac{1}{s^2+s}+r\right).$$

Due to the assumption $r < \frac{1}{s^2+s}$, we conclude that $\alpha > 0$ and $0 < \beta < 1$. Let $x_0 \in X$ be arbitrary and $x_1 \in Fx_0$. Owing to (2.1), we have

$$\sigma(r)d(x_0, Fx_0) \leq \sigma(r)d(x_0, x_1) \leq sd(x_0, x_1),$$

which yields that

$$H(Fx_0, Fx_1) \leq rd(x_0, x_1).$$

By Lemma 3, there exists $x_2 \in Fx_1$. Now, by using the previous inequality, we obtain

$$d(x_1, x_2) \leq H(Fx_0, Fx_1) + \alpha d(x_0, x_1) \leq rd(x_0, x_1) + \alpha d(x_0, x_1) = \beta d(x_0, x_1),$$

where $\beta = r + \alpha$. On the other hand, we have

$$\begin{aligned}\sigma(r)d(x_1, Fx_1) &\leq \sigma(r)d(x_1, x_2)\\ &\leq d(x_1, x_2)\\ &\leq sd(x_1, x_2).\end{aligned}$$

Thus, we derive that

$$H(Fx_1, Fx_2) \leq rd(x_1, x_2)$$

by condition (2.1). Employing Lemma 3 again, there exists $x_3 \in Fx_2$ such that

$$d(x_2, x_3) \leq H(Fx_1, Fx_2) \leq rd(x_1, x_2) + \alpha d(x_1, x_2) \leq \beta d(x_1, x_2).$$

Continuing in this way, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Fx_n$ and

$$d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1) \tag{2.2}$$

for all $n \in \mathbb{N}$. Having in mind $s \geq 1$ together with $\beta = \frac{1}{2}(\frac{1}{s^2+s} + r)$ and $r < \frac{1}{s^2+s}$, one can easily obtain that $s\beta < 1$. Taking Lemma 4 into account, we conclude that the sequence $\{x_n\}$ is a Cauchy sequence in (X, d). Since the b-metric space (X, d) is complete, there exists $u \in X$ such that $\lim_{n\to+\infty} d(x_n, u) = 0$. Due to fact that $\beta < 1$, we can easily observe that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0,$$

by using inequality (2.2). Notice that the condition (bms_3) yields

$$d(x_{n+1}, u) \leq s\big(d(x_{n+1}, x_n) + d(x_n, u)\big).$$

Consequently, we have

$$\lim_{n\to+\infty} d(x_{n+1}, u) = 0.$$

In what follows, we shall show that

$$d(u,Fx) \leq srd(u,x)$$

for all $x \in X \setminus \{u\}$. Since $d(x_n,u) \to 0$ as $n \to +\infty$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n,u) \leq \frac{1}{3}d(u,x)$$

for all $n \in \mathbb{N}$ with $n \geq n_0$. Then we have

$$\begin{aligned}\sigma(r)d(x_n,Fx_n) \leq d(x_n,Fx_n) \leq d(x_n,x_{n+1}) &\leq s\bigl(d(x_n,u)+d(u,x_{n+1})\bigr)\\ &\leq \frac{2s}{3}d(u,x) \leq sd(u,x)-sd(x_n,u)\\ &\leq sd(x_n,x),\end{aligned}$$

and hence by assumption (2.1) we get $H(Fx_n,Fx) \leq rd(x_n,x)$. Further, we have

$$\begin{aligned}d(u,Fx) &\leq s\bigl(d(u,x_{n+1})+d(x_{n+1},Fx)\bigr)\\ &\leq s\bigl(d(u,x_{n+1})+H(Fx_n,Fx)\bigr)\\ &\leq s\bigl(d(u,x_{n+1})+rd(x_n,x)\bigr).\end{aligned}$$

Letting $n \to +\infty$ in the inequality above, we obtain

$$d(u,Fx) \leq rsd(u,x) \tag{2.3}$$

for all $x \in X \setminus \{u\}$.

Next, we prove that

$$H(Fx,Fu) \leq rd(x,u)$$

for all $x \in X$ with $x \neq u$. For all $n \in \mathbb{N}$, we choose $v_n \in Fx$ such that

$$d(u,v_n) \leq d(u,Fx)+\frac{1}{n}d(x,u).$$

Then, using (2.3) and the previous inequality, we get

$$\begin{aligned}d(x,Fx) \leq d(x,v_n) &\leq s\bigl(d(x,u)+d(u,v_n)\bigr)\\ &\leq s\left(d(x,u)+d(u,Fx)+\frac{1}{n}d(x,u)\right)\\ &\leq \left(d(x,u)+srd(u,x)+\frac{1}{n}d(x,u)\right)\\ &= s\left(1+sr+\frac{1}{n}\right)d(x,u).\end{aligned}$$

Hence, for all $n \in \mathbb{N}$, we obtain $\sigma(r)d(x,Fx) \leq sd(x,u)$. So, we have

$$H(Fx,Fu) \leq rd(x,u).$$

Finally, if for some $n \in \mathbb{N}$ we have $x_n = x_{n+1}$, then x_n is a fixed point of F. Consequently, throughout the proof we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. This implies that there exists an infinite subset J of $\mathbb{N}$ such that $x_n \neq u$ for all $n \in J$. By Lemma 1, we have

$$\begin{aligned} d(u,Fu) &\leq s\bigl(d(u,x_{n+1}) + d(x_{n+1},Fu)\bigr) \\ &\leq s\bigl(d(u,x_{n+1}) + H(Fx_n,Fu)\bigr) \\ &\leq s\bigl(d(u,x_{n+1}) + rd(x_n,u)\bigr). \end{aligned}$$

Letting $n \to +\infty$ in the inequality above, with $n \in J$, we find that

$$d(u,Fu) = 0.$$

By Remark 1, we deduce that $u \in Fu$ and hence u is a fixed point of F. □

Remark 3 Taking $s = 1$ in Theorem 2 (it corresponds to the case of metric spaces), the condition on $r < \frac{1}{2}$, $\sigma(r) = \frac{1}{1+r}$, we find Theorem 1.2 of Kikkawa and Suzuki. Hence, Theorem 2 is an extension of the result of Kikkawa *et al.* [13], which itself improves the theorem of Nadler [7].

In the case where $T : X \to X$ is a single-valued mapping on a b-metric space, we have the following corollary (it is a consequence of Theorem 2).

Corollary 1 *Let (X,d) be a complete b-metric space and let $F : X \to X$ be a single-valued mapping. Define a strictly decreasing function σ from $[0,1)$ onto $(\frac{1}{2},1]$ by $\sigma(rs) = \frac{1}{1+sr}$, $r < \frac{1}{s^2+s} < 1$ such that*

$$\sigma(rs)d(x,Fx) \leq sd(x,y) \quad \Longrightarrow \quad d(Fx,Fy) \leq rd(x,y) \tag{2.4}$$

for all $x,y \in X$. Then there exists $u \in X$ such that $u = Fu$.

Proof It follows by applying Theorem 2 and the fact that $H(Fx,Fy) = d(Fx,Fy)$. □

Remark 4 Corollary 1 implies the corresponding result of Suzuki [23] if we take $s = 1$.

The following theorem is a result of Reich type [8] as well as a generalization of Kikkawa and Suzuki type in the framework of b-metric spaces.

Theorem 3 *Let (X,d) be a complete b-metric space and let $F : X \to CB(X)$ be a multi-valued mapping. If for $s \geq 1$ there exist nonnegative numbers a, b, c with $s(a+b+c) \in [0,1)$ and $\theta = \frac{1-sb-sc}{1+sa}$ such that*

$$\theta d(x,Fx) \leq sd(x,y) \quad \Longrightarrow \quad H(Fx,Fy) \leq ad(x,y) + bd(x,Fx) + cd(y,Fy) \tag{2.5}$$

for all $x,y \in X$, then F has a fixed point.

Proof Let $x_0 \in X$ be arbitrary and $x_1 \in Fx_0$, then we have

$$\theta d(x_0, Fx_0) \leq \theta d(x_0, x_1) \leq sd(x_0, x_1).$$

By condition (2.5) we get

$$H(Fx_0, Fx_1) \leq ad(x_0, x_1) + bd(x_0, Fx_0) + cd(x_1, Fx_1).$$

Let $h \in (1, \frac{1}{s(a+b+c)})$, then by Lemma 2 there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq hH(Fx_0, Fx_1),$$

which yields

$$\begin{aligned} d(x_1, x_2) &\leq hH(Fx_0, Fx_1) \leq h\bigl(ad(x_0, x_1) + bd(x_0, Fx_0) + cd(x_1, Fx_1)\bigr) \\ &\leq h(a + b)d(x_0, x_1) + hcd(x_1, x_2) \\ &\leq \frac{h(a + b)}{1 - hc} d(x_0, x_1). \end{aligned}$$

Now, we have

$$\theta d(x_1, Fx_1) \leq \theta d(x_1, x_2) \leq sd(x_1, x_2).$$

Due to assumption (2.1), we get

$$H(Fx_1, Fx_2) \leq ad(x_1, x_2) + bd(x_1, Fx_1) + cd(x_2, Fx_2).$$

Taking Lemma 2 into account, we conclude that there exists $x_3 \in Fx_2$ such that

$$d(x_2, x_3) \leq hH(Fx_1, Fx_2).$$

Consequently, we have

$$\begin{aligned} d(x_2, x_3) &\leq hH(Fx_1, Fx_2) \leq h\bigl(ad(x_1, x_2) + bd(x_1, Fx_1) + cd(x_2, Fx_2)\bigr) \\ &\leq h(a + b)d(x_1, x_2) + hcd(x_2, x_3) \\ &\leq \frac{h(a + b)}{1 - hc} d(x_1, x_2). \end{aligned}$$

Continuing in a similar way, we can obtain a sequence $\{x_n\}$ of successive approximations for F, starting from x_0, satisfying the following:

(a) $x_{n+1} \in Fx_n$ for all $n \in \mathbb{N}$;

(b) $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$ for all $n \in \mathbb{N}$,

where $k = \frac{h(a+b)}{1-hc} < 1$. Now, following the lines in the proof of Theorem 2, we deduce that the sequence $\{x_n\}$ converges to some $u \in X$ with respect to the metric d, that is, $\lim_{n\to+\infty} d(x_n, u) = 0$.

For this purpose, we first claim that

$$d(u,Fx) \leq s\left(a+\frac{b}{\theta}\right)d(u,x)+scd(x,Fx)$$

for all $x \in X \setminus \{u\}$. Since $d(x_n,u) \to 0$ as $n \to +\infty$ under the metric d, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n,u) \leq \frac{1}{3}d(u,x)$$

for each $n \geq n_0$. Then we have

$$\begin{aligned}\theta d(x_n,Fx_n) &\leq d(x_n,Fx_n) \leq d(x_n,x_{n+1}) \\ &\leq s\big(d(x_n,u)+d(u,x_{n+1})\big) \\ &\leq s\left(\frac{2}{3}d(u,x)\right) \leq s\big(d(u,x)-d(x_n,u)\big) \\ &\leq sd(x_n,x),\end{aligned}$$

which implies that

$$\begin{aligned}H(Fx_n,Fx) &\leq ad(x_n,x)+bd(x_n,Fx_n)+cd(x,Fx) \\ &\leq ad(x_n,x)+\frac{b}{\theta}d(x_n,x)+cd(x,Fx) \\ &= \left(a+\frac{b}{\theta}\right)d(x_n,x)+cd(x,Fx)\end{aligned}$$

for all $n \geq n_0$. Thus we have

$$\begin{aligned}d(u,Fx) &\leq s\big(d(u,x_{n+1})+d(x_{n+1},Fx)\big) \\ &\leq s\big(d(u,x_{n+1})+H(Fx_n,Fx)\big) \\ &\leq s\left(d(u,x_{n+1})+\left(a+\frac{b}{\theta}\right)d(x_n,x)+cd(x,Fx)\right)\end{aligned}$$

for all $n \geq n_0$. Letting $n \to +\infty$, we get

$$d(u,Fx) \leq s\left(a+\frac{b}{\theta}\right)d(u,x)+scd(x,Fx)$$

for all $x \in X \setminus \{u\}$.

Next, we show that

$$H(Fx,Fu) \leq \left(a+\frac{bs}{\theta}\right)d(x,u)+cd(u,Fu)$$

for all $x \in X$ with $x \neq u$. Now, for all $n \in \mathbb{N}$, there exists $y_n \in Fx$ such that

$$d(u,y_n) \leq d(u,Fx)+\frac{1}{n}d(x,u).$$

On the other hand, we have

$$\begin{aligned} d(x,Fx) &\le d(x,y_n) \le s\bigl(d(x,u)+d(u,y_n)\bigr) \\ &= s\bigl(d(x,u)+d(u,y_n)\bigr) \\ &\le s\left(d(x,u)+d(u,Fx)+\frac{1}{n}d(x,u)\right) \\ &\le s\left(d(x,u)+s\left(a+\frac{b}{\theta}\right)d(u,x)+cd(x,Fx)+\frac{1}{n}d(x,u)\right) \\ &= s\left(1+sa+\frac{sb}{\theta}+\frac{s}{n}\right)d(x,u)+scd(x,Fx) \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in the inequality above, we derive that

$$(1-sc)d(x,Fx) \le s\left(1+sa+\frac{sb}{\theta}\right)d(x,u).$$

Hence, we have $\theta d(x,Fx) \le sd(x,u)$, which implies

$$\begin{aligned} H(Fx,Fu) &\le ad(x,u)+bd(x,Fx)+cd(u,Fu) \\ &\le \left(a+\frac{bs}{\theta}\right)d(x,u)+cd(u,Fu) \end{aligned}$$

for all $x \in X \setminus \{u\}$.

Finally, if for some $n \in \mathbb{N}$ we have $x_n = x_{n+1}$, then x_n is a fixed point of F. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Thus, there exists an infinite subset J of $\mathbb{N}$ such that $x_n \neq u$ for all $n \in J$. Now, for all $n \in J$, we have

$$\begin{aligned} d(u,Fu) &\le s\bigl(d(u,x_{n+1})+d(x_{n+1},Fu)\bigr) \\ &\le s\bigl(d(u,x_{n+1})+H(Fx_n,Fu)\bigr) \\ &\le s\left(d(u,x_{n+1})+\left(a+\frac{sb}{\theta}\right)d(x_n,u)+cd(u,Fu)\right). \end{aligned}$$

Letting $n \to +\infty$ with $n \in J$, we get

$$d(u,Fu) = 0.$$

By Remark 1, we deduce that $u \in Fu$ and hence u is a fixed point of F. □

Remark 5 Taking $s = 1$ in Theorem 3 (it corresponds to the case of metric spaces), with $a+b+c \in [0,1)$, $\theta = \frac{1-b-c}{1+a}$, we get Theorem 6.6 of Mot and Petrusel [14] which itself is an extension of the theorem given in Reich [8], p.5, as well as a generalization of Kikkawa-Suzuki's Theorem 1.1.

If $T : X \to X$ is a single-valued mapping on a b-metric space, we have the following corollary which is a consequence of Theorem 3.

Corollary 2 *Let (X,d) be a complete b-metric space and let $F : X \to X$ be a single-valued mapping. If for $s \geq 1$ there exist nonnegative numbers a, b, c with $s(a+b+c) \in [0,1)$ and $\theta = \frac{1-sb-sc}{1+sa}$ such that*

$$\theta d(x,Fx) \leq sd(x,y) \implies d(Fx,Fy) \leq ad(x,y) + bd(x,Fx) + cd(y,Fy) \tag{2.6}$$

for all $x,y \in X$, then F has a fixed point.

Remark 6 If we take $s = 1$ in Corollary 2, we immediately get a Kikkawa-Suzuki type fixed point theorem for a Reich-type single-valued operator, see [8, 24].

Example 2 Let $X = [1,\infty)$ and $d(x,y) = |x-y|^2$ for all $x,y \in X$. Then d is a b-metric on X with $s = 2$ and (X,d) is complete. Also, d is not a metric on X. Define $F : X \to CB(X)$ by

$$Fx = \left[2, 2 + \frac{x}{3}\right]$$

for all $x,y \in X$. Consider $H(Fx,Fy) = \frac{1}{9}(x-y)^2 = \frac{1}{9}d(x,y)$, where $r = \frac{1}{9} < \frac{1}{6} = \frac{1}{s^2+s} < 1$. So all the conditions of Theorem 2 are satisfied. Moreover, 2 and 3 are the two fixed points of F.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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