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SOME FIXED POINT THEOREMS IN BANACH SPACES

BY

K. GOEBEL AND E. ZŁOTKIEWICZ (LUBLIN)

Let B denote a Banach space with the norm $\| \ \|$ and let C be a closed subset of B. The transformation $F\colon C \to C$ is called *contraction* if there exists a constant $0 \leqslant k < 1$ such that for arbitrary $x, y \in C$ the inequality $\|Fx - Fy\| \leqslant k\|x-y\|$ holds. It is called *non-expansive* if the same condition with $k = 1$ holds. By the Banach contraction principle each contraction of C has exactly one fixed point. The same is true if we assume that only some powers of F are contractions, but it is not true for non-expansive mappings. However, Browder [1] has proved that every non-expansive mapping of a closed, bounded, convex subset of a uniformly convex Banach space has at least one fixed point. Kirk [3] proved similar theorem in the space with the so called normal structure.

It is natural to raise the question whether these results can be extended to the case of transformations with a non-expansive iteration. The answer is in general negative. Klee [4] showed that even in Hilbert space some convex sets admit continuous transformations without fixed points and even such that $F^2 = I$, where I denotes identity mapping.

In the present note we shall give the proofs of some fixed point theorems for involutions and a generalisation of Browder's result.

THEOREM 1. *If C is a closed and convex subset of B and if $F\colon C \to C$ satisfies conditions* 1° $F^2 = I$ *and* 2° $\|Fx - Fy\| \leqslant k\|x-y\|$, *where* $0 \leqslant k < 2$, *then F has at least one fixed point.*

Proof. Let $G = \frac{1}{2}(F+I)$ and let x be an arbitrary point of C. Put $y = Gx$, $z = Fy$, and $z^* = 2y - z$. We have

$$\|z-x\| = \|Fy - F^2x\| \leqslant k\|y-Fx\| = \frac{k}{2}\|x-Fx\|$$

and

$$\|z^* - x\| = \|2y - Fy - x\| = \|Fx - Fy\| \leqslant k\|x-y\| = \frac{k}{2}\|x-Fx\|;$$

thus z and z^* are contained in the ball of radius $\frac{1}{2}k\|x-Fx\|$, centered at x. Consequently,

$$\|z-z^*\| \leqslant k\|x-Fx\| \quad \text{and} \quad \|y-Fy\| = \frac{1}{2}\|z-z^*\| \leqslant \frac{k}{2}\|x-Fx\|,$$

hence

$$\|G^2x-Gx\| \leqslant \frac{k}{2}\|Gx-x\|.$$

Since by assumptions we have $0 \leqslant k/2 < 1$, distances between consecutive powers of transformation G converge to zero as rapidly as the sequence $(k/2)^n$ does and so the sequence $x_n = G^n x$ converges. Putting $x^* = \lim_{n\to\infty} x_n$ we obtain $x^* = Gx^*$. Hence $x^* = Fx^*$.

Suppose now that B is uniformly convex. Then there exists an increasing function $\delta(\varepsilon)$: $\langle 0, 2\rangle \to \langle 0, 1\rangle$ such that for each pair of points $x, y \in B$ which are subject to conditions $\|x\| \leqslant R$, $\|y\| \leqslant R$, and $\|x-y\| \geqslant R\varepsilon$, the inequality

$$\left\|\frac{x+y}{2}\right\| \leqslant \big(1-\delta(\varepsilon)\big)R$$

holds.

In this case we can obtain a stronger result.

THEOREM 2. *If B is a uniformly convex Banach space and if C is a closed and convex subset of B, then each transformation F: $C \to C$ satisfying conditions 1° and 2° of Theorem 1 with k such that $k\delta^{-1}(1-1/k) < 4$ has at least one fixed point.*

LEMMA. *If B is a uniformly convex Banach space and if $x, z, z^* \in B$ are subject to conditions*

$$\|z-x\| \leqslant R,\ \|z^*-x\| \leqslant R \quad \textit{and} \quad \left\|\frac{z+z}{2}-x\right\| \geqslant r,$$

then

$$\|z-z^*\| \leqslant R\delta^{-1}\left(\frac{R-r}{R}\right).$$

Proof of Lemma follows immediately from the inequality

$$\left\|\frac{(x-z)+(x-z)}{2}\right\| \geqslant \left(1-\frac{R-r}{R}\right)R$$

and from the above described property of the function δ.

Proof of Theorem 2. Let G, x, y, z, z^* be the same as in Theorem 1.

We have

$$\|z-x\| \leqslant \frac{k}{2}\|x-Fx\|,$$

$$\|z^*-x\| = \frac{k}{2}\|x-Fx\|,$$

$$\left\|\frac{z^*+z}{2}-x\right\| = \frac{1}{2}\|x-Fx\|.$$

Putting

$$R = \frac{k}{2}\|x-Fx\| \quad \text{and} \quad r = \frac{1}{2}\|x-Fx\|$$

we obtain, in view of our Lemma, the inequality

$$\|z-z^*\| \leqslant \frac{k}{2}\delta^{-1}\left(1-\frac{1}{k}\right)\|x-Fx\|.$$

Hence

$$\|G^2x-Gx\| \leqslant \frac{k}{4}\delta^{-1}\left(1-\frac{1}{k}\right)\|Gx-x\|,$$

and the same argument as in Theorem 1 gives the result.

Remark 1. The spaces l^p, L^p $(p>1)$ are uniformly convex [2] and so

$$\delta(\varepsilon) = 1-\left(1-\left(\frac{\varepsilon}{2}\right)^s\right)^{1/s},$$

where $s = \max(p,q)$, $p^{-1}+q^{-1} = 1$. In this case the assumptions of Theorem 2 are satisfied for $k < (1+2^s)^{1/s}$.

Remark 2. The following problem is still open: are the evaluations for k in both theorems best possible? Or, does there exist an involution of a convex closed set in Banach space which has no fixed points and which satisfies condition 2° of Theorem 1 with $k = 2$ (or, in uniformly convex space, with k such that $k\delta^{-1}(1-1/k) = 4$)? (**P 732**)

THEOREM 3. *Suppose B is uniformly convex and C is closed, bounded and convex subset of B. If $F\colon C \to C$ satisfies conditions* 1° $\|F^2x-F^2y\| \leqslant \|x-y\|$ *and* 2° $\|Fx-Fy\| \leqslant k\|x-y\|$, *where k is such as in Theorem* 2, *then F has at least one fixed point.*

Proof. By Browder's theorem [1], the transformation F^2 has at least one fixed point in C. It is easy to verify that the set C^* of fixed points of F^2 in C is convex and closed. Moreover, $F(C^*) = C^*$ and $F^2 = I$ on C^*. Hence Theorem 2 implies that F has a fixed point in C^*.

REFERENCES

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MARIA CURIE-SKŁODOWSKA UNIVERSITY, LUBLIN

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