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Some framed f-structures on transversally Finsler foliations

Abstract. Some problems concerning to Liouville distribution and framed f-structures are studied on the normal bundle of the lifted Finsler foliation to its normal bundle. It is shown that the Liouville distribution of transversally Finsler foliations is an integrable one and some natural framed $f(3,\varepsilon)$-structures of corank 2 exist on the normal bundle of the lifted Finsler foliation.

1. Introduction and preliminaries. The study of structures on manifolds defined by a tensor field satisfying $f^3 \pm f = 0$ has the origin in a paper by K. Yano [15]. Later on, these structures have been generically called f-structures. On the tangent manifold of a Finsler space, the notion of framed $f(3,1)$-structure was defined and studied by M. Anastasiei in [2]. Further developments concerning framed $f(3,-1)$-structure on such manifold was studied in [5, 6]. In a paper by A. Miernowski and W. Mozgawa [9] was defined the notion of transversally Finsler foliation and there it is proved that the normal bundle of the lifted Finsler foliation to its normal bundle has a local model of tangent manifold and it is the Riemannian one. Thus, some problems specific for tangent manifolds can be extended and studied on the normal bundle of the lifted Finsler foliation. Firstly, following [4], we define a Liouville distribution in the vertical bundle and we prove that it is integrable. Next, by analogy with [2], some framed $f(3,\varepsilon)$-structures

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on the normal bundle of the lifted Finsler foliation are defined and studied in the paper.

In this section we briefly recall some basic facts about transversally Finsler foliations (see [9]).

Let (M, F) be a n-dimensional Finsler manifold, where $F : TM \to \mathbb{R}$ is a Finsler metric (for necessary definitions, see [3, 11]).

Definition 1.1 ([9]). A diffeomorphism $f : M \to M$ is said to be a Finsler isometry if $F(f(x), f_*(y)) = F(x, y)$ for each $x \in M$ and $y \in T_xM$.

Definition 1.2 ([9]). A foliated cocycle $\{U_i, f_i, \gamma_{ij}\}$ on a $(n+m)$-dimensional manifold $\mathcal{M}$ is said to be a Finslerian foliation $\mathcal{F}$ if

(i) $\{U_i\}$, $i \in I$ is an open covering of $\mathcal{M}$;
(ii) $f_i : U_i \to M$ is a submersion, where (M, F) is a Finsler manifold;
(iii) γ_{ij} is a local Finslerian isometry of (M, F) such that for each $u \in U_i \cap U_j$

$$f_i(u) = (\gamma_{ij} \circ f_j)(u).$$

The Finsler manifold (M, F) will be called the transversal manifold of foliation $\mathcal{F}$.

The local submersions $\{f_i\}$ define by pull-back a Finsler metric in the normal bundle $Q = Q(\mathcal{M})$ of the foliation $\mathcal{F}$, denoted by F_Q and given by

$$F_Q(u, p(X_u)) = F(f_i(u), (f_i)_*(X_u)) \tag{1.1}$$

for any $u \in \mathcal{M}$, $X_u \in T_u\mathcal{M}$, where $p : T\mathcal{M} \to Q$ is the natural projection (for Lagrangian case, see [12]).

We denote by $u = (x^i, x^\alpha)$, $\alpha = 1, \ldots, n$, $i = n+1, \ldots, n+m = \dim \mathcal{M}$ the adapted coordinates in a local foliated chart on $\mathcal{M}$ and let $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\alpha}\}$ be a local frame of $T\mathcal{M}$. If we denote by $\{\frac{\overline{\partial}}{\partial x^\alpha}\}$ the corresponding local frame of Q, then we can induce a chart $(x^i, x^\alpha, y^\alpha)$ on Q, where $y^\alpha \frac{\overline{\partial}}{\partial x^\alpha}$ is a transversal vector at a point (x^i, x^α). Note that in this coordinate system the metric F_Q in Q does not depend on (x^i).

According to [9], the distribution spanned by $\{\frac{\partial}{\partial x^i}\}$, $i = n+1, \ldots, n+m$ defines a foliation $\mathcal{F}_Q$ on Q called *the natural lift of* $\mathcal{F}$ *to* Q.

Let us consider $Q(Q)$ to be the bundle over Q transversal to the foliation $\mathcal{F}_Q$. The canonical projection $\pi : Q \to \mathcal{M}$, $\pi(x^i, x^\alpha, y^\alpha) = (x^i, x^\alpha)$ induces another projection $\pi_* : TQ \to T\mathcal{M}$ which maps the tangent vectors to $\mathcal{F}_Q$ in the tangent vectors to $\mathcal{F}$. Thus, π_* induces a mapping $\overline{\pi}_* : Q(Q) \to Q$ defined by $\overline{\pi}_* = p \circ \pi_* \circ \overline{p}^{-1}$, where $\overline{p} : TQ \to Q(Q)$ is the natural projection. If we denote by $V := V(Q) = \ker \overline{\pi}_*$, it is a vertical bundle spanned by the vectors $\{\frac{\overline{\partial}}{\partial y^\alpha}\}$, $\alpha = 1, \ldots, n$.

Let us put $G_Q = F_Q^2$ and $G_\alpha = \frac{\overline{\partial} G_Q}{\partial y^\alpha}$, $G_{\alpha\beta} = \frac{\overline{\partial}^2 G_Q}{\partial y^\alpha \partial y^\beta}$, etc. Then, by the same arguments as in [3, 11] we have

$$G_{\alpha\beta} y^\alpha = G_\beta\,;\; G_{\alpha\beta} y^\alpha y^\beta = 2G_Q\,;\; G_{\alpha\beta\gamma} y^\gamma = 0. \tag{1.2}$$

According to [9], for any vertical vectors $X = \dot{X}^\alpha \frac{\overline{\partial}}{\partial y^\alpha}$ and $Y = \dot{Y}^\beta \frac{\overline{\partial}}{\partial y^\beta}$ the formula

$$\mathcal{G}^v(X, Y) = G_{\alpha\beta} \dot{X}^\alpha \dot{Y}^\beta \tag{1.3}$$

defines a Riemannian metric in the vertical bundle V.

An important global vector field is defined by

$$\Gamma : Q \to V\,;\; \Gamma(x^i, x^\alpha, y^\alpha) = y^\alpha \frac{\overline{\partial}}{\partial y^\alpha} \tag{1.4}$$

and it is called *the Liouville vector field* (or radial vertical vector field). Moreover, by the second equality of (1.2), we have

$$G_Q = \frac{1}{2} \mathcal{G}^v(\Gamma, \Gamma) > 0. \tag{1.5}$$

Also, by using the technique of good vertical connection often used in Finsler geometry (see [1]), in [9] it is proved that the normal bundle $Q(Q)$ has a local model of tangent manifold. Thus we have the spliting $Q(Q) = H(Q) \oplus V(Q)$, where the horizontal bundle $H := H(Q)$ is spanned by the vectors $\{\frac{\delta}{\delta x^\alpha} = \frac{\overline{\partial}}{\partial x^\alpha} - N_\alpha^\beta \frac{\overline{\partial}}{\partial y^\beta}\}$, where the coefficients N_α^β are related only in terms of Finsler metric G_Q.

In the sequel we will use the adapted basis $\{\delta_\alpha := \frac{\delta}{\delta x^\alpha}, \dot{\partial}_\alpha := \frac{\overline{\partial}}{\partial y^\alpha}\}$ as well as its dual $\{dx^\alpha, \delta y^\alpha := dy^\alpha + N_\beta^\alpha dx^\beta\}$.

We notice that $\mathcal{G}^v$ induces a Riemannian metric on the horizontal bundle denoted by $\mathcal{G}^h$ and we consider $\mathcal{G} = \mathcal{G}^h + \mathcal{G}^v$ the Sasaki type lift of the fundamental tensor $G_{\alpha\beta}$, locally given by

$$\mathcal{G} = G_{\alpha\beta} dx^\alpha \otimes dx^\beta + G_{\alpha\beta} \delta y^\alpha \otimes \delta y^\beta. \tag{1.6}$$

2. A Liouville distribution. Let us consider $\xi = \frac{\Gamma}{F_Q\sqrt{2}}$ to be the unit Liouville vector field with respect to $\mathcal{G}^v$, i.e.

$$\mathcal{G}^v(\xi, \xi) = 1. \tag{2.1}$$

Using $\mathcal{G}^v$ and ξ, we define a vertical 1-form $\eta \in \Gamma(V^*)$ by

$$\eta(X) = \mathcal{G}^v(X, \xi),\; \forall\, X \in \Gamma(V). \tag{2.2}$$

Denote by $\{\xi\}$ the line vector bundle over Q spanned by ξ and we define *the Liouville distribution* as the complementary orthogonal distribution SQ to $\{\xi\}$ in V with respect to $\mathcal{G}^v$, namely $V = SQ \oplus \{\xi\}$. Hence, SQ is defined by η, that is

$$\Gamma(SQ) = \{X \in \Gamma(V)\,;\, \eta(X) = 0\}. \tag{2.3}$$

Thus, any vertical vector field $X \in \Gamma(V)$ can be expressed as

$$X = \Phi X + \eta(X)\xi, \tag{2.4}$$

where Φ is the projection morphism of V on SQ. By direct calculations, one gets

Proposition 2.1. *For any vertical vector fields $X, Y \in \Gamma(V)$, we have*

$$\mathcal{G}^v(X, \Phi Y) = \mathcal{G}^v(\Phi X, \Phi Y) = \mathcal{G}^v(X, Y) - \eta(X)\eta(Y). \tag{2.5}$$

Then the local components of η and Φ with respect to the basis $\{\delta y^\alpha\}$ and $\{\delta y^\alpha \otimes \dot{\partial}_\alpha\}$, respectively, are given by

$$\eta_\alpha = \frac{G_\alpha}{F_Q\sqrt{2}} \tag{2.6}$$

and

$$\Phi_\alpha^\beta = \delta_\alpha^\beta - \frac{\eta_\alpha y^\beta}{F_Q\sqrt{2}}, \tag{2.7}$$

where δ_α^β denotes the Kronecker symbol.

Theorem 2.1. *The Liouville distribution SQ is integrable.*

Proof. Let $X, Y \in \Gamma(SQ)$. As V is an integrable distribution on Q, it is sufficient to prove that $[X, Y]$ has no component with respect to ξ. By using (1.3) and (2.2), we obtain that $X \in \Gamma(SQ)$ if and only if

$$G_{\alpha\beta} y^\alpha \dot{X}^\beta = 0, \tag{2.8}$$

where $\dot{X}^\beta$ are the components of X. Differentiating (2.8) with respect to y^γ, we get

$$G_{\alpha\beta\gamma} y^\alpha \dot{X}^\beta + G_{\gamma\beta}\dot{X}^\beta + G_{\alpha\beta} y^\alpha \,\dot{\partial}_\gamma\, \dot{X}^\beta = 0\,,\ \forall\, \gamma = 1, \ldots, n \tag{2.9}$$

and taking into account the last equality of (1.2), we get

$$G_{\gamma\beta}\dot{X}^\beta + G_{\alpha\beta} y^\alpha \,\dot{\partial}_\gamma\, \dot{X}^\beta = 0\,,\ \forall\, \gamma = 1, \ldots, n. \tag{2.10}$$

Then, by direct calculations using (1.3), (1.4) and (2.10), we have

$$\begin{aligned}
\mathcal{G}^v([X, Y], \xi) &= \frac{1}{F_Q\sqrt{2}} G_{\alpha\beta} y^\alpha \left[\dot{\partial}_\gamma\, (\dot{Y}^\beta)\dot{X}^\gamma - \dot{\partial}_\gamma\, (\dot{X}^\beta)\dot{Y}^\gamma\right] \\
&= -\frac{1}{F_Q\sqrt{2}}\left(G_{\gamma\beta}\dot{Y}^\beta\dot{X}^\gamma - G_{\gamma\beta}\dot{X}^\beta\dot{Y}^\gamma\right) \\
&= 0
\end{aligned}$$

which completes the proof. □

Let us consider $\nabla : \mathcal{X}(V) \to \mathcal{X}(T^*Q \otimes V)$ to be the unique good vertical Bott connection introduced in [9]. We notice that for vertical vector fields, it is locally given by

$$\nabla_{\dot{\partial}_\alpha} \dot{\partial}_\beta = C^\gamma_{\alpha\beta} \dot{\partial}_\gamma, \tag{2.11}$$

where the local vertical coefficients are given by

$$C^\gamma_{\alpha\beta} = \frac{1}{2} G^{\delta\gamma} \dot{\partial}_\alpha (G_{\beta\delta}), \tag{2.12}$$

where $(G^{\delta\gamma})$ denotes the inverse of $(G_{\gamma\delta})$.

Now, contracting (2.12) by y^α, we deduce

$$C^\gamma_{\alpha\beta} y^\alpha = 0. \tag{2.13}$$

By straightforward calculations using (2.4), (2.5), (2.6), (2.7) and (2.13) we obtain

Proposition 2.2. *The vertical covariant derivatives, with respect to ∇ of ξ, η and Φ, are*

$$\nabla_X \xi = \frac{1}{F_Q\sqrt{2}} \Phi X \tag{2.14}$$

$$(\nabla_X \eta) Y = \frac{1}{F_Q\sqrt{2}} \mathcal{G}^v(\Phi X, \Phi Y) \tag{2.15}$$

$$(\nabla_X \Phi) Y = -\frac{1}{F_Q\sqrt{2}} [\mathcal{G}^v(\Phi X, \Phi Y)\xi + \eta(Y)\Phi X] \tag{2.16}$$

for any $X, Y \in \Gamma(V)$.

3. Some framed $f(3,\varepsilon)$-structures on $Q(Q)$. A framed $f(3,1)$-structure of corank s on a $(2n+s)$-dimensional manifold N is a natural generalization of an almost contact structure on N and it is a triplet $(f, (\xi_a), (\omega^a))$, $a = 1, \ldots, s$, where f is a tensor field of type $(1,1)$, (ξ_a) are vector fields and (ω^a) are 1-forms on N such that

$$\omega^a(\xi_b) = \delta^a_b\,;\ f(\xi_a) = 0\,;\ \omega^a \circ f = 0\,;\ f^2 = -I + \sum_a \omega^a \otimes \xi_a, \tag{3.1}$$

where I denotes the Kronecker tensor field on N. The name of $f(3,1)$-structure was suggested by the identity $f^3 + f = 0$. For an account of this kind of structures we refer to [10].

The linear operator ϕ given in the local adapted basis by

$$\phi(\delta_\alpha) = \dot{\partial}_\alpha\,;\ \phi(\dot{\partial}_\alpha) = -\delta_\alpha \tag{3.2}$$

defines an almost complex structure on $Q(Q)$ and it is easy to see that

$$\mathcal{G}(\phi(X), \phi(Y)) = \mathcal{G}(X, Y),\ \forall X, Y \in \Gamma(Q(Q)). \tag{3.3}$$

Let us put $\xi_1 = \frac{y^\alpha}{F_Q\sqrt{2}}\delta_\alpha$ and $\xi_2 = \frac{y^\alpha}{F_Q\sqrt{2}}\dot{\partial}_\alpha$. From the definition of ϕ, it follows:

Proposition 3.1. *We have $\phi(\xi_1) = \xi_2$ and $\phi(\xi_2) = -\xi_1$.*

Now, let us consider the 1-forms $\omega^1 = \frac{y_\alpha}{F_Q\sqrt{2}}dx^\alpha$ and $\omega^2 = \frac{y_\alpha}{F_Q\sqrt{2}}\delta y^\alpha$, where $y_\alpha = G_{\alpha\beta}y^\beta$. By the second equality of (1.2) we have $y_\alpha y^\alpha = 2G_Q$ and so $\omega^a(\xi_b) = \delta^a_b$. By a direct calculation, we obtain

Proposition 3.2. *We have $\omega^1 \circ \phi = -\omega^2$ and $\omega^2 \circ \phi = \omega^1$.*

Proposition 3.3. *We have $\omega^1(X) = \mathcal{G}(X, \xi_1)$ and $\omega^2(X) = \mathcal{G}(X, \xi_2)$, for any $X \in \Gamma(Q(Q))$.*

Now, we define a tensor field f of type $(1,1)$ on $Q(Q)$ by

$$f(X) = \phi(X) - \omega^1(X)\xi_2 + \omega^2(X)\xi_1 \tag{3.4}$$

for any $X \in \Gamma(Q(Q))$.

Theorem 3.1. *The triplet $(f, (\xi_a), (\omega^a))$, $a = 1, 2$ provides a framed $f(3,1)$-structure on $Q(Q)$, namely*

(i) $\omega^a(\xi_b) = \delta^a_b$; $f(\xi_a) = 0$; $\omega^a \circ f = 0$;
(ii) $f^2(X) = -X + \omega^1(X)\xi_1 + \omega^2(X)\xi_2$, *for any* $X \in \Gamma(Q(Q))$;
(iii) f *is of rank* $2n-2$ *and* $f^3 + f = 0$.

Proof. Using (3.4) and Propositions 3.1 and 3.2, by direct calculations we get (i) and (ii). Applying f on the equality (ii) and taking into account the equality (i), we obtain $f^3 + f = 0$. Now, from the second equations in (i), we see that $\mathrm{span}\{\xi_1, \xi_2\} \subseteq \ker f$. If $X = X^\alpha\delta_\alpha + \dot{X}^\alpha\dot{\partial}_\alpha$ belongs to $\ker f$ and it is not in $\mathrm{span}\{\xi_1, \xi_2\}$, by using (3.4), we have

$$f(X) = \left(X^\beta - \frac{X^\alpha y_\alpha}{2G_Q}y^\beta\right)\dot{\partial}_\beta - \left(\dot{X}^\beta - \frac{\dot{X}^\alpha y_\alpha}{2G_Q}y^\beta\right)\delta_\beta = 0.$$

Thus, $X = \frac{X^\alpha y_\alpha}{F_Q\sqrt{2}}\xi_1 + \frac{\dot{X}^\alpha y_\alpha}{F_Q\sqrt{2}}\xi_2 \in \mathrm{span}\{\xi_1, \xi_2\}$, contradiction. Hence $\ker f = \mathrm{span}\{\xi_1, \xi_2\}$ and $\mathrm{rank} f = 2n-2$. □

Theorem 3.2. *The Riemannian metric $\mathcal{G}$ verifies*

$$\mathcal{G}(f(X), f(Y)) = \mathcal{G}(X,Y) - \omega^1(X)\omega^1(Y) - \omega^2(X)\omega^2(Y) \tag{3.5}$$

for any $X, Y \in \Gamma(Q(Q))$.

Proof. Since $\mathcal{G}(\xi_1, \xi_2) = 0$ and $\mathcal{G}(\xi_1, \xi_1) = \mathcal{G}(\xi_2, \xi_2) = 1$, by using (3.4) and Propositions 3.2 and 3.3 we get (3.5). □

Remark 3.1. In the local basis $\{\delta_\alpha, \dot{\partial}_\alpha\}$, we have

$$f(\delta_\alpha) = \left(\delta^\beta_\alpha - \frac{y_\alpha y^\beta}{2G_Q}\right)\dot{\partial}_\beta\,;\ f(\dot{\partial}_\alpha) = \left(-\delta^\beta_\alpha + \frac{y_\alpha y^\beta}{2G_Q}\right)\delta_\beta \tag{3.6}$$

and using (3.6) one finds

$$(3.7)\qquad \begin{aligned} \mathcal{G}(f(\delta_\alpha), f(\delta_\beta)) &= G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q} \\ \mathcal{G}(f(\delta_\alpha), f(\dot{\partial}_\beta)) &= 0 \\ \mathcal{G}(f(\dot{\partial}_\alpha), f(\dot{\partial}_\beta)) &= G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}. \end{aligned}$$

Now, from (3.7) easily follows (3.5).

Theorem 3.2 says that $(f, \mathcal{G})$ is a Riemannian framed $f(3,1)$-structure on $Q(Q)$.

Let us put $\varphi(X,Y) = \mathcal{G}(f(X), Y)$ for any $X, Y \in \Gamma(Q(Q))$. We have

Theorem 3.3. *φ is a 2-form on $Q(Q)$ and the annihilator of φ is spanned by $\{\xi_1, \xi_2\}$.*

Proof. φ is bilinear since $\mathcal{G}$ is bilinear. Now, using Proposition 3.3 and Theorems 3.1 and 3.2, by direct calculations we have $\varphi(Y,X) = -\varphi(X,Y)$ which says that φ is a 2-form on $Q(Q)$.

By the second equality (i) from Theorem 3.1 it follows that $\varphi(\xi_1, \xi_1) = \varphi(\xi_1, \xi_2) = \varphi(\xi_2, \xi_2) = 0$, hence $\mathrm{span}\{\xi_1, \xi_2\}$ is contained in the null space of φ. Conversely, if $X = X^\alpha \delta_\alpha + \dot{X}^\alpha \dot{\partial}_\alpha \in \Gamma(Q(Q))$ such that $\varphi(X,X) = 0$, by direct calculations, we get $X = \frac{X^\alpha y_\alpha}{F_Q\sqrt{2}}\xi_1 + \frac{\dot{X}^\alpha y_\alpha}{F_Q\sqrt{2}}\xi_2 \in \mathrm{span}\{\xi_1, \xi_2\}$. □

Remark 3.2. Locally, we have

$$(3.8)\qquad \varphi = \left(G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}\right) dx^\alpha \wedge \delta y^\beta - \left(G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}\right) \delta y^\alpha \wedge dx^\beta$$

and it appears as a deformation of the symplectic structure $\varphi^{'}(X,Y) = \mathcal{G}(\phi X, Y)$ for any $X, Y \in \Gamma(Q(Q))$.

Finally, we make a similar study concerning framed $f(3,-1)$-structures on $Q(Q)$.

A framed $f(3,-1)$-structure of corank s on a $(2n+s)$-dimensional manifold N is a natural generalization of an almost paracontact structure on N and it consists of a triplet $(\widetilde{f}, (\xi_a), (\omega^a))$, $a = 1, \ldots, s$, where $\widetilde{f}$ is a tensor field of type $(1,1)$, (ξ_a) are vector fields and (ω^a) are 1-forms on N such that

$$(3.9)\qquad \omega^a(\xi_b) = \delta^a_b\,;\ \widetilde{f}(\xi_a) = 0\,;\ \omega^a \circ \widetilde{f} = 0\,;\ \widetilde{f}^2 = I - \sum_a \omega^a \otimes \xi_a.$$

The name of $f(3,-1)$-structure was suggested by the identity $\widetilde{f}^3 - \widetilde{f} = 0$. This is in some sense dual to the framed $f(3,1)$-structure on N.

Let us consider the linear operator $\widetilde{\phi}$ given in the local adapted basis by

$$(3.10)\qquad \widetilde{\phi}(\delta_\alpha) = \delta_\alpha\,;\ \widetilde{\phi}(\dot{\partial}_\alpha) = -\,\dot{\partial}_\alpha\,.$$

It is easy to check that $\widetilde{\phi}^2 = I$ and

$$\mathcal{G}(\widetilde{\phi}(X), \widetilde{\phi}(Y)) = \mathcal{G}(X, Y) \tag{3.11}$$

for any $X, Y \in \Gamma(Q(Q))$. From the definition of $\widetilde{\phi}$ it follows

Proposition 3.4. *We have* $\widetilde{\phi}(\xi_1) = \xi_1$ *and* $\widetilde{\phi}(\xi_2) = -\xi_2$.

Proposition 3.5. *We have* $\omega^1 \circ \widetilde{\phi} = \omega^1$ *and* $\omega^2 \circ \widetilde{\phi} = -\omega^2$.

Now, we define a tensor field $\widetilde{f}$ of type $(1,1)$ on $Q(Q)$ by

$$\widetilde{f}(X) = \widetilde{\phi}(X) - \omega^1(X)\xi_1 + \omega^2(X)\xi_2 \tag{3.12}$$

for any $X \in \Gamma(Q(Q))$.

Theorem 3.4. *The triplet* $(\widetilde{f}, (\xi_a), (\omega^a))$, $a = 1, 2$ *provides a framed* $f(3, -1)$*-structure on* $Q(Q)$*, namely*

(i) $\omega^a(\xi_b) = \delta^a_b$; $\widetilde{f}(\xi_a) = 0$; $\omega^a \circ \widetilde{f} = 0$;
(ii) $\widetilde{f}^2(X) = X - \omega^1(X)\xi_1 - \omega^2(X)\xi_2$, *for any* $X \in \Gamma(Q(Q))$;
(iii) $\widetilde{f}$ *is of rank* $2n - 2$ *and* $\widetilde{f}^3 - \widetilde{f} = 0$.

Proof. Using (3.12) and Propositions 3.4 and 3.5, by direct calculations we get (i) and (ii). Applying $\widetilde{f}$ on the equality (ii) and taking into account the equality (i), we obtain $\widetilde{f}^3 - \widetilde{f} = 0$. The equality (iii) follows by the same argument as in the proof of Theorem 3.1. □

Theorem 3.5. *The Riemannian metric* $\mathcal{G}$ *verifies*

$$\mathcal{G}(\widetilde{f}(X), \widetilde{f}(Y)) = \mathcal{G}(X, Y) - \omega^1(X)\omega^1(Y) - \omega^2(X)\omega^2(Y) \tag{3.13}$$

for any $X, Y \in \Gamma(Q(Q))$.

Proof. It follows by direct calculations by using (3.12) and Propositions 3.3 and 3.5. □

Remark 3.3. In the local basis $\{\delta_\alpha, \dot{\partial}_\alpha\}$, we have

$$\widetilde{f}(\delta_\alpha) = \left(\delta^\beta_\alpha - \frac{y_\alpha y^\beta}{2G_Q}\right)\delta_\beta\,;\ \widetilde{f}(\dot{\partial}_\alpha) = \left(-\delta^\beta_\alpha + \frac{y_\alpha y^\beta}{2G_Q}\right)\dot{\partial}_\beta \tag{3.14}$$

and using (3.14) one finds

$$\begin{aligned}
&\mathcal{G}(\widetilde{f}(\delta_\alpha), \widetilde{f}(\delta_\beta)) = G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}\\
&\mathcal{G}(\widetilde{f}(\delta_\alpha), \widetilde{f}(\dot{\partial}_\beta)) = 0\\
&\mathcal{G}(\widetilde{f}(\dot{\partial}_\alpha), \widetilde{f}(\dot{\partial}_\beta)) = G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}.
\end{aligned} \tag{3.15}$$

Now, (3.13) easily follows from (3.15).

Theorem 3.5. says that $(\widetilde{f}, \mathcal{G})$ is a Riemannian framed $f(3,-1)$-structure on $Q(Q)$.

Let us put $\widetilde{\varphi}(X,Y) = \mathcal{G}(\widetilde{f}(X), Y)$ for any $X, Y \in \Gamma(Q(Q))$. We have

Theorem 3.6. *$\widetilde{\varphi}$ is a symmetric bilinear form on $Q(Q)$ and the annihilator of $\widetilde{\varphi}$ is spanned by $\{\xi_1, \xi_2\}$.*

Proof. It follows in a similar manner with the proof of Theorem 3.3, by using Proposition 3.3 and Theorems 3.4 and 3.5. □

Locally, we have

$$\widetilde{\varphi} = \left(G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}\right) dx^\alpha \otimes dx^\beta - \left(G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}\right) \delta y^\alpha \otimes \delta y^\beta \tag{3.16}$$

with $\det(G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q}) = 0$, since $(G_{\alpha\beta} - \frac{y_\alpha y_\beta}{2G_Q})y^\beta = y_\alpha - y_\alpha = 0$.

Remark 3.4. The map $\widetilde{\varphi}$ is a singular pseudo-Riemannian metric on $Q(Q)$.

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References

[1] Abate, M., Patrizio, G., *Finsler Metrics – A Global Approach*, Lecture Notes in Math., **1591**, Springer-Verlag, Berlin, 1994.

[2] Anastasiei, M., *A framed f-structure on tangent bundle of a Finsler space*, An. Univ. Bucureşti, Mat.-Inf., **49** (2000), 3–9.

[3] Bao, D., Chern, S. S. and Shen, Z., *An Introduction to Riemannian Finsler Geometry*, Graduate Texts in Math., **200**, Springer-Verlag, New York, 2000.

[4] Bejancu, A., Farran, H. R., *On the vertical bundle of a pseudo-Finsler manifold*, Int. J. Math. Math. Sci. **22** (3) (1997), 637–642.

[5] Gîrţu, M., *An almost paracontact structure on the indicatrix bundle of a Finsler space*, Balkan J. Geom. Appl. **7**(2) (2002), 43–48.

[6] Gîrţu, M., *A framed $f(3,-1)$-structure on the tangent bundle of a Lagrange space*, Demonstratio Math. **37**(4) (2004), 955–961.

[7] Hasegawa, I., Yamaguchi, K. and Shimada, H., *Sasakian structures on Finsler manifolds*, Antonelli, P. L., Miron R. (eds.), *Lagrange and Finsler Geometry*, Kluwer Acad. Publ., Dordrecht, 1996, 75–80.

[8] Miernowski, A., *A note on transversally Finsler foliations*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **60** (2006), 57–64.

[9] Miernowski, A., Mozgawa, W., *Lift of the Finsler foliations to its normal bundle*, Differential Geom. Appl. **24** (2006), 209–214.

[10] Mihai, I., Roşca, R. and Verstraelen, L., *Some aspects of the differential geometry of vector fields*, PADGE, Katholieke Univ. Leuven, vol. **2** (1996).

[11] Miron, R., Anastasiei, M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., Dordrecht, 1994.

[12] Popescu, P., Popescu, M., *Lagrangians adapted to submersions and foliations*, Differential Geom. Appl. **27** (2009), 171–178.

[13] Singh, K. D., Singh, R., *Some $f(3,\varepsilon)$-structure manifold*, Demonstratio Math. **10** (3–4) (1977), 637–645.

[14] Vaisman, I., *Lagrange geometry on tangent manifolds*, Int. J. Math. Math. Sci. **51** (2003), 3241–3266.
[15] Yano, K., *On a structure defined by a tensor field of type* $(1,1)$ *satisfying* $f^3+f=0$, Tensor (N.S.) **14** (1963), 99–109.

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