SOME FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract. Sufficient conditions are given for the solution of the functional differential equations with associated boundary conditions

$$dy/dx = \sum_{n=0}^{\infty} a_n y(\mu^n x), \qquad y(0) = 1,$$

$$dy/dx = \int_0^{\infty} a(u) y(\mu^u x)\, du, \qquad y(0) = 1.$$

A discussion is also given of some possible solutions to the differential equations which do not satisfy the boundary conditions.

1. Functional differential equations involving a parameter μ of the forms

$$dy/dx = \sum_{n=0}^{\infty} a_n y(\mu^n x), \qquad y(0) = 1 \tag{1}$$

and

$$dy/dx = \int_0^{\infty} a(u) y(\mu^u x)\, du, \qquad y(0) = 1 \tag{2}$$

do not seem to have been considered so far. It is the purpose of this paper to obtain some solutions for these equations together with sufficient conditions for their existence. A short discussion will also be given of sufficient conditions for the nonuniqueness of the solutions. It will be seen that a number of different types of solution to the system (1) exist, and that analogous solutions exist to the system (2). Unless otherwise mentioned, all quantities are real, and if the series $\sum_{n=0}^{\infty} a_n t^n$ converges with nonzero radius of convergence r, its sum will be written $A(t)$. $A(t)$ may be termed the generating function.

2. Before proceeding to solutions of the system (1) when the set $\{a_n\}$ is arbitrary, and of the system (2) when the function $a(u)$ is arbitrary, it is worth noting the following obvious solutions: if $\sum_{n=0}^{\infty} a_n = 0$, then a solution to the system (1) is $y = 1$.

Similarly, if $\int_0^{\infty} a(u)\, du = 0$, then a solution to the system (2) is $y = 1$.

3. The obvious first form of solution to look for is a power series. When such a series converges, it will provide a unique solution.

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THEOREM 1. The power-series solution of the differential equation (1) is given by

$$y(x) = \sum_{m=0}^{\infty} \prod_{s=1}^{m} A(\mu^{s-1})x^m/(m!),$$

the empty product being unity whenever the right-hand side has a meaning.

Consider now the system (1) and look for a solution of the form

$$y(x) = \sum_{m=0}^{\infty} y_m x^m/(m!). \tag{3}$$

Note that $y_0 = y(0) = 1$.

Substituting the relation (3) in the differential equation (1), it follows that

$$\sum_{m=0}^{\infty} y_{m+1}x^m/(m!) = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} y_m\mu^{mn}x^m/(m!)$$

$$= \sum_{m=0}^{\infty} y_m \sum_{n=0}^{\infty} a_n\mu^{mn}x^m/(m!) \tag{4}$$

$$= \sum_{m=0}^{\infty} y_m A(\mu^m)x^m/(m!). \tag{5}$$

If the radius of convergence r of the power series associated with $A(t)$ is infinite, the relation (5) is always valid. This will always be the case if $A(t)$ is a polynomial. If, however, r is finite, the relation will be valid when $|\mu^m| < r$ for all positive or zero m. The condition for this to hold is clearly $|\mu| < \min(r, 1)$.

From Eq. (5) it follows, on comparing powers of x on both sides of the equation, that

$$y_{m+1} = A(\mu^m)y_m, \tag{6}$$

whence

$$y_m = \prod_{s=1}^{m} A(\mu^{s-1}) \tag{7}$$

(the empty product being, as always, unity), and so

$$y(x) = \sum_{m=0}^{\infty} \prod_{s=1}^{m} A(\mu^{s-1})x^m/(m!). \tag{8}$$

The series on the right-hand side terminates at the term x^p if $A(\mu^p)$ vanishes where p is an integer; furthermore, the series converges by D'Alembert's test if the ratio of the modulus of the ratio of the $(m + 1)$th term to the mth term is less than unity. This condition can be written

$$\lim_{m\to\infty} \left| \frac{x}{m+1} A(\mu^m) \right| < 1. \tag{9}$$

Clearly, if $|\mu|$ is less than unity, this is the case, and the solution is valid for all x, as then $\lim_{m\to\infty} A(\mu^m) = a_0$. If $|\mu|$ exceeds unity, there will not be convergence if r is finite. If the radius of convergence is infinite, there may or may not be convergence depending upon $A(t)$. There remain two cases to be considered. If $\mu = 1$, the differential equation reduces to $dy/dx = \sum_{n=0}^{\infty} a_n y(x)$, which requires no further discussion. If $\mu = -1$, the differential equation reduces to

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_{2n} y(x) + \sum_{n=1}^{\infty} a_{2n-1} y(-x).$$

This can be rewritten in the form

$$dy/dx = py(x) - qy(-x). \tag{10}$$

It can easily be verified that the solution of this under the given initial condition is

$$y = \cosh (p^2 - q^2)^{1/2} x + \{(p + q)/(p - q)\} \sinh (p^2 - q^2)^{1/2} x, \qquad p > q \tag{11a}$$

$$= 1 + 2px, \qquad p = q \tag{11b}$$

$$= \cos (q^2 - p^2)^{1/2} x + \{(q + p)/(q - p)\}^{1/2} \sin (p^2 - p^2)^{1/2} x, \qquad p < q. \tag{11c}$$

It may be remarked that if the a_n and $|\mu|$ are complex, the results given above for $|\mu| > 1$ and $|\mu| < 1$ will still hold. If, however $|\mu| = 1$ the problem is more complicated. If $\mu = \exp \{i2\pi m/n\}$, where m and n are integers, it is not difficult to see that the differential equation (1) assumes the form

$$\frac{dy}{dx} = \sum_{t=0}^{n=1} A_t y(x \exp \{2\pi i t/n\}), \tag{12}$$

a solution to which can be found by writing

$$y = \sum_{s=0}^{n-1} y_s(x \exp \{2\pi i s/n\}) \tag{13}$$

and solving n simultaneous equations for the y_s. If, however, $\mu = \exp \{i2\pi\lambda\}$, where λ is irrational, $\lim_{n\to\infty} y(x \exp \{i2\pi\lambda n\})$ does not exist and the sum of the infinite series is not defined.

4. A power-series solution may also be obtained for the system (2).

THEOREM 2. The power-series solution of the differential equation (2) is given by

$$y(x) = \sum_{m=0}^{\infty} \prod_{s=0}^{m-1} \bar{a}(s\alpha) x^m/(m!)$$

(where $1 > \mu = \exp \{-\alpha\} > 0$, $\bar{a}(p)$ is the Laplace transformation of $a(u)$, and the empty product is unity) whenever the right-hand side has a meaning.

Consider now the system (2) and write

$$y(x) = \sum_{m=0}^{\infty} \eta_m x^m/(m!) \qquad (\eta_0 = 1). \tag{14}$$

Substituting in Eq. (2), it follows that

$$\sum_{m=0}^{\infty} \eta_{m+1} x^m/(m!) = \int_0^{\infty} a(u) \sum_{m=0}^{\infty} \eta_m \mu^{um} x^m/(m!). \tag{15}$$

On equating powers of x, it follows that

$$\eta_{m+1} = \left[\int_0^{\infty} a(u) \mu^{um}\, du\right] \eta_m \tag{16}$$

$$= \bar{a}(\alpha m) \tag{17}$$

where $\bar{a}(p) = \int_0^\infty a(u) \exp\{-pu\}\, du$ and $\mu = \exp(-\alpha)$. Thus

$$\eta_m = \prod_{s=0}^{m-1} \bar{a}(\alpha s), \tag{18}$$

and

$$y(x) = \sum_{m=0}^{\infty} \prod_{s=0}^{m-1} \bar{a}(\alpha s) x^m/(m!). \tag{19}$$

The series (14) will converge, giving a unique solution of the differential equation, if

$$\lim_{m\to\infty} |(\eta_{m+1}x)/\{\eta_m(m+1)\}| < 1,$$

that is,

$$\lim_{m\to\infty} |x\bar{a}(\alpha m)/(m+1)| < 1. \tag{20}$$

A formula for the behavior of $\bar{a}(p)$ for certain types of functions $a(u)$ has been given by Doetsch [1]. Suppose that $a(u)$ is a regularly increasing function, that is $a(u) \sim u^\beta L(u)$ as $u \to 0$, where $\beta + 1 > 0$, and $L(u)$ is a slowly increasing function, that is

$$\lim_{u\to 0} (L(\gamma u)/L(u)) = 1 \tag{21}$$

for all positive γ. Then, under these conditions

$$\bar{a}(p) \sim \Gamma(\beta + 1) p^{-\beta-1} L(p^{-1}) \tag{22}$$

for $p \to \infty$. By using the result (22), the convergence condition (20) becomes

$$\lim_{m\to\infty} |xm^{-(\beta+2)} L(\alpha^{-1} m^{-1})| < 1.$$

It can easily be seen that the limit is in fact zero, and so there is convergence. It is necessary for this that $0 < \mu < 1$, as otherwise α (and α^{-1}) is not real and positive.

Thus if $0 < \mu < 1$, the power series provides a solution to the differential equation (2). If $\mu = 0$, the solution is

$$y(x) = 1 + x \int_0^\infty A(u)\, du \tag{23a}$$

and if $\mu = 1$, the solution is

$$y(x) = \exp\left\{x \int_0^\infty A(u)\, du\right\}. \tag{23b}$$

If $1 < \mu$, the quantity $\int_0^\infty a(u)\mu^{um}\, du$ will increase indefinitely with m and there will not be convergence. If μ is not real and positive, and consequently the integral on the right-hand side of (2) is not defined uniquely, the problem is not defined.

5. A solution of the differential equation (1) in terms of exponentials can also be obtained.

THEOREM 3. A solution of problem (1) is given by

$$y(x) = \sum_{m=0}^{\infty} Y_m \exp\{a_0 \mu^m x\}, \tag{24}$$

where Y_m is defined as the coefficient of t^m in the power-series expansion of Y

$$Y(t) = \prod_{s=0}^{\infty} \{A(\mu^s)/A(\mu^s t)\} \tag{25}$$

whenever $|\mu| < 1$.

The way this solution is obtained is as follows. Look for a solution of the form

$$y(x) = \sum_{m=0}^{\infty} Y_m \exp \{\alpha\mu^m x\} \tag{26}$$

where the Y_m are functions of μ and α is to be determined. On substituting the expression (26) into the differential equation (1), it will be necessary, if the solution is of the form (26), for the following equation to hold:

$$\sum_{m=0}^{\infty} \alpha\mu^m Y_m \exp \{\alpha\mu^m x\} = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} Y_m \exp \{\alpha\mu^{m+n} x\}. \tag{27}$$

Equating terms in $\exp \{\alpha\mu^{m+n}x\}$ in Eq. (27), there follows the recurrence relation

$$\alpha\mu^m Y_m = \sum_{s=0}^{m} a_{m-s} Y_s\,, \qquad 0 \le m. \tag{28}$$

Putting $m = 0$, it follows that

$$\alpha = a_0 = A(0). \tag{29}$$

Suppose that

$$Y(t) = \sum_{m=0}^{\infty} Y_m t^m. \tag{30}$$

The relation (28) is equivalent to the result

$$a_0 Y(t) = A(t) Y(t). \tag{31}$$

From Eq. (26), it follows that

$$y(0) = \sum_{m=0}^{\infty} Y_m = Y(1). \tag{32}$$

Thus the solution of the differential equation (1) is given by the solution of the functional relationship (31).

The solution of the functional relationship (31) will, however, be indefinite by a factor $B(t)$, where

$$B(\mu t) = B(t). \tag{33}$$

If $\mu = \exp \beta$, it is easy to see that B is of the form

$$B(t) = \sum_{n=-\infty}^{\infty} b_n t^{(2\pi i n)/\beta}. \tag{34}$$

This, however, is not a power series in t, except for the trivial case $B(t) = b_0$. Thus, the solution of the relation (31) *as a power series* will be unique, apart from a constant factor which can be taken so as to make the solution satisfy the initial condition

$$y(0) = Y(1) = 1. \tag{35}$$

A formal solution of the relation (31) is given by

$$Y(t) = b_0 \prod_{s=0}^{\infty} \{a_0/A(\mu^s t)\} \tag{36}$$

and so

$$y(0) = b_0 \prod_{s=0}^{\infty} \{a_0/A(\mu^s)\}. \tag{37}$$

Using the condition (35), it follows that

$$Y(t) = \prod_{s=0}^{\infty} \{A(\mu^s)/A(\mu^s t)\}. \tag{38}$$

If $|\mu|$ is less than unity,

$$\lim_{s\to\infty} A(\mu^s t) = \lim_{s\to\infty} A(\mu^s) = A(0) = a_0 . \tag{39}$$

It may be verified that this infinite product converges, for the sth term is, if s is large, given to the first order by

$$(A(0) + A'(0)\mu^s)/\{A(0) + A'(0)\mu^s t\} = 1 + \{A'(0)/A(0)\}\mu^s(1 - t), \tag{40}$$

as μ^s will be small. The convergence of the infinite product will depend on the convergence of the series whose sth term is

$$\{A'(0)/A(0)\}\mu^s(1 - t). \tag{41}$$

This is the sth term of a geometric series, and as $|\mu|$ is less than unity, the series converges. Thus, if $|\mu|$ is less than unity, the generating function $Y(t)$ can be determined uniquely, provided that the series (34) converges. This result will hold, in fact, even if μ is complex.

6. It may be shown that a solution to problem (2) analogous to that given in Sec. 5 for problem (1) is not possible. If the possibility of a solution of the form

$$y(x) = \int_0^\infty \eta(t) \exp \{\beta\mu^t x\}\, dt \tag{42}$$

where β is to be found, be considered, then

$$\int_0^\infty \beta\mu^t \eta(t) \exp \{\beta\mu^t x\}\, dt = \int_0^\infty a(u) \int_0^\infty \eta(t) \exp \{\beta\mu^{t+u} x\}\, dt\, du. \tag{43}$$

Writing $t + u = w$, it follows that the right-hand side of (43) becomes

$$\int_0^\infty \exp \{\beta\mu^w x\} \int_0^w a(u)\eta(w - u)\, du\, dw. \tag{44}$$

t and w are dummy variables and so it follows that for a solution of the type (42) to be possible,

$$\beta\mu^w \eta(w) = \int_0^w a(w - u)\eta(u)\, du, \tag{45}$$

where a property of the Faltung has been used on the right-hand side. Writing $\mu^w \eta(w) = F(w)$ and $\mu^{-u} a(w - u) = K(w, u)$, we can rewrite Eq. (45) as

$$\beta F(w) = \int_0^w K(w, u) F(u) \, du. \tag{46}$$

β is as yet unspecified, and would be the eigenvalue of the system defined by Eq. (46). This, however, is a Volterra integral equation with a regular kernel and has no eigenvalues [2]. It follows therefore that a solution of the type (42) does not exist.

7. It is also possible to obtain solutions in terms of descending powers of x. Such series will in general be asymptotic rather than convergent.

THEOREM 4. A formal solution of the differential equation (1) without any boundary condition is given by

$$y(x) = \sum_{n=0}^{\infty} \left\{ \prod_{s=1}^{n} A(\xi \mu^{-s}) \right\}^{-1} \frac{x^{c-n}}{\Gamma(c - n + 1)}$$

where ξ is a zero of $A(t)$ and c is defined by the relation $\xi = \mu^c$, whenever the expression is meaningful.

Consider now the possibility of a solution of the differential equation (1) of the form

$$y = \sum_{n=0}^{\infty} z_n \frac{x^{c-n}}{\Gamma(c - n + 1)}, \tag{47}$$

where c is as yet undefined. If c is a positive integer the series will terminate and there will simply be a power series of the type discussed previously:

$$\begin{aligned} \frac{dy}{dx} &= \sum_{n=0}^{\infty} z_n \frac{x^{c-n-1}}{\Gamma(c - n)} \\ &= \sum_{n=1}^{\infty} z_{n-1} \frac{x^{c-n}}{\Gamma(c - n + 1)} \\ y(\mu^n x) &= \sum_{n=0}^{\infty} z_n \mu^{m(c-n)} \frac{x^{c-n}}{\Gamma(c - n + 1)}. \end{aligned} \tag{48}$$

The differential equation (1) now assumes the form

$$\begin{aligned} \sum_{n=1}^{\infty} z_{n-1} \frac{x^{c-n}}{\Gamma(c - m + 1)} &= \sum_{m=0}^{\infty} a_m \sum_{n=0}^{\infty} \frac{z_n \mu^{m(c-n)} x^{c-n}}{\Gamma(c - n + 1)} \\ &= \sum_{n=0}^{\infty} A(\mu^{c-n}) z_n \frac{x^{c-n}}{\Gamma(c - n + 1)}. \end{aligned} \tag{49}$$

For Eq. (49) to hold,

$$0 = A(\mu^c) z_0 , \tag{50}$$

$$z_{n-1} = A(\mu^{c-n}) z_n . \tag{51}$$

Thus, if ξ is a zero of $A(t)$,

$$z_n = \left\{ \prod_{s=1}^{n} A(\xi \mu^{-s})^{-1} z_0 \right\} \tag{52}$$

and so an asymptotic solution is given by

$$y = z_0 \sum_{n=0}^{\infty} \left\{ \prod_{s=0}^{n} A(\xi\mu^{-s} \right\}^{-1} \frac{x^{c-n}}{\Gamma(c-n+1)} \tag{53}$$

where

$$\xi = \mu^c. \tag{54}$$

The ratio of the nth to the $(n - 1)$th term is given by

$$(c - n + 1)/\{xA(\xi\mu^{-n})\}. \tag{55}$$

If $|\mu| > 1$, this behaves for large n as $-nxA(0)$, the nth term behaving as $(-1)^n n!/\{xA(0)\}^n$, and the series is asymptotic. When $|\mu| < 1$, the behavior of expression (55) is not so easily determined.

Suppose that ρ is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n t^n$. Then Eq. (51) can only be meaningful so long as $|\mu^{c-n}| < \rho$, i.e. $|\xi|\ |\mu|^{-n} < \rho$. Suppose that now N is the positive integer such that

$$|\mu|^N > |\xi|/\rho > |\mu|^{N+1}. \tag{56}$$

Then the relation (51) will only be meaningful if $z_n = 0$ for $n > N$, and in this case the series terminates. If, however, the series $\sum_{n=0}^{\infty} a_n t^n$ is convergent for all t, then the ratio (55) will depend upon the behavior of $A(t)$ for $|t|$ large. This will depend upon the phase of t and every case must be discussed separately, the outcome depending upon the phases of ξ and μ. When μ is real and positive, it will be the phase of ξ that will be relevant. It should be noted that this method of solution will give rise in fact to an infinite set of possible solutions. For Eq. (54) has the infinite set of solutions for c given by $\log \xi = c \log \mu - 2\pi pi$ where p is an arbitrary integer; that is,

$$c = (\log \xi + 2\pi pi)/(\log \mu). \tag{57}$$

Thus the series of Eq. (54) will in general involve complex c, the value of the series will be complex, and both the real and imaginary parts will provide solutions. This phenomenon will occur with a number of other solutions discussed in this paper and will not be mentioned again.

8. THEOREM 5. A formal solution of the differential equation (2) without any initial condition is given by

$$y(x) = \sum_{n=0}^{\infty} \left\{ \prod_{s=1}^{n} \bar{a}(\zeta + s \log \mu) \right\}^{-1} \frac{x^{\beta-n}}{\Gamma(\beta - n + 1)}$$

where $\bar{a}$ is the Laplace transform of a, ζ is a zero of $\bar{a}$, and β is defined by the relation $\zeta = -\beta \log \mu$, whenever the expression is meaningful.

Consider now the possibility of finding a solution of the form

$$y = \sum_{n=0}^{\infty} q_n \frac{x^{\beta-1}}{\Gamma(\beta - n + 1)} \tag{58}$$

to the differential equation (2). It is not difficult to see that substituting the expression (58) into (2) gives the relation

$$\sum_{n=1}^{\infty} q_{n-1} \frac{x^{\beta-n}}{\Gamma(\beta - n + 1)} = \sum_{n=0}^{\infty} \int_0^{\infty} a(u)\mu^{u(\beta-n)} \frac{x^{\beta-n}\, du}{\Gamma(\beta - n + 1)}. \tag{59}$$

(59) holds when

$$0 = \left[\int_0^\infty a(u)\mu^{u\beta}\, du\right] q_0 \tag{60}$$

and

$$q_{n-1} = \left[\int_0^\infty a(u)\mu^{u(\beta-n)}\, du\right] q_n . \tag{61}$$

If we write $\mu = \exp\{\log \mu\}$, the relations (60) and (61) become

$$0 = \bar{a}(-\beta \log \mu), \tag{62}$$

$$q_{n-1} = \bar{a}(n - \beta \log \mu) q_n , \tag{63}$$

where $\bar{a}$ denotes the Laplace transform of a. If ζ is a zero of $\bar{a}$, the recurrence relation (63) can be written as

$$q_{n-1} = \bar{a}(\zeta + n \log \mu) q_n \tag{64}$$

and the formal solution is given by

$$q_n = q_0 \prod_{s=1}^{n} [\bar{a}(\zeta + s \log \mu)]^{-1}, \tag{65}$$

$$y(x) = q_0 \sum_{n=0}^{\infty} \left[\prod_{s=1}^{n} \bar{a}(\zeta + s \log \mu)\right]^{-1} \frac{x^{\beta-n}}{\Gamma(\beta - n + 1)}. \tag{66}$$

The ratio of the nth to the $(n - 1)$th term is given by

$$(c - n + 1)/[x\bar{a}(n \log \mu + \zeta)]. \tag{67}$$

As before, it is now necessary to determine the conditions under which the solution exists.

The first point which arises is the condition that $\bar{a}(\zeta)$ exists. This is equivalent to the statement that there exists a number β such that $\int_0^\infty a(u)\mu^{u\beta}\, du$ exists and has value zero. The cases $0 < \mu < 1$ and $\mu > 1$ require separate consideration. If $\mu > 1$, $\log \mu$ is positive, and so $n \log \mu + \zeta \geq \zeta$ always. Now the Laplace transform

$$\bar{f}(p) = \int^\infty e^{-pt} f(t)\, dt$$

either for all p or for all p such that Re $p \geq$ some real number. This depends upon the nature of the function $f(t)$. Thus if $\bar{a}(\zeta)$ exists, so also does $\bar{a}(n \log \mu + \zeta)$. Using the formula (22) for the asymptotic behavior of $\bar{a}(p)$ for large p, it follows that

$$\bar{a}(n \log \mu + \zeta) \sim \Gamma(\beta + 1)(n \log \mu + \zeta)^{-\beta-1} L([n \log \mu + \zeta]^{-1}), \tag{68}$$

where $\beta + 1 > 0$, and the expression (67) becomes asymptotically equal to

$$(c - n + 1)(n \log \mu + \zeta)^{\beta+1}/\{x\Gamma(\beta + 1)L(n \log \mu + \zeta]^{-1}.$$

The behavior of this is effectively

$$-n^{\beta+2}(\log \mu)^{\beta+1}/x\{\Gamma(\beta + 1)L(0)\}. \tag{69}$$

It follows therefore that the series (66) can never converge, but is asymptotic. If, however, $0 < \mu < 1$, and so $\log \mu$ is negative, the previous analysis does not hold. If, on the one hand, $a(u)$ is a function whose Laplace transform is defined for all p, the recurrence relation (61) is always meaningful and the behavior of the series (66) can be determined for each individual function $a(u)$. If, on the other hand, this is not the case, there will be a value of N of n such that $\int_0^\infty a(u)\mu^{u(\beta-N)}\,du$ exists and $\int_0^\infty a(u)\mu^{u(\beta-N-1)}\,du$ diverges. The recurrence relation (61) thus implies that q_n vanishes for $n \geq N+1$; that is, there are only a finite number of terms in the series.

9. Formal solutions to the equations can also be obtained in terms of a set of generalized functions defined as follows. Let

$$\phi_\alpha(x) = \frac{x^\alpha}{\Gamma(\alpha+1)} H(x) \qquad (\mathrm{Re}\,\alpha \geq 0) \tag{70}$$

where $H(x) = 1, x > 0; = \frac{1}{2}, x = 0; = 0, x < 0$; and let $\phi_\alpha(x)$ be defined when α has a negative real part by as many applications as are necessary of the recurrence relation

$$\phi_\alpha'(x) = \phi_{\alpha-1}(x). \tag{71}$$

This recurrence relation is clearly satisfied for Re $\alpha \geq 0$ also. If α is an integer n, then

$$\begin{aligned} \phi_n(x) &= (x^n/n!)H(x), & n \geq 0 \\ &\delta^{(1-n)}(x), & n < 0, \end{aligned} \tag{72}$$

where $\delta^m(x)$ is the mth derivative of the delta function. A further property of the $\{\phi_\alpha\}$ is that

$$\phi_\alpha(\lambda x) = \lambda^\alpha \phi_\alpha(x). \tag{73}$$

THEOREM 6. A formal solution of the differential equation (1) without any boundary condition is given by

$$y(x) = \sum_{n=0}^{\infty} \left\{ \prod_{s=1}^{n} A(\xi\mu^{-s}) \right\}^{-1} \phi_{c-n}(x)$$

under the same conditions as in Theorem 4.

The proof is almost identical with that of Theorem 4, and will not be given here.

THEOREM 7. A formal solution of the differential equation (2) without any boundary condition is given by

$$y(x) = \sum_{n=0}^{\infty} \left\{ \prod_{s=1}^{n} \bar{a}(\zeta + s \log \mu) \right\} \phi_{\beta-n}(x)$$

under the same conditions as in Theorem 4.

The proof is almost identical with that of Theorem 5 and will not be given here.

Although the coefficients in the series of Theorems 4 and 6 and Theorems 5 and 7 are respectively the same, the nature of the series given by Theorems 6 and 7 are different from those given by Theorems 4 and 5. The series given by Theorems 4 and 6 are in general infinite series in descending powers of x, and are valid for positive or negative x. The series given by Theorems 5 and 7 have a zero value for negative x, and very sharp singularities at the origin.

10. The differential equation

$$\frac{dy}{dx} = y(x) + \alpha y(ux) \quad y(0) = 1 \tag{74}$$

occurs, when $0 < \mu < 1$, in the problem of the dynamics of a current collection system for an electric locomotive [3]. Apart from the two obvious cases where $\mu = 1$ and $\alpha = 0$, two other cases can be solved by inspection. If $\alpha = -1$, $y = 1$ and if $\mu = -1$,

$$y = \cosh (1 - \alpha^2 x)^{1/2} + \frac{1 + \alpha}{1 - \alpha} \sinh (1 - \alpha^2 x)^{1/2}. \tag{75}$$

The generating function is $1 + \alpha t$. The power series solution for Eq. (74) in the general case follows from Sec. 3 and is

$$y = \sum_{m=0}^{\infty} \frac{x^m}{m!} \prod_{s=1}^{m} (1 + \alpha\mu^{m-1}). \tag{76}$$

The condition for convergence is

$$\lim_{m\to\infty} \left| \frac{x}{m + 1} (1 + \alpha\mu^{m-1}) \right| < 1.$$

If $|\mu| \leq 1$, $|1 + \alpha\mu^{m-1}| \leq 1 + |\alpha|$ and

$$\lim_{m\to\infty} \frac{x(1 + \alpha\mu^{m-1})}{m + 1} = 0$$

If $|\mu| > 1$,

$$\left| \frac{1 + \alpha\mu^{m-1}}{m + 1} \right| \to \infty .$$

Thus the power series converges for $|\mu| \leq 1$, and diverges for $|\mu| > 1$. If a solution of the form $\sum_{m=0}^{\infty} y_m \exp \{\mu^m x\}$ is looked, for the recurrence relation becomes

$$\mu^m y_m = y_m + \alpha y_{m-1} . \tag{77}$$

This can be solved simply, and it is not necessary to use the result of Sec. 3:

$$y_m(\mu^m - 1) = \alpha y_{m-1} \tag{78}$$

and so

$$y_m = \alpha^m \prod_{s=1}^{m} \frac{1}{(\mu^s - 1)}. \tag{79}$$

Thus

$$y = \frac{\sum_{m=0}^{\infty} \prod_{s=1}^{m} \frac{1}{(\mu^s - 1)} \alpha^m \exp \{\mu^m x\}}{\left\{ \sum_{m=0}^{\infty} \prod_{s=1}^{m} \frac{1}{(\mu^s - 1)} \alpha^m \right\}}. \tag{80}$$

This solution is clearly that which would be obtained by looking for an expression in powers of α, the division being necessary to make $y(0) = 1$. The convergence criterion is again given through D'Alembert's test. The ratio is given by

$$\frac{\alpha}{\mu^{m+1} - 1} \{\exp \{(\mu^{m+1} - \mu^m)x\}\}.$$

If $|\mu| < 1$, the ratio is $-\alpha$ and there is accordingly convergence if $|\alpha| < 1$, and diverges if $|\alpha| > 1$. If $|\mu| > 1$ the ratio is infinite, and the series does not converge.

Consider now the differential equation

$$dy/dx = y(x) - \mu^{-p} y(\mu x) \tag{81}$$

This is a special case of Eq. (74) with $\alpha = -\mu^{-p}$. Look for a solution of the form

$$y = \sum_{n=0}^{\infty} g_n \frac{x^{p-n}}{\Gamma(p-n+1)}. \tag{82}$$

The series terminates if p is a positive integer. (The same is true of the solution given by the series (76).) It is easy to see that they are effectively the same solution, and will be valid for all values of μ and x. Substituting the expansion (82) into the differential equation (81), it follows that

$$\sum_{n=1}^{\infty} g_{n-1} \frac{x^{p-n}}{\Gamma(p+n-1)} = \sum_{n=0}^{\infty} g_n \frac{x^{p-n}}{\Gamma(p-n+1)} - \sum_{n=0}^{\infty} g_n \frac{x^{p-n}}{\Gamma(p-n+1)} \mu^{-n}.$$

Comparing terms in x^{p-n}, it follows that $g_{n-1} = g_n(1 - \mu^{-n})$ and so

$$g_n = g_0 \prod_{s=0}^{n} (1 - \mu^{-s})^{-1}.$$

g_0 represents the "scale" of the solution and may be assumed unity. (It is fairly easy to see that the initial condition $y(0) = 1$ is not possible unless p is a positive integer.) The D'Alembert ratio is

$$x^{-1}(p-n) \cdot \frac{1}{1 - \mu^{-(n+1)}}.$$

If $|\mu| > 1$, this behaves as $x^{-1}(p-n)$, and so there is divergence. If $|\mu| < 1$, this behaves as $x^{-1}(p-n)\mu^{(n+1)}$ and so there is convergence. Finally, a formal solution of Eq. (81) in terms of the infinite set of functions $\phi_\alpha(x)$ defined by Eq. (70) will give rise to a series with exactly the same coefficients:

$$y = \sum_{n=0}^{\infty} \prod_{s=1}^{n} (1 - \mu^{-s})^{-1} \phi_{p-n}(x). \tag{83}$$

It will be observed that all the different methods of solution for the differential equation (74) discussed here give infinite series which converge when $|\mu| < 1$. It seems impossible to find solutions which converge for $|\mu| > 1$. This is presumably linked up with the fact that the solution is not unique for $|\mu| > 1$. A fairly full discussion of the differential equation (74) has been given by Kato and McLeod [4]. There is, however, no overlap with the treatment here.

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