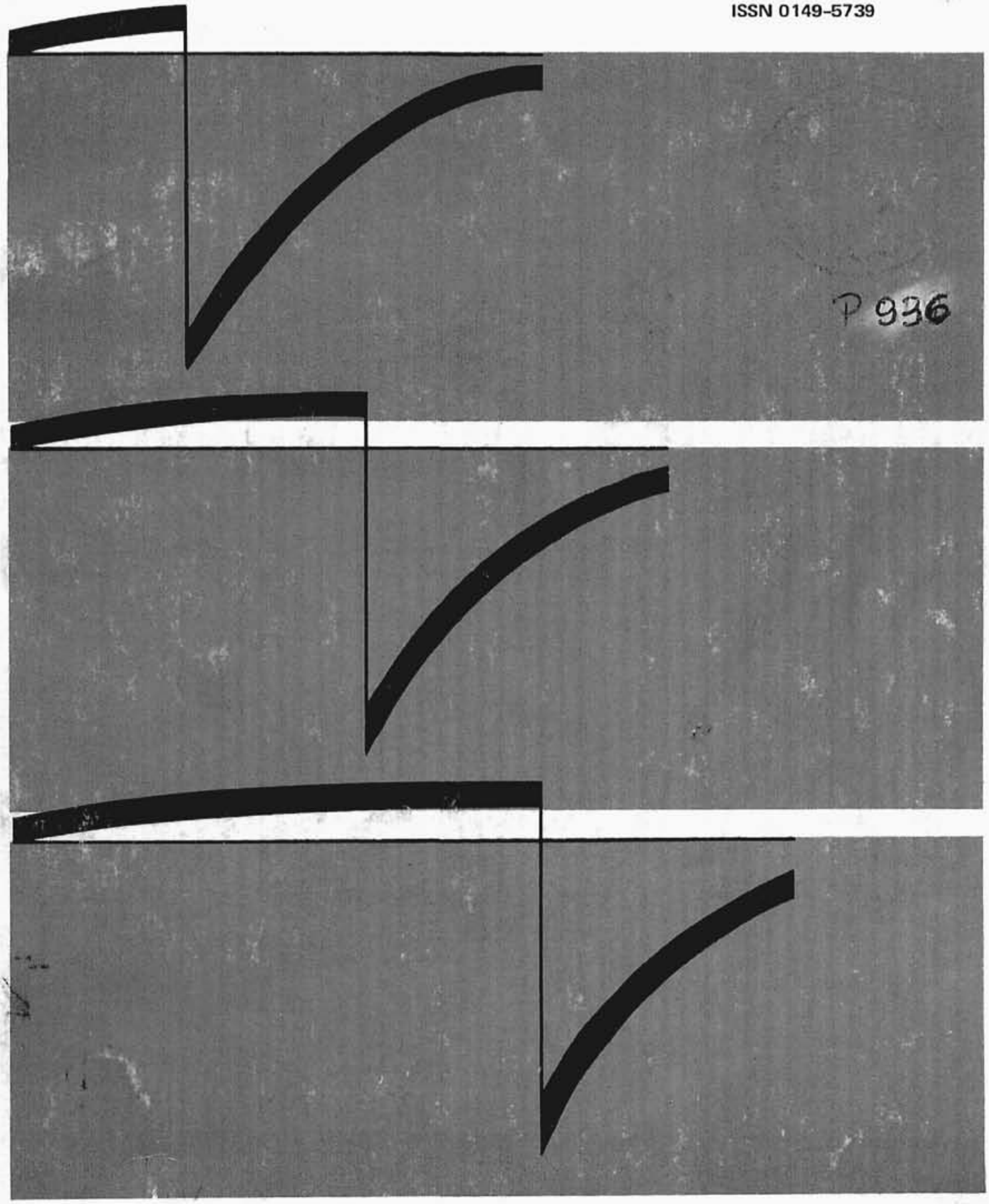
JOURNAL OF THERMAL STRESSES

EDITORIAL BOARD

AN INTERNATIONAL QUARTERLY

VOLUME 1 • NUMBER 2 • OCTOBER 1978

JTSTDA 1(2) 149-249 (1978)
ISSN 0149-5739



SOME GENERAL THEOREMS OF THERMOPIEZOELECTRICITY

W. Nowacki
Institute of Mechanics
University of Warsaw
Warsaw, Poland

In this paper a uniqueness theorem for the solutions of the differential equations of thermopiezoelectricity is given, on the basis of the energy balance. The generalized Hamilton principle and the theorem of reciprocity of work are also deduced.

INTRODUCTION

The equations governing small vibrations of piezoelectric crystals, including the coupling among deformation, temperature, and electric field, have been derived by Mindlin [1]. The coupled problem under consideration consists of determining the stresses $\sigma_{ij}(\mathbf{x}, t)$ and strains $\epsilon_{ij}(\mathbf{x}, t)$, displacement $u_i(\mathbf{x}, t)$, temperature $\theta(\mathbf{x}, t)$ and electric potential $\varphi(\mathbf{x}, t)$ for $\mathbf{x} \in B$ and $t > 0$.

In the region B and for $t > 0$, the following equations should be satisfied: the equation of motion

$$\sigma_{ji,j} + X_i = \rho \ddot{u}_i \qquad i,j = 1, 2, 3 \tag{1}$$

the generalized heat equation

$$T\dot{S} = k_{ij}\theta_{,ji} + W \qquad \theta = T - T_0 \tag{2}$$

and the equation of the quasistationary electric field

$$D_{i,i} = 0 \tag{3}$$

where X_i = components of mass forces
S = entropy per unit volume
D_i = components of electric displacement
T = absolute temperature
T_0 = temperature of natural state in which stresses and strains are zero

Journal of Thermal Stresses, 1:171–182, 1978

0149-5739/78/020171-12$2.75

W = heat source intensity, referred to a unit volume of the body and unit time
k_{ij} = coefficients of heat conduction for an anisotropic body
ρ = density

These equations should be completed with boundary and initial conditions. The following quantities may be assigned at the surface ∂B of the body: the displacements or loads

$$u_i = U_i(\mathbf{x}, t) \qquad \text{on} \qquad \partial B_1$$
$$\sigma_{ji} n_j = p_i(\mathbf{x}, t) \qquad \text{on} \qquad \partial B_2 \tag{4}$$
$$\partial B = \partial B_1 \cup \partial B_2 \qquad \partial B_1 \cap \partial B_2 = 0$$

the temperature or heat flux

$$\theta = \vartheta(\mathbf{x}, t) \qquad \text{on} \qquad \partial B_3$$
$$-k_{ij}\theta_{,j} n_i = k(\mathbf{x}, t) \qquad \text{on} \qquad \partial B_4 \tag{5}$$
$$\partial B = \partial B_3 \cup \partial B_4 \qquad \partial B_3 \cap \partial B_4 = 0$$

and the electric field or the electric charge

$$\varphi = \phi(\mathbf{x}, t) \qquad \text{on} \qquad \partial B_5$$
$$D_i n_i = -\sigma(\mathbf{x}, t) \qquad \text{on} \qquad \partial B_6 \tag{6}$$
$$\partial B = \partial B_5 \cup \partial B_6 \qquad \partial B_5 \cap \partial B_6 = 0$$

The initial conditions have the form

$$u_i(\mathbf{x}, 0) = f_i(\mathbf{x}) \qquad \dot{u}_i(\mathbf{x}, 0) = g_i(\mathbf{x}) \qquad \theta(\mathbf{x}, 0) = h(\mathbf{x}) \qquad \mathbf{x} \in B \qquad t = 0 \tag{7}$$

In addition we have constitutive equations in the form [1]

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} - e_{kij}E_k - \gamma_{ij}\theta \tag{8}$$

$$D_i = e_{ikl}\epsilon_{kl} + \epsilon_{ik}E_k + g_i\theta \tag{9}$$

$$S = \gamma_{ji}\epsilon_{ji} + g_i E_i + \frac{c_\epsilon}{T_0}\theta \tag{10}$$

where

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad E_i = -\varphi_{,i} \tag{11}$$

Equation (8) is the Duhamel-Neumann equation, generalized to piezoelectricity. Equation (9) is an expression for the electric displacement, and Eq. (10) is an expression for the entropy in terms of the independent variables ϵ_{ij}, E_k, and θ.

Observe that Eqs. (8)-(10) lead to the constitutive relations

$$\frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \epsilon_{ij}} \qquad \frac{\partial D_i}{\partial E_j} = \frac{\partial D_j}{\partial E_i} \tag{12}$$

and

$$\frac{\partial \sigma_{ij}}{\partial E_k} = -\frac{\partial D_k}{\partial \epsilon_{ij}} \qquad \frac{\partial \sigma_{ij}}{\partial T} = -\frac{\partial S}{\partial \epsilon_{ij}} \qquad \frac{\partial S}{\partial E_i} = \frac{\partial D_i}{\partial T} \tag{13}$$

Relations (12) imply that

$$c_{ijkl} = c_{klij} \qquad \epsilon_{ij} = \epsilon_{ji} \tag{14}$$

The symmetry of the tensors σ_{ij} and ϵ_{ij} leads to the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{ijlk} \qquad \gamma_{ij} = \gamma_{ji} \qquad e_{kij} = e_{kji} \tag{15}$$

In the case of general anisotropy, we have 21 constants c_{ijkl}, 18 piezoelectric constants e_{kij}, 6 constants γ_{ij} and ϵ_{ij} and 3 constants g_i. There also appears the constant c_ϵ, which is the specific heat at constant strain and constant electric field E_k. Altogether there are 55 material constants. Introducing the constitutive relations (8)-(10) into the equations of motion (1), the heat equation (2), and the equation of the electric field (3), we have

$$c_{ijkl} u_{k,lj} + e_{kij} \varphi_{,kj} - \gamma_{ij} \theta_{,j} = \rho \ddot{u}_i \tag{16}$$

$$k_{ij} \theta_{,ij} - c_\epsilon \dot{\theta} - T_0 (\gamma_{ij} \dot{\epsilon}_{ij} - g_i \dot{\varphi}_{,i}) = -W \tag{17}$$

$$e_{ikl} u_{k,li} - \epsilon_{ik} \varphi_{,ki} + g_i \theta_{,i} = 0 \qquad E_i = -\varphi_{,i} \tag{18}$$

Assuming that $|\theta / T_0| \ll 1$, we arrive at the linear heat-conduction equation (17).

Equations (16)-(18) constitute the complete set of equations of thermopiezoelectricity. This set of equations is coupled. In a stationary problem, Eq. (17) becomes the Poisson equation

$$k_{ij} \theta_{,ij} = -W \tag{19}$$

while Eqs. (16) and (17) are still coupled. In this case the function θ is known from Eq. (19).

UNIQUENESS OF THE SOLUTION OF THE DIFFERENTIAL EQUATIONS OF THERMOPIEZOELECTRICITY

To prove the uniqueness of the solution of the differential equations of thermopiezoelectricity, we need a modified energy balance. It follows from the principle of virtual work

$$\int_B (X_i - \rho\ddot{u}_i)\delta u_i \, dv + \int_{\partial B} p_i \delta u_i \, da = \int_B \sigma_{ij}\delta\epsilon_{ij} \, dv \tag{20}$$

in which the virtual increments have been replaced by the real increments

$$\delta u_i = \frac{\partial u_i}{\partial t} dt = v_i \, dt \qquad \delta\epsilon_{ij} = \frac{\partial \epsilon_{ij}}{\partial t} dt = \dot{\epsilon}_{ij} \, dt \qquad \cdots$$

Thus, we obtain the fundamental energy equation

$$\int_B (X_i - \rho\dot{v}_i)v_i \, dv + \int_{\partial B} p_i v_i \, da = \int_B \sigma_{ij}\dot{\epsilon}_{ij} \, dv \tag{21}$$

into which we introduce the constitutive relations

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} - e_{kij}E_k - \gamma_{ij}\theta \tag{22}$$

Hence

$$\frac{d}{dt}(\mathcal{W} + \mathcal{K}) = \int_B X_i v_i \, dv + \int_{\partial B} p_i v_i \, da + \int_B (\gamma_{ij}\dot{\epsilon}_{ij} + e_{kij}E_k\dot{\epsilon}_{ij}) \, dv \tag{23}$$

where $\mathcal{K}$ is the kinetic energy and $\mathcal{W}$ the work of deformation:

$$\mathcal{K} = \frac{\rho}{2}\int_B v_i v_i \, dv \qquad \mathcal{W} = \frac{1}{2}\int_B c_{ijkl}\epsilon_{ij}\epsilon_{kl} \, dv$$

To eliminate the term $\int_B \gamma_{ij}\dot{\epsilon}_{ij}\theta \, dv$, we consider the heat-conduction equation

$$\frac{1}{T_0}(k_{ij}\theta_{,ij} - c_\epsilon\dot{\theta}) = \gamma_{ij}\dot{\epsilon}_{ij} + g_k\dot{E}_k - \frac{W}{T_0} \tag{24}$$

Multiplying it by θ and integrating over the region of the body, after simple transformations we obtain

$$\int_B \gamma_{ij}\dot{\epsilon}_{ij}\theta\, dv = \frac{k_{ij}}{T_0}\int_{\partial B}\theta\theta_{,j}n_i\, da - g_k\int_B \theta\dot{E}_k\, dv + \frac{1}{T_0}\int_B W\theta\, dv - \frac{d\mathcal{P}}{dt} - \chi_\theta \quad (25)$$

where

$$\mathcal{P} = \frac{c_\epsilon}{2T_0}\int_B \theta^2\, dv \qquad \chi_\theta = \frac{k_{ij}}{T_0}\int_B \theta_{,i}\theta_{,j}\, dv$$

Substituting Eq. (25) into Eq. (23), we are led to the equation

$$\frac{d}{dt}(\mathcal{K} + \mathcal{W} + \mathcal{P}) + \chi_\theta = \int_B X_i v_i\, dv + \int_{\partial B} p_i v_i\, da + \frac{k_{ij}}{T_0}\int_{\partial B}\theta\theta_{,j}n_i\, da$$
$$+ \frac{1}{T_0}\int_B W\theta\, dv + \int_B (e_{kij}\dot{\epsilon}_{ij}E_k - g_k\dot{E}_k\theta)\, dv \quad (26)$$

To eliminate the term $e_{kij}\dot{\epsilon}_{ij}E_k$ from the last integral of Eq. (26), we make use of the constitutive relations

$$D_k = e_{kij}\epsilon_{ij} + g_k\theta + \epsilon_{kj}E_j \quad (27)$$

Finally, let us make use of the equation of the electric field $\dot{D}_{k,k} = 0$. Multiplying the equation by φ and integrating over the region of the body, we obtain

$$\int_{\partial B}\dot{D}_k n_k \varphi\, da + \int_B \dot{D}_k E_k\, dv = 0 \quad (28)$$

Using relation (27), after simple transformations we obtain

$$\int_B (e_{kij}E_k\dot{\epsilon}_{ij} - g_k\theta\dot{E}_k)\, dv = -\int_{\partial B}\dot{D}_k n_k\varphi\, da - \frac{d\mathcal{E}}{dt} - \frac{d}{dt}\left(g_k\int_B \theta E_k\, dv\right) \quad (29)$$

where

$$\mathcal{E} = \frac{1}{2}\epsilon_{ij}\int_B E_i E_j\, dv$$

In view of Eqs. (26)-(28), we arrive at the modified energy balance

$$\frac{d}{dt}\left(\mathfrak{K}+\mathfrak{W}+\mathfrak{P}+g_k\int_B \theta E_k\,dv\right)+\chi_\theta=\int_B X_i v_i\,dv+\int_{\partial B} p_i v_i\,da$$

$$+\frac{k_{ij}}{T_0}\int_{\partial B}\theta\theta_{,j}n_i\,da-\int_{\partial B}\dot{D}_i\varphi n_i\,da+\frac{1}{T_0}\int_B W\theta\,dv \tag{30}$$

The energy balance (30) makes possible the proof of the uniqueness of the solution.

We assume that two distinct solutions $(u_i', \varphi', \theta')$ and $(u_i'', \varphi'', \theta'')$ satisfy Eqs. (1)-(3) and the appropriate boundary and initial conditions. Their difference $\hat{u}_i=u_i'-u_i''$, $\hat{\varphi}=\varphi'-\varphi''$, $\hat{\theta}=\theta'-\theta''$ therefore satisfies the homogeneous equations (1)-(3) and the homogeneous boundary and initial conditions. Equation (30) holds for the solutions $\hat{u}_i$, $\hat{\varphi}$, $\hat{\theta}$.

In view of the homogeneity of the equations and the boundary conditions, the right-hand side of Eq. (30) vanishes. Hence,

$$\frac{d}{dt}\left(\hat{\mathfrak{K}}+\hat{\mathfrak{W}}+\hat{\mathfrak{P}}+\hat{\mathcal{E}}+g_k\int_B \hat{\theta}\hat{E}_k\,dv\right)=-\hat{\chi}_\theta\leqslant 0 \tag{31}$$

where we have made use of the fact that the integrand of the energy-dissipation function χ_θ is a positive-definite quadratic form. The integral in the left-hand side of Eq. (31) vanishes at the initial instant, since the functions $\hat{u}_i$, $\hat{\theta}$, $\hat{\varphi}$ satisfy the homogeneous initial conditions. On the other hand, the inequality in Eq. (31) proves that its left-hand side is either negative or zero. The second possibility occurs if the integrand is a sum of squares.

Consequently, we assume that

$$\hat{\mathfrak{K}}=0 \qquad \hat{\mathfrak{W}}=0 \qquad \hat{\mathfrak{P}}+\hat{\mathcal{E}}+g_k\int_B \hat{\theta}\hat{E}_k\,dv\geqslant 0 \tag{32}$$

These results imply that

$$\hat{\vartheta}_i=0 \qquad \hat{\epsilon}_{ij}=0 \qquad \hat{\theta}=0 \qquad \hat{E}_k=0 \tag{33}$$

J. Ignaczak* deduced the following sufficient condition. Assume that ϵ_{ij} is a known positive-definite symmetric tensor, $\mathbf{g}_i$ is a vector, and $c=c_\epsilon/2T_0>0$. Consider the function

$$A(\theta, E_k)=c\theta^2+2\theta g_k E_k+\epsilon_{ij}E_iE_j$$

*Private communication.

A is nonnegative ($A \geqslant 0$) for every real pair (θ, E_k), provided

$$|g_i|^2 \leqslant c\lambda_m$$

where λ_m is the smallest positive eigenvalue of the tensor ϵ_{ij}. Equations (33) imply the uniqueness of the solutions of the thermopiezoelectricity equations, i.e.,

$$u_i' = u_i'' \qquad \theta' = \theta'' \qquad E_k' = E_k'' \tag{34}$$

Moreover, it follows from the constitutive relations that

$$\sigma_{ij}' = \sigma_{ij}'' \qquad D_i' = D_i'' \qquad S' = S'' \tag{35}$$

THE GENERALIZED HAMILTON'S PRINCIPLE

We define two functionals

$$\Pi = \int_B (H + ST - X_i u_i)\, dv - \int_{\partial B} (p_i u_i - \sigma\varphi)\, da \tag{36}$$

$$\Psi = \int_B (\Gamma - S\dot{T} - WT)\, dv + \int_{\partial B} TQ_i n_i\, da \qquad H = F - D_i E_i \tag{37}$$

where H = electric enthalpy
φ = electric potential
F = free energy
σ = electric charge on ∂B
Γ = potential of the heat flow

$$\Gamma = \frac{1}{2} k_{ij} T_{,i} T_{,j} \qquad q_i = \frac{\partial \Gamma}{\partial T_{,i}} = -k_{ij} T_{,j} \tag{38}$$

The generalized Hamilton principle has the form

$$\delta \int_{t_1}^{t_2} (\varkappa - \Pi)\, dt = 0 \qquad \delta \int_{t_1}^{t_2} \Psi\, dt = 0 \tag{39}$$

This form of Hamilton's principle was first stated for the problem of thermoelasticity by Parkus [2], and for the adiabatic problem of piezoelectricity by Tiersten [3].

The admissible motions of the body must be compatible with the conditions restricting the motion of the body. Moreover, the following conditions must be satisfied:

$$\delta u_i(\mathbf{x}, t_1) = \delta u_i(\mathbf{x}, t_2) = 0 \qquad \delta\theta(\mathbf{x}, t_1) = \delta\theta(\mathbf{x}, t_2) = 0 \tag{40}$$

The quantities subject to variations are the displacement u_i, the temperature θ, and the electric potential φ. Performing the variations in accordance with the first of Eqs. (39) and bearing in mind that

$$\begin{aligned} \delta H &= \frac{\partial H}{\partial \epsilon_{ij}} \delta\epsilon_{ij} + \frac{\partial H}{\partial T} \delta T + \frac{\partial H}{\partial E_i} \delta E_i \\ &= \sigma_{ij}\, \delta\epsilon_{ij} - S\, \delta T - D_i\, \delta\varphi_{,i} \end{aligned} \tag{41}$$

we obtain the equation

$$\begin{aligned} \int_{t_1}^{t_2} dt \left\{ \int_B [(X_i + \sigma_{ji,j} - \rho\ddot{u}_i)\, \delta u_i + D_{i,i}\, \delta\varphi]\, dv + \int_{\partial B} [(\sigma_{ji} n_j - p_i)\, \delta u_i \right. \\ \left. + (D_i n_i + \sigma)\, \delta\varphi]\, da \right\} = 0 \end{aligned} \tag{42}$$

Since the variations δu_i and $\delta\varphi$ are arbitrary, we obtain from Eq. (42) the equations governing the motion and the electric field, completed by the appropriate boundary conditions. These equations and boundary conditions are identical with those presented in the introduction.

Performing the required variation in the second of Eqs. (39),

$$\begin{aligned} \delta \int_{t_1}^{t_2} \Psi\, dt = \int_{t_1}^{t_2} dt \left[\int_B \left(\frac{\partial \Gamma}{\partial T_{,i}} \delta T_{,i} - S\dot{T}\, \delta T - ST\, \delta\dot{T} - W\, \delta T \right) dv \right. \\ \left. + \int_{\partial B} \theta_i n_i\, \delta T\, da \right] \end{aligned} \tag{43}$$

and taking into account that

$$q_i = -\frac{\partial \Gamma}{\partial T_{,i}} \qquad q_i\, \delta T_{,i} = -q_{i,i}\, \delta T + (q_i\, \delta T)_{,i}$$

we transform Eq. (43) to the form

$$\int_{t_1}^{t_2} dt \left\{ \int_B [(q_{i,i} - W + \dot{S}T)\, \delta T - (\dot{\overline{ST\delta T}})]\, dv - \int_{\partial B} (q_i - Q_i) n_i\, \delta T\, da \right\} = 0 \tag{44}$$

In view of the second of Eqs. (40), we have

$$\int_{t_1}^{t_2} (\dot{\overline{ST\delta T}})\, dt = |ST\,\delta T|_{t_1}^{t_2} = 0 \tag{45}$$

We still have the equation

$$\int_{t_1}^{t_2} dt \left[\int_B (q_{i,i} - W + T\dot{S})\,\delta T\, dv + \int_{\partial B} (Q_i - q_i)n_i\,\delta T\, da \right] = 0 \tag{46}$$

valid for arbitrary variation δT satisfying condition (45). Equation (46) yields the entropy balance

$$T\dot{S} = -q_{i,i} + W \qquad \mathbf{x} \in B \tag{47}$$

and the boundary condition for the heat flow

$$q_i = Q_i \qquad \mathbf{x} \in \partial B \tag{48}$$

THEOREM OF RECIPROCITY OF WORK

Consider two sets of causes and effects. The causes are the actions of body forces, heat sources, prescribed displacements, tractions and temperatures on the boundary, electric potentials or electric charges on ∂B, and finally the actions of initial conditions. The effects are the displacements u_i, the electric potential φ, and the temperature θ. The second set of causes and effects will be denoted by primes.

Based on the equations of motion, for both sets of causes and effects,

$$\sigma_{ji,j} + X_i = \rho\ddot{u}_i \tag{49}$$

$$\sigma'_{ji,j} + X'_i = \rho\ddot{u}'_i \tag{50}$$

Taking the Laplace transforms of both equations, assuming that the initial conditions for the displacements are homogeneous, we obtain the equation

$$\int_B [(\bar{\sigma}_{ji,j} + \bar{X}_i)\bar{u}'_i - (\bar{\sigma}'_{ji,j} + \bar{X}'_i)\bar{u}_i]\; dv = 0 \tag{51}$$

or

$$\int_B (\bar{X}_i\bar{u}'_i - \bar{X}'_i\bar{u}_i)\, dv + \int_{\partial B} (\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i)\, da = \int_B (\sigma_{ji}\epsilon'_{ji} - \sigma'_{ji}\epsilon_{ji})\, dv \tag{52}$$

where

$$\bar{u}_i(\mathbf{x}, p) = \int_0^\infty e^{-pt} u_i(\mathbf{x}, t)\, dt \qquad \cdots$$

is the Laplace transform.

In view of the constitutive equations

$$\bar{\sigma}_{ij} = c_{ijkl}\bar{\epsilon}_{kl} - \gamma_{ij}\bar{\theta} - e_{kij}\bar{E}_k \qquad \bar{\sigma}'_{ij} = c_{ijkl}\bar{\epsilon}'_{kl} - \gamma_{ij}\bar{\theta}' - e_{kij}\bar{E}'_k \tag{53}$$

we obtain

$$\int_B (\bar{X}_i\bar{u}'_i - \bar{X}'_i\bar{u}_i)\, dv + \int_{\partial B} (\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i)\, da + \int_B [\gamma_{ij}(\bar{\theta}\bar{\epsilon}'_{ij} - \bar{\theta}'\bar{\epsilon}_{ij})$$

$$+ e_{kij}(\bar{E}_k\bar{\epsilon}'_{ij} - \bar{E}'_k\bar{\epsilon}_{ij})]\, dv = 0 \tag{54}$$

In what follows we shall make use of the heat-conduction equation for both systems of loadings

$$\frac{1}{T_0}(k_{ij}\bar{\theta}_{,ij} - c_\epsilon p\bar{\theta}) - p(\gamma_{ij}\bar{\epsilon}_{ij} + g_i\bar{E}_i) = -\frac{\bar{W}}{T_0} \tag{55}$$

$$\frac{1}{T_0}(k_{ij}\bar{\theta}'_{,ij} - c_\epsilon p\bar{\theta}') - p(\gamma_{ij}\bar{\epsilon}'_{ij} + g_i\bar{E}'_i) = -\frac{\bar{W}'}{T_0} \tag{56}$$

We now multiply Eq. (55) by $\bar{\theta}'$ and Eq. (56) by $\bar{\theta}$, substract the results, and integrate over the region of the body. After transformations, we obtain the equation

$$\frac{1}{T_0} k_{ij} \int_{\partial B} (\bar{\theta}'\bar{\theta}_{,j} - \bar{\theta}\bar{\theta}'_{,j})n_i\, da - p\int_B [(\gamma_{ij}\bar{\epsilon}_{ij} + g_k\bar{E}_k)\bar{\theta}'$$

$$- (\gamma_{ij}\bar{\epsilon}'_{ij} + g_k\bar{E}'_k)\bar{\theta}]\, dv + \frac{1}{T_0}\int_B (\bar{W}\bar{\theta}' - \bar{W}'\bar{\theta})\, dv = 0 \tag{57}$$

Finally, we make use of the equations for the electric field

$$\bar{D}_{k,k} = 0 \qquad \bar{D}'_{k,k} = 0 \tag{58}$$

Multiplying the first by $\bar{\varphi}'$ and the second by $\bar{\varphi}$, subtracting the results, and integrating over the region of the body, we have

$$\int_{\partial B} (\bar{D}_k \bar{\varphi}' - \bar{D}'_k \bar{\varphi}) n_k \, da + \int_B (\bar{D}_k \bar{E}'_k - \bar{D}'_k \bar{E}_k) \, dv = 0 \tag{59}$$

Introducing the constitutive relation

$$\bar{D}_k = e_{kij} \bar{\epsilon}_{ij} + g_k \bar{\theta} + \epsilon_{kj} \bar{E}_j \tag{60}$$

and a similar relation for $\bar{D}'_k$ into the volume integral, we transform Eq. (60) to the form

$$\int_{\partial B} (\bar{D}_k \bar{\varphi}' - \bar{D}'_k \bar{\varphi}) n_k \, da + \int_B [e_{kij}(\bar{\epsilon}_{ij} \bar{E}'_k - \bar{\epsilon}'_{ij} \bar{E}_k) + g_k(\bar{\theta} \bar{E}'_k - \bar{\theta}' \bar{E}_k)] \, dv = 0 \tag{61}$$

Eliminating the common terms from Eqs. (54), (57), and (61), we arrive at one common equation of the reciprocity of work containing all causes and effects:

$$T_0 p \left\{ \int_B (\bar{X}_i \bar{u}'_i - \bar{X}'_i \bar{u}_i) \, dv + \int_{\partial B} \left[\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i + (\bar{D}_k \bar{\varphi}' - \bar{D}'_k \bar{\varphi}) n_k \right] da \right\}$$

$$+ \int_B (\bar{W}' \bar{\theta} - \bar{W} \bar{\theta}') \, dv + k_{ij} \int_{\partial B} (\bar{\theta} \bar{\theta}_{,j} - \bar{\theta}' \bar{\theta}_{,j}) n_i \, da = 0 \tag{62}$$

By taking the inverse Laplace transform of this equation, we obtain [4]

$$T_0 \left\{ \int_B (X_i \odot u'_i - X'_i \odot u_i) \, dv + \int_{\partial B} [p_i \odot u'_i - p'_i \odot u_i \right.$$

$$\left. + (D_k \odot \varphi' - D'_k \odot \varphi) n_k] \, da \right\} + \int_B (W * \theta' - W' * \theta) \, dv$$

$$+ k_{ij} \int_{\partial B} (\theta * \theta'_{,j} - \theta' * \theta_{,j}) n_i \, da = 0 \tag{63}$$

where we have introduced the notation

$$X_i \odot u'_i = \int_0^t X_i(\mathbf{x}, t - \tau) \frac{\partial u'_i(\mathbf{x}, \tau)}{\partial \tau} \, d\tau, \ldots$$

$$W * \theta' = \int_0^t W(\mathbf{x}, t - \tau)\, \theta'(\mathbf{x}, \tau)\, d\tau, \ldots$$

As for thermoelasticity [5], we can investigate the action of instantaneous and moving concentrated sources, we can derive the Somigliana and Green formulas generalized to thermopiezoelectricity, and so on. It is a simple matter to deduce the theorem of the reciprocity of work for harmonic vibrations and stationary problems.

REFERENCES

1. R. D. Mindlin, On the Equations of Motion of Piezoelectric Crystals, *Problems of Continuous Media,* SIAM, Philadelphia, 1961.
2. H. Parkus, Über die Erweiterung des Hamilton'shen Principes auf thermoelastische Vorgänge, Federhofer-Girkmann Festschrift, Verlag F. Deuticke, Wien, 1950.
3. H. F. Tiersten, *Linear Piezoelectric Plate Vibrations,* Plenum, New York, 1969.
4. W. Nowacki, A Reciprocity Theorem for Coupled Mechanical and Thermoelectric Fields in Piezoelectric Crystals, *Proc. Vibrations Probl.,* vol. 6, no. 1, 1965.
5. W. Nowacki, in P. H. Francis and R. B. Hetnarski (eds.), *Dynamic Problems of Thermoelasticity,* Noordhoff, Leyden, Netherlands, 1975.

Received February 15, 1978

Request reprints from W. Nowacki.