Some generalizations of Darbo fixed point theorem and applications

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Abstract

In the paper we provide a few generalizations of Darbo fixed point theorem. Several interconnections among assumptions imposed in the proved theorems are indicated. We also show the applicability of obtained results to the theory of functional integral equations. A concrete example illustrating the mentioned applicability is also included.

1 Introduction

In the fixed point theory an important role is played by the concept of a measure of noncompactness. This concept was initiated by the fundamental paper of Kuratowski [14]. In 1955 G. Darbo, using the concept of a measure of noncompactness, proved a theorem guaranteeing the existence of fixed points of the so-called condensing operators [10]. That theorem found an abundance of applications in proving the existence of solutions for a wide class of differential and integral equations (cf. [2-4, 7-9, 11], for example).
It is worthwhile mentioning that Darbo theorem extends both the classical Banach contraction principle and the Schauder fixed point theorem [8].

The aim of this paper is to obtain some generalizations of the above mentioned Darbo fixed point theorem and to indicate the applicability of the obtained results to existence theorems for some functional integral equations.

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At the beginning we provide notation, definition and some auxiliary facts which will be needed in the sequel. To this end assume that E is a given Banach space with the norm $||\cdot||$ and zero element θ. Denote by $B(x,r)$ the closed ball in E centered at x and with radius r. We write B_r to denote $B(\theta,r)$. If X is a subset of E then the symbols $\overline{X}$, Conv X stand for the closure and the closed convex hull of X, respectively. The algebraic operations on sets will be denoted by $X+Y$ and λX $(\lambda \in \mathbb{R})$.
Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

Definition 1.1 ([8]). A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1^o The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.

2^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

3^o $\mu(\overline{X}) = \mu(X)$.

4^o $\mu(\mathrm{Conv}X) = \mu(X)$.

5^o $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6^o If (X_n) is a nested sequence of closed sets from $\mathfrak{M}_E$ such that $\lim\limits_{n\to\infty} \mu(X_n) = 0$ then the intersection set $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described in 1^o is called the kernel of the measure of noncompactness μ.
Observe that the intersection set X_∞ from axiom 6^o is a member of the kernel $\ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we have that $\mu(X_\infty) = 0$. This yields that $X_\infty \in \ker\mu$.

Now we recall three important theorems playing a key role in the fixed point theory (cf. [1, 8, 12]).

Theorem 1.1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $F : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

Obviously the above formulated theorem constitutes the well known Schauder fixed point principle. Its generalization, called the Darbo fixed point theorem, is formulated below.

Theorem 1.2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that*

$$\mu(TX) \leq k\mu(X)$$

for any nonempty subset X of Ω, where μ is a measure of noncompactness defined in E. Then T has a fixed point in the set Ω.

Next, we present the following theorem due to Caristi [12].

Theorem 1.3. *Let (M, ρ) be a complete metric space and let $\varphi : M \to \mathbb{R}$ be a lower semicontinuous function which is bounded from below. Suppose $T : M \to M$ is an arbitrary mappings such that*

$$\rho(u, Tu) \leq \varphi(u) - \varphi(Tu)$$

for any $u \in M$. Then T has a fixed point in M.

2 Main results

This section is devoted to prove a few generalizations of Darbo fixed point theorem (cf. Theorem 1.2).

Theorem 2.1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that*

$$\psi(\mu(TX)) \leq \psi(\mu(X)) - \varphi(\mu(X)) \tag{2.1}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions such that φ is lower semicontinuous and ψ is continuous on $\mathbb{R}_+$. Moreover, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Then T has at least one fixed point in Ω.

Proof. Consider the sequence (Ω_n) defined as $\Omega_0 = \Omega$ and $\Omega_n = \text{Conv}T\Omega_{n-1}$ for $n = 1, 2, \ldots$. If there exists a natural number n_0 such that $\mu(\Omega_{n_0}) = 0$, then Ω_{n_0} is compact. In this case Theorem 1.1 implies that T has a fixed point in Ω. Next, assume that $\mu(\Omega_n) > 0$ for $n = 1, 2, \ldots$. By our assumptions, we get

$$\psi(\mu(\Omega_{n+1})) = \psi(\mu(\text{Conv}T\Omega_n)) = \psi(\mu(T\Omega_n)) \leq \psi(\mu(\Omega_n)) - \varphi(\mu(\Omega_n)) \,. \tag{2.2}$$

Since the sequence $(\mu(\Omega_n))$ is nonincreasing and nonnegative, we infer that $\mu(\Omega_n) \to r$ when n tends to infinity, where $r \geq 0$ is a nonnegative real number. On the other hand, in view of (2.2) we obtain

$$\limsup_{n\to\infty} \psi(\mu(\Omega_{n+1})) \leq \limsup_{n\to\infty} \psi(\mu(\Omega_n)) - \liminf_{n\to\infty} \varphi(\mu(\Omega_n)) \,.$$

This yields $\psi(r) \leq \psi(r) - \varphi(r)$. Consequently $\varphi(r) = 0$ so $r = 0$. Hence we deduce that $\mu(\Omega_n) \to 0$ as $n \to \infty$. Now, taking into account that $\Omega_{n+1} \subset \Omega_n$, on the base of axiom 6^o of Definition 1.1 we derive that the set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed, convex and $\Omega_\infty \subset \Omega$. Moreover, the set Ω_∞ is invariant under the operator T and $\Omega_\infty \in \ker \mu$. Thus, applying Theorem 1.1 we complete the proof. ∎

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator satisfying the inequality*

$$\mu(TX) \leq \varphi(\mu(X)) \tag{2.3}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing functions such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. Then T has at least one fixed point in the set Ω.

Proof. Similarly as in the proof of the preceding theorem, we define by induction the sequence (Ω_n), where $\Omega_0 = \Omega$ and $\Omega_n = \mathrm{Conv}T\Omega_{n-1}$ for $n = 1, 2, \ldots$. Moreover, in the same way as before we can assume that $\mu(\Omega_n) > 0$ for all $n = 1, 2, \ldots$. Further, taking into account our assumptions, we have

$$\mu(\Omega_{n+1}) = \mu(\mathrm{Conv}T\Omega_n) = \mu(T\Omega_n) \leq \varphi(\mu(\Omega_n)) \leq \varphi^2(\mu(\Omega_{n-1})) \leq \ldots \leq \varphi^n(\mu(\Omega)) .$$

This implies that $\mu(\Omega_n) \to 0$ as $n \to \infty$. Since the sequence (Ω_n) is nested, in view of axiom 6^o of Definition 1.1 we deduce that the set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty, closed and convex subset of the set Ω. Hence we get that Ω_∞ is a member of the kernel $\ker\mu$. So, Ω_∞ is compact. Next, keeping in mind that T maps Ω_∞ into itself and taking into account Schauder fixed point principle (cf. Theorem 1.1) we infer that the operator T has a fixed point x in the set Ω_∞. Obviously $x \in \Omega$. This completes the proof. ∎

Now, let us pay attention to the following corollary from the above theorem which belongs to the classical metric fixed point theory.

Corollary 2.1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be an operator such that*

$$||Tx - Ty|| \leq \varphi(||x - y||) \tag{2.4}$$

for all $x, y \in \Omega$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$. Then T has a fixed point in the set Ω.

Proof. Let $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ be a set quantity defined by the formula

$$\mu(X) = \mathrm{diam}X ,$$

where $\mathrm{diam}X = \sup\{||x - y|| : x, y \in X\}$ stands for the diameter of X. It is easily seen that μ is a measure of noncompactness in a space E in the sense of Definition 1.1 (cf. [6, 8]).
Further observe that since the function φ is nondecreasing, then in view of (2.4) we have

$$\sup_{x,y\in X} ||Tx - Ty|| \leq \sup_{x,y\in X} \varphi(||x - y||) \leq \varphi\left(\sup_{x,y\in X} ||x - y||\right) .$$

This yields that

$$\mu(TX) \leq \varphi(\mu(X)) .$$

The application of Theorem 2.2 completes the proof. ■

In what follows, we show that the assumption saying that $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$ can be replaced by other handy requirement.

Lemma 2.1. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then the following two conditions are equivalent:*

(i) $\lim_{n\to\infty} \varphi^n(t) = 0$ *for any* $t > 0$.

(ii) $\varphi(t) < t$ *for any* $t > 0$.

Proof. Let φ satisfy condition (i). Suppose that condition (ii) does not hold i.e., there exists a number $t_0 > 0$ such that $\varphi(t_0) \geq t_0$. Hence, in view of the fact that φ is nondecreasing we infer that $\varphi^2(t_0) = \varphi(\varphi(t_0)) \geq \varphi(t_0) \geq t_0$. By induction we obtain that $\varphi^n(t_0) \geq t_0 > 0$ for $n = 1, 2, \ldots.$ This yields the contradiction and proves that condition (ii) is satisfied. Conversely, assume that φ satisfies condition (ii). Take an arbitrary number $t > 0$ and consider the sequence $(\varphi^n(t))$. Then $\varphi^2(t) = \varphi(\varphi(t)) < \varphi(t)$. Similarly, by induction we can easily seen that the sequence $(\varphi^n(t))$ is decreasing. Thus, there exists the limit $\lim_{n\to\infty} \varphi^n(t) = r$. If $r = 0$ we have the desired conclusion. If $r > 0$, then in view of our assumptions we have that $\varphi(r) < r$. On the other hand, we have that $r < \varphi^n(t)$ for any $n = 1, 2, \ldots.$ In view of the upper semicontinuity of φ, this implies that

$$r \leq \lim_{n\to\infty} \varphi^n(t) = \lim_{n\to\infty} \varphi(\varphi^{n-1}(t)) \leq \varphi(r) < r .$$

The obtained contradiction completes the proof. ■

Observe that, the assumptions on the upper semicontinuity of the function φ is exploited only in the proof of the implication (ii)⇒(i). Moreover, in the proof of this implication we do not utilize the fact that, φ is nondecreasing.
On the other hand it is easily seen that if φ is not upper semicontinuous on $\mathbb{R}_+$ and satisfies condition (ii) then φ has not to satisfy condition (i).
Indeed, consider the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formula

$$\varphi(t) = \begin{cases} 0, \text{ if } t = 0 \\ \frac{1}{n} + \ln\left(t + \frac{n-1}{n}\right), \text{ if } t \in \left(\frac{1}{n}, \frac{1}{n-1}\right] \text{ and } n \text{ is a natural number, } n \geq 3 \\ \frac{1}{2} + \ln\left(t + \frac{1}{2}\right), \text{ if } t > \frac{1}{2} . \end{cases}$$

It is easy to check that φ is increasing on $\mathbb{R}_+$, $\varphi(t) < t$ for $t > 0$ but φ is not upper semicontinuous (obviously φ is lower semicontinuous). On the other hand we can easily seen that φ does not satisfy condition (i).

Now we provide a remark concerning an interconnection between Theorems 2.1 and 2.2.

Remark 2.1. Observe that if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ appearing in Theorem 2.1 is increasing, then Theorem 2.1 can be treated as a special case of Theorem 2.2 provided we assume additionally that $\varphi(t) > 0$ for $t > 0$ and ψ is continuous on $\mathbb{R}_+$.

To prove this fact let us first observe that from inequality (2.1) we infer that $\psi(t) - \varphi(t) \geq 0$ for $t \geq 0$. Thus, since the function ψ is invertible and the inverse function ψ^{-1} is defined and continuous on an subinterval of $\mathbb{R}_+$, we can write equivalently inequality (2.1) in the form

$$\mu(TX) \leq \psi^{-1}(\psi(\mu(X)) - \varphi(\mu(X))) \tag{2.5}$$

for any $X \in \mathfrak{M}_E$.
Further, let us consider the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formula

$$\phi(t) = \psi^{-1}(\psi(t) - \varphi(t))\ .$$

Observe that ϕ is continuous on $\mathbb{R}_+$. Moreover, inequality (2.5) can be written in the form

$$\mu(TX) \leq \phi(\mu(X))$$

for $X \in \mathfrak{M}_E$, which has the same form as inequality (2.3) from Theorem 2.2. Notice that in view of the fact that the function ψ^{-1} is increasing on $\mathbb{R}_+$ we deduce that for $t > 0$ the following inequality holds

$$\phi(t) = \psi^{-1}(\psi(t) - \varphi(t)) < \psi^{-1}(\psi(t)) = t\ .$$

Thus, in view of Lemma 2.1, the function ϕ satisfies the requirement $\lim_{n\to\infty} \phi^n(t) = 0$ from Theorem 2.2. This shows that we can apply Theorem 2.2 which justifies our above stated assertion.

3 An application to a functional integral equation

In this section we provide applications of the generalization of Darbo fixed point theorem contained in Theorem 2.2 to prove the existence of solutions of a functional integral equation of Volterra type.
We will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is furnished with the standard supremum norm i.e., the norm defined by the formula

$$||x|| = \sup\{|x(t)| : t \geq 0\}\ .$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which corresponds to the asymptotic stability of solutions of considered integral equations (cf. [3, 7, 9, 11]). In order to define this measure of noncompactness let us fix a nonempty, bounded subset X of $BC(\mathbb{R}_+)$ and a positive number $L > 0$. For $x \in X$ and $\varepsilon > 0$ denote by $\omega^L(x, \varepsilon)$ the modulus of continuity of the function x on the interval $[0, L]$:

$$\omega^L(x,\varepsilon) = \sup\{|x(t) - x(s)| :\ t, s \in [0, L],\ |t - s| \leq \varepsilon\}\ .$$

Moreover, let us put

$$\omega^L(X,\varepsilon) = \sup\{\omega^L(x,\varepsilon) : x \in X\} ,$$

$$\omega_0^L(X) = \lim_{\varepsilon\to 0} \omega^L(X,\varepsilon) ,$$

$$\omega_0(X) = \lim_{L\to\infty} \omega_0^L(X) .$$

Further, for a fixed number $t \in \mathbb{R}_+$, let us denote

$$X(t) = \{x(t) : x \in X\} .$$

Finally, let us define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ but putting

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \mathrm{diam} X(t) ,$$

where $\mathrm{diam}X(t)$ is understood as

$$\mathrm{diam}X(t) = \sup\{|x(t) - y(t)| : x, y \in X\} .$$

It may be shown [8] (cf. also [9]) that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ (in the sense of Definition 1.1). The kernel $\ker\mu$ of this measure contains nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by the graphs of functions belonging to X tends to zero at infinity. This property will be further used to deduce the asymptotic stability of solutions of the equations investigated by ourselves.

Further on, denote by ϕ the family of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ being nondecreasing on $\mathbb{R}_+$ and such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$.
Then we can formulate the following theorem.

Theorem 3.1. *Assume that the following conditions are satisfied:*

(i) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Moreover, the function* $t \to f(t,0)$ *is a member of the space* $BC(\mathbb{R}_+)$.

(ii) There exists an upper semicontinuous function $\varphi \in \phi$ *such that any* $t \in \mathbb{R}_+$ *and for all* $x, y \in \mathbb{R}$ *we have that*

$$|f(t,x) - f(t,y)| \le \varphi(|x-y|) .$$

Additionally we assume that φ *is superadditive i.e.,* $\varphi(t) + \varphi(s) \le \varphi(t+s)$ *for all* $t, s \in \mathbb{R}_+$.

(iii) $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and there exist continuous functions* $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\lim_{t\to\infty} a(t) \int_0^t b(s)ds = 0$$

and

$$|g(t,s,x)| \le a(t)b(s)$$

for $t, s \in \mathbb{R}_+$ *such that* $s \le t$, *and for each* $x \in \mathbb{R}$.

(iv) There exists a positive solution r_0 of the inequality

$$\varphi(r) + q \leq r \,,$$

where q is the constant defined by the equality

$$q = \sup \left\{ |f(t,0)| + a(t) \int_0^t b(s)ds : \; t \geq 0 \right\} \,.$$

Then the functional integral equation

$$x(t) = f(t, x(t)) + \int_0^t g(t,s,x(s))ds, \;\; t \in \mathbb{R}_+ \,, \tag{3.1}$$

has at least one solution in the space $BC(\mathbb{R}_+)$.

Remark 3.1. Observe that the constant q defined above is finite in view of assumptions (i) and (iii). On the other hand from Lemma 2.1 follows that $\varphi(r) < r$ for each $r > 0$. This explains that the inequality from assumption (iv) has a sense.

Proof of Theorem 3.1. Consider the operator T defined on the space $BC(\mathbb{R}_+)$ by the formula

$$(Tx)(t) = f(t, x(t)) + \int_0^t g(t,s,x(s))ds \,,$$

for $t \in \mathbb{R}_+$.

In view of the imposed assumptions we have that the function Tx is continuous on $\mathbb{R}_+$. Further, for an arbitrarily fixed function $x \in BC(\mathbb{R}_+)$, using our assumptions, we obtain

$$\begin{aligned} |(Tx)(t)| &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + \int_0^t |g(t,s,x(s))|ds \\ &\leq \varphi(|x(t)|) + |f(t,0)| + a(t) \int_0^t b(s)ds \\ &= \varphi(|x(t)|) + |f(t,0)| + c(t) \,, \end{aligned}$$

where we denoted

$$c(t) = a(t) \int_0^t b(s)ds \,.$$

Since by assumption (ii) the function φ is nondecreasing, in virtue of forth obtained estimate we get

$$||Tx|| \leq \varphi(||x||) + q \,,$$

where q is a constant defined in assumption (iv). Thus T maps the space $BC(\mathbb{R}_+)$ into itself.
Moreover, keeping in mind assumption (iv) we infer that T is a self mapping of the ball B_{r_0}, where r_0 is a constant appearing in assumption (iv).

In what follows we show that T is continuous on the ball B_{r_0}. To this end fix an arbitrary number $\varepsilon > 0$. Then, for $x, y \in B_{r_0}$ such that $||x - y|| \leq \varepsilon$, we obtain

$$|(Tx)(t) - (Ty)(t)| \leq \varphi(|x(t) - y(t)|) + \int_0^t |g(t,s,x(s)) - g(t,s,y(s))|ds$$

$$\leq \varphi(|x(t) - y(t)|) + \int_0^t |g(t,s,x(s))|ds + \int_0^t |g(t,s,y(s))|ds$$

$$\leq \varphi(\varepsilon) + 2c(t)\ , \tag{3.2}$$

for any $t \in \mathbb{R}_+$.
Further, in view of assumption (iii) we deduce that there exists a number $L > 0$ such that

$$2a(t) \int_0^t b(s)ds \leq \varepsilon \tag{3.3}$$

for each $t \geq L$.
Thus, taking into account Lemma 2.1 and linking (3.2) and (3.3), for an arbitrary $t \geq L$ we get

$$|(Tx)(t) - (Ty)(t)| \leq 2\varepsilon\ . \tag{3.4}$$

Now, let us define the quantity $\omega^L(g,\varepsilon)$ by putting

$$\omega^L(g,\varepsilon) = \sup\{|g(t,s,x) - g(t,s,y)| :\ t,s \in [0,L],\ x,y \in [-r_0,r_0],\ |x-y| \leq \varepsilon\}\ .$$

Since the function $g(t,s,x)$ is uniformly continuous on the set $[0,L] \times [0,L] \times [-r_0,r_0]$ we infer, that $\omega^L(g,\varepsilon) \to 0$ as $\varepsilon \to 0$.

Next, taking into account the first part of estimate (3.2), for an arbitrarily fixed $t \in [0,L]$ we obtain

$$|(Tx)(t) - (Ty)(t)| \leq \varphi(\varepsilon) + \int_0^L \omega^L(g,\varepsilon)ds = \varphi(\varepsilon) + L\omega^L(g,\varepsilon)\ . \tag{3.5}$$

Finally, combining estimates (3.4) and (3.5), on the base of the above established fact concerning the quantity $\omega^L(g,\varepsilon)$, we conclude that the operator T is continuous on the ball B_{r_0}.

In what follows let us take an arbitrary nonempty subset X of the ball B_{r_0}. Fix numbers $\varepsilon > 0$ and $L > 0$. Next, choose arbitrarily $t, s \in [0,L]$ such that $|t - s| \leq \varepsilon$. Without loss of generality we may assume that $s < t$. Then, for $x \in X$

we get:

$$
\begin{aligned}
|(Tx)(t)-(Tx)(s)| &\le |f(t,x(t))-f(s,x(s))| + \\
&\quad \left| \int_0^t g(t,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau \right| \\
&\le |f(t,x(t))-f(s,x(t))| + |f(s,x(t))-f(s,x(s))| \\
&+ \left| \int_0^t g(t,\tau,x(\tau))d\tau - \int_0^t g(s,\tau,x(\tau))d\tau \right| + \left| \int_0^t g(s,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau \right| \\
&\le \omega_1^L(f,\varepsilon) + \varphi(|x(t)-x(s)|) + \int_0^t |g(t,\tau,x(\tau)) - g(s,\tau,x(\tau))| d\tau \\
&+ \int_s^t |g(s,\tau,x(\tau))| d\tau \le \omega_1^L(f,\varepsilon) + \varphi(\omega^L(x,\varepsilon)) \\
&+ \int_0^t \omega_1^L(g,\varepsilon) d\tau + a(s)\int_s^t b(\tau)d\tau \le \omega_1^L(f,\varepsilon) + \varphi(\omega^L(x,\varepsilon)) \\
&+ L\omega_1^L(g,\varepsilon) + \varepsilon \sup\{a(s)b(t) : t,s\in[0,L]\}, \qquad (3.6)
\end{aligned}
$$

where we denote

$$\omega_1^L(f,\varepsilon) = \sup\{|f(t,x)-f(s,x)| : t,s\in[0,L],\ x\in[-r_0,r_0],\ |t-s|\le\varepsilon\},$$

$$\omega_1^L(g,\varepsilon) = \sup\{|g(t,\tau,x)-g(s,\tau,x)| : t,s,\tau\in[0,L],\ x\in[-r_0,r_0],\ |t-s|\le\varepsilon\}.$$

Further, observe that in view of the uniform continuity of the function f on the set $[0,L]\times[-r_0,r_0]$ and the function g on the set $[0,L]\times[0,L]\times[-r_0,r_0]$ we infer that $\omega_1^L(f,\varepsilon)\to 0$ and $\omega_1^L(g,\varepsilon)\to 0$ as $\varepsilon\to 0$. Moreover, since the functions $a=a(t)$ and $b=b(t)$ are continuous on $\mathbb{R}_+$ we have that the quantity

$$\sup\{a(s)b(t) : t,s\in[0,L]\}$$

is finite. Hence, from estimate (3.6) we derive that

$$\omega_0^L(TX) \le \lim_{\varepsilon\to 0}\varphi(\omega^L(X,\varepsilon)).$$

Consequently, taking into account the upper semicontinuity of the function φ, we get

$$\omega_0^L(TX) \le \varphi(\omega_0^L(X))$$

and, finally

$$\omega_0(TX) \le \varphi(\omega_0(X)). \qquad (3.7)$$

Now, let us choose two arbitrary functions $x,y\in X$. Then, for $t\in\mathbb{R}$ we get

$$
\begin{aligned}
|(Tx)(t)-(Ty)(t)| &\le |f(t,x(t))-f(t,y(t))| \\
&+\int_0^t |g(t,s,x(s))|ds+\int_0^t |g(t,s,y(s))|ds \\
\le \varphi(|x(t)-y(t)|)+2a(t)\int_0^t b(s)ds &= \varphi(|x(t)-y(t)|)+2c(t)\,.
\end{aligned}
$$

This estimate allows us to derive the following one

$$\operatorname{diam}(TX)(t) \le \varphi(\operatorname{diam}X(t))+2c(t)\,.$$

Consequently, in view of the upper semicontinuity of the function φ we obtain

$$\limsup_{t\to\infty}\operatorname{diam}(TX)(t) \le \varphi\left(\limsup_{t\to\infty}\operatorname{diam}X(t)\right)\,. \tag{3.8}$$

Further, combining (3.7), (3.8) and taking into account the superadditivity of the function φ (cf. assumption (iii)), we get

$$\omega_0(TX)+\limsup_{t\to\infty}\operatorname{diam}(TX)(t) \le \varphi\left(\omega_0(X)+\limsup_{t\to\infty}\operatorname{diam}X(t)\right)$$

or, equivalently

$$\mu(TX) \le \varphi(\mu(X))\,, \tag{3.9}$$

where μ is the measure of noncompactness defined in the space $BC(\mathbb{R}_+)$ at the beginning of this section. Finally, keeping in mind (3.9) and applying Theorem 2.2 we complete the proof. ■

Remark 3.2 Let us notice that as the function φ appearing in Theorem 3.1 (cf. assumption (ii)) we can take an arbitrary function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ which is convex and such that $\varphi(t)<t$ for $t>0$. Indeed, the assumption on convexity of φ and the condition $\varphi(0)=0$ imply that φ is increasing, continuous and superadditive on $\mathbb{R}_+$ (cf. [5]).
As examples of functions φ may serve the function $\varphi(t)=t-\ln(t+1)$ for $t\ge 0$ or the functions defined on $\mathbb{R}_+$ in the following way

$$\varphi(t)=\begin{cases} t^p, \text{ if } t\in\left[0,1/p^{1/(p-1)}\right] \\ t+1/p^{p/(1-p)}-1/p^{1/(p-1)}, \text{ if } t>1/p^{1/(p-1)}\,, \end{cases}$$

where p is an arbitrarily fixed real number such that $p>1$.

Remark 3.3. Applying suitable reasoning we can also show that instead of convex functions φ (cf. Remark 3.2) we can take functions being concave on $\mathbb{R}_+$ and such that $\varphi(t)<t$ for $t>0$. In fact, it is sufficient to require that $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is continuous, $\varphi(t)<t$ for $t>0$ and

$$\varphi\left(\frac{t+s}{2}\right) \ge \frac{\varphi(t)+\varphi(s)}{2} \tag{3.10}$$

for $t,s \in \mathbb{R}_+$. In this case, combining estimates (3.7) and (3.8) and considering the measure of noncompactness μ_1, defined on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu_1(X) = \frac{1}{2}\mu(X)\,,$$

for an arbitrarily fixed nonempty subset X of $\mathfrak{M}_{BC(\mathbb{R}_+)}$ we obtain

$$\begin{aligned}\mu(TX) &= \omega_0(TX) + \limsup_{t\to\infty} \operatorname{diam}(TX)(t)\\ &\le \varphi(\omega_0(X)) + \varphi\left(\limsup_{t\to\infty} \operatorname{diam} X(t)\right)\\ &= 2\frac{\varphi(\omega_0(X)) + \varphi\left(\limsup_{t\to\infty} \operatorname{diam} X(t)\right)}{2}\\ &\le 2\varphi\left(\frac{\omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t)}{2}\right)\,.\end{aligned}$$

This implies the estimate

$$\mu_1(TX) \le \varphi(\mu_1(X))\,,$$

and our assertion follows from Theorem 2.2.

It is worthwhile mentioning that inequality (3.10) in conjunction with the assumption on continuity of the function φ implies that the function φ is concave on the interval $\mathbb{R}_+$ i.e., for all $t,s \in \mathbb{R}_+$ and for arbitrary $\alpha \in [0,1]$ we have that

$$\varphi(\alpha t + (t-\alpha)s) \ge \alpha\varphi(t) + (1-\alpha)\varphi(s)$$

(cf. [13]).
Obviously, if we assume that φ is concave on $\mathbb{R}_+$ and $\varphi(t) < t$ for $t > 0$ then we infer that φ is continuous on $\mathbb{R}_+$ and satisfies inequality (3.10) (see [13]).

Let us also pay attention to the fact that each concave function such that $\varphi(0) = 0$ is subadditive. On the other hand the assumption on subadditivity of the function φ seems to not work in our situation.

Now we illustrate our result contained in Theorem 3.1 (cf. also Theorem 2.2) with help of an example.

Example 3.1. Consider the following functional integral equation

$$x(t) = \frac{t^2}{1+t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\cos x(s)}{1+|\sin x(s)|}ds\,, \tag{3.11}$$

for $t \in \mathbb{R}_+$. Observe that this equation is a special case of Eq. (3.1) if we put

$$f(t,x) = \frac{t^2}{1+t^4}\ln(1+|x|)\,,$$

$$g(t,s,x) = \frac{se^{-t}\cos x}{1+|\sin x|}\,.$$

Indeed, taking $\varphi(t) = \ln(1+t)$ we see that $\varphi(t) < t$ for $t > 0$. Obviously the function φ is increasing and concave on $\mathbb{R}_+$. Further, for arbitrarily fixed $x, y \in \mathbb{R}$ such that $|x| \geq |y|$ and for $t > 0$ we obtain

$$|f(t,x) - f(t,y)| = \frac{t^2}{1+t^4} \ln \frac{1+|x|}{1+|y|} \leq \ln\left(1 + \frac{|x|-|y|}{1+|y|}\right)$$

$$< \ln(1+|x-y|) = \varphi(|x-y|)\ .$$

The case $|y| \geq |x|$ can be treated in the same way.
Thus, keeping in mind Remark 3.3 we infer that the function f satisfies assumption (ii) of Theorem 3.1. It is also easily seen that f satisfies also assumption (i).

Further, notice that the function g acts continuously from the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$. Moreover, we have

$$|g(t,s,x)| \leq e^{-t}s$$

for $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. So, if we put $a(t) = e^{-t}$, $b(s) = s$, then we can see that assumption (iii) is satisfied. Indeed, we have

$$\lim_{t\to\infty} a(t) \int_0^t b(s)ds = \lim_{t\to\infty} e^{-t} \int_0^t sds = 0\ .$$

Now, let us calculate the constant q appearing in assumption (iv). We get

$$q = \sup\{|f(t,0)| + a(t) \int_0^t b(s)ds :\ t \geq 0\} = \sup\{t^2 e^{-t/2} :\ t \geq 0\} = 2e^{-2} = 0.27067\dots\ .$$

Further, let us consider the inequality from assumption (iv), having now the form

$$\ln(1+r) + q \leq r\ .$$

It is easily seen that each number $r \geq 1$ (this estimate can be improved) satisfies the above inequality. Thus, as the number r_0 we can take $r_0 = 1$.
Finally, on the base of Theorem 3.1 we conclude that Eq. (3.11) has at least one solution in the space $BC(\mathbb{R}_+)$, located in the ball B_1.

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