Brendan Goldsmith; Fatemeh Karimi; Ahad Mehdizadeh Aghdam

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SOME GENERALIZATIONS OF TORSION-FREE CRAWLEY GROUPS

Brendan Goldsmith, Dublin, Fatemeh Karimi, Tehran,
Ahad Mehdizadeh Aghdam, Tabriz

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Abstract. In this paper we investigate two new classes of torsion-free Abelian groups which arise in a natural way from the notion of a torsion-free Crawley group. A group G is said to be an *Erdős group* if for any pair of isomorphic pure subgroups H, K with $G/H \cong G/K$, there is an automorphism of G mapping H onto K; it is said to be a *weak Crawley* group if for any pair H, K of isomorphic dense maximal pure subgroups, there is an automorphism mapping H onto K. We show that these classes are extensive and pay attention to the relationship of the Baer-Specker group to these classes. In particular, we show that the class of Crawley groups is strictly contained in the class of weak Crawley groups and that the class of Erdős groups is strictly contained in the class of weak Crawley groups.

Keywords: Abelian group; Crawley group; weak Crawley group; Erdős group

MSC 2010: 20K10, 20K21

1. Introduction

The notion of a torsion-free Crawley group was introduced in [1], where the concept took its definition from an observation of Megibben relating to the more familiar notion of Crawley p-groups. Recall that a torsion-free Abelian group G is said to be a *Crawley* group if, given any pair of pure, dense subgroups of corank 1 in G, there is an automorphism of G mapping one onto the other. The emphasis in [1] was firmly on infinite rank groups and the issues of independence of results from the usual Zermelo-Fraenkel set theory (ZFC). However, it was noted that the concept is a perfectly suitable one for groups of finite rank and some elementary results were obtained in this setting; see [1, Section 2]. A key observation in that work was that as a consequence of a theorem of J. Erdős [3] (or see [4, §51]) a free group is always a Crawley group. Motivated by this result of Erdős, we introduce a new class of torsion-free groups; specifically we say that a torsion-free group G is an *Erdős group* if, given any pair of isomorphic pure subgroups H, K which have isomorphic quotients (i.e. $G/H \cong G/K$), then there exists an automorphism of G mapping H onto K. Similar classes of groups have been investigated by Hill and co-authors in a variety of contexts—see, for example, [8], [9] and [10].

We shall also find it convenient to introduce a somewhat larger class of groups than the Erdős groups which has a clear connection to the notion of the Crawley group; we term such groups *weak Crawley* groups. Specifically we say that a torsion-free group G is a *weak Crawley group* if, given any pair of isomorphic dense maximal pure subgroups H, K of G, there is an automorphism of G mapping H onto K. In other words, we are requiring in the weak Crawley situation that the quotients $G/H, G/K$ are isomorphic to $\mathbb{Q}$, the group of rationals. Denoting the respective classes by $\mathscr{C}, \mathscr{E}$ and $w\mathscr{C}$, it is clear that we have the inclusions $\mathscr{C} \subseteq w\mathscr{C}$ and $\mathscr{E} \subseteq w\mathscr{C}$. The rest of the paper shall be devoted to exploring the various interconnections between these classes, particularly in the situation that the groups have finite rank. In order to avoid repetition, we shall make use of the following *ad hoc* terminology: a group G is said to be *Crawley-like* if G belongs to one of the classes $\mathscr{C}, w\mathscr{C}$ or $\mathscr{E}$ and we shall make statements of the form 'if G is Crawley-like and H is a subgroup satisfying some condition $\mathscr{P}$, then H is Crawley-like' as an abbreviation for the more tedious 'if G is in $\mathscr{C}$ (respectively $w\mathscr{C}, \mathscr{E}$) and H is a subgroup satisfying some condition $\mathscr{P}$, then H is in $\mathscr{C}$ (respectively $w\mathscr{C}, \mathscr{E}$)'.

Throughout, the word group shall mean an additively-written torsion-free Abelian group; the books [5], [6] contain the necessary terminology on Abelian group theory and we follow both that terminology and notation throughout. In particular, we write $A \leqslant_* B$ to denote that A is a pure subgroup of B and we denote the purification of a subgroup H in the group G by writing H_*. The type of a subgroup $X \leqslant \mathbb{Q}$ will be denoted by $t(X)$.

2. Preliminaries

We shall frequently use, without specific reference, the following observation of Tony Corner; for a proof see [6, Exercise 13, p. 141].

Proposition 2.1. *If A is a torsion-free group of finite rank n, and B, C are isomorphic pure subgroups of rank $n-1$, then $A/B \cong A/C$.*

We shall also need the following simple result:

Proposition 2.2. *Suppose that A is a reduced group and $X = A \oplus D$, where D is divisible. If $H \leqslant_* X$, then $H + D_1 \leqslant_* X$ for any divisible subgroup D_1 of D.*

P r o o f. Let $h+d_1 \in (H+D_1)\cap kX$; say $h+d_1 = k(a+d)$ for some $a \in A$, $d \in D$. Since D_1 is divisible, we have $d_1 = kd_1'$ for some $d_1' \in D_1$. So $h = k(a + d - d_1') \in H \cap kX = kH$, i.e. $h = kh_1$ for some $h_1 \in H$ and hence, $h + d_1 = kh_1 + kd_1' = k(h_1 + d_1') \in k(H + D_1)$, as required. □

Our next result gathers together some elementary results relating to groups common to all three classes.

Proposition 2.3. *If G is either* (i) *of rank* 1 (ii) *free or* (iii) *divisible, then $G \in \mathscr{C} \cap w\mathscr{C} \cap \mathscr{E}$.*

P r o o f. (i) Since the only proper pure subgroup of a rank one group is zero, the result follows immediately.

(ii) A free group of arbitrary rank is a Crawley group (see e.g. Example 3.1 in [1]) and hence it is certainly weak Crawley. That a free group of infinite rank is an Erdős group follows from Erdős's celebrated result [3], while the finite rank case follows easily from the fact that pure subgroups of a finite rank free group are necessarily summands.

(iii) The proof is straightforward since any pure subgroup of a divisible group is a direct summand. □

Proposition 2.4. *If G is a Crawley-like group and $G = A \oplus B$, then if* $\mathrm{Hom}(B, A) = 0$, *both A, B are Crawley-like. In particular a characteristic summand of a Crawley-like group is again Crawley-like.*

P r o o f. We give the proof for Erdős groups and shall note any changes required in the other situations. So suppose that $G \in \mathscr{E}$. If C, D are isomorphic pure subgroups of A with $A/C \cong A/D$, then $X = C \oplus B$ and $Y = D \oplus B$ are isomorphic subgroups of G and $G/X \cong G/Y$. Hence there is an automorphism of G mapping X onto Y. Since $\mathrm{Hom}(B, A) = 0$, we can represent any automorphism of G by a 2×2 matrix which is lower triangular, say $\Delta = \left(\begin{smallmatrix} \alpha & 0 \\ \delta & \beta \end{smallmatrix}\right)$ where $\alpha \in \mathrm{End}(A), \beta \in \mathrm{End}(B)$ and $\delta \in \mathrm{Hom}(A, B)$. Moreover, since every endomorphism of G has such a lower triangular form, we can conclude that α, β are actually automorphisms of A, B respectively. Since $\Delta(X) = Y$, we deduce that $\alpha(C) = D$ and so A is an Erdős group. An identical proof works for weak Crawley groups; the situation for Crawley groups is equally straightforward since the only use made of the fact that C was isomorphic to D was to establish that the extensions by the direct summand B were again of the appropriate form.

We now consider the summand B. So suppose that C, D are isomorphic pure subgroups of B with $B/C \cong B/D$, then $X = A \oplus C$ and $Y = A \oplus D$ are isomorphic subgroups of G and $G/X \cong G/Y$. As before we have an automorphism Δ mapping X onto Y, $\begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} A \\ D \end{pmatrix}$. Thus $\delta(A) + \beta(C) = D$; in particular $\delta(a) \in D$ for all $a \in A$.

However, the inverse of Δ maps Y onto X and so $\Delta^{-1} = \begin{pmatrix} \alpha^{-1} & 0 \\ \gamma & \beta^{-1} \end{pmatrix}$ for an appropriate γ. However, Δ^{-1} maps $(0, \delta(a)) \mapsto \beta^{-1}\delta(a)$ and so $\beta^{-1}\delta(a) \in C$ for all $a \in A$ or, equivalently, $\delta(a) \in \beta(C)$. But this then implies that $D = \delta(A) + \beta(C) = \beta(C)$ and since β is an automorphism of B, we conclude that B is an Erdős group. As with the argument for A, this too carries over immediately to the cases of weak Crawley and Crawley groups.

Finally, the specific case follows immediately since if B is a characteristic summand, $\mathrm{Hom}(B, A) = 0$. □

It was observed in [1, Proposition 2.1] that the class of Crawley groups is closed under the formation of direct sums with free groups of finite rank. This property persists for weak Crawley groups but fails for Erdős groups.

Proposition 2.5. *If G is a weak Crawley group and F is free of finite rank, then $G \oplus F$ is again weak Crawley. However, if G is a group of rank* 1 *(and thus trivially an Erdős group) which is not divisible and of type $\tau > 0 = t(\mathbb{Z})$, then $G \oplus \mathbb{Z}$ is not an Erdős group.*

Proof. The first part is similar to the proof of [1, Proposition 2.1] and the second claim is a consequence of Proposition 2.6 below. □

The requirement in the second part of Proposition 2.5 that the rank 1 group not be divisible cannot be dropped. In fact we have:

Proposition 2.6. *If $G = A \oplus B$ is a completely decomposable group of rank two with $t(A) < t(B)$, then G is Erdős if, and only if, $B = \mathbb{Q}$.*

Proof. First let $B \cong \mathbb{Q}$. If H and K are two isomorphic pure subgroups of G, then $t(H)$ is either $t(\mathbb{Q})$ or $t(A)$. In the former case H is a direct summand of G and we get $G = H \oplus X$, where $X \cong G/H$. Similarly $G = K \oplus Y$ and now the direct sum of isomorphisms between the components gives an automorphism of G mapping H onto K. Finally, if $t(H) = t(A)$, then $H \cap \mathbb{Q}$ is pure in $\mathbb{Q}$ and so must be zero. Therefore $H \oplus \mathbb{Q}$ is a pure subgroup of rank 2 in G and so $G = H \oplus \mathbb{Q}$. Similar decomposition holds for K and hence the direct sum of an isomorphism between H and K with $\iota_{\mathbb{Q}}$ yields the required automorphism of G which maps H onto K. So if $B \cong \mathbb{Q}$, then G is Erdős.

Now suppose B is not divisible. Then $t(B) < (\infty, \infty, \ldots) = t(\mathbb{Q})$. So there exist a characteristic $\chi \in t(B)$ and a prime p such that χ_p, the p-component of χ, is finite. Now choose $b \in B$ with $h_p(b) = 0$ and $a \in A$ with $0 < h_p(a)$, and let $H = \langle a \rangle_*$ and $K = \langle a + b \rangle_*$. So H and K are two pure subgroups of G with rank one, where $t(H) = t(K) = t(A)$, and hence they are isomorphic. Moreover, $G/H \cong G/K$, automatically. Now if G were Erdős, then there would exist an automorphism ψ of G for which $\psi(H) = K$. But ψ is of the form

$$\begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$$

for some $\alpha \in \mathrm{Aut}(A)$, $\beta \in \mathrm{Aut}(B)$, $\delta \in \mathrm{Hom}(A, B)$ and $\psi(h) = a + b$ for some $h \in H$. On the other hand, $h \in H$ which means there exist non-zero integers m, n such that $nh = ma$. This yields

$$n(a + b) = n\psi(h) = \psi(nh) = \psi(ma) = m\psi(a) = m(\alpha(a) + \delta(a)),$$

and hence $m\alpha(a) = na$, $m\delta(a) = nb$. But $\alpha \in \mathrm{Aut}(A)$ which implies $h_p(a) = h_p(\alpha(a))$, and therefore $h_p(m) = h_p(n)$. Now consider $\delta(a)$; noting $h_p(\delta(a)) \geqslant h_p(a) > h_p(b)$, we have

$$\begin{aligned} h_p(m\delta(a)) = h_p(m) + h_p(\delta(a)) &\geqslant h_p(m) + h_p(a) > h_p(m) + h_p(b) \\ &= h_p(n) + h_p(b) = h_p(nb); \end{aligned}$$

but this yields a contradiction because $m\delta(a) = nb$, and this completes the proof. □

It follows immediately from Proposition 2.5 above that the containment $\mathscr{E} \subseteq w\mathscr{C}$ is actually strict. We now show that a similar situation pertains for Crawley and weak Crawley groups so that we have $\mathscr{C} \subsetneqq w\mathscr{C}$ and $\mathscr{E} \subsetneqq w\mathscr{C}$.

Proposition 2.7. *The class of weak Crawley groups properly contains the class of Crawley groups.*

P r o o f. Consider the additive group of p-adic integers, J_p. Claim that J_p is not a Crawley group. If $\mathbb{Z}_p$ denotes the group of integers localized at the prime p, then the quotient $J_p/\mathbb{Z}_p$ is isomorphic to the direct sum of continuously many copies of $\mathbb{Q}$. Hence there are $2^{2^{\aleph_0}}$ dense maximal pure subgroups of J_p. Since the endomorphism ring of J_p is just J_p, not all such subgroups can lie in the same orbit and so J_p is certainly not a Crawley group. However, it is a weak Crawley group: let M, N be dense maximal pure subgroups of J_p so that $0 \to M \to J_p \to \mathbb{Q} \to 0$ is an exact sequence. Then taking homomorphisms into J_p, we get

$$0 \to \mathrm{Hom}(J_p, J_p) \to \mathrm{Hom}(M, J_p) \to \mathrm{Ext}(\mathbb{Q}, J_p) = 0,$$

the last equality following since J_p is algebraically compact. Hence every homomorphism from M into J_p is just multiplication by a p-adic integer. Suppose then that $M \cong N$ via the multiplication by $\alpha \in J_p$. Reversing roles we get $\beta \in J_p$ so that $\beta\alpha M = M$. However, $1 \in M$ and so it follows immediately that α is a unit in J_p. Thus there is an automorphism of J_p mapping M onto N and J_p is a weak Crawley group. □

Corollary 2.1. *The group J_p is an Erdős group.*

Proof. Note that if M is pure in J_p then it follows from an observation of Kaplansky that the closure of M is a summand and hence in this case must be J_p itself. So we have that $\mathrm{Hom}(J_p/M, J_p) = 0$ and since J_p/M is torsion-free, the exact sequence displayed in Proposition 2.7 above reduces to $\mathrm{Hom}(J_p, J_p) = \mathrm{Hom}(M, J_p)$ and the rest of the proof carries over *mutatis mutandis.* □

Note that if A is a rank 1 group not isomorphic to $\mathbb{Z}$ or $\mathbb{Q}$, then $A \oplus \mathbb{Z}$ is a Crawley group but not an Erdős group, while the group J_p is Erdős but not Crawley. So we also have that $\mathscr{C} \nsubseteq \mathscr{E}$ and $\mathscr{E} \nsubseteq \mathscr{C}$.

In fact we can establish quite a bit more and in the process show that the class of Erdős groups, and hence the class of weak Crawley groups, are extensive.

Theorem 2.1. *A reduced torsion-free algebraically compact group is an Erdős group.*

Proof. Suppose G is a complete group and M, N are isomorphic pure subgroups with $G/M \cong G/N$. Then as noted above $\overline{M}, \overline{N}$ are summands of G, say $G = \overline{M} \oplus G_M = \overline{N} \oplus G_N$. Note first that $G_M \cong G_N$: this follows because $G/M = \overline{M}/M \oplus X$ where $\overline{M}/M$ is the divisible part of G/M and $X \cong G_M$ with similar results holding for N. Finally, if $\varphi\colon M \to N$ is an isomorphism then φ extends to an isomorphism $\bar{\varphi}\colon \overline{M} \to \overline{N}$ and the (direct) sum of $\bar{\varphi}$ and the isomorphism between G_M and G_N is the required automorphism of G mapping M onto N. □

Our choice of the name Erdős derives from the famous theorem of Erdős on subgroups of free groups—see [4] Corollary 51.5. The extension of Erdős's theorem to homogeneous completely decomposable groups of arbitrary type is reasonably well known but seems not to be explicitly recorded, although it follows easily from recent work of Salce and Strüngman [11]. We present a straightforward proof based on Warfield's duality:

Proposition 2.8. *A homogeneous completely decomposable group is an Erdős group.*

P r o o f. If G is an abelian group, let $N(G)$ denote the subring of $\mathbb{Q}$ generated by 1 and $1/p$ for all primes p such that $pG = G$. Then by a well-known result of Warfield, [12, Proposition 1], we have that if A is torsion-free of rank 1, the natural map $\varphi\colon G \to \operatorname{Hom}(A, A \otimes G)$ is an isomorphism if and only if $N(A) \subseteq N(G)$.

Recall also that Erdős's theorem holds for completey homogeneous groups of reduced (or non-nil) type R: the free groups in the original argument can be replaced by free R-modules and only minor modifications are then required.

Now suppose G is a completely homogeneous group of type S, S arbitrary. Then we can write $S = R + T$ where R is the reduced type of S (i.e. all finite entries are replaced by zeros) and T is the co-reduced type (i.e. all infinities are replaced by zeros). Note then that every entry in the type of T is finite and so $N(T) = \mathbb{Z}$.

So now suppose that M, N are pure subgroups of G such that $M \cong N$ and $G/M \cong G/N$. Then M, N are also homogeneous completely decomposable groups of type S. Hence we may write $G = F_0 \otimes T$, $M = F_1 \otimes T$, $N = F_2 \otimes T$ where each F_i $(i = 0, 1, 2)$ is a direct sum of copies of R. Since $M \cong N$, it follows immediately that $F_1 \cong F_2$.

Consider the short exact sequences

$$(*) \qquad 0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow X \longrightarrow 0,\ 0 \longrightarrow F_2 \longrightarrow F_0 \longrightarrow Y \longrightarrow 0$$

where $F_0/F_1 = X$, $F_0/F_2 = Y$.

Since X is a homomorphic image of F_0, X is p-divisible for every prime p for which $pF_0 = F_0$. Since F_0 is a free R-module this means that $N(R) \subseteq N(X)$; similarly $N(R) \subseteq N(Y)$. Now tensor the exact sequences $(*)$ with T to get

$$(**)$$
$$0 \longrightarrow F_1 \otimes T \longrightarrow F_0 \otimes T \longrightarrow X \otimes T \longrightarrow 0,\ 0 \longrightarrow F_2 \otimes T \longrightarrow F_0 \otimes T \longrightarrow Y \otimes T \longrightarrow 0,$$

noting that the sequences in $(**)$ are exact since T is torsion-free.

Now $X \otimes T \cong F_0 \otimes T/F_1 \otimes T \cong G/M$ and $Y \otimes T \cong F_0 \otimes T/F_2 \otimes T \cong G/N$. Since $G/M \cong G/N$ by assumption, we have $X \otimes T \cong Y \otimes T$. However, since $N(T) = \mathbb{Z} \subseteq N(R) \subseteq N(X)$ we have by Warfield's result that $X \cong \operatorname{Hom}(T, X \otimes T)$; similarly $Y \cong \operatorname{Hom}(T, Y \otimes T)$. But as $X \otimes T \cong Y \otimes T$, we certainly have $\operatorname{Hom}(T, X \otimes T) \cong \operatorname{Hom}(T, Y \otimes T)$ and so $X \cong Y$. Thus in $(*)$ we have that F_1, F_2 are isomorphic pure subgroups of F_0 with $X \cong F_0/F_1 \cong F_0/F_2 \cong Y$. So by Erdős's theorem for homogeneous completely decomposable groups of reduced type (or free R-modules, which amounts to the same thing) there is an automorphism α of F_0 with $\alpha(F_1) = F_2$. Set $\psi = \alpha \otimes 1_T$, so that ψ is an automorphism of $G = F_0 \otimes T$. But then $\psi(M) = (\alpha \otimes 1_T)(F_1 \otimes T) = \alpha(F_1) \otimes T = F_2 \otimes T = N$. □

We can, in fact, extend Proposition 2.6 and show a good deal more about the outcome of adding divisible summands:

Proposition 2.9. (i) *If G is a reduced Crawley group then the direct sum $G \oplus D$, where D is divisible of finite rank, is not necessarily Crawley;*

(ii) *If G is a reduced weak Crawley group then the direct sum $G \oplus D$, where D is divisible of finite rank, is again weak Crawley;*

(iii) *If G is a reduced completely decomposable homogeneous group of finite rank then the direct sum $G \oplus D$, where D is divisible of arbitrary rank, is Erdős, Crawley and weak Crawley.*

Proof. For (i) it suffices to note that if G is a free group of infinite rank, then G is a Crawley group by Proposition 2.3. However, G has a pure subgroup G_1 such that $G/G_1 \cong \mathbb{Q}$ and thus G and $G_1 \oplus \mathbb{Q}$ are maximal pure subgroups of $G \oplus \mathbb{Q}$ which are not even isomorphic.

For (ii) let $X = G \oplus D$ and let H, K be two isomorphic subgroups of X and $X/H \cong X/K \cong \mathbb{Q}$.

If $G \leqslant H$ and $G \leqslant K$, then by the modular law $H = G \oplus (H \cap D)$, $K = G \oplus (K \cap D)$. Now $(H \cap D)$ and $(K \cap D)$ are maximal divisible subgroups of two isomorphic groups and so they are isomorphic. Moreover, $D/(D \cap H) \cong X/H \cong \mathbb{Q} \cong X/K \cong D/(D \cap K)$ and D is weak Crawley. So there exists an automorphism φ of D which maps $(D \cap H)$ onto $(D \cap K)$. Now $\psi = \iota_G \oplus \varphi \in \operatorname{Aut}(X)$ is our desired map which satisfied $\psi(H) = K$.

If $G \leqslant H$, $G \nleqslant K$ then $H = G \oplus (H \cap D)$ and so $K = K_1 \oplus D_1$, where D_1 is a maximal divisible subgroup of K which is isomorphic to $(D \cap H)$ and $K_1 \cong G$. Let φ is the isomorphism between G and K_1. In this case $K <_* K_1 \oplus D \leqslant_* X$, and K is a maximal pure subgroup of X, which implies $X = K_1 \oplus D$. But D is weak Crawley and by the first part of the proof, there exists an automorphism θ of D which maps $(D \cap H)$ onto D_1. Therefore, $\psi = \varphi \oplus \theta \in \operatorname{Aut}(X)$ and $\psi(H) = K$.

If $G \nleqslant H$ and $G \nleqslant K$; then $H = H_1 \oplus (H \cap D)$ and $K = K_1 \oplus (K \cap D)$. But the reduced parts (and divisible parts) of two isomorphic groups are isomorphic. Hence $H_1 \cong K_1, (H \cap D \cong K \cap D)$. In this case if $(H \cap D) = D = (K \cap D)$, then $H = D \oplus (H \cap G)$ and $K = D \oplus (K \cap G)$ and G is weak Crawley, which implies X is weak Crawley.

If $(H \cap D) \neq D, (K \cap D) \neq D$, then $H < H_1 \oplus D \leqslant_* X$ and similar is true for $K_1 \oplus D$. So $X = H_1 \oplus D = K_1 \oplus D$ and this completes the proof.

(iii) By Exc. 8 of [6], every pure subgroup of such a group is a direct summand. Therefore, $G \oplus D \in \mathscr{C} \cap w\mathscr{C} \cap \mathscr{E}$. □

In [1] it was established when a direct sum G of a free group F and a group N with trivial dual is Crawley. There is an interesting dichotomy: if F has infinite rank, G is Crawley if, and only if, N is trivial, while if the rank of F is finite, a necessary and sufficient condition is that N is a Crawley group. At present we cannot prove a result of this generality for either weak Crawley or Erdős groups, but if we make an additional assumption on the group N then we can achieve a parallel result in the infinite rank case:

Proposition 2.10. *If N has trivial dual, $N^* = 0$, and F is a free group then*

(i) *if F has finite rank, then the direct sum $G = F \oplus N$ is a weak Crawley group if, and only if, N is a weak Crawley group;*

(ii) *if N has an isomorphic maximal pure subgroup, then the direct sum $G = F \oplus N$, where F is free of infinite rank, is a weak Crawley (Erdős) group if, and only if, $N = 0$.*

P r o o f. (i) If G is weak Crawley, then N is also weak Crawley by Proposition 2.4. Conversely, if N is weak Crawley, the result follows from Proposition 2.5.

(ii) In part (ii) it suffices, of course, to establish the result for weak Crawley groups.

Certainly if $N = 0$, G is weak Crawley (Erdős) since a free group of infinite rank is both weak Crawley and Erdős. Conversely, assume $N \neq 0$ and choose a nonzero pure subgroup M of N which is isomorphic to N. Then F has a pure subgroup W such that $F/W \cong \mathbb{Q}$ and thus $W \oplus N \cong F \oplus M$. Moreover, $G/(W \oplus N) \cong G/(F \oplus M) \cong \mathbb{Q}$. Hence if G is weak Crawley, then $F \oplus M$ is equivalent to $W \oplus N$ via an automorphism, θ say, of G.

However, any endomorphism of G can be represented as a lower triangular matrix since $N^* = 0$ and so the automorphism θ can be represented as a matrix

$$\theta = \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$$

where α is an automorphism of F. Consequently, the image of $F \oplus M$ under θ must have the form $\alpha(F) \oplus X$ where $X = \delta(F) + \beta(M)$ and thus cannot be $W \oplus N$—a contradiction. Thus we must have $N = 0$ as required. □

Corollary 2.2. *The group $F \oplus V$, where F is free of countable rank and V is is divisible of countable rank, is not Crawley-like.*

Theorem 2.2. *Let G be a reduced Erdős group and D a divisible group of finite rank. Then $X = G \oplus D$ is an Erdős group.*

P r o o f. Let H and K be two isomorphic subgroups of X and $X/H \cong X/K$. We have two cases:

(1) $H \cap D \neq 0$, which implies $K \cap D \neq 0$. So $D \cap H$ and $D \cap K$ are respectively the maximal divisible subgroups of H, K and

$$H = H' \oplus (D \cap H), \quad K = K' \oplus (D \cap K).$$

But $H \cong K$ and hence the divisible parts and the reduced parts of these groups are isomorphic. This means $H' \cong K'$ and $D \cap H \cong D \cap K$. Moreover, D is of finite rank and $D \cap H, D \cap H$ are two isomorphic pure subgroups of D. So they are direct summands of D and their complements in D are isomorph. But D is Erdős and hence there exists an automorphism φ of G which maps $D \cap H$ onto $D \cap K$.

Now let $H_1 = H + D = H' \oplus D$ and $K_1 = K + D = K' \oplus D$. From $H' \cong K'$ we deduce that $H_1 \cong K_1$. Moreover, $D \leqslant H_1, K_1$. Hence by the modular law

$$H_1 = (H_1 \cap G) \oplus D, \ K_1 = (K_1 \cap G) \oplus D.$$

Moreover, $X/H_1 \cong X/K_1$, because $X/H_1 = (X/H)/(H_1/H)$, but $H_1/H \cong D/(D \cap H)$, which is divisible, and so

$$(*) \qquad \frac{X}{H} \cong \frac{H_1}{H} \oplus \frac{A}{H},$$

for an appropriate subgroup A of X. Similarly, $(*)$ is true for X/K, i.e.,

$$\frac{B}{K} \oplus \frac{K_1}{K} \cong \frac{X}{K},$$

for some $B \leqslant X$. But $X/H \cong X/K$ and $H_1/H \cong D/(D \cap H) \cong D/(D \cap K) \cong K_1/K$, which yields $B/K \cong A/H$ and as we claimed $X/K_1 \cong B/K \cong A/H \cong X/H_1$. But $H_1 \cong K_1$ and the reduced parts of these isomorphic subgroups are isomorphic which means $H_1 \cap G \cong K_1 \cap G$. Further, $G/H_1 \cap G \cong X/H_1 \cong X/K_1 \cong G/K_1 \cap G$. But G is Erdős and so there exists an automorphism α of G which maps $H_1 \cap G$ onto $K_1 \cap G$.

Now $H' \subseteq H_1 = (H_1 \cap G) \oplus D$ and hence

$$H' = \{w_i + d_i \colon \ w_i \in H_1 \cap G, \ d_i \in D\}.$$

Moreover, if $w \in H_1 \cap G$, then $w \in H_1 = H' \oplus D$. This means $w = h + d$ for some $h \in H, d \in d$ and $w - d \in H'$. Besides, if $w \in H_1 \cap G$ are such that $d_1 \neq d_2$ and $w + d_1, w + d_2$ are in H', then $(w + d_1) - (w + d_2) \in H'$. Therefore, $H' \cap D \neq 0$, which is a contradiction. This means that for every $w \in H_1 \cap G$ there exists a unique element $d \in D$ such that $w + d \in H'$.

Similarly $K' = \{k_i + d_i' \colon \ k_i \in K_1 \cap G, d_i' \in D\}$ and for every element v of $K_1 \cap G$ there exists a unique element $d' \in D$ such that $v + d' \in K$.

But $\alpha(H_1 \cap G) = K_1 \cap G$ and so every element of K' is of the form $\alpha(w) + r$ for some $w \in H_1 \cap G$ and $r \in D$.

Now let $w \in H_1 \cap G$ and let d be a unique element of D such that $w + d \in H'$ and let r be a unique element of D such that $\alpha(w) + r \in K'$.

Define $\theta\colon H_1 \cap G \to D$ in which $\theta(w) = r - \varphi(d)$. Then θ is a well-defined homomorphism. (Indeed, let $w_1, w_2 \in H_1 \cap G$ and $\theta(w_1) = r_1 - \varphi(d_1)$, $\theta(w_2) = r_2 - \varphi(d_2)$. So $w_1 + d_1, w_2 + d_2 \in H'$, $\alpha(w_1) + r_1, \alpha(w_2) + r_2 \in K'$. Then $w_1 + w_2 + d_1 + d_2 \in H'$ and $\alpha(w_1) + \alpha(w_2) + r_1 + r_2 = \alpha(w_1 + w_2) + r_1 + r_2 \in K'$. So $\theta(w_1+w_2) = (r_1+r_2)-\varphi(d_1+d_2) = (r_1-\varphi(d_1))+(r_2-\varphi(d_2)) = \theta(w_1)+\theta(w_2)$.) But by injectivity of D, there exists a homomorphism $\delta\colon G \to D$ such that $\delta{\restriction}_{H_1\cap G} = \theta$. Now

$$\psi = \begin{pmatrix} \alpha & 0 \\ \delta & \varphi \end{pmatrix}$$

is an automorphism of X which maps H onto K.

(2) $H \cap D = 0$ and hence $K \cap D = 0$.

In this case if $H, K \leqslant G$ then $G/H \oplus D \cong X/H \cong X/K \cong G/K \oplus D$ which yields $G/H \cong G/K$. Moreover, G is Erdős and this completes the proof.

Otherwise, let $H_1 = H \oplus D$ and $K_1 = K \oplus D$. We have $D \subseteq K_1, H_1$ and by the modular law $H_1 = (H_1 \cap G) \oplus D$ and $K_1 = (K_1 \cap G) \oplus D$. Now similarly to the proof of part (1), $X/H_1 \cong X/K_1$ and $H_1 \cong K_1$. Moreover, there exists an automorphism α of G which maps $H_1 \cap G$ onto $K_1 \cap G$.

Now if we set

$$H = \{w_i + d_i\colon\ w_i \in H_1 \cap G,\ d_i \in D\},$$

then for every $w \in H_1 \cap G$ there exists a unique element $d \in D$ such that $w + d \in H$.

Similarly $K = \{k_i + d_i'\colon\ k_i \in K_1 \cap G, d_i' \in D\}$ and for every element v of $K_1 \cap G$ there exists a unique element $d' \in D$ such that $v + d' \in K$.

Moreover, $\alpha(H_1 \cap G) = K_1 \cap G$ and so every element of K is of the form $\alpha(w) + r$ for some $w \in H_1 \cap G$ and $r \in D$. Now let $w \in H_1 \cap G$ and let d be the unique element of D such that $w + d \in H$ and let r be the unique element of D such that $\alpha(w) + r \in K$.

Define $\theta\colon H_1 \cap G \to D$ in which $\theta(w) = r - d$. Then θ is a well-defined homomorphism and by injectivity of D, there exists a homomorphism $\delta\colon G \to D$ such that $\delta{\restriction}_{H_1\cap G} = \theta$. Now

$$\psi = \begin{pmatrix} \alpha & 0 \\ \delta & \iota_D \end{pmatrix}$$

is an automorphism of X which maps H onto K and so X is an Erdős group. □

Corollary 2.3. *Let F be a free group of infinite rank and D a divisible group of finite rank. Then $X = F \oplus D$ is an Erdős group.*

Proposition 2.11. *If $G = \bigoplus_{i\in I} G_i$, where $|I| \geqslant 3$, each G_i is homogeneous and $t(G_1) < t(G_2) < \ldots$, then G is not an Erdős group.*

P r o o f. We have $\operatorname{Hom}\Bigl(\bigoplus_{(1,2\neq)i\in I} G_i,\ G_1 \oplus G_2\Bigr) = 0$ and so the result follows from Proposition 2.4 and Proposition 2.6. □

Our final result shows that both the inclusions $\mathscr{C} \subseteq w\mathscr{C}$ and $\mathscr{E} \subseteq w\mathscr{C}$ are proper inclusions even when one restricts one's attention to $\aleph_1$-free groups.

Theorem 2.3. *The Baer-Specker group* $P = \prod_{\aleph_0} \mathbb{Z}$ *is a weak Crawley group which is neither Crawley nor Erdős.*

P r o o f. That P is not a Crawley group has been established in [1, Example 4.1]. Dugas and Irwin [2, Theorem 18] have shown that P contains $2^{2^{\aleph_0}}$ different basic subgroups. Now if B is a basic subgroup of P, then since $B/pB \cong P/pP$ for any prime p, we deduce that B is free of rank $2^{\aleph_0}$. Furthermore, it follows from Lemma below that P/B is divisible of rank $2^{\aleph_0}$ for any basic subgroup B. Thus, any pair of basic subgroups B, B' satisfies $B \cong B'$ and $P/B \cong P/B'$. However, there cannot be an automorphism interchanging arbitrary pairs of basic subgroups, since there are at most $2^{\aleph_0}$ automorphisms of P but there are $2^{2^{\aleph_0}}$ such pairs. Thus P is not an Erdős group.

Finally, the claim that P is a weak Crawley group follows from [7, Theorem 3.1]. □

Lemma 2.1. *If G is a torsion-free group with a pure free uncountable subgroup F and $|G/F| < |G|$, then G has a free summand of cardinality $|F|$. In particular, if B is a basic subgroup of the Baer-Specker group P, then $|P/B| = |B| = 2^{\aleph_0}$.*

P r o o f. Note that $|G| = |F|$ and let $\mu = |G/F|$. Choose representatives $\{g_\alpha\colon \alpha < \mu\}$ for G/F and let $A = \langle g_\alpha \rangle$, so that $|A| \leqslant \mu < |G|$. Then $|A \cap F| < |F|$ and so we can embed $A \cap F$ in a summand F_0 of F with $\aleph_0 \leqslant \operatorname{rk}(F_0) < \operatorname{rk}(F)$; say $F = F_0 \oplus F_1$. Let $N = A + F_0$ so that $G = N + F_1$ and clearly $\operatorname{rk}(F_1) = \operatorname{rk}(F)$. A straightforward argument shows that $N \cap F_1 = 0$ and so $G = N \oplus F_1$ has a free summand isomorphic to F.

In the particular case of P we have $|P| = |B| = 2^{\aleph_0}$ and so the assumption that $|P/B| < |B|$ would lead to the contradiction that P has a free summand of uncountable rank. □

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Authors' addresses: B. Goldsmith, School of Mathematical Sciences, Dublin Institute of Technology, Aungier Street, Dublin 2, Ireland, e-mail: `brendan.goldsmith@dit.ie`; F. Karimi, Mathematics Department, Payame Noor University, Tehran, Iran, e-mail: `fatemehkarimi63@yahoo.com`; A. M. Aghdam, Faculty of Mathematical Sciences, Tabriz University, Tabriz, Iran, e-mail: `mehdizadeh@tabrizu.ac.ir`.