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Research Article

Some generalized numerical radius inequalities involving Kwong functions

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Abstract

We prove several numerical radius inequalities involving positive semidefinite matrices via the Hadamard product and Kwong functions. Among other inequalities, it is shown that if X is an arbitrary $n \times n$ matrix and A, B are positive semidefinite, then

$$\omega(H_{f,g}(A)) \le k\,\omega(AX + XA),$$

which is equivalent to

$$\begin{aligned}&\omega(H_{f,g}(A,B) \pm H_{f,g}(B,A))\\ &\quad\le k'\left\{\omega((A+B)X + X(A+B)) + \omega((A-B)X - X(A-B))\right\},\end{aligned}$$

where f and g are two continuous functions on $(0,\infty)$ such that $h(t) = \frac{f(t)}{g(t)}$ is Kwong, $k = \max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A)\right\}$ and $k' = \max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A) \cup \sigma(B)\right\}$.

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1. Introduction

Let $\mathcal{M}_n$ be the C^*-algebra of all $n \times n$ complex matrices and $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{C}^n$. A capital letter means an $n \times n$ matrix in $\mathcal{M}_n$. For Hermitian matrices A and B, we write $A \ge 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \ge B$ if $A - B \ge 0$. The numerical radius of $A \in \mathcal{M}_n$ is defined by

$$\omega(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \| x \| = 1\}.$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathcal{M}_n$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathcal{M}_n$, $\frac{1}{2}\|A\| \le \omega(A) \le \|A\|$; see [11]. For further information about numerical radius inequalities we refer the reader to [4, 11, 15, 16] and references therein. We use the notation J for the matrix whose entries are equal to one.

The Hadamard product (Schur product) of two matrices $A, B \in \mathcal{M}_n$ is the matrix $A \circ B$ whose (i,j) entry is $a_{ij}b_{ij}$ $(1 \le i,j \le n)$. The Schur multiplier operator S_A on $\mathcal{M}_n$ is

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defined by $S_A(X) = A \circ X$ $(X \in \mathcal{M}_n)$. The induced norm of S_A with respect to the numerical radius norm will be denoted by

$$\|S_A\|_\omega = \sup_{X \neq 0} \frac{\omega(S_A(X))}{\omega(X)} = \sup_{X \neq 0} \frac{\omega(A \circ X)}{\omega(X)}.$$

A continuous real valued function f on an interval $(a, b) \subseteq \mathbb{R}$ is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all Hermitian matrices $A, B \in \mathcal{M}_n$ with spectra in (a, b). Following [3], a continuous real-valued function f defined on an interval (a, b) with $a > 0$ is called a Kwong function if the matrix $K_f = \left(\frac{f(\lambda_i)+f(\lambda_j)}{\lambda_i+\lambda_j}\right)_{i,j=1,2,\cdots,n}$ is positive semidefinite for any (distinct) $\lambda_1, \cdots, \lambda_n$ in (a, b). It is easy to see that if f is a nonzero Kwong function, then f is positive and $\frac{1}{f}$ is Kwong. Kwong [13] showed that the set of all Kwong functions on $(0, \infty)$ is a closed cone and includes all non-negative operator monotone functions on $(0, \infty)$. Also, Audenaert [3] gave a characterization of Kwong functions by showing that, for given $0 \leq a < b$, a function f on an interval (a, b) is Kwong if and only if the function $g(x) = \sqrt{x} f(\sqrt{x})$ is operator monotone on (a^2, b^2).

The Heinz means are defined as $H_\nu(a, b) = \frac{a^{1-\nu}b^\nu + a^\nu b^{1-\nu}}{2}$ for $a, b > 0$ and $0 \leq \nu \leq 1$. These interesting means interpolate between the geometric and arithmetic means. In fact, the Heinz inequalities assert that $\sqrt{ab} \leq H_\nu(a, b) \leq \frac{a+b}{2}$, where $a, b > 0$ and $0 \leq \nu \leq 1$. There have been obtained several Heinz type inequalities for Hilbert space operators and matrices; see [5] and references therein.

For two continuous functions f and g on $(0, \infty)$ we denote

$$H_{f,g}(A, B) = f(A)Xg(B) + g(A)Xf(B)$$

and

$$H_{f,g}(A) = f(A)Xg(A) + g(A)Xf(A),$$

where $A, B, X \in \mathcal{M}_n$ such that A, B are positive semidefinite. In particular, for $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ $(\alpha \in [0, 1])$, we get $H_\alpha(A, B) = A^\alpha X B^{1-\alpha} + A^{1-\alpha} X B^\alpha$ and $H_\alpha(A) = A^\alpha X A^{1-\alpha} + A^{1-\alpha} X A^\alpha$. A norm $|||\cdot|||$ on $\mathcal{M}_n$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. Let $A, B, X \in \mathcal{M}_n$ such that A and B are positive semidefinite. In [14] it was conjectured a general norm inequality of the Heinz inequality $|||H_{f,g}(A, B)||| \leq |||AX + XB|||$, where f and g are two continuous functions on $(0, \infty)$ such that $f(t)g(t) \leq t$ and the function $h(t) = \frac{f(t)}{g(t)}$ is Kwong. In particular, if $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ $(\alpha \in [0, 1])$, then we state a Heinz type inequality $|||H_\alpha(A, B)||| \leq |||AX + XB|||$, where $A, B, X \in \mathcal{M}_n$ such that A, B are positive semidefinite. For further information, we refer the reader to [5, 6] and references therein.

The numerical radius $\omega(\cdot)$ is a weakly unitarily invariant norm on $\mathcal{M}_n$, that is $\omega(U^*AU) = \omega(A)$ for every $A \in \mathcal{M}_n$ and every unitary $U \in \mathcal{M}_n$. In [1], the authors proved a Heinz type inequality for the numerical radius as follows

$$\omega(H_\alpha(A)) \leq \omega(AX + XA), \tag{1.1}$$

in which $A, X \in \mathcal{M}_n$ such that A is positive semidefinite. They also showed that the inequality $\omega(H_\alpha(A, B)) \leq \omega(AX + XB)$ is not true in general.

Our research aim is to show some numerical radius inequalities via the Hadamard product and Kwong functions. By using some ideas of [8, 10] and [14], we obtain some extensions and generalizations of inequality (1.1), which are generalizations of a Hienz type inequality for the numerical radius. For instance, we prove if $A, X \in \mathcal{M}_n$ such that A is positive semidefinite, then

$$\omega(H_{f,g}(A)) \leq k\,\omega(AX + XA),$$

where f and g are two continuous functions on $(0,\infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong and $k = \max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda \in \sigma(A)\right\}$.

2. Main results

For our purpose we need the following lemmas.

Lemma 2.1 ([18, Theorem 3.4])**.** (*Spectral Decomposition*) *Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then A is normal if and only if there exists a unitary matrix U such that*

$$U^*AU = \operatorname{diag}(\lambda_1, \cdots, \lambda_n).$$

In particular, A is positive definite if and only if the λ_j $(1 \le j \le n)$ are positive.

Lemma 2.2 ([2, Corollary 4])**.** *Let $A = [a_{ij}] \in \mathcal{M}_n$ be positive semidefinite. Then*

$$\|S_A\|_\omega = \max_i a_{ii}.$$

Lemma 2.3. ([12]). *Let $X, Y \in \mathcal{M}_n$. Then*
(i) $\omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X), \omega(Y)\}$;
(ii) $\frac{\max(\omega(X+Y),\omega(X-Y))}{2} \le \omega\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right) \le \frac{\omega(X+Y)+\omega(X-Y)}{2}$.

Now, we are in position to demonstrate the first result of this section by using some ideas of [8, 10, 14].

Theorem 2.4. *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, and let f, g be two continuous functions on $(0,\infty)$ such that $h(t) = \frac{f(t)}{g(t)}$ is Kwong. Then*

$$\omega(H_{f,g}(A)) \le k\,\omega(AX + XA), \tag{2.1}$$

where $k = \max_{\lambda\in\sigma(A)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$.
Moreover, inequality (2.1) *is equivalent to the inequality*

$$\begin{aligned}&\omega(H_{f,g}(A,B) \pm H_{f,g}(B,A))\\ &\qquad \le k'\left\{\omega((A+B)X + X(A+B)) + \omega((A-B)X - X(A-B))\right\},\end{aligned} \tag{2.2}$$

where $k' = \max_{\lambda\in\sigma(A)\cup\sigma(B)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$.

Proof. Assume that A is positive definite. Since the numerical radius is weakly unitarily invariant, we may assume that A is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. It follows from $\frac{f}{g}$ is a Kwong function that

$$Z = [z_{ij}] = \Lambda\left(\frac{f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_i)}{\lambda_i + \lambda_j}\right)_{(i,j=1,\cdots,n)}\Lambda$$

is positive semidefinite, where $\Lambda = \operatorname{diag}(g(\lambda_1), \cdots, g(\lambda_n))$. It follows from Lemma 2.2 that

$$\|S_Z\|_\omega = \max_i z_{ii} = \max_i \frac{f(\lambda_i)g(\lambda_i)}{\lambda_i} \le k$$

or equivalently, $\frac{\omega(Z\circ X)}{\omega(X)} \le k$ $(0 \ne X \in \mathcal{M}_n)$. If we put $E = [\frac{1}{\lambda_i+\lambda_j}]$ and $F = [f(\lambda_i)g(\lambda_j) + f(\lambda_j)g(\lambda_i)] \in \mathcal{M}_n$, then

$$\omega(E \circ F \circ X) = \omega(Z \circ X) \le k\,\omega(X) \qquad (X \in \mathcal{M}_n).$$

Let the matrix C be the entrywise inverse of E, i.e., $C \circ E = J$. Thus

$$\omega(F \circ X) \le k\,\omega(C \circ X) \qquad (X \in \mathcal{M}_n)$$

or equivalently

$$\omega(H_{f,g}(A)) = \omega(f(A)Xg(A) + g(A)Xf(A)) \le k\,\omega(AX + XA). \tag{2.3}$$

Now, if A is positive semidefinite, we may assume that $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $A_1 \in \mathcal{M}_k\,(k < n)$ is a positive definite matrix. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$, where $X_1 \in \mathcal{M}_k$ and $X_4 \in \mathcal{M}_{n-k}$. Then we have

$$\begin{aligned}
\omega(H_{f,g}(A)) &= \omega\left(\begin{bmatrix} f(A_1)X_1g(A_1) + g(A_1)X_1f(A_1) & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\qquad\qquad\qquad \text{(by Lemma 2.3(i))} \\
&\le k\,\omega\left(\begin{bmatrix} A_1X_1 + X_1A_1 & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{(by (2.3))} \\
&= k\,\omega(A_1X_1 + X_1A_1) \qquad \text{(by Lemma 2.3(i))} \\
&\le k\,\omega(AX + XA) \qquad \text{(by [7, Lemma 2.1])}.
\end{aligned} \tag{2.4}$$

Hence, we reach inequality (2.1). Moreover, if we replace A and X by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$ in inequality (2.1), respectively, then

$$\omega\left(\begin{bmatrix} 0 & H_{f,g}(A,B) \\ H_{f,g}(B,A) & 0 \end{bmatrix}\right) \le k'\,\omega\left(\begin{bmatrix} 0 & AX + XB \\ XA + BX & 0 \end{bmatrix}\right),$$

whence

$$\begin{aligned}
&\max\left\{\omega(H_{f,g}(A,B) \pm H_{f,g}(B,A))\right\} \\
&\le 2\,\omega\left(\begin{bmatrix} 0 & f(A)Xg(B) + g(A)Xf(B) \\ g(B)Xf(A) + f(B)Xg(A) & 0 \end{bmatrix}\right) \\
&\qquad\qquad\qquad \text{(by Lemma 2.3(ii))} \\
&\le 2k'\,\omega\left(\begin{bmatrix} 0 & AX + XB \\ XA + BX & 0 \end{bmatrix}\right) \qquad \text{(by (2.4))} \\
&\le k'\,(\omega(AX + XB + XA + BX) + \omega(AX + XB - XA - BX)) \\
&\qquad\qquad\qquad \text{(by Lemma 2.3(ii))}.
\end{aligned}$$

Thus, we have inequality (2.2). Also, if we put $B = A$ in inequality (2.2), then we reach inequality (2.1). □

If we take $f(t) = t^{\alpha}$ and $g(t) = t^{1-\alpha}$ in Theorem 2.4 for each $0 \le \alpha \le 1$, then we get the next result.

Corollary 2.5 ([1, Theorem 2.4])**.** *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, and let $0 \le \alpha \le 1$. Then*

$$\omega(H_\alpha(A)) \le \omega(AX + XA). \tag{2.5}$$

Moreover, inequality (2.5) *is equivalent to the inequality*

$$\begin{aligned}
&\omega(H_\alpha(A,B) \pm H_\alpha(B,A)) \\
&\qquad \le \omega((A+B)X + X(A+B)) + \omega((A-B)X - X(A-B)).
\end{aligned}$$

Corollary 2.6. *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, and let f be a non-negative operator monotone function on $[0,\infty)$ such that $f'(0) = \lim_{x\to 0^+} f'(x) < \infty$ and $f(0) = 0$. Then*

$$\omega(f(A)X + Xf(A)) \leq f'(0)\,\omega(AX + XA). \tag{2.6}$$

Moreover, inequality (2.6) is equivalent to the inequality

$$\begin{aligned}&\omega(X(f(A)+f(B)) + (f(A)+f(B))X)\\ &\leq f'(0)\left(\omega((A+B)X + X(A+B)) + \omega((A-B)X - X(A-B))\right).\end{aligned}$$

Proof. A function g is non-negative operator increasing on $[0,\infty)$ if and only if $\frac{t}{g(t)}$ is non-negative operator increasing on $[0,\infty)$; see [9]. Hence $\frac{t}{f(t)}$ is operator increasing. Then $\frac{f(t)}{t}$ is decreasing. If $0 \leq x \leq t$, then $\frac{f(t)}{t} \leq \frac{f(x)}{x}$. Now, by taking $x \to 0^+$ we have $\frac{f(t)}{t} \leq f'(0)$. If we put $g(t) = 1\ (t \in [0,\infty))$ in Theorem 2.4, it follows from $k = k' \leq f'(0)$ that we get the required result. □

We first cite the following lemma due to Fujii et al. [10], which will be needed in the next theorem.

Lemma 2.7 ([10, Lemma 3.1])**.** *Let $\lambda_1, \cdots, \lambda_n$ be any positive real numbers and $-2 < t \leq 2$. If f and g are two continuous functions on $(0,\infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong, then the $n \times n$ matrix*

$$Y = \left(\frac{f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_i)}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2}\right)_{i,j=1,\cdots,n}$$

is positive semidefinite.

Theorem 2.8. *Let $A, B \in \mathcal{M}_n$ be positive semidefinite, $X \in \mathcal{M}_n$, f, g be two continuous functions on $(0,\infty)$ such that $\frac{f(t)}{g(t)}$ is Kwong, and let $-2 < t \leq 2$. Then*

$$\omega(A^{\frac{1}{2}}(H_{f,g}(A))A^{\frac{1}{2}}) \leq \frac{2k}{t+2}\,\omega(A^2X + tAXA + XA^2), \tag{2.7}$$

where $k = \max_{\lambda\in\sigma(A)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$.
Moreover, inequality (2.7) is equivalent to the inequality

$$\omega(A^{\frac{1}{2}}(H_{f,g}(A,B))B^{\frac{1}{2}}) \leq \frac{4k'}{t+2}\,\omega(A^2X + tAXB + XB^2), \tag{2.8}$$

where $k' = \max_{\lambda\in\sigma(A)\cup\sigma(B)}\left\{\frac{f(\lambda)g(\lambda)}{\lambda}\right\}$.

Proof. First, we show inequality (2.7). It is enough to show the inequality in the case A is positive definite. Since the numerical radius is weakly unitarily invariant, we may assume that A is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Let $\Sigma = \operatorname{diag}\left(\lambda_1^{\frac{1}{2}}g(\lambda_1), \cdots, \lambda_n^{\frac{1}{2}}g(\lambda_n)\right)$. It follows from Lemma 2.7 that

$$Z = [z_{ij}] = \Sigma\Big(\frac{(t+2)\left(f(\lambda_i)g^{-1}(\lambda_j) + f(\lambda_j)g^{-1}(\lambda_j)\right)}{2(\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2)}\Big)_{i,j=1,\cdots,n}\Sigma$$

is positive semidefinite for $-2 < t \leq 2$. In addition, all diagonal entries of Z are no more than k. Therefore,

$$\|S_Z\|_\omega = \max_i z_{ii} = \max_i \frac{f(\lambda_i)g(\lambda_i)}{\lambda_i} \leq k,$$

whence $\frac{\omega(Z\circ X)}{\omega(X)} \le k$ $(0 \ne X \in \mathcal{M}_n)$. Now, let $M = \left[\frac{1}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right]_{i,j=1,\cdots,n}$ and $P = \left[\frac{t+2}{2}\lambda_i^{\frac{1}{2}} f(\lambda_i)g(\lambda_j) + f(\lambda_j)g(\lambda_i)\lambda_j^{\frac{1}{2}}\right]_{i,j=1,\cdots,n}$. Then

$$\omega(M\circ P\circ X) = \omega(Z\circ X) \le k\,\omega(X) \qquad (0\ne X\in\mathcal{M}_n).$$

Let the matrix N be the entrywise inverse of M, i.e., $M\circ N = J$. Hence

$$\omega(P\circ X) \le k\,\omega(N\circ X) \qquad (0\ne X\in\mathcal{M}_n)$$

or equivalently

$$\omega(A^{\frac{1}{2}}\,(H_{f,g}(A))\,A^{\frac{1}{2}}) \le \frac{2k}{t+2}\,\omega(A^2X + tAXA + XA^2),$$

where $X\in\mathcal{M}_n$, $-2<t\le 2$ and $k = \max\left\{\frac{f(\lambda)g(\lambda)}{\lambda} : \lambda\in\sigma(A)\right\}$. Hence we have inequality (2.7).

Now, if we replace A and X by $\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}$ and $\begin{pmatrix} 0 & X\\ 0 & 0\end{pmatrix}$ inequality (2.7), respectively, then

$$\omega\left(\begin{bmatrix} 0 & A^{\frac{1}{2}}\,(H_{f,g}(A,B))\,B^{\frac{1}{2}}\\ 0 & 0\end{bmatrix}\right) \le \frac{2k'}{t+2}\,\omega\left(\begin{bmatrix} 0 & A^2X+tAXB+XB^2\\ 0 & 0\end{bmatrix}\right).$$

Hence

$$\begin{aligned}
\frac{1}{2}\omega(A^{\frac{1}{2}}(H_{f,g}(A,B))B^{\frac{1}{2}}) &\le \omega\left(\begin{bmatrix} 0 & A^{\frac{1}{2}}(H_{f,g}(A,B))B^{\frac{1}{2}}\\ 0 & 0\end{bmatrix}\right)\\
&\qquad\qquad \text{(by Lemma 2.3)}\\
&\le \frac{2k'}{t+2}\,\omega\left(\begin{bmatrix} 0 & A^2X+tAXB+XB^2\\ 0 & 0\end{bmatrix}\right)\\
&\le \frac{2k'}{t+2}\,\omega(A^2X+tAXB+XB^2)\\
&\qquad\qquad \text{(by Lemma 2.3).}
\end{aligned}$$

Thus, we reach inequality (2.8). Also, if we put $B = A$ in inequality (2.7), then we get inequality (2.8). □

Corollary 2.9. *Let $A\in\mathcal{M}_n$ be positive semidefinite. If f is a positive operator monotone function on $(0,\infty)$, then*

$$\begin{aligned}
&\omega(A^{\frac{1}{2}}f(A)Xf(A)^{-1}A^{\frac{3}{2}} + A^{\frac{3}{2}}f(A)^{-1}Xf(A)A^{\frac{1}{2}})\\
&\qquad\qquad \le \frac{4}{t+2}\,\omega(A^2X+tAXA+XA^2),
\end{aligned}$$

where $X\in\mathcal{M}_n$ and $-2<t\le 2$.

Proof. Since f positive operator monotone on $(0,\infty)$, then $g(t) = \frac{t}{f(t)}$ is operator monotone on $(0,\infty)$ and also $\frac{f(t)}{g(t)} = tf^2(t)$ is Kwong function [14]. So f and g satisfy the conditions of Theorem 2.8. Hence we have the desired inequality. □

Example 2.10. The function $f(t) = \log(1+t)$ is operator monotone on $(0,\infty)$; see [9]. If we put $g(t) = 1$, then $\frac{f(t)}{g(t)} = \log(1+t)$ is Kwong [13]. Using Theorem 2.4 we have

$$\begin{aligned}
&\omega(A^{\frac{1}{2}}(\log(I+A)X + X\log(I+A))A^{\frac{1}{2}})\\
&\qquad\qquad \le \frac{2}{t+2}\,\omega(A^2X+tAXA+XA^2),
\end{aligned}$$

where $A, X\in\mathcal{M}_n$ such that A is positive semidefinite and $-2<t\le 2$.

Now, we infer the following lemma due to Zhan [17], which will be needed in the next theorem.

Lemma 2.11 ([17, Lemma 5])**.** *Let $\lambda_1, \cdots, \lambda_n$ be any positive real numbers, $r \in [-1, 1]$ and $-2 < t \leq 2$. Then the $n \times n$ matrix*

$$L = \left(\frac{\lambda_i^r + \lambda_j^r}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2} \right)_{i,j=1,\cdots,n}$$

is positive semidefinite.

Now, we shall show the following result related to [10].

Proposition 2.12. *Let $A, X \in \mathcal{M}_n$ such that A is positive semidefinite, $\beta > 0$ and $1 \leq 2r \leq 3$. Then*

$$\begin{aligned} &\omega(A^r X A^{2-r} + A^{2-r} X A^r) \\ &\leq \omega\left(2(1-2\beta+2\beta r_0)AXA + \frac{4\beta(1-r_0)}{t+2}(A^2X + tAXA + XA^2)\right), \end{aligned}$$

where $-2 < t \leq 2\beta - 2$ and $r_0 = \min\{\frac{1}{2} + |1-r|, 1-|1-r|\}$.

Proof. Since the numerical radius is weakly unitarily invariant, we may assume that A is diagonal matrix with positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Since $1 \leq 2r \leq 3$, then $\frac{1}{2} \leq r_0 \leq \frac{3}{4}$. Let $t_0 = \frac{1-2\beta+2\beta r_0}{2\beta(1-r_0)}(t+2) + t$. It follows from $-2 < t \leq 2\beta - 2$ and $\frac{1}{4} \leq 1 - r_0 \leq \frac{1}{4}$, that $\frac{t+2}{4\beta(1-r_0)} > 0$ and $-2 < t_0 \leq 2$, where $t_0 = \frac{t}{2\beta(1-r_0)} + \frac{1}{\beta(1-r_0)} - 2$. Hence, by using Lemma 2.11, the $n \times n$ matrix

$$W = [w_{ij}] = \frac{t+2}{4\beta(1-r_0)} \Lambda^r \Big(\frac{\lambda_i^{2-2r} + \lambda_j^{2-2r}}{\lambda_i^2 + t_0\lambda_i\lambda_j + \lambda_j^2} \Big)_{i,j=1,\cdots,n} \Lambda^r$$

is positive semidefinite for $\frac{1}{2} \leq r \leq \frac{3}{2}$, where $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$. Therefore,

$$\|S_W\|_\omega = \max_i w_{ii} = \max_i \frac{(t+2)\lambda_i^r(2\lambda_i^{2-2r})\lambda_i^r}{4\beta(1-r_0)(t_0+2)\lambda_i^2} = 1,$$

whence $\frac{\omega(W \circ X)}{\omega(X)} \leq 1$ $(0 \neq X \in \mathcal{M}_n)$. Now, let $O = \left[\lambda_i^2 + t_0\lambda_i\lambda_j + \lambda_j^2\right]_{i,j=1,\cdots,n}$ and

$$M = \left[\frac{1}{2(1-2\beta+2\beta r_0)\lambda_i\lambda_j + \frac{4\beta(1-r_0)}{t+2}(\lambda_i^2 X + t\lambda_i\lambda_j + \lambda_j^2)} \right]_{i,j=1,\cdots,n}.$$

Then

$$\omega(O \circ M \circ X) = \omega(W \circ X) \leq \omega(X) \qquad (0 \neq X \in \mathcal{M}_n).$$

Let the matrix N be the entrywise inverse of M, i.e., $M \circ N = J$. Hence

$$\omega(O \circ X) \leq \omega(N \circ X) \qquad (0 \neq X \in \mathcal{M}_n)$$

or equivalently

$$\begin{aligned} &\omega(A^r X A^{2-r} + A^{2-r} X A^r) \\ &\leq \omega\left(2(1-2\beta+2\beta r_0)AXA + \frac{4\beta(1-r_0)}{t+2}(A^2X + tAXA + XA^2)\right), \end{aligned}$$

where $-2 < t \leq 2\beta - 2$ and $r_0 = \min\{\frac{1}{2} + |1-r|, 1-|1-r|\}$. □

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