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Some Geometric Applications of Dilworth's Theorem*

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Abstract. A geometric graph is a graph drawn in the plane such that its edges are closed line segments and no three vertices are collinear. We settle an old question of Avital, Hanani, Erdős, Kupitz, and Perles by showing that every geometric graph with n vertices and $m > k^4 n$ edges contains $k + 1$ pairwise disjoint edges. We also prove that, given a set of points V and a set of axis-parallel rectangles in the plane, then either there are $k + 1$ rectangles such that no point of V belongs to more than one of them, or we can find an at most $2 \cdot 10^5 k^8$ element subset of V meeting all rectangles. This improves a result of Ding, Seymour, and Winkler. Both proofs are based on Dilworth's theorem on partially ordered sets.

1. Introduction

Ever since Erdős and Szekeres [ES] rediscovered Ramsey's theorem [R] to show that out of $\binom{2n-4}{n-2}$ points in the plane it is possible to select n points which form a convex n-gon, combinatorial geometry has been a rich area of application for Ramsey theory (see [GRS]). Many classes of geometric objects can be equipped with natural ordering relations, and this information is often lost when we apply Ramsey's theorem. For example, given n convex bodies in the plane, Ramsey's theorem only implies that we can choose $\frac{1}{2} \log_2 n$ of them which are either pairwise disjoint or pairwise intersecting. However, as we have shown in a recent paper

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[LMPT], $n^{1/5}$ elements can also be chosen with the above property. Our proof was based on the following well-known result [D].

Dilworth's Theorem. *Let P be a partially ordered set containing no chain (totally ordered subset) of* $k + 1$ *elements. Then P can be covered by k antichains (subsets of pairwise incomparable elements).*

In this note we present some further applications of the same idea.

A *geometric graph G* is a graph drawn in the plane by (possibly crossing) line segments, i.e., it is defined as a pair $(V(G), E(G))$, where $V(G)$ is a set of points in the plane in general position and $E(G)$ is a collection of closed line segments whose endpoints belong to $V(G)$.

The following question was raised by Avital and Hanani [AH], Kupitz [K], and Perles: Determine the smallest number $e_k(n)$ such that any geometric graph with n vertices and $m > e_k(n)$ edges contains $k + 1$ pairwise disjoint edges. An old result of Hopf and Pannwitz [HP] and Erdős [E] implies that $e_1(n) = n$. Alon and Erdős [AE] showed that $e_2(n) \leq 6n$, which was subsequently improved by O'Donnel and Perles [OP] and Goddard *et al.* [GKK] to $e_2(n) \leq 3n$. In the latter paper it has also been established that $e_3(n) \leq 10n$ and, for any fixed k, $e_k(n) < c_k n \log^{k-3} n$ with a suitable constant c_k.

Theorem 1. *Any geometric graph with n vertices and more than* $k^4 n$ *edges contains at least* $k + 1$ *pairwise disjoint edges.*

This means that $e_k(n) \leq k^4 n$ for all k and n. Moreover, we can show that if a geometric graph has much more than $e_k(n)$ edges, then it has many $(k + 1)$-tuples of pairwise disjoint edges.

Theorem 2. *For every k, there exists* $c'_k > 0$ *such that any geometric graph with n vertices and* $m > k^4 n$ *edges has at least* $c'_k m^{2k+1}/n^{2k}$ $(k + 1)$*-tuples of pairwise disjoint edges. Furthermore, this bound is asymptotically tight apart from the exact value of* c'_k.

Our next result can also be established by repeated application of Dilworth's theorem. It is a substantial improvement of a theorem of Ding *et al.* [DSW], whose original proof requires more involved techniques.

Theorem 3. *Let V and* $\mathcal{R}$ *be a set of points and a set of axis-parallel rectangles in the plane, respectively, and suppose that every element of* $\mathcal{R}$ *contains at least one point of V. Then for any natural number k, either*

(i) *there are* $k + 1$ *rectangles in* $\mathcal{R}$ *so that no point of V belongs to more than one of them, or*
(ii) *one can choose at most* $2 \cdot 10^5 k^8$ *points of V so that every member of* $\mathcal{R}$ *contains at least one of them.*

In [DSW] the same result was proved with $(k + 63)^{127}$, instead of $2 \cdot 10^5 k^8$.

2. Geometric Graphs

First we prove Theorem 1. Let G be a geometric graph on n vertices and m edges, containing no $k+1$ pairwise disjoint edges. Let $x(v)$ and $y(v)$ denote the x-coordinate and the y-coordinate of a point v, respectively. For any two edges $e = v_1v_2$, $e' = v'_1v'_2 \in E(G)$, we say that e *precedes* e' (in notation, $e \ll e'$) if $x(v_1) \leq x(v'_1)$ and $x(v_2) \leq x(v'_2)$. Furthermore, e is said to lie *below* e', if there is no vertical line l (parallel to the y-axis) which intersects both e and e' with $y(l \cap e) \geq y(l \cap e')$. If e is below e', then we write $e \prec e'$. Note that $\prec$ is not necessarily a transitive relation on $E(G)$. Finally, let $\pi(e)$ denote the orthogonal projection of e onto the x-axis.

Define four binary relations $\prec_i$ $(i = 1, \ldots, 4)$ on $E(G)$, as follows. Two edges can be related by any of these relations only if they are disjoint. Given two disjoint edges $e, e' \in E(G)$, let

$$
\begin{array}{llll}
e \prec_1 e' & \text{if} & e \ll e' & \text{and} \quad e \prec e', \\
e \prec_2 e' & \text{if} & e \ll e' & \text{and} \quad e \succ e', \\
e \prec_3 e' & \text{if} & \pi(e) \subseteq \pi(e') & \text{and} \quad e \prec e', \\
e \prec_4 e' & \text{if} & \pi(e) \subseteq \pi(e') & \text{and} \quad e \succ e'.
\end{array}
$$

It follows readily from the definitions that:

(a) Each of the relations $\prec_i$ $(i = 1, \ldots, 4)$ is transitive.
(b) Any pair of disjoint edges is comparable by at least one of the relations $\prec_i$ $(i = 1, \ldots, 4)$.

None of the partial orders $(E(G), \prec_i)$ contains a chain of length $k+1$, otherwise G would have $k+1$ pairwise disjoint edges. By Dilworth's theorem, for any i, $E(G)$ can be partitioned into at most k classes so that no two edges belonging to the same class are comparable by $\prec_i$. Superimposing these four partitions, we obtain a decomposition of $E(G)$ into at most k^4 classes E_j $(1 \leq j \leq k^4)$ so that no two elements of E_j are related by any $\prec_i$. Hence, by (b), none of the classes E_j contains two disjoint edges, which implies that $|E_j| \leq e_1(n) = n$. Therefore, $m = |E(G)| = \sum_{j=1}^{k^4} |E_j| \leq k^4 n$, as desired. □

Next we deduce Theorem 2 from Theorem 1, by using an idea of Ajtai *et al.* [ACNS]. Given a geometric graph G with n vertices and $m \geq e_k(n) + n$ edges, let $f(G)$ denote the number of $k+1$-tuples of pairwise disjoint edges in G. We show, by induction on n, that

$$
f(G) \geq (6k)^{-6k} \binom{n}{2k+2} \left(\frac{m}{\binom{n}{2}} \right)^{2k+1}, \tag{$*$}
$$

whenever $m \geq e_k(n) + n$.

It follows from Theorem 1 that $f(G) \geq m - e_k(n)$, which is stronger than (*) if $e_k(n) + n \leq m \leq e_k(n) + 2n$, and it also shows that (*) holds for $n = k^4 + 3$.

Suppose now that $n > k^4 + 3$, $m > e_k(n) + 2n$, and that we have already established the above inequality for all graphs with at most $n - 1$ vertices. For any $v \in V(G)$, let $d(v)$ denote the number of edges of G incident to v, and let $G - v$ stand for the graph obtained from G by the deletion of v. Using the induction hypothesis, we get

$$(n - 2k - 2)f(G) = \sum_{v \in V(G)} f(G - v) \geq (6k)^{-6k} \frac{\binom{n-1}{2k+2}}{\binom{n-1}{2}^{2k+1}} \sum_{v \in V(G)} (m - d(v))^{2k+1}.$$

(Notice that $G - v$ has at least $m - (n - 1) \geq e_k(n - 1) + (n - 1)$ edges.) However, by Jensen's inequality,

$$\sum_{v \in V(G)} (m - d(v))^{2k+1} \geq n^{-2k}\left(\sum_{v \in V(G)} (m - d(v))\right)^{2k+1} = \frac{(n-2)^{2k+1} m^{2k+1}}{n^{2k}},$$

and (*) follows. This proves the first part of Theorem 2.

To show that, for any fixed k, the bound is tight up to a constant factor depending only on k, assume that $n \leq m \leq n^2/16$ and set $t = \lfloor n^2/8m \rfloor$. Divide a circle into $2t$ equal arcs $A_1, \ldots, A_{2t}$, and pick n vertices on the circle as equally distributed among the arcs as possible. Join every vertex belonging to A_i to all vertices in A_{i+t} by line segments $(1 \leq i \leq t)$. The resulting geometric graph has n vertices and at least m edges. Furthermore, for any $(k + 1)$-tuple of pairwise disjoint edges of G, there exists an i such that all endpoints of these edges lie in $A_i \cup A_{i+t}$. Thus, the number of these $(k + 1)$-tuples cannot exceed

$$t\left\lceil \frac{n}{2t} \right\rceil^{2k+2} \leq \frac{(10m)^{2k+1}}{n^{2k}},$$

completing the proof of Theorem 2. □

3. Systems of Points and Rectangles

In this section we establish Theorem 3.

Let V be a set of points and let $\mathscr{R}$ be a set of axis-parallel rectangles in the plane. Two rectangles $R, S \in \mathscr{R}$ are said to be *almost disjoint* if $R \cap S$ contains no element of V. For any $R \in \mathscr{R}$, let $\pi_x(R)$ and $\pi_y(R)$ denote the orthogonal projections of R onto the x-axis and onto the y-axis, respectively. Given two intervals I, J on the x-axis (or on the y-axis), we say that I precedes J (or, in notation, $I \ll J$) if the left and right endpoints of I precede the left and right endpoints of J, respectively.

Define eight binary relations $\prec_i$ $(i = 1, \ldots, 8)$ on $\mathscr{R}$, as follows. Two rectangles can be related by any of these relations only if they are almost disjoint. Given a pair of almost disjoint rectangles $R, S \in \mathscr{R}$, let

$$\begin{aligned}
R \prec_1 S \quad &\text{if} \quad \pi_x(R) \subseteq \pi_x(S) \quad \text{and} \quad \pi_y(R) \ll \pi_y(S),\\
R \prec_2 S \quad &\text{if} \quad \pi_x(R) \subseteq \pi_x(S) \quad \text{and} \quad \pi_y(R) \gg \pi_y(S),\\
R \prec_3 S \quad &\text{if} \quad \pi_x(R) \ll \pi_x(S) \quad \text{and} \quad \pi_y(R) \subseteq \pi_y(S),\\
R \prec_4 S \quad &\text{if} \quad \pi_x(R) \ll \pi_x(S) \quad \text{and} \quad \pi_y(R) \supseteq \pi_y(S),\\
R \prec_5 S \quad &\text{if} \quad \pi_x(R) \ll \pi_x(S) \quad \text{and} \quad \pi_y(R) \ll \pi_y(S),\\
R \prec_6 S \quad &\text{if} \quad \pi_x(R) \ll \pi_x(S) \quad \text{and} \quad \pi_y(R) \gg \pi_y(S),\\
R \prec_7 S \quad &\text{if} \quad \pi_x(R) \subseteq \pi_x(S) \quad \text{and} \quad \pi_y(R) \subseteq \pi_y(S),\\
R \prec_8 S \quad &\text{if} \quad \pi_x(R) \subseteq \pi_x(S) \quad \text{and} \quad \pi_y(R) \supseteq \pi_y(S),
\end{aligned}$$

It follows from the definitions that:

(a) Each of the relations $\prec_i$ $(i = 1, \ldots, 8)$ is transitive.
(b) Any pair of almost disjoint rectangles is comparable by one of the relations $\prec_i$ $(i = 1, \ldots, 8)$.

We can assume without loss of generality that none of the partial orders $(\mathscr{R}, \prec_i)$ contains a chain of length $k + 1$, otherwise condition (i) of the theorem holds. Hence, by the repeated application of Dilworth's theorem, we obtain that $\mathscr{R}$ can be partitioned into k^8 classes $\mathscr{R}_j$ $(1 \leq j \leq k^8)$ so that no pair of rectangles belonging to the same class is comparable by any of the relations $\prec_i$. This means, by (b), that the intersection of any two rectangles belonging to the same class contains at least one point of V.

To prove the theorem, it is sufficient to show that, for any j, at most $2 \cdot 10^5$ points of V can be selected such that every member of $\mathscr{R}_j$ contains at least one of them.

Lemma 1. *Let V be a set of points, and let $\mathscr{R}_0$ be a set of axis-parallel rectangles in the plane such that the intersection of any two members contains at least one element of V. Then one can choose at most $2 \cdot 10^5$ points of V so that every member of $\mathscr{R}_0$ contains at least one of them.*

Proof. By Helly's theorem, $\bigcap \mathscr{R}_0 \neq \emptyset$. Thus, we can assume without loss of generality that the origin $(0, 0)$ is contained in every member of $\mathscr{R}_0$. Suppose, for simplicity, that no point of the coordinate axes belongs to V. Let V_i denote the set of all points of V lying in the ith quadrant of the coordinate system $(i = 1, \ldots, 4)$. Let V_i' consist of all points $(x, y) \in V_i$ for which there is no $(x', y') \in V_i$ with $x' \in [0, x]$,

$y' \in [0, y]$. Finally, set $V' = V'_1 \cup \cdots \cup V'_4$. Notice that:

(a) If we number the elements of V_i according to their (say) x-coordinates, then the intersection of any rectangle $R \in \mathscr{R}_0$ with V'_i consists of a single interval of consecutive points ($i = 1, \ldots, 4$).
(b) The intersection of any two members of $\mathscr{R}_0$ contains at least one element of V'.

A general theorem of Gyárfás and Lehel [GL] states that, for every p and q, there exists an integer $f(p, q) < \infty$ with the following property. Let $\mathscr{F}$ be a family of sets, each of them obtained as the union of p intervals taken from p parallel lines. If $\mathscr{F}$ does not have more than q pairwise disjoint members, then one can find at most $f(p, q)$ points so that every member of $\mathscr{F}$ contains at least one of them. This easily implies, by (a) and (b), that V' has a subset of size $f(4, 1) \leq 2 \cdot 10^5$ meeting every member of $\mathscr{R}_0$. □

4. Concluding Remarks

Given a partially ordered set of m elements, in $O(m^2)$ time a topological sort can be performed and a maximal chain can be found (see [CLR]). Hence, the proof in Section 2 also yields the following result.

Corollary 1. *There is an $O(m^2)$-time algorithm for finding at least $(m/n)^{1/4}$ pairwise disjoint edges in any geometric graph with n vertices and m edges.*

Note that the proof of Theorem 3 can be easily extended to higher dimensions.

Theorem 4. *Let V and $\mathscr{B}$ be a set of points and a set of axis-parallel boxes in $\mathbf{R}^d$, respectively, such that every box contains at least one point. Then, for any natural number k, either*

(i) *there are $k + 1$ boxes in $\mathscr{B}$ so that no point of V belongs to more than one of them, or*
(ii) *one can choose at most $c_d k^{2^{2d-1}}$ points in V so that any member of $\mathscr{B}$ contains at least one of them, where c_d is a constant depending only on d.*

Using some related techniques we can also establish the following strengthening of a theorem of Bárány and Lehel. Given two points $x, y \in \mathbf{R}^d$, let $\mathrm{Box}(x, y)$ denote the smallest box parallel to the axes, which contains x and y.

Theorem 5. *Any set of points $V \subseteq \mathbf{R}^d$ contains at most 2^{2^d} elements x_i $(1 \leq i \leq 2^{2^d})$ such that*

$$\bigcup_{1 \leq i < j \leq 2^{2^d}} \mathrm{Box}(x_i, x_j) \supseteq V.$$

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