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Some Geometric Calculations on Wasserstein Space*

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Abstract: We compute the Riemannian connection and curvature for the Wasserstein space of a smooth compact Riemannian manifold.

1. Introduction

If M is a smooth compact Riemannian manifold then the Wasserstein space $P_2(M)$ is the space of Borel probability measures on M, equipped with the Wasserstein metric W_2. We refer to [21] for background information on Wasserstein spaces. The Wasserstein space originated in the study of optimal transport. It has had applications to PDE theory [16], metric geometry [8,19,20] and functional inequalities [9,17].

Otto showed that the heat flow on measures can be considered as a gradient flow on Wasserstein space [16]. In order to do this, he introduced a certain formal Riemannian metric on the Wasserstein space. This Riemannian metric has some remarkable properties. Using O'Neill's theorem, Otto gave a formal argument that $P_2(\mathbb{R}^n)$ has nonnegative sectional curvature. This was made rigorous in [8, Theorem A.8] and [19, Prop. 2.10] in the following sense: M has nonnegative sectional curvature if and only if the length space $P_2(M)$ has nonnegative Alexandrov curvature.

In this paper we study the Riemannian geometry of the Wasserstein space. In order to write meaningful expressions, we restrict ourselves to the subspace $P^\infty(M)$ of absolutely continuous measures with a smooth positive density function. The space $P^\infty(M)$ is a smooth infinite-dimensional manifold in the sense, for example, of [7]. The formal calculations that we perform can be considered as rigorous calculations on this smooth manifold, although we do not emphasize this point.

In Sect. 3 we show that if c is a smooth immersed curve in $P^\infty(M)$ then its length in $P_2(M)$, in the sense of metric geometry, equals its Riemannian length as computed with Otto's metric. In Sect. 4 we compute the Levi-Civita connection on $P^\infty(M)$. We use it to derive the equation for parallel transport and the geodesic equation.

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In Sect. 5 we compute the Riemannian curvature of $P^\infty(M)$. The answer is relatively simple. As an application, if M has sectional curvatures bounded below by $r \in \mathbb{R}$, one can ask whether $P^\infty(M)$ necessarily has sectional curvatures bounded below by r. This turns out to be the case if and only if $r = 0$.

There has been recent interest in doing Hamiltonian mechanics on the Wasserstein space of a symplectic manifold [1,4,5]. In Sect. 6 we briefly describe the Poisson geometry of $P^\infty(M)$. We show that if M is a Poisson manifold then $P^\infty(M)$ has a natural Poisson structure. We also show that if M is symplectic then the symplectic leaves of the Poisson structure on $P^\infty(M)$ are the orbits of the group of Hamiltonian diffeomorphisms, thereby making contact with [1,5]. This approach is not really new; closely related results, with applications to PDEs, were obtained quite a while ago by Alan Weinstein and collaborators [10,11,22]. However, it may be worth advertising this viewpoint.

2. Manifolds of Measures

In what follows, we use the Einstein summation convention freely.

Let M be a smooth connected closed Riemannian manifold of positive dimension. We denote the Riemannian density by dvol_M. Let $P_2(M)$ denote the space of Borel probability measures on M, equipped with the Wasserstein metric W_2. For relevant results about optimal transport and the Wasserstein metric, we refer to [8, Sects. 1 and 2] and references therein.

Put

$$P^\infty(M) = \{\rho\ \mathrm{dvol}_M : \rho \in C^\infty(M), \rho > 0, \int_M \rho\ \mathrm{dvol}_M = 1\}. \tag{2.1}$$

Then $P^\infty(M)$ is a dense subset of $P_2(M)$, as is the complement of $P^\infty(M)$ in $P_2(M)$. We do not claim that $P^\infty(M)$ is necessarily a totally convex subset of $P_2(M)$, i.e. that if $\mu_0, \mu_1 \in P^\infty(M)$ then the minimizing geodesic in $P_2(M)$ joining them necessarily lies in $P^\infty(M)$. However, the absolutely continuous probability measures on M do form a totally convex subset of $P_2(M)$ [12]. For the purposes of this paper, we give $P^\infty(M)$ the smooth topology. (This differs from the subspace topology on $P^\infty(M)$ coming from its inclusion in $P_2(M)$.) Then $P^\infty(M)$ has the structure of an infinite-dimensional smooth manifold in the sense of [7]. The formal calculations in this paper can be rigorously justified as being calculations on the smooth manifold $P^\infty(M)$. However, we will not belabor this point.

Given $\phi \in C^\infty(M)$, define $F_\phi \in C^\infty(P^\infty(M))$ by

$$F_\phi(\rho\ \mathrm{dvol}_M) = \int_M \phi\, \rho\ \mathrm{dvol}_M. \tag{2.2}$$

This gives an injection $P^\infty(M) \to (C^\infty(M))^*$, i.e. the functions F_ϕ separate points in $P^\infty(M)$. We will think of the functions F_ϕ as "coordinates" on $P^\infty(M)$.

Given $\phi \in C^\infty(M)$, define a vector field V_ϕ on $P^\infty(M)$ by saying that for $F \in C^\infty(P^\infty(M))$,

$$(V_\phi F)(\rho\, \mathrm{dvol}_M) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F\left(\rho\, \mathrm{dvol}_M - \epsilon\, \nabla^i(\rho \nabla_i \phi)\, \mathrm{dvol}_M\right). \tag{2.3}$$

The map $\phi \to V_\phi$ passes to an isomorphism $C^\infty(M)/\mathbb{R} \to T_{\rho\,\mathrm{dvol}_M} P^\infty(M)$. This parametrization of $T_{\rho\,\mathrm{dvol}_M} P^\infty(M)$ goes back to Otto's paper [16]; see [2] for further discussion. Otto's Riemannian metric on $P^\infty(M)$ is given [16] by

$$\begin{aligned} \langle V_{\phi_1}, V_{\phi_2}\rangle(\rho\,\mathrm{dvol}_M) &= \int_M \langle \nabla\phi_1, \nabla\phi_2\rangle\ \rho\ \mathrm{dvol}_M \\ &= -\int_M \phi_1 \nabla^i(\rho\nabla_i\phi_2)\ \mathrm{dvol}_M\,. \end{aligned} \tag{2.4}$$

In view of (2.3), we write $\delta_{V_\phi}\rho = -\nabla^i(\rho\nabla_i\phi)$. Then

$$\langle V_{\phi_1}, V_{\phi_2}\rangle(\rho\,\mathrm{dvol}_M) = \int_M \phi_1\ \delta_{V_{\phi_2}}\rho\ \mathrm{dvol}_M = \int_M \phi_2\ \delta_{V_{\phi_1}}\rho\ \mathrm{dvol}_M\,. \tag{2.5}$$

In terms of the weighted L^2-spaces $L^2(M, \rho\ \mathrm{dvol}_M)$ and $\Omega^1_{L^2}(M, \rho\ \mathrm{dvol}_M)$, let d be the usual differential on functions and let d^*_ρ be its formal adjoint. Then (2.4) can be written as

$$\langle V_{\phi_1}, V_{\phi_2}\rangle(\rho\,\mathrm{dvol}_M) = \int_M \langle d\phi_1, d\phi_2\rangle\ \rho\ \mathrm{dvol}_M = \int_M \phi_1\ d^*_\rho d\phi_2\ \rho\ \mathrm{dvol}_M\,. \tag{2.6}$$

We now relate the function F_ϕ and the vector field V_ϕ.

Lemma 1. *The gradient of F_ϕ is V_ϕ.*

Proof. Letting $\overline{\nabla} F_\phi$ denote the gradient of F_ϕ, for all $\phi' \in C^\infty(M)$ we have

$$\begin{aligned} \langle \overline{\nabla} F_\phi, V_{\phi'}\rangle(\rho\ \mathrm{dvol}_M) &= (V_{\phi'} F_\phi)(\rho\ \mathrm{dvol}_M) = -\int_M \phi\ \nabla^i(\rho\nabla_i\phi')\,\mathrm{dvol}_M \\ &= \langle V_\phi, V_{\phi'}\rangle(\rho\ \mathrm{dvol}_M). \end{aligned} \tag{2.7}$$

This proves the lemma. □

3. Lengths of Curves

In this section we relate the Riemannian metric (2.4) to the Wasserstein metric. One such relation was given in [17], where it was heuristically shown that the geodesic distance coming from (2.4) equals the Wasserstein metric. To give a rigorous relation, we recall that a curve $c : [0, 1] \to P_2(M)$ has a length given by

$$L(c) = \sup_{J\in\mathbb{N}}\ \sup_{0=t_0\le t_1\le\ldots\le t_J=1} \sum_{j=1}^{J} W_2\left(c(t_{j-1}), c(t_j)\right). \tag{3.1}$$

From the triangle inequality, the expression $\sum_{j=1}^{J} W_2\left(c(t_{j-1}), c(t_j)\right)$ is nondecreasing under a refinement of the partition $0 = t_0 \le t_1 \le \ldots \le t_J = 1$.

If $c : [0, 1] \to P^\infty(M)$ is a smooth curve in $P^\infty(M)$ then we write $c(t) = \rho(t)\,\mathrm{dvol}_M$ and let $\phi(t)$ satisfy $\frac{\partial\rho}{dt} = -\nabla^i\,(\rho\nabla_i\phi)$, where we normalize ϕ by requiring for example that $\int_M \phi\ \rho\ \mathrm{dvol}_M = 0$. If c is immersed then $\nabla\phi(t) \neq 0$. The Riemannian length of c, as computed using (2.4), is

$$\int_0^1 \langle c'(t), c'(t)\rangle^{\frac{1}{2}}\, dt = \int_0^1 \left(\int_M |\nabla\phi(t)|^2(m)\ \rho(t)\ \mathrm{dvol}_M\right)^{\frac{1}{2}} dt. \tag{3.2}$$

The next proposition says that this equals the length of c in the metric sense.

Proposition 1. *If $c : [0,1] \to P^\infty(M)$ is a smooth immersed curve then its length $L(c)$ in the Wasserstein space $P_2(M)$ satisfies*

$$L(c) = \int_0^1 \langle c'(t), c'(t) \rangle^{\frac{1}{2}} \, dt. \tag{3.3}$$

Proof. We can parametrize c so that $\int_M |\nabla \phi(t)|^2 \, \rho(t) \, \mathrm{dvol}_M$ is a constant $C > 0$ with respect to t.

Let $\{S_t\}_{t \in [0,1]}$ be the one-parameter family of diffeomorphisms of M given by

$$\frac{\partial S_t(m)}{\partial t} = (\nabla \phi(t))(S_t(m)) \tag{3.4}$$

with $S_0(m) = m$. Then $c(t) = (S_t)_*(\rho(0) \, \mathrm{dvol}_M)$.

Given a partition $0 = t_0 \le t_1 \le \ldots \le t_J = 1$ of $[0,1]$, a particular transference plan from $c(t_{j-1})$ to $c(t_j)$ comes from the Monge transport $S_{t_j} \circ S_{t_{j-1}}^{-1}$. Then

$$\begin{aligned} W_2\left(c(t_{j-1}), c(t_j)\right)^2 &\le \int_M d(m, S_{t_j}(S_{t_{j-1}}^{-1}(m)))^2 \, \rho(t_{j-1}) \, \mathrm{dvol}_M \\ &= \int_M d(S_{t_{j-1}}(m), S_{t_j}(m))^2 \, \rho(0) \, \mathrm{dvol}_M \\ &\le \int_M \left(\int_{t_{j-1}}^{t_j} |\nabla \phi(t)|(S_t(m)) \, dt \right)^2 \rho(0) \, \mathrm{dvol}_M \\ &\le (t_j - t_{j-1}) \int_M \int_{t_{j-1}}^{t_j} |\nabla \phi(t)|^2 (S_t(m)) \, dt \, \rho(0) \, \mathrm{dvol}_M \\ &= (t_j - t_{j-1}) \int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t)|^2(m) \, \rho(t) \, \mathrm{dvol}_M \, dt, \end{aligned} \tag{3.5}$$

so

$$\begin{aligned} W_2\left(c(t_{j-1}), c(t_j)\right) &\le (t_j - t_{j-1})^{\frac{1}{2}} \left(\int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t)|^2(m) \, \rho(t) \, \mathrm{dvol}_M \, dt \right)^{\frac{1}{2}} \\ &= (t_j - t_{j-1}) \left(\int_M |\nabla \phi(t_j')|^2(m) \, \rho(t_j') \, \mathrm{dvol}_M \right)^{\frac{1}{2}} \end{aligned} \tag{3.6}$$

for some $t_j' \in [t_{j-1}, t_j]$. It follows that

$$L(c) \le \int_0^1 \langle c'(t), c'(t) \rangle^{\frac{1}{2}} \, dt. \tag{3.7}$$

Next, from [8, Lemma A.1],

$$\begin{aligned} &(t_j - t_{j-1}) \left| \int_M \phi(t_{j-1}) \, \rho(t_j) \, \mathrm{dvol}_M - \int_M \phi(t_{j-1}) \, \rho(t_{j-1}) \, \mathrm{dvol}_M \right|^2 \\ &\le W_2(c(t_{j-1}), c(t_j))^2 \int_{t_{j-1}}^{t_j} \int_M |\nabla \phi(t_{j-1})|^2 \, d\mu_t \, dt, \end{aligned} \tag{3.8}$$

where $\{\mu_t\}_{t\in[t_{j-1},t_j]}$ is the Wasserstein geodesic between $c(t_{j-1})$ and $c(t_j)$. Now

$$\begin{aligned}
&\int_M \phi(t_{j-1})\,\rho(t_j)\ \mathrm{dvol}_M - \int_M \phi(t_{j-1})\,\rho(t_{j-1})\ \mathrm{dvol}_M \\
&= -\int_M \int_{t_{j-1}}^{t_j} \phi(t_{j-1})\,\nabla^i\,(\rho(t)\nabla_i\phi(t))\ dt\ \mathrm{dvol}_M \\
&= \int_{t_{j-1}}^{t_j} \int_M \langle \nabla\phi(t_{j-1}), \nabla\phi(t)\rangle\,\rho(t)\ \mathrm{dvol}_M\ dt,
\end{aligned} \tag{3.9}$$

so (3.8) becomes

$$\begin{aligned}
&(t_j - t_{j-1}) \left(\int_{t_{j-1}}^{t_j} \int_M \langle \nabla\phi(t_{j-1}), \nabla\phi(t)\rangle\,\rho(t)\ \mathrm{dvol}_M\ dt \right)^2 \\
&\leq W_2(c(t_{j-1}), c(t_j))^2 \int_{t_{j-1}}^{t_j} \int_M |\nabla\phi(t_{j-1})|^2\, d\mu_t\, dt.
\end{aligned} \tag{3.10}$$

Thus

$$L(c) \geq \sum_{j=1}^{J} \frac{\frac{\int_{t_{j-1}}^{t_j} \int_M \langle \nabla\phi(t_{j-1}), \nabla\phi(t)\rangle\,\rho(t)\ \mathrm{dvol}_M\ dt}{t_j - t_{j-1}}}{\sqrt{\frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M |\nabla\phi(t_{j-1})|^2\, d\mu_t\, dt}}\,(t_j - t_{j-1}). \tag{3.11}$$

As the partition of [0, 1] becomes finer, the term $\frac{\int_{t_{j-1}}^{t_j} \int_M \langle \nabla\phi(t_{j-1}), \nabla\phi(t)\rangle\,\rho(t)\ \mathrm{dvol}_M\ dt}{t_j - t_{j-1}}$ uniformly approaches the constant C.

The Wasserstein geodesic $\{\mu_t\}_{t\in[t_{j-1},t_j]}$ has the form $\mu_t = (F_t)_*\mu_{t_{j-1}}$ for measurable maps $F_t : M \to M$ with $F_{t_{j-1}} = \mathrm{Id}$ [12]. Then

$$\begin{aligned}
&\left| \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M |\nabla\phi(t_{j-1})|^2\, d\mu_t\, dt - C \right| \\
&= \left| \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \left(\int_M |\nabla\phi(t_{j-1})|^2\, d\mu_t - \int_M |\nabla\phi(t_{j-1})|^2\, d\mu_{t_{j-1}} \right) dt \right| \\
&= \left| \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M \left(|\nabla\phi(t_{j-1})|^2 \circ F_t - |\nabla\phi(t_{j-1})|^2 \right) d\mu_{t_{j-1}}\, dt \right| \\
&\leq \frac{1}{t_j - t_{j-1}} \parallel \nabla|\nabla\phi(t_{j-1})|^2 \parallel_\infty \int_{t_{j-1}}^{t_j} \int_M d(m, F_t(m))\, d\mu_{t_{j-1}}(m)\, dt \\
&\leq \frac{1}{t_j - t_{j-1}} \parallel \nabla|\nabla\phi(t_{j-1})|^2 \parallel_\infty \int_{t_{j-1}}^{t_j} \sqrt{\int_M d(m, F_t(m))^2\, d\mu_{t_{j-1}}(m)}\, dt \\
&= \frac{1}{t_j - t_{j-1}} \parallel \nabla|\nabla\phi(t_{j-1})|^2 \parallel_\infty \int_{t_{j-1}}^{t_j} W_2(\mu_{t_{j-1}}, \mu_t)\, dt \\
&\leq \parallel \nabla|\nabla\phi(t_{j-1})|^2 \parallel_\infty\ W_2(c(t_{j-1}), c(t_j)).
\end{aligned} \tag{3.12}$$

Now continuity of a 1-parameter family of smooth measures in the smooth topology implies continuity in the weak-$*$ topology, which is metricized by W_2 (as M is compact). It follows that as the partition of $[0, 1]$ becomes finer, the term $\frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \int_M |\nabla\phi(t_{j-1})|^2 \, d\mu_t \, dt$ uniformly approaches the constant C. Thus from (3.11),

$$L(c) \geq \sqrt{C} = \int_0^1 \langle c'(t), c'(t) \rangle^{\frac{1}{2}} \, dt. \tag{3.13}$$

This proves the proposition. □

Remark 1. Let X be a finite-dimensional Alexandrov space and let R be its set of nonsingular points. There is a continuous Riemannian metric g on R so that lengths of curves in R can be computed using g [15]. (Note that in general, R and $X - R$ are dense in X.) This is somewhat similar to the situation for $P^\infty(M) \subset P_2(M)$.

In fact, there is an open dense subset $O \subset X$ with a Lipschitz manifold structure and a Riemannian metric of bounded variation that extends g [18]. We do not know if there is a Riemannian manifold structure, in some appropriate sense, on an open dense subset of $P_2(M)$. Other approaches to geometrizing $P_2(M)$, with a view toward gradient flow, are in [2,3]; see also [14].

4. Levi-Civita Connection, Parallel Transport and Geodesics

In this section we compute the Levi-Civita connection of $P^\infty(M)$. We derive the formula for parallel transport in $P^\infty(M)$ and the geodesic equation for $P^\infty(M)$.

We first compute commutators of our canonical vector fields $\{V_\phi\}_{\phi \in C^\infty(M)}$.

Lemma 2. *Given $\phi_1, \phi_2 \in C^\infty(M)$, the commutator $[V_{\phi_1}, V_{\phi_2}]$ is given by*

$$\begin{aligned} &\big([V_{\phi_1}, V_{\phi_2}]F\big)(\rho \, \mathrm{dvol}_M) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} F\Big(\rho \, \mathrm{dvol}_M - \epsilon \nabla_i \Big[\rho \Big((\nabla^i \nabla^j \phi_2)\nabla_j \phi_1 - (\nabla^i \nabla^j \phi_1)\nabla_j \phi_2\Big)\Big] \mathrm{dvol}_M\Big) \end{aligned} \tag{4.1}$$

for $F \in C^\infty(P^\infty(M))$.

Proof. We have

$$\begin{aligned} &\big([V_{\phi_1}, V_{\phi_2}]F\big)(\rho \, \mathrm{dvol}_M) = \big(V_{\phi_1}(V_{\phi_2}F)\big)(\rho \, \mathrm{dvol}_M) - \big(V_{\phi_2}(V_{\phi_1}F)\big)(\rho \, \mathrm{dvol}_M) \\ &= \frac{d}{d\epsilon_1}\Big|_{\epsilon_1=0} (V_{\phi_2}F)\Big(\rho \, \mathrm{dvol}_M - \epsilon_1 \nabla^i(\rho \nabla_i \phi_1) \, \mathrm{dvol}_M\Big) \\ &\quad - \frac{d}{d\epsilon_2}\Big|_{\epsilon_2=0} (V_{\phi_1}F)\Big(\rho \, \mathrm{dvol}_M - \epsilon_2 \nabla^i(\rho \nabla_i \phi_2) \, \mathrm{dvol}_M\Big) \\ &= \frac{d}{d\epsilon_1}\Big|_{\epsilon_1=0} \frac{d}{d\epsilon_2}\Big|_{\epsilon_2=0} F\Big((\rho - \epsilon_1 \nabla^i(\rho \nabla_i \phi_1)) \, \mathrm{dvol}_M \\ &\quad - \epsilon_2 \nabla^j((\rho - \epsilon_1 \nabla^i(\rho \nabla_i \phi_1))\nabla_j \phi_2) \, \mathrm{dvol}_M\Big) \end{aligned}$$

$$\frac{d}{d\epsilon_2}\Big|_{\epsilon_2=0}\ \frac{d}{d\epsilon_1}\Big|_{\epsilon_1=0}\ F\Big((\rho\ -\ \epsilon_2\,\nabla^i(\rho\nabla_i\phi_2))\,\mathrm{dvol}_M$$
$$-\epsilon_1\,\nabla^j((\rho-\epsilon_2\nabla^i(\rho\nabla_i\phi_2))\nabla_j\phi_1)\,\mathrm{dvol}_M\Big)$$
$$=\frac{d}{d\epsilon}\Big|_{\epsilon=0}F\Big(\rho\,\mathrm{dvol}_M+\epsilon\nabla^j(\nabla^i(\rho\nabla_i\phi_1)\nabla_j\phi_2)\,\mathrm{dvol}_M-\epsilon\nabla^j(\nabla^i(\rho\nabla_i\phi_2)\nabla_j\phi_1)\,\mathrm{dvol}_M\Big). \tag{4.2}$$

One can check that

$$\begin{aligned}\nabla^j(\nabla^i(\rho\nabla_i\phi_1)\nabla_j\phi_2)\ -\ \nabla^j(\nabla^i(\rho\nabla_i\phi_2)\nabla_j\phi_1) =\\ -\nabla_i\left[\rho\left((\nabla^i\nabla^j\phi_2)\nabla_j\phi_1\ -\ (\nabla^i\nabla^j\phi_1)\nabla_j\phi_2\right)\right],\end{aligned} \tag{4.3}$$

from which the lemma follows. □

We now compute the Levi-Civita connection.

Proposition 2. *The Levi-Civita connection $\overline{\nabla}$ of $P^\infty(M)$ is given by*

$$((\overline{\nabla}_{V_{\phi_1}}V_{\phi_2})F)(\rho\ \mathrm{dvol}_M)\ =\ \frac{d}{d\epsilon}\Big|_{\epsilon=0}F\left(\rho\ \mathrm{dvol}_M\ -\epsilon\ \nabla_i\left(\rho\ \nabla_j\phi_1\ \nabla^i\nabla^j\phi_2\right)\ \mathrm{dvol}_M\right) \tag{4.4}$$

for $F\in C^\infty(P^\infty(M))$.

Proof. Define a vector field $D_{V_{\phi_1}}V_{\phi_2}$ by

$$((D_{V_{\phi_1}}V_{\phi_2})F)(\rho\ \mathrm{dvol}_M)\ =\ \frac{d}{d\epsilon}\Big|_{\epsilon=0}F\left(\rho\ \mathrm{dvol}_M\ -\epsilon\ \nabla_i\left(\rho\ \nabla_j\phi_1\ \nabla^i\nabla^j\phi_2\right)\ \mathrm{dvol}_M\right) \tag{4.5}$$

for $F\in C^\infty(P^\infty(M))$. We also write

$$\delta_{D_{V_{\phi_1}}V_{\phi_2}}\rho\ =\ -\nabla_i\left(\rho\ \nabla_j\phi_1\ \nabla^i\nabla^j\phi_2\right). \tag{4.6}$$

It is clear from Lemma 2 that

$$D_{V_{\phi_1}}V_{\phi_2}\ -\ D_{V_{\phi_2}}V_{\phi_1}\ =\ [V_{\phi_1},V_{\phi_2}]. \tag{4.7}$$

Next,

$$\begin{aligned}\left(V_{\phi_1}\langle V_{\phi_2},V_{\phi_3}\rangle\right)(\rho\ \mathrm{dvol}_M)\ &=\ -\int_M\nabla^i\phi_2\ \nabla_i\phi_3\ \nabla^j(\rho\nabla_j\phi_1)\ \mathrm{dvol}_M\\ &=\int_M\nabla_j\phi_1\ \nabla^i\nabla^j\phi_2\ \nabla_i\phi_3\ \rho\ \mathrm{dvol}_M\\ &\quad+\int_M\nabla_j\phi_1\ \nabla^i\nabla^j\phi_3\ \nabla_i\phi_2\ \rho\ \mathrm{dvol}_M\\ &=-\int_M\phi_3\ \nabla_i(\rho\ \nabla_j\phi_1\ \nabla^i\nabla^j\phi_2)\ \mathrm{dvol}_M\\ &\quad-\int_M\phi_2\ \nabla_i(\rho\ \nabla_j\phi_1\ \nabla^i\nabla^j\phi_3)\ \mathrm{dvol}_M\\ &=\int_M\phi_3\ \delta_{D_{V_{\phi_1}}V_{\phi_2}}\rho\ \mathrm{dvol}_M+\int_M\phi_2\ \delta_{D_{V_{\phi_1}}V_{\phi_3}}\rho\ \mathrm{dvol}_M\\ &=\langle D_{V_{\phi_1}}V_{\phi_2},V_{\phi_3}\rangle(\rho\ \mathrm{dvol}_M)+\langle V_{\phi_2},D_{V_{\phi_1}}V_{\phi_3}\rangle(\rho\ \mathrm{dvol}_M).\end{aligned} \tag{4.8}$$

Thus

$$V_{\phi_1}\langle V_{\phi_2}, V_{\phi_3}\rangle = \langle D_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3}\rangle + \langle V_{\phi_2}, D_{V_{\phi_1}} V_{\phi_3}\rangle. \tag{4.9}$$

As

$$2\langle \overline{\nabla}_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3}\rangle = V_{\phi_1}\langle V_{\phi_2}, V_{\phi_3}\rangle + V_{\phi_2}\langle V_{\phi_3}, V_{\phi_1}\rangle - V_{\phi_3}\langle V_{\phi_1}, V_{\phi_2}\rangle + \langle V_{\phi_3}, [V_{\phi_1}, V_{\phi_2}]\rangle - \langle V_{\phi_2}, [V_{\phi_1}, V_{\phi_3}]\rangle - \langle V_{\phi_1}, [V_{\phi_2}, V_{\phi_3}]\rangle, \tag{4.10}$$

substituting (4.7) and (4.9) into the right-hand side of (4.10) shows that

$$\langle \overline{\nabla}_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3}\rangle = \langle D_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3}\rangle \tag{4.11}$$

for all $\phi_3 \in C^\infty(M)$. The proposition follows. □

Lemma 3. *The connection coefficients at ρ dvol_M are given by*

$$\langle \overline{\nabla}_{V_{\phi_1}} V_{\phi_2}, V_{\phi_3}\rangle = \int_M \nabla_i \phi_1 \, \nabla_j \phi_3 \, \nabla^i \nabla^j \phi_2 \, \rho \, \mathrm{dvol}_M . \tag{4.12}$$

Proof. This follows from (2.5) and (4.4). □

Let G_ρ be the Green's operator for $d^*_\rho d$ on $L^2(M, \rho \, \mathrm{dvol}_M)$. (More explicitly, if $\int_M f \, \rho \, \mathrm{dvol}_M = 0$ and $\phi = G_\rho f$ then ϕ satisfies $-\frac{1}{\rho} \nabla^i(\rho \nabla_i \phi) = f$ and $\int_M \phi \, \rho \, \mathrm{dvol}_M = 0$, while $G_\rho 1 = 0$.) Let Π_ρ denote orthogonal projection onto $\overline{\mathrm{Im}(d)} \subset \Omega^1_{L^2}(M, \rho \, \mathrm{dvol}_M)$.

Lemma 4. *At ρ dvol_M, we have $\overline{\nabla}_{V_{\phi_1}} V_{\phi_2} = V_\phi$, where $\phi = G_\rho d^*_\rho(\nabla_i \nabla_j \phi_2 \nabla^j \phi_1 \, dx^i)$.*

Proof. Given $\phi_3 \in C^\infty(M)$, we have

$$\begin{aligned}
\langle V_{\phi_3}, V_\phi\rangle(\rho \, \mathrm{dvol}_M) &= \int_M \langle d\phi_3, dG_\rho d^*_\rho(\nabla_i \nabla_j \phi_2 \, \nabla^j \phi_1 \, dx^i)\rangle \, \rho \, \mathrm{dvol}_M \\
&= \int_M \langle d\phi_3, \Pi_\rho(\nabla_i \nabla_j \phi_2 \, \nabla^j \phi_1 \, dx^i)\rangle \, \rho \, \mathrm{dvol}_M \\
&= \int_M \langle d\phi_3, \nabla_i \nabla_j \phi_2 \, \nabla^j \phi_1 \, dx^i\rangle \, \rho \, \mathrm{dvol}_M \\
&= \langle V_{\phi_3}, \overline{\nabla}_{V_{\phi_1}} V_{\phi_2}\rangle(\rho \, \mathrm{dvol}_M).
\end{aligned} \tag{4.13}$$

The lemma follows. □

To derive the equation for parallel transport, let $c : (a, b) \to P^\infty(M)$ be a smooth curve. As before, we write $c(t) = \rho(t) \, \mathrm{dvol}_M$ and define $\phi(t) \in C^\infty(M)$, up to a constant, by $\frac{dc}{dt} = V_{\phi(t)}$. Let $V_{\eta(t)}$ be a vector field along c, with $\eta(t) \in C^\infty(M)$. If $\{\phi_\alpha\}_{\alpha=1}^\infty$ is a basis for $C^\infty(M)/\mathbb{R}$ then $\{V_{\phi_\alpha}\}_{\alpha=1}^\infty$ is a global basis for $TP^\infty(M)$ and we can write $\eta(t) = \sum_\alpha \eta_\alpha(t) \, V_{\phi_\alpha}\big|_{c(t)}$. The condition for V_η to be parallel along c is

$$\sum_\alpha \frac{d\eta_\alpha}{dt} V_{\phi_\alpha}\Big|_{c(t)} + \sum_\alpha \eta_\alpha(t) \, \overline{\nabla}_{V_{\phi(t)}} V_{\eta_\alpha}\Big|_{c(t)} = 0, \tag{4.14}$$

or

$$V_{\frac{\partial \eta}{\partial t}} + \overline{\nabla}_{V_{\phi(t)}} V_{\eta(t)} = 0. \tag{4.15}$$

Proposition 3. *The equation for V_η to be parallel along c is*

$$\nabla_i \left(\rho \left(\nabla^i \frac{\partial \eta}{\partial t} + \nabla_j \phi \, \nabla^i \nabla^j \eta \right) \right) = 0. \tag{4.16}$$

Proof. This follows from (2.3), (4.4) and (4.15). □

As a check on Eq. (4.16), we show that parallel transport along c preserves the inner product.

Lemma 5. *If V_{η_1} and V_{η_2} are parallel vector fields along c then $\int_M \langle \nabla \eta_1, \nabla \eta_2 \rangle \, \rho \, \mathrm{dvol}_M$ is constant in t.*

Proof. We have

$$\begin{aligned}
\frac{d}{dt} \int_M \langle \nabla \eta_1, \nabla \eta_2 \rangle \, \rho \, \mathrm{dvol}_M &= \int_M \nabla^i \frac{\partial \eta_1}{\partial t} \nabla_i \eta_2 \, \rho \, \mathrm{dvol}_M + \int_M \nabla_i \eta_1 \, \nabla^i \frac{\partial \eta_2}{dt} \, \rho \, \mathrm{dvol}_M \\
&\quad - \int_M \nabla_i \eta_1 \nabla^i \eta_2 \, \nabla^j (\rho \nabla_j \phi) \, \mathrm{dvol}_M \\
&= \int_M \nabla^i \frac{\partial \eta_1}{\partial t} \nabla_i \eta_2 \, \rho \, \mathrm{dvol}_M + \int_M \nabla_i \eta_1 \, \nabla^i \frac{\partial \eta_2}{dt} \, \rho \, \mathrm{dvol}_M \\
&\quad + \int_M \left(\nabla^i \nabla^j \eta_1 \, \nabla_i \eta_2 + \nabla_i \eta_1 \, \nabla^i \nabla^j \eta_2 \right) \nabla_j \phi \, \rho \, \mathrm{dvol}_M \\
&= - \int_M \eta_2 \, \nabla_i \left(\rho \left(\nabla^i \frac{\partial \eta_1}{\partial t} + \nabla_j \phi \, \nabla^i \nabla^j \eta_1 \right) \right) \mathrm{dvol}_M \\
&\quad - \int_M \eta_1 \, \nabla_i \left(\rho \left(\nabla^i \frac{\partial \eta_2}{\partial t} + \nabla_j \phi \, \nabla^i \nabla^j \eta_2 \right) \right) \mathrm{dvol}_M \\
&= 0.
\end{aligned} \tag{4.17}$$

This proves the lemma. □

Finally, we derive the geodesic equation.

Proposition 4. *The geodesic equation for c is*

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 = 0, \tag{4.18}$$

modulo the addition of a spatially-constant function to ϕ.

Proof. Taking $\eta = \phi$ in (4.16) gives

$$\nabla_i \left(\rho \, \nabla^i \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \right) = 0. \tag{4.19}$$

Thus $\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2$ is spatially constant. Redefining ϕ by adding to it a function of t alone, we can assume that (4.18) holds. □

Remark 2. Equation (4.18) has been known for a while, at least in the case of $\mathbb{R}^n$, to be the formal equation for Wasserstein geodesics. For general Riemannian manifolds M, it was formally derived as the Wasserstein geodesic equation in [17] by minimizing lengths of curves. For $t > 0$, it has the Hopf-Lax solution

$$\phi(t, m) = \inf_{m' \in M} \left(\phi(0, m') + \frac{d(m, m')^2}{2t} \right). \tag{4.20}$$

Given $\mu_0, \mu_1 \in P^\infty(M)$, it is known that there is a unique minimizing Wasserstein geodesic $\{\mu_t\}_{t \in [0,1]}$ joining them. It is of the form $\mu_t = (F_t)_* \mu_0$, where $F_t \in \mathrm{Diff}(M)$ is given by $F_t(m) = \exp_m(-t \nabla_m \phi_0)$ for an appropriate Lipschitz function ϕ_0 [12]. If ϕ_0 happens to be smooth then defining $\rho(t)$ by $\mu_t = \rho(t) \operatorname{dvol}_M$ and defining $\phi(t) \in C^\infty(M)/\mathbb{R}$ as above, it is known that ϕ satisfies (4.18), with $\phi(0) = \phi_0$ [21, Sect. 5.4.7]. In this way, (4.18) rigorously describes certain geodesics in the Wasserstein space $P_2(M)$.

5. Curvature

In this section we compute the Riemannian curvature tensor of $P^\infty(M)$.

Given $\phi, \phi' \in C^\infty(M)$, define $T_{\phi\phi'} \in \Omega^1_{L^2}(M)$ by

$$T_{\phi\phi'} = (I - \Pi_\rho) \left(\nabla^i \phi \, \nabla_i \nabla_j \phi' \, dx^j \right). \tag{5.1}$$

(The left-hand side depends on ρ, but we suppress this for simplicity of notation.)

Lemma 6. $T_{\phi\phi'} + T_{\phi'\phi} = 0$.

Proof. As

$$\nabla^i \phi \, \nabla_i \nabla_j \phi' \, dx^j + \nabla^i \phi' \, \nabla_i \nabla_j \phi \, dx^j = d\langle \nabla \phi, \nabla \phi' \rangle, \tag{5.2}$$

and $I - \Pi_\rho$ projects away from $\mathrm{Im}(d)$, the lemma follows. □

Theorem 1. *Given $\phi_1, \phi_2, \phi_3, \phi_4 \in C^\infty(M)$, the Riemannian curvature operator $\overline{R}$ of $P^\infty(M)$ is given by*

$$\begin{aligned} \langle \overline{R}(V_{\phi_1}, V_{\phi_2}) V_{\phi_3}, V_{\phi_4} \rangle = &\int_M \langle R(\nabla \phi_1, \nabla \phi_2) \nabla \phi_3, \nabla \phi_4 \rangle \, \rho \operatorname{dvol}_M - 2 \langle T_{\phi_1\phi_2}, T_{\phi_3\phi_4} \rangle \\ &+ \langle T_{\phi_2\phi_3}, T_{\phi_1\phi_4} \rangle - \langle T_{\phi_1\phi_3}, T_{\phi_2\phi_4} \rangle, \end{aligned} \tag{5.3}$$

where both sides are evaluated at $\rho \operatorname{dvol}_M \in P^\infty(M)$.

Proof. We use the formula

$$\begin{aligned} \langle \overline{R}(V_{\phi_1}, V_{\phi_2}) V_{\phi_3}, V_{\phi_4} \rangle = &V_{\phi_1} \langle \overline{\nabla}_{V_{\phi_2}} V_{\phi_3}, V_{\phi_4} \rangle - \langle \overline{\nabla}_{V_{\phi_2}} V_{\phi_3}, \overline{\nabla}_{V_{\phi_1}} V_{\phi_4} \rangle \\ &- V_{\phi_2} \langle \overline{\nabla}_{V_{\phi_1}} V_{\phi_3}, V_{\phi_4} \rangle + \langle \overline{\nabla}_{V_{\phi_1}} V_{\phi_3}, \overline{\nabla}_{V_{\phi_2}} V_{\phi_4} \rangle \\ &- \langle \overline{\nabla}_{[V_{\phi_1}, V_{\phi_2}]} V_{\phi_3}, V_{\phi_4} \rangle. \end{aligned} \tag{5.4}$$

First, from (2.3) and (3),

$$
\begin{aligned}
V_{\phi_1} \langle \overline{\nabla}_{V_{\phi_2}} V_{\phi_3}, V_{\phi_4} \rangle &= - \int_M \nabla_i \phi_2 \, \nabla_j \phi_4 \, \nabla^i \nabla^j \phi_3 \, \nabla^k (\rho \nabla_k \phi_1) \, \mathrm{dvol}_M \\
&= \int_M \nabla^k \nabla_i \phi_2 \, \nabla_j \phi_4 \, \nabla^i \nabla^j \phi_3 \, \nabla_k \phi_1 \, \rho \, \mathrm{dvol}_M \\
&\quad + \int_M \nabla_i \phi_2 \, \nabla^k \nabla_j \phi_4 \, \nabla^i \nabla^j \phi_3 \, \nabla_k \phi_1 \, \rho \, \mathrm{dvol}_M \\
&\quad + \int_M \nabla_i \phi_2 \, \nabla_j \phi_4 \, \nabla^k \nabla^i \nabla^j \phi_3 \, \nabla_k \phi_1 \, \rho \, \mathrm{dvol}_M \, .
\end{aligned} \tag{5.5}
$$

Similarly,

$$
\begin{aligned}
V_{\phi_2} \langle \overline{\nabla}_{V_{\phi_1}} V_{\phi_3}, V_{\phi_4} \rangle &= \int_M \nabla^k \nabla_i \phi_1 \, \nabla_j \phi_4 \, \nabla^i \nabla^j \phi_3 \, \nabla_k \phi_2 \, \rho \, \mathrm{dvol}_M \\
&\quad + \int_M \nabla_i \phi_1 \, \nabla^k \nabla_j \phi_4 \, \nabla^i \nabla^j \phi_3 \, \nabla_k \phi_2 \, \rho \, \mathrm{dvol}_M \\
&\quad + \int_M \nabla_i \phi_1 \, \nabla_j \phi_4 \, \nabla^k \nabla^i \nabla^j \phi_3 \, \nabla_k \phi_2 \, \rho \, \mathrm{dvol}_M \, .
\end{aligned} \tag{5.6}
$$

Next, using (2.4), Lemma 4 and (5.1),

$$
\begin{aligned}
\langle \overline{\nabla}_{V_{\phi_2}} V_{\phi_3}, \overline{\nabla}_{V_{\phi_1}} V_{\phi_4} \rangle &= \langle d G_\rho d^*_\rho (\nabla_i \nabla_j \phi_3 \, \nabla^j \phi_2 \, dx^i), d G_\rho d^*_\rho (\nabla_k \nabla_l \phi_4 \, \nabla^l \phi_1 \, dx^k) \rangle_{L^2} \\
&= \langle \Pi_\rho (\nabla_i \nabla_j \phi_3 \, \nabla^j \phi_2 \, dx^i), \Pi_\rho (\nabla_k \nabla_l \phi_4 \, \nabla^l \phi_1 \, dx^k) \rangle_{L^2} \\
&= \langle \nabla_i \nabla_j \phi_3 \, \nabla^j \phi_2 \, dx^i, \nabla_k \nabla_l \phi_4 \, \nabla^l \phi_1 \, dx^k \rangle_{L^2} - \langle T_{\phi_2 \phi_3}, T_{\phi_1 \phi_4} \rangle \\
&= \int_M \nabla_i \nabla_j \phi_3 \, \nabla^j \phi_2 \, \nabla^i \nabla_l \phi_4 \, \nabla^l \phi_1 \, \rho \, \mathrm{dvol}_M - \langle T_{\phi_2 \phi_3}, T_{\phi_1 \phi_4} \rangle .
\end{aligned} \tag{5.7}
$$

Similarly,

$$
\langle \overline{\nabla}_{V_{\phi_1}} V_{\phi_3}, \overline{\nabla}_{V_{\phi_2}} V_{\phi_4} \rangle = \int_M \nabla_i \nabla_j \phi_3 \nabla^j \phi_1 \nabla^i \nabla_l \phi_4 \nabla^l \phi_2 \rho \, \mathrm{dvol}_M - \langle T_{\phi_1 \phi_3}, T_{\phi_2 \phi_4} \rangle . \tag{5.8}
$$

Finally, we compute $\langle \overline{\nabla}_{[V_{\phi_1}, V_{\phi_2}]} V_{\phi_3}, V_{\phi_4} \rangle$. From (4.1), we can write $[V_{\phi_1}, V_{\phi_2}] = V_\phi$, where

$$
\phi = G_\rho \, d^*_\rho \left(\nabla_i \nabla_j \phi_2 \, \nabla^j \phi_1 \, dx^i - \nabla_i \nabla_j \phi_1 \, \nabla^j \phi_2 \, dx^i \right). \tag{5.9}
$$

Then from (4.12),

$$
\begin{aligned}
\langle \overline{\nabla}_{[V_{\phi_1}, V_{\phi_2}]} V_{\phi_3}, V_{\phi_4} \rangle &= \int_M \nabla_i \phi \, \nabla_j \phi_4 \, \nabla^i \nabla^j \phi_3 \, \rho \, \mathrm{dvol}_M = \langle d\phi, \nabla^j \phi_4 \, \nabla_i \nabla_j \phi_3 \, dx^i \rangle_{L^2} \\
&= \langle d G_\rho \, d^*_\rho \, (\nabla_i \nabla_j \phi_2 \, \nabla^j \phi_1 \, dx^i - \nabla_i \nabla_j \phi_1 \, \nabla^j \phi_2 \, dx^i), \\
&\qquad \nabla^j \phi_4 \, \nabla_i \nabla_j \phi_3 \, dx^i \rangle_{L^2} \\
&= \langle \Pi_\rho \left(\nabla_i \nabla_j \phi_2 \, \nabla^j \phi_1 \, dx^i - \nabla_i \nabla_j \phi_1 \, \nabla^j \phi_2 \, dx^i \right), \\
&\qquad \Pi_\rho \left(\nabla^j \phi_4 \, \nabla_i \nabla_j \phi_3 \, dx^i \right) \rangle_{L^2}
\end{aligned}
$$

$$= \int_M \left(\nabla_i \nabla_j \phi_2 \nabla^j \phi_1 - \nabla_i \nabla_j \phi_1 \nabla^j \phi_2 \right) \nabla_k \phi_4 \nabla^i \nabla^k \phi_3 \rho \, \mathrm{dvol}_M$$
$$- \langle T_{\phi_1\phi_2}, T_{\phi_4\phi_3} \rangle + \langle T_{\phi_2\phi_1}, T_{\phi_4\phi_3} \rangle$$
$$= \int_M \left(\nabla_i \nabla_j \phi_2 \nabla^j \phi_1 - \nabla_i \nabla_j \phi_1 \nabla^j \phi_2 \right) \nabla_k \phi_4 \, \nabla^i \nabla^k \phi_3 \, \rho \, \mathrm{dvol}_M$$
$$+ 2 \langle T_{\phi_1\phi_2}, T_{\phi_3\phi_4} \rangle. \tag{5.10}$$

The theorem follows from combining Eqs. (5.4)-(5.10). □

Corollary 1. *Suppose that* $\phi_1, \phi_2 \in C^\infty(M)$ *satisfy* $\int_M |\nabla \phi_1|^2 \rho \, \mathrm{dvol}_M = \int_M |\nabla \phi_2|^2 \rho \, \mathrm{dvol}_M = 1$ *and* $\int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \, \rho \, \mathrm{dvol}_M = 0$. *Then the sectional curvature at* $\rho \, \mathrm{dvol}_M \in P^\infty(M)$ *of the* 2*-plane spanned by* V_{ϕ_1} *and* V_{ϕ_2} *is*

$$\overline{K}(V_{\phi_1}, V_{\phi_2}) = \int_M K(\nabla \phi_1, \nabla \phi_2) \left(|\nabla \phi_1|^2 |\nabla \phi_2|^2 - \langle \nabla \phi_1, \nabla \phi_2 \rangle^2 \right) \rho \, \mathrm{dvol}_M + 3 |T_{\phi_1\phi_2}|^2, \tag{5.11}$$

where $K(\nabla \phi_1, \nabla \phi_2)$ *denotes the sectional curvature of the* 2*-plane spanned by* $\nabla \phi_1$ *and* $\nabla \phi_2$.

Corollary 2. *If* M *has nonnegative sectional curvature then* $P^\infty(M)$ *has nonnegative sectional curvature.*

Remark 3. One can ask whether the condition of M having sectional curvature bounded below by $r \in \mathbb{R}$ implies that $P^\infty(M)$ has sectional curvature bounded below by r. This is not the case unless $r = 0$. The reason is one of normalizations. The normalizations on ϕ_1 and ϕ_2 are $\int_M |\nabla \phi_1|^2 \rho \, \mathrm{dvol}_M = \int_M |\nabla \phi_2|^2 \rho \, \mathrm{dvol}_M = 1$ and $\int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \rho \, \mathrm{dvol}_M = 0$. One cannot conclude from this that $\int_M \left(|\nabla \phi_1|^2 \, |\nabla \phi_2|^2 - \langle \nabla \phi_1, \nabla \phi_2 \rangle^2 \right) \rho \, \mathrm{dvol}_M$ is ≥ 1 or ≤ 1.

More generally, if M has nonnegative sectional curvature then $P_2(M)$ is an Alexandrov space with nonnegative curvature [8, Theorem A.8], [19, Prop. 2.10(iv)]. On the other hand, if M does not have nonnegative sectional curvature then one sees by an explicit construction that $P_2(M)$ is not an Alexandrov space with curvature bounded below [19, Prop. 2.10(iv)].

Remark 4. The formula (5.3) has the structure of the O'Neill formula for the sectional curvature of the base space of a Riemannian submersion. In the case $M = \mathbb{R}^n$, Otto argued that $P^\infty(\mathbb{R}^n)$ is formally the quotient space of $\mathrm{Diff}(\mathbb{R}^n)$, with an L^2-metric, by the subgroup that preserves a fixed volume form [16]. As $\mathrm{Diff}(\mathbb{R}^n)$ is formally flat, it followed that $P^\infty(\mathbb{R}^n)$ formally had nonnegative sectional curvature.

6. Poisson Structure

Let M be a smooth connected closed manifold. We do not give it a Riemannian metric. In this section we describe a natural Poisson structure on $P^\infty(M)$ arising from a Poisson structure on M. If M is a symplectic manifold then we show that the symplectic leaves in $P^\infty(M)$ are orbits of the action of the group $\mathrm{Ham}(M)$ of Hamiltonian diffeomorphisms acting on $P^\infty(M)$. We recover the symplectic structure on the orbits that was considered in [1,5].

Let M be a smooth manifold and let $p \in C^\infty(\wedge^2 TM)$ be a skew bivector field. Given $f_1, f_2 \in C^\infty(M)$, one defines the Poisson bracket $\{f_1, f_2\} \in C^\infty(M)$ by $\{f_1, f_2\} =$

$p(df_1 \otimes df_2)$. There is a skew trivector field $\partial p \in C^\infty(\wedge^3 TM)$ so that for $f_1, f_2, f_3 \in C^\infty(M)$,

$$(\partial p)(df_1, df_2, df_3) = \{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}. \tag{6.1}$$

One says that p defines a Poisson structure on M if $\partial p = 0$. We assume hereafter that p is a Poisson structure on M.

Definition 1 *Define a skew bivector field* $P \in C^\infty(\wedge^2 TP^\infty(M))$ *by saying that its Poisson bracket is* $\{F_{\phi_1}, F_{\phi_2}\} = F_{\{\phi_1,\phi_2\}}$*, i.e.*

$$\{F_{\phi_1}, F_{\phi_2}\}(\mu) = \int_M \{\phi_1, \phi_2\}\, d\mu \tag{6.2}$$

for $\mu \in P^\infty(M)$.

The map $\phi \to dF_\phi \big|_\mu$ passes to an isomorphism $C^\infty(M)/\mathbb{R} \to T^*_\mu P^\infty(M)$. As the right-hand side of (6.2) vanishes if ϕ_1 or ϕ_2 is constant, Eq. (6.2) does define an element of $C^\infty(\wedge^2 TP^\infty(M))$.

Proposition 5. *P is a Poisson structure on $P^\infty(M)$.*

Proof. It suffices to show that ∂P vanishes. This follows from the equation

$$\begin{aligned}(\partial P)(dF_{\phi_1}, dF_{\phi_2}, dF_{\phi_3}) &= \{\{F_{\phi_1}, F_{\phi_2}\}, F_{\phi_3}\} + \{\{F_{\phi_2}, F_{\phi_3}\}, F_{\phi_1}\} + \{\{F_{\phi_3}, F_{\phi_1}\}, F_{\phi_2}\} \\ &= F_{\{\{\phi_1,\phi_2\},\phi_3\} + \{\{\phi_2,\phi_3\},\phi_1\} + \{\{\phi_3,\phi_1\},\phi_2\}} = 0. \end{aligned} \tag{6.3}$$

□

A finite-dimensional Poisson manifold has a (possibly singular) foliation with symplectic leaves [6]. The leafwise tangent vector fields are spanned by the vector fields W_f defined by $W_f h = \{f, h\}$. The symplectic form Ω on a leaf is given by saying that $\Omega(W_f, W_g) = \{f, g\}$.

Suppose now that (M, ω) is a closed $2n$-dimensional symplectic manifold. Let $\mathrm{Ham}(M)$ be the group of Hamiltonian symplectomorphisms of M [13, Chap. 3.1].

Proposition 6. *The symplectic leaves of $P^\infty(M)$ are the orbits of the action of* $\mathrm{Ham}(M)$ *on $P^\infty(M)$. Given $\mu \in P^\infty(M)$ and $\phi_1, \phi_2 \in C^\infty(M)$, let $\widehat{H}_{\phi_1}, \widehat{H}_{\phi_2} \in T_\mu P^\infty(M)$ be the infinitesimal motions of μ under the flows generated by the Hamiltonian vector fields H_{ϕ_1}, H_{ϕ_2} on M. Then $\Omega(\widehat{H}_{\phi_1}, \widehat{H}_{\phi_2}) = \int_M \{\phi_1, \phi_2\}\, d\mu$.*

Proof. Write $\mu = \rho\, \omega^n$. We claim that $(W_{F_\phi} \widehat{F})(\mu) = \frac{d}{d\epsilon}\big|_{\epsilon=0} \widehat{F}(\mu - \epsilon\, \{\phi, \rho\}\, \omega^n)$ for $\widehat{F} \in C^\infty(P^\infty(M))$. To show this, it is enough to check it for each $\widehat{F} = F_{\phi'}$, with $\phi' \in C^\infty(M)$. But

$$(W_{F_\phi} F_{\phi'})(\mu) = F_{\{\phi,\phi'\}}(\mu) = \int_M \{\phi, \phi'\}\, \rho\, \omega^n = - \int_M \phi'\, \{\phi, \rho\}\, \omega^n, \tag{6.4}$$

from which the claim follows. This shows that $W_{F_\phi} = \widehat{H}_\phi$.

Next, at $\mu \in P^\infty(M)$ we have

$$\Omega(\widehat{H}_{\phi_1}, \widehat{H}_{\phi_2}) = \Omega(W_{F_{\phi_1}}, W_{F_{\phi_2}}) = \{F_{\phi_1}, F_{\phi_2}\}(\mu) = \int_M \{\phi_1, \phi_2\}\, d\mu. \tag{6.5}$$

This proves the proposition. □

Remark 5. As a check on Proposition 6, suppose that $\phi_2 \in C^\infty(M)$ is such that $\widehat{H}_{\phi_2}$ vanishes at $\mu = \rho\,\omega^n$. Then $\{\phi_2, \rho\} = 0$, so by our formula we have

$$\Omega(\widehat{H}_{\phi_1}, \widehat{H}_{\phi_2}) = \int_M \{\phi_1, \phi_2\}\, d\mu = \int_M \{\phi_1, \phi_2\}\, \rho\, \omega^n = \int_M \phi_1\, \{\phi_2, \rho\}\, \omega^n = 0. \quad (6.6)$$

Remark 6. The Poisson structure on $P^\infty(M)$ is the restriction of the Poisson structure on $(C^\infty(M))^*$ considered in [10,11,22]. Here the Poisson structure on $(C^\infty(M))^*$ comes from the general construction of a Poisson structure on the dual of a Lie algebra, considering $C^\infty(M)$ to be a Lie algebra with respect to the Poisson bracket on $C^\infty(M)$. The cited papers use the Poisson structure on $(C^\infty(M))^*$ to show that certain PDE's are Hamiltonian flows.

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