REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 9, número suplementario, 1996
http://dx.doi.org/10.5209/rev_REMA.1996.v9.17537

Some geometric properties concerning fixed point theory.

T. DOMÍNGUEZ BENAVIDES

Dedicated to Prof. Rodríguez Salinas in occasion of his 70th birthday

Abstract

The Fixed Point Theory for nonexpansive mappings is strongly based upon the geometry of the ambient Banach space. In Section 1 we state the role which is played by the multidimensional convexity and smoothness in this theory. In Section 2 we study the computation of the normal structure coefficient in finite dimensional ℓ_p–spaces and its connection with several classic geometric problems.

The most known and important fixed point theorem is the contractive mapping principle, which assures that every contraction T from a complete metric space X into itself has a unique fixed point. The simplicity of its proof and the possibility of attaining the fixed point by using succesive approximations let this theorem become a very useful tool in Analysis and in Applied Mathematics. A translation in R^n is a simple example showing that the Banach theorem does not hold if we replace the condition T is contractive by the weaker condition T is nonexpansive, i.e. $d(Tx,Ty) \leq d(x,y)$ for every x, y in X. Even the "middle" condition $d(Tx,Ty) < d(x,y)$ for every x, y in X (in this case T is usually called *weakly contractive*) does not assure the existence of a fixed point (consider, for instance, the mapping $Tx = x + 1/x$ defined in the

---

1991 Mathematics Subject Classification: 46B20, 47H10
Servicio Publicaciones Univ. Complutense. Madrid, 1996.
The research of the author is partially supported by the D.G.I.C.Y.T. under project PB93-1177-C01 and the Junta de Andalucia under project JA 1241.

complete metric space $[1, +\infty)$). In this situation, it is not surprising that for almost forty years the problem of the existence fixed points for nonexpansive mappings was relegated. However in 1965, Browder [Br1] proved that every nonexpansive mapping T from a convex bounded closed subset C of a Hilbert space X into C has a fixed point. In the same year Browder [Br2] and Kirk [K] proved that this result could be improved assuming the weaker condition X is a uniform convex space or X is a reflexive Banach space with normal structure. We recall that a bounded set A is called *diametral* if for every point $x \in A$ we have diam $A = \sup\{\|x - y\| : y \in A\}$. A Banach space X is said to have *normal structure* if every bounded convex diametral subset A of X is a singleton. It is clear that in the cartesian plane every convex set with more than one element is not diametral. However the set $\mathrm{co}\{e_n\}$ in c_0 is clearly a diametral bounded convex set. These results are noteworthy regarding the imposed conditions on C, which look more suitable in the compact fixed point theory (Schauder's Theorem) and the "geometric" conditions which X must satisfy. From this point a very wide theory has been developed in trying to find more general condition on the Banach space X and on the subset C which still assure the existence of fixed points. To simplify we shall say that a Banach space X has the fixed point property (f.p.p.) if every nonexpansive mapping T defined from a convex bounded closed subset C of X into X has a fixed point. Since Kakutani showed a simple example of a nonexpansive mapping from the unit ball B of c_0 into B without fixed points, it is clear that Banach spaces exist which do not have the f.p.p. The failure of the f.p.p. in this example is a consequence of the weakly noncompactness of B. However it can be proved that every nonexpansive mapping from a weakly compact convex set C of c_0 into C has a fixed point. When such a condition is satisfied we shall say that the Banach space X has the weak fixed point property (w.f.p.p.). Obviously, the f.p.p. and the w.f.p.p. are identical if X is reflexive. For a long time an open question was: Does every Banach space X have the w.f.p.p.? The answer to this question was given by Alspach [Al] in 1981, proving that $L_1[0,1]$ fails to have the w.f.p.p. Since Maurey [Ma] proved that every reflexive subspace of L_1 has the f.p.p., another question becomes very important: Does any reflexive Banach space have the f.p.p.? Until now nobody has been able to answer this question.

We shall show in this paper different geometric properties connected with metric fixed point theory. In Section 1 we see that multidimensional convexity and smoothness give conditions which assure the fixed point property. In Section 2 we show the relationship between the problem of computing the normal structure coefficient and some "classic frame" geometric problems.

1 Convexity and smoothness in fixed point theory

We start recalling some definitions. Let X be a Banach space, A and B bounded subsets of X. The Chebyshev radius of the set A with respect to the set B is defined by

$$r(A,B) = \inf\{\sup\{\|x-y\| : x \in A\} : y \in B\},$$

that is, roughly speaking, $r(A,B)$ is the least radius such that a ball centered in B with this radius contains A; when $B = \operatorname{co} A$, we denote $r(A) = r(A,\operatorname{co} A)$; the Chebyshev center of A with respect to B is defined by

$$Z(A,B) = \{y \in B : \sup\{\|x-y\| : x \in A\} = r(A,B)\};$$

denoting $Z(A) = Z(A,\operatorname{co} A)$. The set $Z(A,B)$ can be empty. However if B is a weakly compact and convex set, the Chebyshev center $Z(A,B)$ is nonempty. With this notation a convex closed bounded set A is diametral if $\operatorname{diam} A = r(A)$ and X has normal structure if $\operatorname{diam} A/r(A) > 1$ for every closed bounded convex subset A of X with $\operatorname{diam} A > 0$. We say that X has weak normal structure if $\operatorname{diam} A/r(A) > 1$ for every weakly compact and convex subset A of X. We recall a "classic" result in metric fixed point theory:

Theorem 1 [K]. *Let X be a Banach space with weak normal structure, C a weakly compact convex subset of X and $T : C \to C$ a nonexpansive mapping. Then T has a fixed point.*

One method to assure that a Banach space satisfies the fixed point property (f.p.p.) can be to prove that this space is near (in the Banach-Mazur distance) to another Banach space with the f.p.p. This method needs to use a "measure" in the sense: To what degree does a Banach space have the f.p.p.? A technique in this way was initiated by Bynum [By] with the definition of some normal structure coefficients.

The most simple normal structure coefficient is the following:

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r(A)} : A \subset X\right.$$

$$\left.\text{convex closed and bounded with diam } (A) > 0\right\}$$

It is obvious from the definition that X has normal structure if $N(X) > 1$. It can be proved that the converse result does not hold. We shall say (ref. [Ma]) that X has uniform normal structure if $N(X) > 1$. There are several geometric properties of the Banach spaces which assures either the normal structure or the f.p.p. It is possible even to give bounds from below for $N(X)$ or some other similar coefficients.

We start by showing that every uniform convex space has uniform normal structure and $N(X)$ can be bounded from below using the Clarkson modulus. We recall that a Banach space X is called *uniformly convex* (U.C.) if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|x+y\|/2 < 1-\delta$ for every $x, y \in X$ such that $\|x\| \leq 1, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$. The function

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x-y\| \geq \epsilon\right\}$$

is called the (Clarkson) modulus of uniform convexity and the number $\epsilon_0(X) = \sup\{\epsilon \geq 0 : \delta_X(\epsilon) = 0\}$ is called the characteristic of convexity of X. It is clear that X is U.C. if and only if $\epsilon_0(X) = 0$.

Theorem 2 [By]. *If X is a Banach space with modulus of convexity δ, then $N(X) \geq (1-\delta(1))^{-1}$.*

Then we see that $N(X) > 1$ if the characteristic of convexity of X is less than 1. A similar result can be proved now concerning the modulus of uniform smoothness. We recall that the dual notion of uniform convexity is the concept of uniform smoothness. The space X is said to be uniformly smooth (U.S.) if $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$ where

$$\rho(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in B_X\right\}.$$

The function $\rho(t)$ is called modulus of uniform smoothness. From the Lindenstrauss formula

$$\rho_{X^*}(t) = \sup_{0\leq\epsilon\leq 2} \frac{t\epsilon}{2} - \delta_X(\epsilon)$$

it easily follows that X is U.C. if and only if X^* is U.S.

Theorem 3 [Pr2]. *Let X be a Banach space with modulus of smoothness $\rho_X(\cdot)$. Denote*

$$\rho = \inf\{\rho_X(\tau) - \frac{\tau}{2} + 1 : \tau \in (0, 1/2]\}.$$

Then $N(X) \geq 1/\rho$. In particular X has normal structure if $\rho'_X(0) < 1/2$.

The notion of uniform convexity and the corresponding modulus only depends on the two dimensional subspaces of X. A natural generalization of this concept is the *$k-$uniform convexity.* Let X be a Banach space. The modulus of k-uniform rotundity associated of space X is defined as

$$\delta_X^k(\epsilon) = \inf\left\{1 - \|\sum_{i=1}^{k+1} \frac{x_i}{k+1}\| : \|x_i\| = 1\, i = 1, ..., k+1 \right.$$
$$\left. \text{and} \quad A(x_1, ..., x_{k+1}) \geq \epsilon \right\}$$

where $A(x_1, ..., x_{k+1})$ is the k-dimensional volume of co $(x_1, ..x_{k+1})$,i. e.

$$\sup\left\{\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ f_1(x_1) & f_1(x_2) & \dots & f_1(x_{k+1}) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_{k+1}) \end{pmatrix} : f_i \in X \right.$$
$$\left. \text{and } \|f_i\| \leq 1 \text{ for } i = 1....., k. \right\}$$

The space is said to be k-uniform convex (k-UC) if for all $\epsilon > 0$, $\delta_X^k(\epsilon) > 0$.

The characteristic of convexity ϵ_0^k of X is defined as follows:

$$\epsilon_0^k(X) = \sup\{\epsilon : \delta_X^k(\epsilon) = 0\}$$

The dual notion of k-UC is the concept of *k-uniform smoothness.* A Banach space X is said to be k-uniformly smooth if: For all $\epsilon > 0$, there exists η such that for all $x \in X$ and for all t , $0 < t < \eta$, if V

is a k-dimensional subspace, there exists a norm one vector $y \in V$ such that:

$$\frac{1}{2}(\|x+ty\| + \|x-ty\|) < 1 + \epsilon t.$$

It can be proved:

Theorem 4 [MP]. *A Banach space X is k-uniformly convex if and only if X^* is k- uniformly smooth.*

Sullivan [Su] proved that every k-UC space has normal structure. The following result is a "quantitave improvement" of this fact:

Theorem 5 [Am]. *For every k and every $\epsilon > 0$, we have*

$$N(X) \geq \frac{1}{\max\{1 - \frac{1-\epsilon}{\epsilon n!}, 1 - \delta_X^k(\epsilon)\}},$$

so that if $\delta_X^k(1) > 0$ then X has uniform normal structure.

What is the situation for k-U.S. spaces? The following class of spaces, usually called Bynum's spaces, gives a first answer to this question.

Example 1 Let $\{x_n\}$ a vector in $\ell_p, 1 \leq p < \infty$. Denote x^+ and x^- the vectors whose components are $x^+(i) = \max\{x(i), 0\}$, $x^-(i) = \max\{-x(i), 0\}$. For any $q \in [1, +\infty)$ and for $x \in \ell_p$ set

$$\|x\|_{p,q} = (\|x^+\|_p^q + \|x^-\|_p^q)^{1/q}; \ \|x\|_{p,\infty} = \max\{\|x^+\|_p, \|x^-\|_p\}.$$

It is easy to check that all norms are equivalent to the usual norm in ℓ_p. The corresponding spaces will be denoted by $\ell_{p,q}, \ell_{p,\infty}$. We know that $\ell_{p,1}$ is 2-UC [ST]. Thus its dual $\ell_{q,\infty}$ is 2-US but this space fails to have normal structure. Indeed the set co $\{e_n : n \in N\}$ is a diametral set because diam $\{e_n : n \in N\} = 1$ and for every point $c = \sum_{i=1}^n \alpha_i e_i$, $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$ we have $\|e_{n+1} - c\|^p = \sup\{1, \sum_{i=1}^n \alpha_i^p\} = 1$.

This example shows that we need a different method to prove that $k-$US spaces have the f.p.p. García Falset [Ga] has recently proved that these spaces have this property. We shall follow his approach in a more general setting, giving a "quantitative" version of his result. For a Banach space X, $[X]$ will denote, as usual, the quotient space $\ell_\infty(X)/c_0(X)$ endowed with the norm $\|[z_n]\| = \limsup \|z_n\|$, where $[z_n]$ denotes the equivalent class of $\{z_n\} \in \ell_\infty(X)$. By identifying $x \in X$ with the class $[x, x, ...]$ we can consider X as a subset of $[X]$. If K is a subset of X we can consider the set $[K] = \{[z_n] \in [X] : z_n \in K$ for every $n \in N\}$. If T is a mapping from K into K we define $[T] : [K] \to [K]$ by $[T]([x_n]) = [Tx_n]$.

Definition 1 *Let X be a Banach space. For any nonnegative number a we define the coefficient*

$$S(a, b, X) = \sup\{\liminf \|x_n + x\|\}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences in B_X such that $\lim_{n,m;n\neq m} \|x_n - x_m\| \leq 1$ and $\lim \|x_n\| \geq b$. We define

$$R(a, X) = \inf\{S(a, b, X) : 0 < b < 1/WCS(X)\}$$

where $WCS(X)$ is the Bynum's coefficient of weak normal structure which can be defined (see, for instance [DL]) by

$$WCS(X) = \inf\left\{\frac{\lim_{n,m;n\neq m} \|x_n - x_m\|}{\lim \|x_n\|}\right\}$$

where the infimum is taken over all weakly null sequences in B_X such that $\lim_{n,m;n\neq m} \|x_n - x_m\|$ and $\lim \|x_n\|$ exist.

Theorem 6 *Let X be a Banach space and assume that for some $a \geq 0$ we have $R(a, X) < 1 + a$. Then X has the weak fixed point property.*

Proof. Assume that X fails to have the f.p.p. Then we can find a weakly compact and convex subset K of X such that $\operatorname{diam}(K) = 1$ and K is minimal invariant for a nonexpansive mapping T which has no fixed point and we can also find a weakly null approximated fixed point sequence $\{x_n\}$ of T in K. We consider the set

$$[W] = \{[z_n] \in [K] : \|[z_n] - [x_n]\| \leq 1 - t$$

and

$$\limsup_n \limsup_m \|z_n - z_m\| \leq t\}$$

where $t = 1/(1 + a)$. It is easy to check that $[W]$ is a closed,convex and $[T]$-invariant set. Furthermore $[W]$ is non-empty because it contains $[tx_n]$. Therefore, from Lin's Lemma [Li] we know that

$$\sup\{\|[w_n] - x\| : [w_n] \in [W]\} = 1$$

for every $x \in K$. We take $[z_n] \in [W]$ and choose a weakly converging subsequence (say to y) $\{y_n\}$ of $\{z_n\}$ such that $\limsup \|z_n\| = \lim \|y_n\|$ and both and $\lim \|y_n - y\|$ exist. In this way we have

$$\lim_{n,m;n\neq m} \| y_n - y_m \| = \limsup_n \limsup_m \| y_n - y_m \| \leq$$

$$\limsup_n \limsup_m \| Z_n - Z_m \| \leq t$$

and for every $n \in N$

$$\|y_n - y\| \leq \liminf_m \|y_n - y_m\|$$

which implies $\lim_n \|y_n - y\| \leq t$.

Since $R(a, X) < 1 + a$ there exists $b \in (0,1)$ such that $S(a, b, X) < 1 + a$. We choose η such that $\eta S(a, b, X) < 1 - S(a, b, X)/(1 + a)$ and $\eta < t(1 - b)/b$. Assume $\lim \|y_n - y\| < b(t + \eta)$. Thus

$$\limsup \|x_n - y\| \leq \limsup \|x_n - y_n\| + \lim \|y_n - y\| \leq 1 - t + b(t + \eta) < 1$$

which is a contradiction bearing Karlovitz's lemma (see [GK], for instance). Thus we can asume $\lim \|y_n - y\| \geq b(t + \eta)$. For a large enough n we have $\|y_n - y\| \leq t + \eta$. Furthermore $\|y\| \leq \liminf \|y_n - x_n\| \leq 1 - t$. Hence

$$\left\| \frac{y_n}{t + \eta} \right\| = \left\| \frac{y_n - y}{t + \eta} + \frac{y}{t + \eta} \right\| \leq S\left(\frac{1 - t}{t}, b, X\right) = S(a, b, X).$$

Thus $\limsup \|z_n\| = \lim \|y_n\| \leq S(a, b, X)(t + \eta) < 1$ which is contradiction with Lin's Lemma.

Definition 2 *Let X be a Banach space. We define the coefficient $M(X)$ as*

$$\sup \left\{ \frac{1 + a}{R(a, X)} : a \geq 0 \right\}.$$

The following result is a direct consequence of Theorem 6

Theorem 7 *Let X be a Banach space. If $M(X) > 1$ then X has the w.f.p.p.*

Bearing in mind the definition of k-US spaces we can define the following modulus for this property:

$$\beta_X^k(t) = \sup_{x \in S_X} \sup_{dim\, V = k} \inf_{y \in S_V} \{\frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1\}$$

It is clear that X is k-US if and only if $\lim_{t\to 0}\beta(t)/t = 0$.

Theorem 8 *Let X be a Banach space and denote*

$$\beta = \inf\left\{1 + \beta_X^k(s) - \frac{s}{2k} : s \in [0,1]\right\}.$$

Then $M(X) > (1+2k)/(1+2k\beta)$. In particular X has the fixed point property if $\lim_{t\to 0}\beta(t)/t < 1/2k$.

Proof. Assume $a \leq 2k$ and $b \in (0, 1/WCS(X))$. An arbitrary positive number η can be chosen such that

$$1 - \frac{t}{r} + \beta_X^k\left(\frac{2kt}{r}\right) < \beta + \eta.$$

Let $\{x_n\}$ be a weakly null sequence in B_X and $x \in X$ such that $\|x\| = r \leq a$, $\lim\|x_n + x\|$ exists and $S(a,b,X) < \lim\|x + x_n\| + \eta/2$. We can assume that $S(a,b,X) < \|x + x_n\| + \eta$ for every n. Since $\lim\|x_n\| \geq b > 0$, we can assume that $\{x_n\}$ is a basic sequence. We choose $x_n^\star \in S_{X^\star}$ such that $x_n^\star(x + tx_n) = \|x + tx_n\|$. Taking subsequences we can also asume $|x_n^\star(x_m)| \leq \delta$ if $n \neq m$. Writing $y_n = (x_{2n} - x_{2n+1})/2$ we know that the vectors $y_n, n = 1, ..., k$ are linearly independent. Therefore a normalized vector $y = \sum_{n=1}^k \alpha_n y_n$ exists such that $(1/2)(\|(x + 2tky)/r\| + \|(x - 2tky)/r\|) \leq 1 + \beta_X^k(2tk/r) + \eta$. Assume $|\alpha_m| = \max\{|\alpha_n| : n = 1, ..., k\}$. It is clear that $\alpha_m \geq 1/k$. Therefore we have

$$S(a,b,X) - \eta \leq \frac{1}{2}(\|x + x_{2m}\| + \|x + x_{2m+1}\|) \leq$$

$$\frac{1}{2}(\|x + tx_{2m}\| + \|x + tx_{2m+1}\|) + (1-t) =$$

$$\frac{1}{2}r\left(\left\|\frac{x}{r} + \frac{tx_{2m}}{r}\right\| + \left\|\frac{x}{r} + \frac{tx_{2m+1}}{r}\right\|\right) + (1-t) =$$

$$\frac{1}{2}r\left[x_{2m}^\star\left(\frac{x}{r} + \frac{tx_{2m}}{r}\right) + x_{2m+1}^\star\left(\frac{x}{r} + \frac{tx_{2m+1}}{r}\right)\right] + (1-t) \leq$$

$$\frac{1}{2}r\left[x_{2m}^\star\left(\frac{x}{r}\right) + x_{2m}^\star\left(\frac{2ty}{r\alpha_m}\right) + x_{2m+1}^\star\left(\frac{x}{r}\right) - x_{2m+1}^\star\left(\frac{2ty}{r\alpha_m}\right)\right]$$

$$+2\eta\sum_{n=1}^{k}\frac{|\alpha_n|}{|\alpha_m|}+(1-t)\le\frac{1}{2}r\left(\left\|\frac{x}{r}+\frac{2ty}{r\alpha_m}\right\|+\left\|\frac{x}{r}-\frac{2ty}{r\alpha_m}\right\|\right)+(1-t)+2\eta k\le$$

$$r\left(1+\beta_X^k\left(\frac{2tk}{r}y\right)-\frac{t}{r}\right)+\eta(2k+1)+1\le r\beta+1+2\eta(k+1+r).$$

Letting $\eta\to 0$, we obtain $R(a,X)\le 1+\beta a$. If $a\ge 2k$ we have

$$\|x+x_n\|\le(r-2k)+\left\|\frac{2kx}{r}+x_n\right\|.$$

Applying the above argument for the sequence $2kx/r+x_n$ we have $R(a,X)\le(a-2k)+1+2k\beta=a-(2k-1)+2k\beta$. For $a=2k$ we obtain $M(X)\ge(1+2k)/(1+2k\beta)$.

We see that the above technique solves a problem which was several years open. The introduction of the coefficient $M(X)$ raises up a new open problem: Is there any reflexive Banach space such that $M(X)=1$? A negative answer to this question would solve the basic problem in metric fixed point theory. Unfortunately, it is not difficult to find a reflexive Banach space such that $M(X)=1$

Example 2 Denote $\ell_{p,\infty}$ the Bynum's space defined in Example 1 and let $\{p_n\}$ be a sequence in $(1,\infty)$ converging to 1. Consider the reflexive Banach space

$$X=\{(x_n)\in\Pi_{n=1}^{\infty}\ell_{p_n,\infty}:\sum_{n=1}^{\infty}\|x_n\|_{p_n,\infty}^2<\infty\}$$

with the norm $\|(x_n)\|=\sqrt{\sum_{n=1}^{\infty}\|x_n\|_{p_n,\infty}^2<\infty\}}$. Since $M(\ell_{p,\infty})=2^{1-1/p}$ (see [DB2]) and $\ell_{p_n,\infty}\subset X$ for every $n\in N$ we have

$$M(X)\le\inf\{M(\ell_{p_n,\infty};n\in N\}=\inf\{2^{1-1/p_n}:n\in N\}=1.$$

2 Computation of $N(X)$ and related problems

Although the coefficient $N(\ell_2)$ was calculated by Bynum [By] the value of $N(\ell_p)$ and $N(L_p)$ was a problem ten years open. In 1990 the value of these coefficients was obtained [Pr1, DB1] using some convexity inequalities derived from the interpolation theory. To simplify the problem we shall start with a lemma by D. Amir

Lemma 1 [Am]. *Let X be a reflexive Banach space. Then*

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r(A)} : A \subset X \text{ finite}\right\}.$$

A second simplification can be made. We shall prove in the following lemma that we can be restricted to consider finite set whose points are all equidistant from a Chebyshev center.

Lemma 2 *Let X be a Banach space and A finite subset of X. Then there exists a subset B of A such that*

(i) $r(B) \geq r(A)$.

(ii) $\|b - x\| = r(B)$ *for every $x \in B$ where b is a Chebyshev center of B.*

Proof. Since A is finite, co A lies in a finite dimensional space. Thus $Z(A) \neq \emptyset$ and the same occurs for any subset of A. Let y_0 be a Chebyshev center of A and define the set $A_1 = \{x \in A : \|x - y_0\| = r(A)\}$. We shall prove that $r(A_1) \geq r(A)$. Indeed, otherwise choose a positive real number ϵ such that $\|x - y_0\| + \epsilon < r(A)$ for every $x \in A \setminus A_1$. Let y_1 be a Chebyshev center of A_1 and λ a real number, $0 < \lambda < 1$, such that $\lambda\|y_0 - y_1\| < \epsilon/2$. For every $x \in A_1$ we have

$$\|x - y_0 + \lambda(y_0 - y_1)\| \leq \lambda\|x - y_1\| + (1 - \lambda)\|x - y_0\| \leq$$

$$\lambda r(A_1) + (1 - \lambda) r(A) < r(A)$$

If x belongs to $A \setminus A_1$ one has

$$\|x - y_0 + \lambda(y_0 - y_1)\| \leq \|x - y_0\| + \lambda\|y_0 - y_1\| < r(A) - \epsilon/2 < r(A)$$

Thus $\|x - y_0 + \lambda(y_0 - y_1)\| < r(A)$ for every $x \in A$, contradicting the minimality of y_0 because $y_0 + \lambda(y_1 - y_0)$ belongs to co A.

Since A is a finite set, there exists B which is minimal in the family of those non-empty subsets of A which satisfy (i). Such B must satisfy (ii) because otherwise the above argument lets us construct $B_1 \subset B$, $B_1 \neq B$ such that B_1 satisfies (i).

Finally we recall some convexity inequalities [WW]

Lemma 3 *Let (Ω, μ) be a σ-finite measure space, $1 < p < +\infty$, $x_1, x_2, ..., x_n$ vectors in $L^p(\Omega))$ and $t_1, t_2, ..., t_n$ nonnegative numbers such that $\sum_{j=1}^n t_j = 1$. Put $\gamma = \max\{1 - t_j : 1 \le j \le n\}$. Then the following inequality holds:*

$$\gamma^{\alpha-2} \sum_{j,k=1}^{n} t_j t_k \|x_j - x_k\|^\alpha \ge 2 \sum_{j=1}^{n} t_j \|x_j - \sum_{k=1}^{n} t_k x_k\|^\alpha$$

where $\alpha = p$ if $2 \le p < +\infty$ and $\alpha = \frac{p}{p-1}$ if $1 < p \le 2$.

Using these lemmas it is not difficult to compute the values of $N(L^p(\Omega))$ (see [DB1]):

Theorem 9 *Let (Ω, μ) be a σ-finite measure space, $1 \le p < +\infty$ and assume that $L^p(\Omega)$ is infinite dimensional. Then $N(L^p(\Omega)) = \min\{2^{1-1/p}, 2^{1/p}\}$.*

Remark It is clear from lemma 1 that the normal structure coefficient of a reflexive Banach space is determined by the finite subsets of the space. Since ℓ^p is isometrically embedded in L^p and L^p is finitely represented in ℓ^p , the coefficients $N(L^p)$ and $N(\ell^p)$ must be equal, as checked in Theorem 9. These considerations are also useful to obtain an upper bound for $N(X)$:

Corollary *Let X be an infinite dimensional Banach space. Then*

$$N(X) \le \sqrt{2}.$$

Proof. Since ℓ_2 is finitely representable in every infinite dimensional Banach space X we obtain from lemma 1 that $N(X) \le N(\ell_2) = \sqrt{2}$.

What is the situation for finite-dimensional Banach space? Is it also $\sqrt{2}$ the maximum value for $N(X)$? What is the value of ℓ_p^n? The argument and techniques used to compute $N(\ell_p)$ can also be applied to obtain lower bound for $N(\ell_p^n)$. The computation of $N(X)$ for finite dimensional space was iniciated by Jung [J] in 1901, obtaining the value of $N(\ell_2^n)$. Bearing in mind that we only need to consider finite subsets of X to compute $N(X)$ and that in n-dimensional spaces every point that is in convex hull of a set formed by m vectors $x_1, ...x_m$ is also in the convex hull of a subset formed by at most $n+1$ vectors we conclude that we only need to consider finite sets of X formed by at most $n+1$ vectors. Our previous results imply that we can also assume that these vectors are equidistant of their Chebyshev center. So the problem can be formulate in the following way: What is the hyperpolyedron (with at most $n+1$ vertices) inscriptible in the unit sphere, containing zero inside, with minimal diameter? Let us study more carefully this problem.

Assume that $A = \{x_1, ...x_N\}, N \leq n+1$ is a set in ℓ_p^n. By traslation we can assume that zero is the Chebyshev center, which is in the convex hull of A and that $\|x_i\| = r(A), i = 1, ..., N$. Applying the convexity inequalities (lemma 3) and noting that $\gamma \geq 1 - 1/N$ we obtain

$$\left(1 - \frac{1}{N}\right)^{\alpha-2} \operatorname{diam} A^\alpha (1 - \sum_{j=1}^{N} t_j^2) \geq 2r(A)^\alpha.$$

Using the Lagrange multiplier theorem it is easy to check that the function $1 - \sum_{j=1}^N t_j^2$ under the restriction $\sum_{j=1}^N t_j = 1$ attains a maximum if $t_j = 1/N$, $j = 1, ..., N$. Thus we have

$$\left(1 - \frac{1}{N}\right)^{\alpha-2} \left(1 - \frac{1}{N}\right) \operatorname{diam} A^\alpha \geq 2r(A)^\alpha$$

that is

$$\frac{\operatorname{diam} A}{r(A)} \geq 2^{1/\alpha} \left(\frac{n+1}{n}\right)^{1-\frac{1}{\alpha}}$$

For $n = p = 2$ this is the exact value of $N(\ell_2^2)$ because $\sqrt{3}$ is the ratio between the diameter of a equilateral triangle and the radius of the circle where the triangle is inscript. It is easy to see that the above bound is also the exact value for ℓ_2^n for every n. What happens if $p \neq 2$? Of course the lower bound only can be attained at any hyperpolyedron if all inequalities in the computation become equality. Thus all t_j must be equal to $1/(n+1)$ and all distances $\|x_j - x_k\|, j \neq k$ must be equal to $\operatorname{diam} A$. In the most simple case, for $n = 2$ this fact means that the equality can only hold for equilateral triangle with vertices in the unit sphere such that the geometric center is the origin. Now this is the question:

Is there any equilateral triangle in ℓ_p^2 satisfying such condition? If this triangle exists, is $2^{2/\alpha-1}3^{1-1/\alpha}$ the length of its side? We shall see an answer for special cases.

Definition 3 *A Hadamard matrix H of order n is an $n \times n$ matrix of +1's and -1's such that $HH^t = nI$, i.e. the inner product of any two distinct rows of H is zero, and the inner product on any row with itself is n.*

It is easy to see that multiplying any row or column by -1 changes H into another Hadamard matrix. By this means we can change the first row and column into =1's. Such a Hadamard matrix is called normalized. It is an open problem to determine the values of n such that a Hadamard matrix of order n exists. It is not difficult to check that such a n must be 1,2 or a multiple of 4. Using quadratic residues it is possible to construct (Paley construction) a Hadamard matrix of any order $n = p+1$ if p is prime and $p+1$ is a multiple of 4. It is conjectured that Hadamard matrices exist whenever the order is a multiple of 4, although this has not yet been proved. A large number of constructions are known, and Hadamard matrix have been constructed for any order that is multiple of 4 less than 268. Hadamard matrix are widely used, for instance, to construct nonlinear codes, maximal determinants, weighing designs, and in communications and physics.

Assume that there exists a Hadamard matrix of order $n+1$ and let

$$\begin{pmatrix} 1 & v_1 \\ 1 & v_2 \\ \vdots & \vdots \\ 1 & v_{n+1} \end{pmatrix}$$

be this matrix, where $v_1, ..., v_{n+1}$ are vectors in R^n. Consider the set $A = \{v_1, ..., v_{n+1}\}$ in ℓ_p^n for $p < 2$. It is clear that

$$\|v_i - v_j\| = 2\left(\frac{n+1}{2}\right)^{1/p}$$

if $i \neq j$ because two distict rows of the Hadamard matrix have $(n+1)/2$ 1's or -1's in the same position. Furthermore $\|x_i\| = n^{1/p}$ for $i = 1, ..., n+1$. Thus

$$\frac{\operatorname{diam} A}{r(A)} = 2^{1/q}\left(\frac{n+1}{n}\right)^{1/p}$$

that is the value corresponding to the above lower bound. Hence, in this case

$$2^{1/q}\left(\frac{n+1}{n}\right)^{1/p}$$

is the exact value of $N(\ell_p^n)$. This is an open problem to calculate $N(\ell_p^n)$ if $p > 2$ or there is not a Hadamard matrix of order $n + 1$.

References

[Al] D.E. Alspach, A fixed point free nonexpansive map, *Proc. Amer. Math. Soc.* **82**, 423-424 (1981).

[Am] D. Amir, On Jung's constant and related constants in normed linear spaces, *Pacific J. Math.* **118** (1), 1-15 (1985).

[Br1] E.F. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, P*Proc. Nat. Acad. Sci. USA***43**, 1272-1276 (1965).

[Br2] E.F. Browder, Nonexpansive nonlinear operators in a Banach space, *Proc. Nat. Acad. Sci. USA***54**, 1041-1044 (1965).

[By] W.L. Bynum, Normal structure coefficients for Banach spaces, *Pacific J. Math.* **86**, 427-426 (1980).

[DB1] T. Domínguez Benavides. Normal structure coefficient in L_p-spaces. *Proc. Royal Soc. Edinburgh* **117A**, 299-303 (1991).

[DB2] T. Domínguez Benavides. A new geometrical constant implying the fixed point property and stability results, *preprint.*

[DL] T. Domínguez Benavides; G. López Acedo, Lower bounds for normal structure coefficients, *Proc. Royal Soc. Edinburgh* **121A**, 245-252, (1992).

[Ga] J. García Falset, The fixed point property in Banach spaces with NUS-property, *preprint.*

[GK] K. Goebel and W.A. Kirk, Topics in Metric Fixed Point Theory, *Cambridge Univ. Press*, 1990.

[J] H.W.E. Jung, Uber die kleinste Kügel, die eine raumliche Figur einschliest, *J. Reine Angew Math.* **123**, 241-257 (1901).

[K] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly* **72**, 1004-1006 (1965).

[Li] P.K. Lin, Unconditional bases and fixed points of nonexpansive mappings *Pacific J. Math.* **116**, 69-76 (1985).

[Ma] E. Maluta. Uniformly normal structure and related coefficients. *Pac. J. Math.* 111(2) (1984), 357-369.

[MP] E. Maluta; S. Prus. Banach spaces which are dual to k-uniformly convex spaces, *preprint*

[M] B. Maurey, Points fixes des contractions de certains faiblement compacts de L^1. **Seminaire d'Analyse Fonctionell,Exposé VIII**, École Polytechnique, Centre de Mathématiques (1980-81).

[Pr1] S. Prus On Bynum's fixed point theorem *Atti. Sem. Mat. Fis. Univ. Modena* **38**, 535-545 (1990).

[Pr2] S. Prus, Some estimates for the normal structure coefficient in Banach spaces *Rend. Circ. Mat. Palermo***XL**, 128-135, (1991).

[Pi] S. A. Pichugov. Jung's constant for the space L_p. Mat. Zametki 43, 348-354 (1988).

[Su] F. Sullivan. A generalization of uniformly rotund Banach spaces, *Can. J. Math.* **31**, 628-636 (1979).

[ST] M.A. Smith; B. Turett. Some examples concerning normal and uniform normal structure in Banach spaces. *J. Austral. Math. Soc.* **48A**, 223-234 (1990).

[WW] J.H.Wells y L.R.Williams,*Embeddings and Extensions in Analysis*, Springer Verlag (Berlin, 1975).

Departamento de Análisis Matemático.
Facultad de Matemáticas.
Universidad de Sevilla.
PO. Box 1.160.
41080-Sevilla.

Recibido: 23 de Mayo de 1996