# SOME GEOMETRIC PROPERTIES OF CONVEX BODIES. II 

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#### Abstract

Topological means are used for the study of approximation of 2-dimensional sections of a 3-dimensional convex body by affine-regular pentagons and approximation of a centrally symmetric convex body by a prism. Also, the problem of estimating the relative surface area of the sphere in a normed 3-space, the problem on universal covers for sets of unit diameter in Euclidean space, and some related questions are considered.


Throughout, by a convex body $K \subset \mathbb{R}^{n}$ (a figure for $n=2$ ) we mean a compact convex subset of $\mathbb{R}^{n}$ with nonempty interior.

We denote by $G_{k}\left(\mathbb{R}^{n}\right)$ (respectively, $G_{k}^{+}\left(\mathbb{R}^{n}\right)$ ) the Grassmann manifold of nonoriented (respectively, oriented) $k$-planes in $\mathbb{R}^{n}$ passing through $O \in \mathbb{R}^{n}$. We let

$$
\gamma_{k}^{n}: E_{k}\left(\mathbb{R}^{n}\right) \rightarrow G_{k}\left(\mathbb{R}^{n}\right) \quad \text { and } \quad\left(\gamma_{k}^{n}\right)^{+}: E_{k}^{+}\left(\mathbb{R}^{n}\right) \rightarrow G_{k}^{+}\left(\mathbb{R}^{n}\right)
$$

be the tautological fiber bundles, where the fiber over an (oriented) $k$-plane $\alpha \in G_{k}\left(\mathbb{R}^{n}\right)$ is $\alpha$ itself regarded as a $k$-dimensional vector space.

We say that a field of convex bodies (or figures; a $C B$ - or a $C F$-field) is given in a vector bundle $\gamma$ if in each fiber $\alpha$ of $\gamma$ we mark a convex body $K(\alpha)$ depending continuously on $\alpha$. A CB-field is pointed if for each $\alpha$ we also mark a point $x(\alpha) \in K(\alpha)$ depending continuously on $\alpha$. (In other words, $x(\alpha)$ is a section of $\gamma$.)

If $\lambda \in \mathbb{R}$ and $P$ is an affine-regular polygon (e.g., a pentagon or a parallelogram) with center $O(P)$, then $\lambda P$ denotes the polygon homothetic to $P$ with homothety ratio $\lambda$ and homothety center $O(P)$.

We denote by $S(K)$ the area of a figure $K \subset \mathbb{R}^{2}$.

> §1. FIELDS OF CONVEX FIGURES IN $\gamma_{2}^{3}$ and $\left(\gamma_{2}^{3}\right)^{+}$,
> AND 2 -DIMENSIONAL SECTIONS OF CONVEX BODIES IN $\mathbb{R}^{3}$

First, we prove two corollaries to the following known result.
Theorem [2]. Each CF-field in $\gamma_{2}^{3}$ contains a figure circumscribed about an affine-regular octagon.

Corollary 1. Suppose $C$ is the bounded component of a cubic surface in $\mathbb{R}^{3}$. Then each inner point $O$ of $C$ lies in a plane intersecting $C$ along an ellipse.

Proof. Indeed, $C$ is convex automatically, because otherwise $C$ intersects some line at 4 points. Consequently, the section of $C$ by some 2-plane through $O$ is circumscribed about an affine-regular octagon. Then this section is a component of a cubic, intersects an ellipse at 8 points, and, consequently, is an ellipse by the Bézout theorem.

[^0]Remark. Certainly, this follows from standard facts of algebraic geometry: any plane that passes through a point inside $C$ and a real line lying on the cubic surface intersects $C$ along an ellipse. But actually we have proved more: each field of bounded components of cubics in $\gamma_{2}^{3}$ contains an ellipse.
Corollary 2. Each $C F$-field in $\gamma_{2}^{3}$ contains a figure $K$ containing a parallelogram $P$ such that

$$
P \subset K \subset\left(1+\frac{\sqrt{2}}{2}\right) P
$$

Proof. Indeed, the figure $K$ circumscribed about an affine-regular octagon $\Omega=A_{1} \ldots A_{8}$ possesses the required property. In this case, $K$ is contained in the octagonal star $\Sigma$ bounded by segments of the rays that extend the sides of $\Omega$. Letting $P$ be the parallelogram $A_{2} A_{4} A_{6} A_{8}$, we easily see that $P \subset K \subset \Sigma \subset\left(1+\frac{\sqrt{2}}{2}\right) P$.

Remark. Considering fields of disks, we see that $1+\frac{\sqrt{2}}{2}$ cannot be replaced by a constant smaller than $\sqrt{2}$.
Theorem 1. Each pointed CF-field $(K(\alpha), x(\alpha))$ in $\left(\gamma_{2}^{3}\right)^{+}$contains a figure $K(\alpha)$ circumscribed about an affine-regular pentagon with center at the marked point $x(\alpha)$.
Corollary 3. Each field of centrally symmetric convex figures in $\left(\gamma_{2}^{3}\right)^{+}$or $\gamma_{2}^{3}$ contains a figure circumscribed about an affine-regular decagon.
Corollary 4. If a field of centrally symmetric convex figures in $\left(\gamma_{2}^{3}\right)^{+}$consists of bounded components of curves of degree at most 4 , then the field contains an ellipse.

Indeed, by the Bézout theorem, the figure circumscribed about an affine-regular decagon is an ellipse.
Corollary 5. Each convex bounded centrally symmetric quartic in $\mathbb{R}^{3}$ has a planar central section that is an ellipse.

The proof of Theorem 1 uses two topological lemmas.
Lemma 1. Suppose $n \in \mathbb{N}$, $W$ is a compact oriented $2 n$-manifold, the cyclic group $\mathbb{Z}_{2 n+1}$ freely acts on $W$, and the boundary $\partial W$ of $W$ with standard orientation is the union of two closed $\mathbb{Z}_{2 n+1}$-invariant manifolds $M$ and $M^{\prime}$. We let $\mathbb{Z}_{2 n+1}$ act on $\mathbb{R}^{2 n+1}$ by cyclic permutations of coordinates of points and denote by $l \subset \mathbb{R}^{2 n+1}$ the line determined by the equations $x_{1}=x_{2}=\cdots=x_{2 n+1}$.

Suppose that $\mathcal{F}: W \rightarrow \mathbb{R}^{2 n+1}$ is a continuous $\mathbb{Z}_{2 n+1}$-equivariant mapping such that $\mathcal{F}(M)$ and $\mathcal{F}\left(M^{\prime}\right)$ do not intersect $l$. Then

$$
\operatorname{deg}\left(\mathcal{F} \mid: M \rightarrow \mathbb{R}^{2 n+1} \backslash l\right)+\operatorname{deg}\left(\mathcal{F} \mid: M^{\prime} \rightarrow \mathbb{R}^{2 n+1} \backslash l\right) \equiv 0 \quad \bmod 2 n+1
$$

Proof. After a slight perturbation, we can assume that $\mathcal{F}$ is smooth and transversal to $l$. Then, $\mathcal{F}^{-1}(l)$ consists of a finite number of orbits of points of $W$. We surround the points by small balls $D_{1}, \ldots, D_{N}$ such that they are mutually disjoint and disjoint from $M$ and $M^{\prime}$ and, furthermore, the balls with centers in one $\mathbb{Z}_{2 n+1}$-orbit are mapped to one another under the action of $\mathbb{Z}_{2 n+1}$.

The mapping $\mathcal{F}$ takes the $2 n$-manifold $\hat{W}:=W \backslash \bigcup_{i=1}^{N} \stackrel{\circ}{D}_{i}$ with boundary to $\mathbb{R}^{2 n+1} \backslash l \simeq$ $S^{2 n-1}$. Consequently, $\operatorname{deg}\left(\mathcal{F} \mid: \partial \hat{W} \rightarrow \mathbb{R}^{2 n+1} \backslash l\right)=0$. We have

$$
\partial \hat{W}=M \cup M^{\prime} \cup \bigcup_{i=1}^{N} \partial D_{i}
$$

By construction, the $\mathbb{Z}_{2 n+1}$-equivariance of $\mathcal{F}$ implies that $\operatorname{deg}\left(\mathcal{F} \mid: \bigcup_{i=1}^{N} \partial D_{i} \rightarrow \mathbb{R}^{2 n+1} \backslash l\right)$ is divisible by $2 n+1$, which completes the proof.

Remark. For prime $2 n+1$, Stiefel manifolds yield interesting examples in which the degree under consideration is nonzero (see [1] and below). The author does not know whether this is possible in the case where $2 n+1$ is composite.

We denote by $S O(3)$ the group of orientation-preserving rotations of $\mathbb{R}^{3}$ about some point.

Lemma 2. Suppose $W$ is a compact 4-manifold, the cyclic group $\mathbb{Z}_{5}$ acts freely on $W$, and the boundary $\partial W$ of $W$ with standard orientation is the union of two closed $\mathbb{Z}_{5}$ invariant 3-manifolds $M$ and $M^{\prime}$, where $M^{\prime}$ is $S O(3)$ with standard action of $\mathbb{Z}_{5}$. We let $\mathbb{Z}_{5}$ act on $\mathbb{R}^{5}$ by cyclic permutations of the coordinates of points.

Suppose that $\mathcal{F}: W \rightarrow \mathbb{R}^{5}$ is a continuous $\mathbb{Z}_{5}$-equivariant mapping. Then $\mathcal{F}(M)$ contains a point of the form $(x, x, x, x, y)$, where $y \leq x$ (or, optionally, $y \geq x$ ).

Proof. Let $l \subset \mathbb{R}^{5}$ be the line determined by the equations $x_{1}=x_{2}=\cdots=x_{5}$. If $F(M) \cap l \neq \varnothing$, then we are done. Otherwise, it suffices to prove the following assertion (see [1]).

Assertion. If $F(M) \cap l=\varnothing$, then the degree of the mapping

$$
F \mid: M \rightarrow \mathbb{R}^{5} \backslash l \simeq S^{3}
$$

is not divisible by 5 .
Proof. After a slight perturbation, we can assume that the mapping $F^{\prime}: M^{\prime} \rightarrow \mathbb{R}^{5}$ is a $\mathbb{Z}_{5}$-equivariant mapping with image in $\mathbb{R}^{5} \backslash l$. It is well known that in this case $\operatorname{deg}\left(F^{\prime} \mid: M^{\prime} \rightarrow \mathbb{R}^{5} \backslash l\right.$ ) is not divisible by 5 (see [1] actually, $\operatorname{deg}\left(F^{\prime} \mid\right)=-1$ ). Now, the required result follows from Lemma 1.

Proof of Theorem 1. We consider positively oriented affine-regular pentagons inscribed in the figures $K(\alpha)$. If $K(\alpha)$ is a generic smooth field of smooth convex figures in $\left(\gamma_{2}^{3}\right)^{+}$, then the pentagons constitute a compact oriented smooth 3-manifold $M$, on which the cyclic group $\mathbb{Z}_{5}$ acts by cyclic permutations of the vertices of the pentagons.

If all figures $K(\alpha)$ are ( $C^{1}$-close to) disks, then, obviously, $M \cong S O(3)$.
We define a continuous mapping

$$
F: M \rightarrow \mathbb{R}^{5}
$$

as follows. If $P \subset \alpha$ is a pentagon $A_{1} \ldots A_{5}$ inscribed in $K(\alpha)$ and with center $O(P)$, then the $i$ th coordinate of $F(P)$ is the orthogonal projection of $x(\alpha)$ to the oriented axis $O(P) A_{i}$ with origin at $O(P), i=1, \ldots, 5$.

By construction, $F$ is $\mathbb{Z}_{5}$-equivariant if $\mathbb{Z}_{5}$ acts on $\mathbb{R}^{5}$ by cyclic permutations of the coordinates of points.

Using a smooth generic deformation $K_{t}(\alpha), t \in[0,1]$, we deform the initial field $K_{0}(\alpha):=K(\alpha)$ into a field of figures $K_{1}(\alpha)$ close to disks. Then the oriented affineregular pentagons inscribed in the figures $K_{t}(\alpha)$ form a cobordism between $M$ and $M^{\prime} \cong$ $S O(3)$. By Lemma 2, $F(M)$ contains a point $F(P)=(x, x, x, x, y)$. Simple geometric arguments show that $x=0$, whence $O(P)=x(\alpha)$.

Theorem 2. Each CF-field $K(\alpha)$ in $\left(\gamma_{2}^{3}\right)^{+}$contains a convex figure $K$ circumscribed about an affine-regular pentagon $P$ and such that

$$
\begin{equation*}
P \subset K \subset\left(1+\frac{1}{2 \sin 54^{\circ}}\right) P \subset 1.6181 P \tag{*}
\end{equation*}
$$

This estimate is sharp for the field consisting of the sections of a tetrahedron $T$ that pass through an inner point of $T$.


Figure 1.
Proof. If $P$ is a pentagon $A_{1} \ldots A_{5}$ inscribed in $K(\alpha)$, then we draw the support lines of $K(\alpha)$ parallel to $A_{1} A_{2}, A_{2} A_{3}, \ldots, A_{5} A_{1}$ and denote by $B_{1}, \ldots, B_{5}$ the points where these support lines touch $K(\alpha)$. We define

$$
F(P):=\left(S\left(\triangle A_{1} A_{2} B_{1}\right), \ldots, S\left(\triangle A_{5} A_{1} B_{5}\right)\right)
$$

As in the proof of Theorem 1, we see that there is a 2-plane $\alpha$ such that for a certain pentagon $P$ inscribed in $K(\alpha)$ we have

$$
f(P)=(x, x, x, x, y)
$$

where $y \leq x$. We show that $P$ is the required pentagon.
After an affine transformation, we may assume that $P=A_{1} \ldots A_{5}$ is a regular pentagon. Obviously, the altitude of the triangles $\triangle A_{1} A_{2} B_{1}, \ldots, \triangle A_{4} A_{5} B_{4}$ is maximal possible if $S\left(\triangle A_{5} A_{1} B_{5}\right)=0$, and the figure $K(\alpha)$ circumscribed about $P$ is the isosceles trapezoid shown in Figure 1.

Simple calculations show that in this case we have $(*)$.
Remark. From [11] it follows that each CF-field in $\left(\gamma_{2}^{3}\right)^{+}$contains a figure $K$ circumscribed about a regular pentagon $P$, in which case we have

$$
P \subset K \subset(1+\sqrt{5}) P
$$

(Certainly, the same is true for any affine-regular pentagon inscribed in $K$.) In the general case, the constant $1+\sqrt{5}=1+\tan (\pi / 5) \cot (\pi / 10)$ here cannot be made smaller. Indeed, suppose that the point $O$ lies near the apex of a regular triangular pyramid $T$ the lateral edge of which is many times longer than the edge of the base. We consider the CFfield in $\left(\gamma_{2}^{3}\right)^{+}$consisting of sections of $T$ by planes through $O$. In this case, the sections intersecting the base of $T$ are very prolate, while for the other triangular sections the above constant, obviously, cannot be improved, because two sides of a pentagon inscribed into a triangle lie on the sides of the triangle.

If in Theorem 2 and in the situation considered above we lift the condition that the pentagon $P$ is inscribed, then the sharp values of the constants are not known to the author.

## §2. The relative surface area of the sphere in a normed 3-space

Definition. Suppose $P$ is a polyhedron in a finite-dimensional normed space with unit ball $K$. For each hyperface $F$ of $P$, we take the ratio of the area of $F$ and the area of the central section of $K$ parallel to $F$. The sum of these ratios over all hyperfaces of $P$ is the relative surface area of $P$.

In [10], the author constructed a one-parameter family of affine images of a cubeoctahedron which are inscribed in the unit ball of a normed 3 -space and have relative surface area of at least $5 / 2$.


Figure 2.

To obtain an upper estimate, we approximate the ball of unit diameter in a normed 3 -space by a circumscribed hexagonal prism.

We need the following result.
Theorem [10]. Suppose $\Pi=A_{1} \ldots A_{12}$ is a regular hexagonal prism, and $A_{13}$ and $A_{14}$ are points that lie outside $\Pi$ on the symmetry axis and are symmetric to each other with respect to the center. Each centrally symmetric convex body $K \subset \mathbb{R}^{3}$ centered at $O$ is circumscribed about an affine image of the 14 -tope $P_{14}=A_{1} \ldots A_{14}$ with the same center and such that the parallel support planes of $K$ at the images of $A_{13}$ and $A_{14}$ are parallel to the images of the base planes of $\Pi$.

Corollary 6. Each centrally symmetric convex body $K \subset \mathbb{R}^{3}$ is circumscribed about an affine-regular hexagonal bipyramid $A_{1} \ldots A_{8}$ with the same center and such that the support planes of $K$ at $A_{7}$ and $A_{8}$ are parallel to the plane of the base $A_{1} \ldots A_{6}$, while the support planes of $K$ at $A_{1}, \ldots, A_{6}$ are parallel to the axis $A_{7} A_{8}$ of the bipyramid.

Proof. We apply the above theorem. As the lateral edge of $\Pi$ in $P_{14}$ tends to zero while the length of $A_{13} A_{14}$ remains constant, in the limit we obtain the required bipyramid.

Theorem 3. The unit ball $K$ in a normed 3 -space is inscribed in a centrally symmetric hexagonal prism with relative surface area not exceeding $\frac{32}{3}$.

Corollary 7. The relative surface area of $K$ is at most $32 / 3$.
Remark. If $K$ is a Euclidean ball, then the regular prism is a hexagonal prism circumscribed about $K$ with minimal relative surface area, which is equal to $12 \sqrt{3} / \pi$.

The proof involves the following lemma.
Lemma 2. Suppose that the unit disk in a 2-dimensional normed space is an affineregular hexagon $H$. Then the area of any centrally symmetric convex figure $K$ circumscribed about $H$ is at most $\frac{4}{3} S(H)$, and the perimeter of $K$ (with respect to the norm) is at most 8. Both estimates are sharp.

Proof. Drawing the support lines of $K$ at the vertices of $H$, we reduce the proof to the case where $K$ is a centrally symmetric hexagon.

1) Simple variational arguments show that the area of $K$ is maximal if the vertices of $H$ are the midpoints of the sides of $K$, or $K$ is a parallelogram. In both cases, we have $S(K)=\frac{4}{3} S(H)$.
2) Let $E D$ and $E^{\prime} D^{\prime}$ be a pair of parallel sides of $K$, and let $C C^{\prime}$ be the diameter of $H$ parallel to them (see Figure 2). Continuing the opposite sides $A B$ and $A^{\prime} B^{\prime}$ that contain (respectively) $C$ and $C^{\prime}$, to the intersection with the lines $E D$ and $E^{\prime} D^{\prime}$, we obtain a
parallelogram $F G F^{\prime} G^{\prime}$ circumscribed about $H$. Obviously,

$$
\frac{E D+E^{\prime} D^{\prime}}{O C}=8 \frac{S(O E D)+S\left(O E^{\prime} D^{\prime}\right)}{S\left(F G F^{\prime} G^{\prime}\right)} \leq 8 \frac{S(O E D)+S\left(O E^{\prime} D^{\prime}\right)}{S(K)}
$$

Writing this inequality for each pair of parallel sides of $K$ and summing up, we obtain the assertion of the lemma concerning the perimeter of $K$.

Remark. If in the lemma we do not assume the central symmetry of $K$, then it is well known that $S(K) \leq \frac{3}{2} S(H)$, and arguments similar to ours show that the perimeter of $K$ is at most 9 .

Proof of Theorem 3. We assume that $K$ is smooth. In the general case, the result is obtained by passage to the limit.

Let $A_{1} \ldots A_{8}$ be the affine-regular hexagonal bipyramid constructed in Corollary 6. Obviously, the support planes of $K$ at $A_{1}, \ldots, A_{8}$ bound a hexagonal prism $\Pi$ with centrally symmetric base.

1) The relative areas of the bases of $\Pi$ are at most $4 / 3$ by Lemma 2, because they are circumscribed about the affine-regular hexagon $A_{1} \ldots A_{6}$ with unit side, which is inscribed in the central section of $K$ parallel to the bases.
2) Suppose $P$ is one of the lateral faces of $\Pi$. Then the area of the parallelogram $P$ does not exceed the length of the base of $P$. (Indeed, the area of the central section of $K$ parallel to $P$ cannot be smaller than the area of the parallelogram $P^{\prime}$ having the same directions of sides and such that the lateral side of $P^{\prime}$ is equal to that of $P$, while the base of $P$ is a unit radius of the central section.)

Thus, by Lemma 2, the relative area of the lateral surface of $\Pi$ is at most 8 , and, consequently, the complete relative surface area of $\Pi$ is at most $8+2 \cdot \frac{4}{3}=\frac{32}{3}$.

Theorem 4. Each convex body $K \subset \mathbb{R}^{3}$ is circumscribed about an affine-regular hexagonal bipyramid $A_{1} \ldots A_{8}$ such that the support planes of $K$ at $A_{7}$ and $A_{8}$ are parallel to the plane of the base $A_{1} \ldots A_{6}$.

Proof. We prove this theorem for strictly convex smooth bodies $K$; in the general case, the theorem is obtained by passage to the limit.

For $\alpha \in G_{2}\left(\mathbb{R}^{3}\right)$, we draw the support planes $\alpha_{1}$ and $\alpha_{2}$ of $K$ parallel to $\alpha$, and also the secant plane $\alpha_{3}$ equidistant from $\alpha_{1}$ and $\alpha_{2}$ and parallel to them. We join the points of tangency of $\alpha_{1}$ and $\alpha_{2}$ with $K$ by a segment $I$ and denote by $A(\alpha)$ the orthogonal projection of the point $I \cap \alpha_{3}$ to the plane $\alpha$. Let $B(\alpha)$ be the orthogonal projection to $\alpha$ of the set of the centers of the affine-regular hexagons inscribed in $\alpha_{3} \cap K$.

It suffices to prove that for some $\alpha \in G_{2}\left(\mathbb{R}^{3}\right)$ we have $A(\alpha) \in B(\alpha)$. By construction,

$$
C_{1}:=\left\{A(\alpha) \mid \alpha \in G_{2}\left(\mathbb{R}^{3}\right)\right\}
$$

is the image of a section of $\gamma_{2}^{3}$ that realizes the generator of $H_{2}\left(E_{2}\left(\mathbb{R}^{3}\right) ; \mathbb{Z}_{2}\right)$.
We denote by $\Omega$ the 8 -manifold of affine-regular hexagons lying in the planes $\alpha_{3}$. If $K$ is generic, then the affine-regular hexagons inscribed into all possible sections $\alpha_{3} \cap K$ constitute a compact smooth 2 -manifold $\mathcal{H}$ in $\Omega$. Then

$$
C_{2}:=\bigcup\left\{B(\alpha) \mid \alpha \in G_{2}\left(\mathbb{R}^{3}\right)\right\}
$$

is a continuous image of $\mathcal{H}$ that intersects a generic fiber $\alpha \in E_{2}\left(\mathbb{R}^{3}\right)$ at an odd number of points, because a generic convex figure is circumscribed about an odd number of affine-regular hexagons (see [2]). Thus, $C_{2}$ also realizes the generator of $H_{2}\left(E_{2}\left(\mathbb{R}^{3}\right) ; \mathbb{Z}_{2}\right)$. Consequently, the $\mathbb{Z}_{2}$ intersection number of $C_{1}$ and $C_{2}$ is equal to 1 , whence $C_{1} \cap C_{2} \neq$ $\varnothing$.

Remark. It is easily seen that the volume of any inscribed affine-regular bipyramid in Theorem 4 is at least $\operatorname{Vol}(K) / 6$.

## §3. Universal covers for sets of unit diameter in Euclidean space

Definitions and examples. A subset $A \subset \mathbb{R}^{n}$ is a universal cover for sets of unit diameter if each subset of $\mathbb{R}^{n}$ of diameter not exceeding 1 is contained in a congruent image of $A$.

A universal cover $A$ is rigid if the set of bodies of constant unit width and such that each of them is contained in finitely many congruent images of $A$ is dense in the Hausdorff metric.

Rigid covers are well known in dimensions 2 and 3. Each centrally symmetric hexagon of unit width is a rigid cover in dimension 2. Borsuk's solution of the Borsuk problem in the plane involved the cover having the form of a regular hexagon of unit width (see [3]).

A regular rhombo-dodecahedron of unit width is a rigid cover in 3 -space (see [47]). The one-parameter families of rigid covers constructed in 7, 8 consist of centrally symmetric dodecahedra circumscribed about a ball of unit diameter. Is it true that each universal cover contains a rigid universal cover?

A universal cover $A$ is an $s$-cover if there is a $C^{1}$ open and dense set of smooth bodies having constant unit width and such that each of them is contained in an odd number of congruent images of $A$.

By definition, all $s$-covers are rigid. All rigid covers mentioned above are $s$-covers. For $n \geq 3$, the author knows no examples of rigid covers that are not $s$-covers.

The following theorem yields infinite series of $s$-covers in Euclidean spaces.
Theorem 5. Suppose that $A_{n} \subset \mathbb{R}^{n}$ is a centrally and mirror-symmetric s-cover bounded by a finite number of regular hypersurfaces. Let $\Pi$ be the right prism in $\mathbb{R}^{n+1}$ with base $A_{n}$ and unit hight, and let $A_{n+1}$ denote the intersection of $\Pi$ with two unit balls centered at the centers of the bases of $\Pi$. Then $A_{n+1}$ is an s-cover in $\mathbb{R}^{n+1}$.

Proof. Suppose that $K \subset \mathbb{R}^{n+1}$ is a generic smooth body of constant unit width. In the total space $E_{n}\left(\mathbb{R}^{n+1}\right)$ of $\gamma_{n}^{n+1}$, we construct two $n$-dimensional cycles $C_{1}$ and $C_{2}$ intersecting the generic fiber at an odd number of points.

For $\alpha \in G_{n}\left(\mathbb{R}^{n+1}\right)$, we denote by $s(\alpha)$ the point of intersection of $\alpha$ with the line containing the diameter of $K$ perpendicular to $\alpha$. Obviously, $s$ is a section of $\gamma_{n}^{n+1}$, and its image $C_{1}=s\left(G_{n}\left(\mathbb{R}^{n+1}\right)\right)$ intersects each fiber at a unique point.

Consider the fiber bundle $\xi: E(\xi) \rightarrow G_{n}\left(\mathbb{R}^{n+1}\right)$ such that the fiber over a hyperplane $\alpha \in G_{n}\left(\mathbb{R}^{n+1}\right)$ is the set of the congruent images of $A_{n}$ that lie in $\alpha$. (By the mirror symmetry of $A_{n}$, no problems with orientation arise.) For $\alpha \in G_{n}\left(\mathbb{R}^{n+1}\right)$, let $B_{\alpha}$ be the set of the congruent images of $A_{n}$ that contain the orthogonal projection of $K$ to $\alpha$, and let $B=\bigcup_{\alpha} B_{\alpha} \subset E(\xi)$. If $K$ is a generic smooth body, then $B$ is a smooth compact n-manifold in $E(\xi)$ intersecting the generic fiber at an odd number of points.

We consider the fiberwise mapping $p: E(\xi) \rightarrow E_{n}\left(\mathbb{R}^{n+1}\right)$ that takes the congruent image of $A$ lying in $\alpha$ to its center in $\alpha$. Then $C_{2}=p(B)$ is an $n$-dimensional cycle in $E_{n}\left(\mathbb{R}^{n+1}\right)$ intersecting the generic fiber at an odd number of points. Consequently, $C_{2}$ is $\mathbb{Z}_{2}$-homologous to $C_{1}$.

As before, the $\mathbb{Z}_{2}$ intersection number of $C_{1}$ and $C_{2}$ in $E_{n}\left(\mathbb{R}^{n+1}\right)$ is nonzero, i.e., in the generic situation they intersect at an odd number of points, which precisely means that the initial body $K$ is contained in an odd number of congruent images of $A_{n+1}$.

Corollary 8 (to the proof). Suppose $A_{n} \subset \mathbb{R}^{n}$ is a centrally and mirror-symmetric $s$-cover bounded by a finite number of regular hypersurfaces. Then each pointed $C B$ field $(K(\alpha), x(\alpha))$ of constant unit width in $\gamma_{n}^{n+1}$ contains a body $K(\alpha)$ that is contained
in a congruent image of $A_{n}$ centered at $x(\alpha)$. If $K(\alpha)$ is a generic field consisting of smooth convex bodies of constant unit width, then the number of such fibers (and covers) is odd.

Remarks. 1. The $s$-cover $A_{n+1}$ itself satisfies the assumptions of Theorem 5, which allows us to use it for constructing an $s$-cover $A_{n+2} \subset \mathbb{R}^{n+2}$, etc.
2. If we take one of the 1 -, 2 -, or 3 -dimensional covers mentioned above as a "basis" cover, then Theorem 5 yields an infinite series of $s$-covers that are intersections of halfspaces, cylinders, and spheres of unit diameter.
3. In the text above, we presented polygonal and polyhedral $s$-covers in dimensions $\leq 3$. The author does not know of any polyhedral $s$-covers in dimensions $4,5, \ldots$.

In [9], it was proved that each centrally symmetric 14-hedron $P$ circumscribed about a ball of unit diameter is a universal cover. However, these covers are certainly not rigid, because for each $P$ and each body $K$ of constant unit width in $\mathbb{R}^{4}$ there is a 3-parametric family of congruent images of $P$ each of which is circumscribed about $K$.

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