Some global properties of mixed super quasi-Einstein manifolds

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Abstract. Within the framework of mixed super quasi-Einstein manifolds, are proved three theorems: the first one shows that the compact orientable manifolds MS(QE)$_n$ ($n \geq 3$) without boundary do not admit non-isometric conformal vector fields, the second one provides a sufficiency condition for non-existence of nontrivial Killing vector fields, and the last one characterizes the harmonic vector fields under certain assumptions.

M.S.C. 2000: 53C25.
Key words: quasi-Einstein manifolds conformal vector fields, harmonic vector fields, Killing vector fields.

1 Introduction

The notion of quasi-Einstein manifold and generalized quasi-Einstein manifold were introduced in [7] and [9].

In [8], Chaki introduced super quasi-Einstein manifold, denoted by $S(QE)_n$, where the Ricci-tensor S of type $(0,2)$ which is not identically zero satisfies the condition

$$\begin{aligned} S(X,Y) = \quad & ag(X,Y) + bA(X)A(Y) + \\ & + c[A(X)B(Y) + A(Y)B(X)] + dD(X,Y), \end{aligned} \tag{1.1}$$

where a, b, c and d are scalars such that b, c, d are nonzero, A, B are two nonzero 1-forms

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \tag{1.2}$$

U and V being unit vectors which are orthogonal, i.e.,

$$g(U,V) = 0. \tag{1.3}$$

and D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \ \ \forall X \in \mathcal{X}(M). \tag{1.4}$$

Differential Geometry - Dynamical Systems, Vol.11, 2009, pp. 105-111.

Here a, b, c, d are called the associated scalars, A, B are called the associated main and auxiliary 1-forms respectively, U, V are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

In [6], A. Bhattacharyya and T. De introduced the notion of mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold is called *mixed generalized quasi-Einstein manifold* if its Ricci tensor is non-zero and satisfies the condition

$$\begin{aligned} S(X,Y) = \ & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)+ \\ & +d[A(X)B(Y) + B(X)A(Y)], \end{aligned} \tag{1.5}$$

where a, b, c, d are non-zero scalars,

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \tag{1.6}$$

$$g(U,V) = 0, \tag{1.7}$$

A, B are two non-zero 1-forms, U and V are unit vector fields corresponding to the 1-forms A and B respectively. If $d = 0$, then the manifold reduces to a $G(QE)_n$. This type of manifold is denoted by $MG(QE)_n$.

Recently in [5], A. Bhattacharyya, M. Majumdar and D. Debnath introduced the notion of mixed super quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is called *mixed super quasi-Einstein manifold* if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = \ & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ & +d[A(X)B(Y) + B(X)A(Y)] + eD(X,Y), \end{aligned} \tag{1.8}$$

where a, b, c, d, e are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and A, B are two non zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \ \ \forall \ \ X \in \mathcal{X}(M) \tag{1.9}$$

U, V being mutually orthogonal unit vector fields, D is a symmetric $(0, 2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0, \ \ \forall \ \ X \in \mathcal{X}(M). \tag{1.10}$$

Here a, b, c, d, e are called the associated scalars, A, B are called the associated main and auxiliary 1-forms, U, V are called the main and the auxiliary generators and D is called the associated tensor of the manifold. We denote this type of manifold $MS(QE)_n$.

2 Preliminaries

Throughout the paper we shall assume that the dimension of the considered real manifolds is $n \geq 3$. From (1.8) and (1.9), we get

$$\begin{aligned} S(X,X) = \ & a|X|^2 + b|g(X,U)|^2 + c|g(X,V|^2 \\ & +2d|g(X,U)g(X,V)| + e|D(X,X)|, \ \ \forall \ \ X \in \mathcal{X}(M). \end{aligned} \tag{2.1}$$

Let θ_1 be the angle between U and X;Let θ_2 be the angle between V and X. Then

$$\cos\theta_1 = \frac{g(X,U)}{\sqrt{g(U,U)}\sqrt{g(X,X)}} = \frac{g(X,U)}{\sqrt{g(X,X)}}, \qquad \text{(as } g(U,U)=1\text{)}$$

and $$\cos\theta_2 = \frac{g(X,V)}{\sqrt{g(X,X)}}$$

If $b > 0$, $c > 0$, $d > 0$ and $e > 0$, we have from (2.1)

$$\begin{aligned}(a+b+c+2d+e)|X|^2 &\geq a|X|^2 + b|g(X,U)|^2 + c|g(X,V)|^2 \\ &+ 2d|g(X,U)g(X,V)| + e|D(X,X)| \\ &= S(X,X)\end{aligned} \tag{2.2}$$

Contracting (1.8) over X and Y, we get

$$r = na + b + c \tag{2.3}$$

where r is the scalar curvature.

Again from (1.8) we have

$$S(U,U) = a + b \tag{2.4}$$

$$S(V,V) = a + c + eD(V,V) \tag{2.5}$$

and

$$S(U,V) = d. \tag{2.6}$$

If X is a unit vector field, then $S(X,X)$ is the Ricci-curvature in the direction of X. Hence from (2.4) and (2.5) we can state that $a+b$ and $a+c+eD(V,V)$ are the Ricci curvature in the directions of U and V respectively. Let L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S, where

$$g(LX,Y) = S(X,Y) \;\; \forall\, X,Y \in \mathcal{X}(M). \tag{2.7}$$

We also consider

$$g(\ell X,Y) = D(X,Y). \tag{2.8}$$

Further, let d_1^2 and d_2^2 denote the squares of the lengths of the Ricci-tensor S and the associated tensor D. Then

$$d_1^2 = S(Le_i, e_i) \tag{2.9}$$

and

$$d_2^2 = D(\ell e_i, e_i) \tag{2.10}$$

where $\{e_i\}$, $i = 1,2,...,n$ is an orthonormal basis of the tangent space at a point of $MS(QE)_n$.

Now from (1.8) we get

$$S(Le_i, e_i) = (n-1)a^2 + a^2 + b^2 + c^2 + 2ab + 2bc + ceD(V,V) + eS(\ell e_i, e_i). \tag{2.11}$$

Now from (1.8), we have

$$S(\ell e_i, e_i) = eD(\ell e_i, e_i). \tag{2.12}$$

From (2.11) and (2.12) it follows that

$$S(Le_i, e_i) = (n-1)a^2 + a^2 + b^2 + c^2 + 2ab + 2bc + ceD(V,V) + e^2 D(\ell e_i, e_i). \tag{2.13}$$

Hence

$$d_1^2 = (n-1)a^2 + (a+b+c)^2 - 2ca + ceD(V,V) + e^2(d_2)^2. \tag{2.14}$$

Considering $eD(V,V) = 2a$, *i.e.*, $D(V,V) = \frac{2a}{e} = k.$(say)

From (2.14) we can write

$$d_1^2 - e^2(d_2)^2 = (n-1)a^2 + (a+b+c)^2 \tag{2.15}$$

These result will be used in the sequel.

3 Compact orientable manifolds MS(QE)$_n$

In this section we consider a compact, orientable $MS(QE)_n$ without boundary having constant associated scalars a, b, c, d and e. Then from (2.3) and (2.15), it follows that the scalar curvature is constant and so also is the length of the Ricci-tensor.

We further suppose that $MS(QE)_n$ under consideration admits a non-isometric conformal motion generated by a vector field X. Since $(d_1^2 - e^2(d_2)^2)$ is constant, it follows that

$$\pounds_X(d_1^2 - e^2(d_2)^2) = 0. \tag{3.1}$$

where $\pounds_X$ denotes Lie differentiation with respect to X.

Now, it is known ([10]$p.57, Theorem(4.6)$) that if a compact Riemannian manifold M of dimension $n > 2$ with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\pounds_X(d_1^2 - e^2(d_2)^2)$=0 then M is isometric to a sphere. But a sphere is Einstein so that b, c, d and e vanish which is a contradiction. This leads to the following theorem.

Theorem 1. *A compact orientable mixed super quasi Einstein manifold MS(QE)$_n$ ($n \geq 3$) without boundary does not admit non-isometric conformal vector fields.*□

4 Killing vector fields in compact orientable MS(QE)$_n$

In this section, we consider a compact, orientable $MS(QE)_n$ ($n \geq 3$) without boundary with a, b, c, d, e as associated scalars and U and V as the generators.

It is known [9] that in such a manifold M, the following relation holds

$$\int_M [S(X,X) - |\nabla X|^2 - (divX)^2]dv \leq 0 \qquad \forall X \in \mathcal{X}(M). \tag{4.1}$$

If X is a Killing vector field, then $divX = 0$ [9].

Hence (4.1) takes the form

$$\int_M [S(X,X) - |\nabla X|^2]dv = 0 \tag{4.2}$$

Let $b > 0$, $c > 0$, $d > 0$ and $e > 0$ then by (2.2)

$$(a + b + c + 2d + e)|X|^2 \geq S(X,X) \tag{4.3}$$

Therefore,

$$(a + b + c + 2d + e)|X|^2 - |\nabla X|^2 \geq S(X,X) - |\nabla X|^2 \tag{4.4}$$

Consequently,

$$\int_M [(a + b + c + 2d + e)|X|^2 - |\nabla X|^2]dv \geq \int_M [S(X,X) - |\nabla X|^2]dv \tag{4.5}$$

and by (4.2)

$$\int_M [(a + b + c + 2d + e)|X|^2 - |\nabla X|^2]dv \geq 0 \tag{4.6}$$

If $a + b + c + 2d + e$ <0, then

$$\int_M [(a + b + c + 2d + e)|X|^2 - |\nabla X|^2]dv = 0. \tag{4.7}$$

Therefore, X=0. This leads to the following.

Theorem 2. *If in a compact, orientable MS(QE)$_n$ ($n \geq 3$) without boundary the associated scalars are such that $b > 0, c > 0, d > 0, e > 0$ and $a + b + c + 2d + e < 0$ then there exists no nontrivial Killing vector fields in this manifold.*□

5 Harmonic vector fields in compact orientable $MS(QE)_n$

Let us assume $\theta_2 \leq \theta_1$, where θ_1 is the angle between U and any vector X; θ_2 is the angle between V and any vector X, then we have

$$\cos\theta_2 \geq \cos\theta_1$$

and $\quad g(X, V) \geq g(X, U)$

Therefore, from (2.1), we have

$$S(X, X) \geq (a + b + c + 2d + e)\{g(X, U)\}^2 \tag{5.1}$$

when a, b, c, d, e are positive.

A vector field V_1 in a Riemannian manifold M is said to be harmonic [10], if

$$d\omega = 0 \qquad and \qquad \delta\omega = 0 \tag{5.2}$$

where $\quad \omega(X) = g(X, V_1) \qquad \forall X \in \mathcal{X}(M).$

It is known [9] that in a compact orientable Riemannian manifold M, the following relation holds for any vector field X.

$$\int_M [S(X, X) - \frac{1}{2}|d\omega|^2 + |\nabla X|^2 - (\delta\omega)^2] dV_1 = 0 \tag{5.3}$$

where dV_1 denotes the volume element of M. Now let us consider a compact, orientable $MS(QE)_n$ $(n \geq 3)$ without boundary. If in such a manifold, X is a harmonic vector field, then by (5.2), (5.3) reduces to

$$\int_M [S(X, X) + |\nabla X|^2] dV_1 = 0 \tag{5.4}$$

Hence if each of the associated scalars a, b, c, d, e of $MG(QE)_n$ is greater than zero, then using (5.1), it follows from (5.4) that

$$\int_M [(a + b + c + 2d)\{g(X, U)\}^2 + |\nabla X|^2] dV_1 \leq 0. \tag{5.5}$$

Since, $a + b + c + 2d + e > 0$ from (5.5) we get

$$g(X, U) = 0 \qquad and \qquad \nabla X = 0 \qquad \forall X \in \mathcal{X}(M). \tag{5.6}$$

From (5.6) it follows that X is orthogonal to U and from the second part it follows that the vector field X is parallel.

Similarly, if $\theta_1 \leq \theta_2$ then for all X we infer $g(X, V_1) = 0$ and $\nabla X = 0$.

Hence we have the following

Theorem 3. *If in a compact, orientable MS(QE)$_n$ $(n \geq 3)$ without boundary, each of the associated scalars a, b, c, d, e is greater than zero, then any harmonic vector field X in the MS(QE)$_n$ is parallel and orthogonal to one of the generators of the manifold which makes largest angle with the vector field X.*

6 Conclusions

In the present paper we study compact orientable manifolds of type $MS(QE)_n$ ($n \geq 3$) without boundary. It is shown that such manifolds do not admit non-isometric conformal vector fields. A sufficiency condition for non-existence of nontrivial Killing vector fields is provided, and a property of for harmonic vector fields is pointed out.

Acknowledgement. The authors are thankful to referees and editors for their valuable suggestions and cooperation.

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