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SOME GROUPS OF FIBONACCI TYPE

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1.

The recent resurgence of interest in the groups introduced in Conway (1967), together with their analogues, is recorded in Johnson, Wamsley and Wright (to appear).

Let $\bar{\gamma}$ be the automorphism of the free group $F_n = \langle x_1, \cdots, x_n | \rangle$ induced by permutation of subscripts in accordance with the cycle $\gamma = (12 \cdots n) \in S_n$. Given any word $w \in F_n$, we define $G_n(w)$ to be the group with generators $x_1, \cdots, x_n$ and relators $w\bar{\gamma}^i$, $0 \leqq i \leqq n-1$. By taking $w = x_1 x_2 x_3^{-1}$, we obtain the groups studied in Conway (1967), while those of Johnson, Wamsley and Wright (to appear) and Campbell and Robertson (to appear) are given by $w = x_1 x_2 \cdots x_r x_{r+1}^{-1}$ and $w = x_1 x_2 \cdots x_r x_{r+k}^{-1}$ respectively (all subscripts being reduced modulo n to lie in the set $\{1, 2, \cdots, n\}$). Dunwoody (to appear) and Mawdesley (1973) are concerned with the cases $w = x_2 x_n x^{-1}$ and $w = x_i x_j x_1^{-1}$ respectively. In this article, which is largely a paraphrase of Mawdesley (1973), we give a description of these groups for all values of i, j with $n \leqq 6$.

2.

LEMMA. (i) *Given $w \in F_n$, let w' be the word obtained by reversing the order of the generators and their inverses as they appear in w. Then $G_n(w) \cong G_n(w')$.*

(ii) *For any $w \in F_n$, $\alpha \in S_n$, we have $G_n(w) \cong G_n(w\bar{\alpha})$.*

PROOF. The required isomorphisms are induced by

(i) $x_i \mapsto x_i^{-1}$, (ii) $x_i \mapsto x_{i\alpha}$, $1 \leqq i \leqq n$.

This lemma enables us to reduce substantially the n^2 groups $G_n(x_i x_j x_1^{-1})$.

We first exclude the cases $i = 1$ and $j = 1$, as they yield only the trivial group. In view of part (i), we need only consider one of $G_n(x_i x_j x_1^{-1})$ and $G_n(x_j x_i x_1^{-1})$ for all i, j. Finally, the groups identified in accordance with part (ii) are grouped together in the following table, together with the relevant values of α.

3.

THEOREM. *The groups $G_n(x_i x_j x^{-1})$ for $n \leqq 6$ are given by the following table.*

$x_1 =$	[ ]	α	$G_n(w)$
x_2^2			Z_3

$n = 2$

$x_1 =$	[ ]	α	$G_n(w)$
x_2^2			Z_7
x_3^2		(23)	
$x_2 x_3$	4		Q_8

$n = 3$

$x_1 =$	[ ]	α	$G_n(w)$
x_2^2			Z_{15}
x_4^2		(24)	
x_3^2			$Z_3 * Z_3$
$x_3 x_4$	4		Z_5
$x_3 x_2$		(24)	
$x_2 x_4$			$SL(2, 3)$

$n = 4$

$x_1 =$	[ ]	α	$G_n(w)$
x_2^2			Z_{31}
x_3^2		(2354)	
x_5^2		(25)(34)	
x_2 x_4		(2453)	
$x_4 x_5$	2		Z_{11}
$x_2 x_4$		(2354)	
$x_3 x_2$		(25)(34)	
$x_5 x_3$		(2453)	
$x_2 x_5$	3		$SL(2, 5)$
$x_3 x_4$		(2354)	

$n = 5$

$x_1 =$	[ ]	α	$G_n(w)$
x_2^2			Z_{63}
x_6^2		(26)(35)	
x_3^2			$Z_7 * Z_7$
x_5^2		(26)(35)	
x_4^2			$Z_3 * Z_3 * Z_3$
$x_5 x_6$	2		∞
$x_3 x_2$		(26)(35)	
$x_2 x_4$			Z_9
$x_6 x_4$		(26)(35)	
$x_2 x_6$	3		∞
$x_3 x_4$	1		Z_7
$x_5 x_4$		(26)(35)	
$x_3 x_5$			$Q_8 * Q_8$
$x_2 x_5$			56
$x_6 x_3$		(26)(35)	

$n = 6$

4.

REMARKS. (i) Z_n stands for the cyclic group of order n, and Q_8 for the quaternions of order 8.

(i) The reference numbers in the second column indicate when the group in question is considered elsewhere in the literature.

(iii) Free products clearly arise whenever $i-1$ and $j-1$ have a factor in common with n.

(iv) Of the remaining two infinite groups, one is dealt with in Conway (1967) and in Johnson, Wamsley and Wright (to appear), while the other is easily seen to have derived factor group isomorphic to $Z \times Z$.

(v) We attempt no general explanation of the rather unbalanced incidence of the members of S_n in the third column.

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(vi) The groups Z_7 and Z_9 arising when $j = 6$ are easily obtained.

In the next section we derive the two linear groups, and obtain the rather interesting group of order 56 in section 6.

5.

(i) Eliminating x_2 and x_4 from the relations

$$x_1 = x_2x_4, \quad x_2 = x_3x_1, \quad x_3 = x_4x_2, \quad x_4 = x_1x_3,$$

we obtain

$$x_1 = x_3x_1^2x_3 \quad x_3 = x_1x_3^2x_1.$$

Substituting the second of these in the first, and manipulating the second, we obtain the equivalent relations,

$$x_1 = x_3x_1^2(x_1x_3^2x_1), \quad x_3x_1^{-1}x_3x_1^{-1} = x_3^3,$$

which under the substitution $x_1 \mapsto a$, $x_3^{-1} \mapsto b$ yield

$$a^3 = b^3 = (ab)^2,$$

which define $SL(2,3)$.

(ii) Eliminating x_3 and x_5 from the relations $x_1 = x_2x_5$, $x_2 = x_3x_1$ $x_3 = x_4x_2$, $x_4 = x_5x_3$, $x_5 = x_1x_4$, we obtain

$$x_2x_1x_4 = x_1, \quad x_4x_2x_1 = x_2, \quad x_1x_4^2x_2 = x_4.$$

Eliminating x_4 using the second relation, we have

$$x_2x_1x_2x_1^{-1}x_2^{-1} = x_1, \quad x_1x_2x_1^{-2} = x_2x_1^{-1}x_2^{-1}.$$

Postmultiplying the second of these by $x_2x_1x_2$ and using the first, we obtain

$$x_2x_1x_2 = x_1x_2x_1, \quad x_1x_2x_1^{-1}x_2x_1 = x_2^2,$$

whence, using the first relation again,

$$x_2x_1x_2 = x_1x_2x_1, \quad x_1x_2^2x_1 = x_2^3.$$

The substitution $x_2 \mapsto a$, $x_1x_2 \mapsto b$ now yields

$$ab = b^2a^{-1}, \quad baba^{-1} = a^3,$$

that is,

$$a^5 = b^3 = (ab)^2,$$

which define $SL(2,5)$.

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6.

We divide the proof that $G = G_6(x_2x_5x_1^{-1})$ has order 56 into a number of steps.

(i)
$$\begin{aligned} x_6 &= x_1x_4, \\ x_3 &= x_4x_1, \\ x_2 &= x_3x_6 = x_4x_1^2x_4, \\ x_5 &= x_6x_3 = x_1x_4^2x_1, \\ x_4 &= x_5x_2 = x_1x_4^2x_1x_4x_1^2x_4, \\ x_1 &= x_2x_5 = x_4x_1^2x_4x_1x_4^2x_1. \end{aligned}$$

Thus, G is generated by $x = x_1$, $y = x_4$ subject to therelations

(a) $x^2yxy^3 = 1$,

(b) $y^2xyx^3 = 1$.

(ii) A relation matrix for G/G' is thus

$$\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix},$$

showing that $|G: G'| = 7$. Furthermore, standard coset enumeration with respect to the subgroup $\langle x \rangle$ of G yields the permutation representation

$$\begin{aligned} G &\to S_8 \\ x &\mapsto (2457863) \\ y &\mapsto (1268473), \end{aligned}$$

the transitivity of which yields that $|G| \geqq 56$. It remains only to show that G' is a homomorphic image of $Z_2 \times Z_2 \times Z_2$, and this we now embark on.

(iii) Using both (a) and (b), we have

$$yxyx = y^{-1}x^{-2} = y^{-1}yxy^3,$$

so that if
$$k = [x, y] = x^{-1}y^{-1}xy,$$

$$k^y = xy^{-1}.$$

Since transposition of x and y is an automorphism of G, we also have

$$(k^{-1})^x = yx^{-1},$$

whence:

(c)
$$k^x = xy^{-1} = k^y,$$

so that k commutes with k^x (which we write $k \sim k^x$).

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Clearly,

(d) $k^{x^2} = k^{xy} = k^{yx} = k^{y^2} = y^{-1}x$.

(iv) Thus,

$$\begin{aligned} k^{x^3} &= x^{-1}(y^{-1}x)x \\ &= [x, y]y^{-1}x \\ &= kk^{x^2}, \end{aligned}$$

and so, since $k \sim k^x$, $k^{x^2} \sim k^{x^3} = kk^{x^2}$, and thus $k \sim k^{x^2}$.

$$\begin{aligned} k^{y^3} &= y^{-1}(y^{-1}x)y \\ &= y^{-1}x[x, y] \\ &= k^{x^2}k \\ &= kk^{x^2}, \text{ since } k \sim k^{x^2}, \\ &= k^{x^3}. \end{aligned}$$

It follows from this, together with (c) and (d), that all conjugates of k by *positive* words in x and y commute, and that the value of the conjugate depends only on the length of the conjugating element.

(v) Squaring relation (b),

$$\begin{aligned} x^{-6} &= y^2xy^3xy \\ &= y^2x(x^{-1}y^{-1}x^{-2})xy, \text{ using (a)}, \\ &= yx^{-1}y, \end{aligned}$$

so that $yx^6 = xy^{-1}$ and, transposing x and y, $xy^6 = yx^{-1}$. Hence, the element

(e) $$x^7 = y^{-7} = (y^{-1}x)^2 = (k^{x^2})^2 = (k^2)^{x^2}$$

belongs to $Z(G)$. Thus, $k^2 \sim x$, and so

$$\begin{aligned} k^2 = (k^{x^3})^2 &= (kk^{x^2})^2 \\ &= k^2(k^{x^3})^2, \text{ since } k \sim k^{x^2}, \\ &= k^4, \end{aligned}$$

whence $k^2 = 1$. Finally, since x^7, $y^7 \in Z(G)$, any $g \in G$ has the form $g = zp$, where $z \in Z(G)$ and p is a positive word in z and y. From the above, it now follows that G' is generated by three commuting elements of order at most 2, and this completes the proof.

7.

The group of order 56 just described has, in common with the series (see below) to which it belongs, a number of interesting properties.

(i) For any Mersenne prime $p = 2^n - 1$, the group $GL(n,2)$ of automorphiphisms of $V(n,2) \cong Z_2{}^{\times n}$ contains an element of order p. Denote by M_p the resulting split extension of $V(n,2)$ by Z_p. The first two members of this series of groups are respectively A_4 and our group of order 56. It is somewhat surprising that A_4 has deficiency -1 (and multiplicator Z_2), while M_7 has deficiency zero. It would be interesting to know the multiplicators and deficiencies of the higher Mersenne groups.

(ii) Each M_p has a unique non-trivial normal subgroup.

(iii) Each M_p has elements of order 2 and p only.

(iv) Each M_p is a Frobenius group (see for example the representation in 6(ii)).

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