Filomat 27:8 (2013), 1575–1582
DOI 10.2298/FIL1308575L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Hermite-Hadamard type inequalities for functions whose n-th derivatives are (α, m)-convex

Wen-Hui Li[a], Feng Qi[a]

[a] *Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300160, China*

Abstract. In the paper, the authors establish some new integral inequalities of Hermite-Hadamard type for functions whose n-th derivatives are of (α, m)-convexity and, from these, deduce some known results.

1. Introduction

The following definition is well known in the literature that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on $I \neq \emptyset$ if the inequality

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$; If the inequality (1.1) reverses, then f is said to be concave on I.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This double inequality is well known in the literature as Hermite-Hadamard integral inequality for convex functions. If f is concave, both inequalities in (1.2) hold in the reversed direction. For more information, see [4, 8] and closely related references therein.

The concept of usually used convexity has been generalized by a number of mathematicians. Some of them can be recited as follows.

Definition 1.1 ([14]). *Let $f : [0, b] \to \mathbb{R}$ be a function and $m \in (0, 1]$. If the inequality*

$$f(\lambda x + m(1-\lambda)y) \le \lambda f(x) + m(1-\lambda)f(y)$$

holds for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that the function $f(x)$ is m-convex on $[0, b]$.

2010 *Mathematics Subject Classification*. Primary 26D07; Secondary 26D15
Keywords. Hermite-Hadamard integral inequality, differentiable function, convex function, (α, m)-convex function
Received: 13 March 2013; Accepted: 13 August 2013
Communicated by Pietro Cerone
Email addresses: wen.hui.li102@gmail.com, wen.hui.li@foxmail.com (Wen-Hui Li), qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com (Feng Qi)
URL: http://qifeng618.wordpress.com (Feng Qi)

Definition 1.2 ([7]). *Let $f : [0, b] \to \mathbb{R}$ be a function and $(\alpha, m) \in (0, 1] \times (0, 1]$. If the inequality*

$$f(\lambda x + m(1-\lambda)y) \le \lambda^{\alpha} f(x) + m(1-\lambda^{\alpha}) f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0, b]$.

For $(\alpha, m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$, we can obtain the following classes of functions: increasing, α-star-shaped, star-shaped, m-convex, convex, and α-convex functions.

In past recent years, a few of inequalities of Hermite-hadamard type for (α, m)-convex functions were presented, some of them can be recited as the following theorems.

Theorem 1.1 ([3, Theorem 2]). *Let $f : I^{\circ} \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L([a, b])$ for $a, b \in I$ with $a < b$. If $|f''(x)|^q$ is m-convex on $[a, b]$ for some fixed $q \ge 1$ and $m \in (0, 1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{12}\left[\frac{\left|f''(a)\right|^q + m\left|f''(b/m)\right|^q}{2}\right]^{1/q}. \tag{1.3}$$

Theorem 1.2 ([12, Theorem 4]). *Let $I \subseteq \mathbb{R}$ be an open interval and $a, b \in I$ with $a < b$, and let $f : I \to \mathbb{R}$ be a twice differentiable mapping such that $f''(x)$ is integrable. If $0 \le \lambda \le 1$ and $|f''(x)|$ is convex on $[a, b]$, then*

$$\begin{aligned}&\left|(\lambda-1) f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2} + \int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \begin{cases}\dfrac{(b-a)^2}{24}\left\{\left[\lambda^4 + (1+\lambda)(1-\lambda)^3 + \dfrac{5\lambda-3}{4}\right]|f''(a)| + \left[\lambda^4 + (2-\lambda)\lambda^3 + \dfrac{1-3\lambda}{4}\right]|f''(b)|\right\}, & 0 \le \lambda \le \dfrac{1}{2};\\ \dfrac{(b-a)^2}{48}(3\lambda-1)\big(|f''(a)| + |f''(b)|\big), & \dfrac{1}{2} \le \lambda \le 1.\end{cases}\end{aligned}$$

Remark 1.1. In Theorem 1.2, when $\lambda = 1$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2}\right| \le \frac{(b-a)^2}{24}[|f''(a)| + |f''(b)|]. \tag{1.4}$$

Theorem 1.3 ([10, Theorem 3]). *Let $b^* > 0$ and $f : [0, b^*] \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L([a, b])$ for $a, b \in [0, b^*]$ with $a < b$. If $|f''(x)|^q$ is (α, m)-convex on $[a, b]$ for $(\alpha, m) \in (0, 1] \times (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{2} - \frac{1}{mb-a}\int_a^{mb} f(x)\,\mathrm{d}x\right| \\ &\qquad \le \frac{(mb-a)^2}{2}\left(\frac{1}{6}\right)^{1-1/q}\left\{\frac{1}{(\alpha+2)(\alpha+3)}|f''(a)|^q + m\left[\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)}\right]|f''(b)|^q\right\}^{1/q}.\end{aligned} \tag{1.5}$$

For more information on this topic, please refer to [1, 2, 5, 9, 11, 13, 17–23] and plenty of references cited therein.

In this paper, we will establish some new inequalities of Hermite-Hadamard type for functions whose n-th derivatives are of (α, m)-convexity and deduce some known results in the form of corollaries.

2. A Lemma

For establishing new integral inequalities of Hermite-Hadamard type for (α, m)-convex functions, we need the following lemma.

Lemma 2.1 ([16, Lemma 2.1]). *For $n \in \mathbb{N}$, let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-time differentiable. If $a, b \in I$ with $a < b$ and $f^{(n)}(x) \in L([a,b])$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b) = \frac{(-1)^n}{2}\frac{(b-a)^n}{n!}\int_0^1 (2t+n-2)(1-t)^{n-1} f^{(n)}(ta+(1-t)b)\,\mathrm{d}t, \quad (2.1)$$

where the sum takes 0 *when $n = 1$.*

Remark 2.1. Similar integral identities to (2.1), produced by replacing $f^{(k)}(b)$ in (2.1) by $f^{(k)}(a)$ or by $f^{(k)}\left(\frac{a+b}{2}\right)$, and corresponding integral inequalities of Hermite-Hadamard type have been established in [6, 15].

3. Inequalities of Hermite-Hadamard type

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for functions which are n-time differentiable and (α, m)-convex.

Theorem 3.1. *Let $(\alpha, m) \in (0,1] \times (0,1]$ and $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a n-time differentiable function such that $f^{(n)}(x) \in L\left(\left[a, \frac{b}{m}\right]\right)$, where $0 < a < b < \infty$ and $n \ge 2$. If $\left|f^{(n)}(x)\right|^p$ is (α, m)-convex on $\left[a, \frac{b}{m}\right]$ for $p \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right| \le \frac{1}{2}\frac{(b-a)^n}{n!}\left(\frac{n-1}{n+1}\right)^{1-1/p} \times \left\{(n+\alpha-1)\mathrm{B}(\alpha+1,n+1)\left|f^{(n)}(a)\right|^p + m\left[\frac{n-1}{n+1} - (n+\alpha-1)\mathrm{B}(\alpha+1,n+1)\right]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\right\}^{1/p}, \quad (3.1)$$

where B *is the classical Beta function which may be defined for $\Re(x) > 0$ and $\Re(y) > 0$ by*

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (3.2)$$

and $\Gamma(z)$ the classical Euler gamma function which may be defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t. \quad (3.3)$$

Proof. It follows from Lemma 2.1 that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right| \le \frac{1}{2}\frac{(b-a)^n}{n!}\int_0^1 (2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta + \frac{m(1-t)b}{m}\right)\right|\mathrm{d}t.$$

When $p = 1$, since $\left|f^{(n)}(x)\right|$ is (α, m)-convex on $\left[a, \frac{b}{m}\right]$, we have

$$\left|f^{(n)}\left(ta + \frac{m(1-t)b}{m}\right)\right| \le t^\alpha\left|f^{(n)}(a)\right| + m(1-t^\alpha)\left|f^{(n)}\left(\frac{b}{m}\right)\right|.$$

Multiplying by the factor $(2t+n-2)(1-t)^{n-1}$ on both sides of the above inequality and integrating with respect to $t \in [0,1]$ lead to

$$\begin{aligned}
&\int_0^1 (2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right| \,\mathrm{d}\,t \\
&\le \int_0^1 (2t+n-2)(1-t)^{n-1}\left[t^\alpha\left|f^{(n)}(a)\right| + m(1-t^\alpha)\left|f^{(n)}\left(\frac{b}{m}\right)\right|\right] \mathrm{d}\,t \\
&= \left|f^{(n)}(a)\right| \int_0^1 (1-t)^{n-1}t^\alpha(2t+n-2)\,\mathrm{d}\,t + m\left|f^{(n)}\left(\frac{b}{m}\right)\right| \int_0^1 (2t+n-2)(1-t)^{n-1}(1-t^\alpha)\,\mathrm{d}\,t \\
&= [(n-2)\mathrm{B}(\alpha+1,n)+2\mathrm{B}(\alpha+2,n)]\left|f^{(n)}(a)\right| + m[(n-2)\mathrm{B}(1,n)+2\mathrm{B}(2,n) \\
&\quad -(n-2)\mathrm{B}(\alpha+1,n)-2\mathrm{B}(\alpha+2,n)]\left|f^{(n)}\left(\frac{b}{m}\right)\right|.
\end{aligned}$$

The proof for the case $p=1$ is complete.

When $p>1$, by the well-known Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^1 (2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right| \,\mathrm{d}\,t \\
&\qquad \le \left[\int_0^1 (2t+n-2)(1-t)^{n-1}\,\mathrm{d}\,t\right]^{1-1/p} \left[\int_0^1 (2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right|^p \mathrm{d}\,t\right]^{1/p}. \quad (3.4)
\end{aligned}$$

Utilizing the (α,m)-convexity of $\left|f^{(n)}(x)\right|^p$ reveals

$$\begin{aligned}
&\int_0^1 (2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right|^p \mathrm{d}\,t \\
&\le \int_0^1 (2t+n-2)(1-t)^{n-1}\left[t^\alpha\left|f^{(n)}(a)\right|^p + m(1-t^\alpha)\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\right] \mathrm{d}\,t \\
&= [(n-2)\mathrm{B}(\alpha+1,n)+2\mathrm{B}(\alpha+2,n)]\left|f^{(n)}(a)\right|^p + m[(n-2)\mathrm{B}(1,n)+2\mathrm{B}(2,n) \\
&\quad -(n-2)\mathrm{B}(\alpha+1,n)-2\mathrm{B}(\alpha+2,n)]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p.
\end{aligned} \quad (3.5)$$

Making use of the identity (3.2) and the property

$$\mathrm{B}(x,y+1) = \frac{y}{x}\mathrm{B}(x+1,y) = \frac{y}{x+y}\mathrm{B}(x,y) \quad (3.6)$$

yields

$$(n-2)\mathrm{B}(1,n)+2\mathrm{B}(2,n) = \frac{n-1}{n+1} \quad \text{and} \quad (n-2)\mathrm{B}(\alpha+1,n)+2\mathrm{B}(\alpha+2,n) = (n+\alpha-1)\mathrm{B}(\alpha+1,n+1). \quad (3.7)$$

Substituting the identities in (3.7) into (3.5), and then combining (3.5) with (3.4), yield the inequality (3.1). This completes the proof of Theorem 3.1. □

Corollary 3.1. *Under the conditions of Theorem 3.1,*

1. *when* $m=1$,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right| \le \frac{1}{2}\frac{(b-a)^n}{n!}\left(\frac{n-1}{n+1}\right)^{1-1/p} \\
&\qquad \times\left\{(n+\alpha-1)\mathrm{B}(\alpha+1,n+1)\left|f^{(n)}(a)\right|^p + \left[\frac{n-1}{n+1}-(n+\alpha-1)\mathrm{B}(\alpha+1,n+1)\right]\left|f^{(n)}(b)\right|^p\right\}^{1/p};
\end{aligned}$$

2. *when* $n = 2$,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-1/p}\left\{\frac{1}{(\alpha+2)(\alpha+3)}\left|f''(a)\right|^p + m\left[\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)}\right]\left|f''\left(\frac{b}{m}\right)\right|^p\right\}^{1/p};$$

3. *when* $\alpha = 1$ *and* $n = 2$, *the inequality* (1.3) *is valid;*
4. *when* $m = \alpha = p = 1$ *and* $n = 2$, *the inequality* (1.4) *holds;*
5. *when* $m = \alpha = 1$ *and* $p = n = 2$,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{12}\left[\frac{|f''(a)|^2 + |f''(b)|^2}{2}\right]^{1/2}.$$

Theorem 3.2. *Let* $(\alpha, m) \in (0,1] \times (0,1]$ *and* $f : \mathbb{R}_0 \to \mathbb{R}$ *be a n-time differentiable function such that* $f^{(n)}(x) \in L\left(\left[a, \frac{b}{m}\right]\right)$, *where* $0 < a < b < \infty$ *and* $n \ge 2$. *If* $\left|f^{(n)}(x)\right|^p$ *is* (α, m)*-convex on* $\left[a, \frac{b}{m}\right]$ *for* $p > 1$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right| \le \frac{1}{2}\frac{(b-a)^n}{n!}\left[\frac{n^{q+1}-(n-2)^{q+1}}{2(q+1)}\right]^{1/q}$$
$$\times\left\{\mathrm{B}(\alpha+1, p(n-1)+1)\left|f^{(n)}(a)\right|^p + m\left[\frac{1}{p(n-1)+1} - \mathrm{B}(\alpha+1, p(n-1)+1)\right]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\right\}^{1/p}, \quad (3.8)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* B *is defined by* (3.2).

Proof. It follows from Lemma 2.1 that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right|$$
$$\le \frac{1}{2}\frac{(b-a)^n}{n!}\int_0^1 (2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta + \frac{m(1-t)b}{m}\right)\right|\mathrm{d}t. \quad (3.9)$$

By Hölder integral inequality, we have

$$\begin{aligned}&\int_0^1 (2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta + \frac{m(1-t)b}{m}\right)\right|\mathrm{d}t\\ &\le \left[\int_0^1 (2t+n-2)^q\,\mathrm{d}t\right]^{1/q}\left[\int_0^1 (1-t)^{p(n-1)}\left|f^{(n)}\left(ta + \frac{m(1-t)b}{m}\right)\right|^p\mathrm{d}t\right]^{1/p}\\ &= \left[\frac{n^{q+1}-(n-2)^{q+1}}{2(q+1)}\right]^{1/q}\left[\int_0^1 (1-t)^{p(n-1)}\left|f^{(n)}\left(ta + \frac{m(1-t)b}{m}\right)\right|^p\mathrm{d}t\right]^{1/p}.\end{aligned} \quad (3.10)$$

Using the (α, m)-convexity of $\left|f^{(n)}(x)\right|^p$, we have

$$\begin{aligned}&\int_0^1 (1-t)^{p(n-1)}\left|f^{(n)}\left(ta + \frac{m(1-t)b}{m}\right)\right|^p\mathrm{d}t\\ &\le \int_0^1 (1-t)^{p(n-1)}\left[t^\alpha\left|f^{(n)}(a)\right|^p + m(1-t^\alpha)\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\right]\mathrm{d}t\\ &= \left|f^{(n)}(a)\right|^p\int_0^1 t^\alpha(1-t)^{p(n-1)}\,\mathrm{d}t + m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\int_0^1 (1-t^\alpha)(1-t)^{p(n-1)}\,\mathrm{d}t\end{aligned}$$

$$= \mathrm{B}(\alpha+1,p(n-1)+1)\left|f^{(n)}(a)\right|^p + m[\mathrm{B}(1,p(n-1)+1) - \mathrm{B}(\alpha+1,p(n-1)+1)]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p$$

$$= \mathrm{B}(\alpha+1,p(n-1)+1)\left|f^{(n)}(a)\right|^p + m\left[\frac{1}{p(n-1)+1} - \mathrm{B}(\alpha+1,p(n-1)+1)\right]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p.$$

Substituting the above inequality into (3.10), and then substituting (3.10) into (3.9), result in the inequality (3.8). This completes the proof of Theorem 3.2. □

Corollary 3.2. *Under the conditions of Theorem 3.2,*

1. *when* $m = 1$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right| \le \frac{1}{2}\frac{(b-a)^n}{n!}\left[\frac{n^{q+1}-(n-2)^{q+1}}{2(q+1)}\right]^{1/q}$$
$$\times\left\{\mathrm{B}(\alpha+1,p(n-1)+1)\left|f^{(n)}(a)\right|^p + \left[\frac{1}{p(n-1)+1} - \mathrm{B}(\alpha+1,p(n-1)+1)\right]\left|f^{(n)}(b)\right|^p\right\}^{1/p};$$

2. *when* $n = 2$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le \frac{(b-a)^2}{2}\left(\frac{1}{q+1}\right)^{1/q}\left\{\mathrm{B}(\alpha+1,p+1)\left|f''(a)\right|^p + m\left[\frac{1}{p+1} - \mathrm{B}(\alpha+1,p+1)\right]\left|f''\left(\frac{b}{m}\right)\right|^p\right\}^{1/p};$$

3. *when* $\alpha = 1$ *and* $n = 2$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{2}\left(\frac{1}{q+1}\right)^{1/q}\left[\frac{\left|f''(a)\right|^p + m(p+1)\left|f''(b/m)\right|^p}{(p+1)(p+2)}\right]^{1/p};$$

4. *when* $m = \alpha = 1$ *and* $p = n = 2$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{12}\left[\left|f''(a)\right|^2 + 3\left|f''(b)\right|^2\right]^{1/2}.$$

Theorem 3.3. *Let* $(\alpha, m) \in (0,1] \times (0,1]$ *and* $f : \mathbb{R}_0 \to \mathbb{R}$ *be a* n*-time differentiable function such that* $f^{(n)}(x) \in L\left(\left[a, \frac{b}{m}\right]\right)$, *where* $0 < a < b < \infty$ *and* $n \ge 2$. *If* $\left|f^{(n)}(x)\right|^p$ *is* (α, m)*-convex on* $\left[a, \frac{b}{m}\right]$ *for* $p \ge 1$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right|$$
$$\le \frac{(n-1)^{1-1/p}}{2}\frac{(b-a)^n}{n!}\left\{L(p,\alpha)\left|f^{(n)}(a)\right|^p + m[Q(p,\alpha) - L(p,\alpha)]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\right\}^{1/p}, \quad (3.11)$$

where B *is defined by* (3.2),

$$L(p,\alpha) = \left[\frac{p(n-2)(n-1)+n(\alpha+2)-2}{p(n-1)+\alpha+2}\right]\mathrm{B}(\alpha+1,p(n-1)+1), \quad (3.12)$$

and

$$Q(p,\alpha) = \frac{(n-1)[p(n-2)+2]}{[p(n-1)+1][p(n-1)+2]}. \quad (3.13)$$

Proof. It follows from Lemma 2.1 that

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{2}\sum_{k=1}^{n-1}\frac{k-1}{(k+1)!}(a-b)^k f^{(k)}(b)\right| \\ \le \frac{1}{2}\frac{(b-a)^n}{n!}\int_0^1(2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right|\mathrm{d}t. \quad (3.14)$$

By the well-known Hölder integral inequality, we have

$$\int_0^1(2t+n-2)(1-t)^{n-1}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right|\mathrm{d}t \\ \le \left[\int_0^1(2t+n-2)\,\mathrm{d}t\right]^{1-1/p}\left[\int_0^1(2t+n-2)(1-t)^{p(n-1)}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right|^p\mathrm{d}t\right]^{1/p}. \quad (3.15)$$

Using the (α,m)-convexity of $\left|f^{(n)}(x)\right|^p$ shows

$$\begin{aligned}&\int_0^1(2t+n-2)(1-t)^{p(n-1)}\left|f^{(n)}\left(ta+\frac{m(1-t)b}{m}\right)\right|^p\mathrm{d}t\\ &\le\int_0^1(2t+n-2)(1-t)^{p(n-1)}\left[t^\alpha\left|f^{(n)}(a)\right|^p+m(1-t^\alpha)\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\right]\mathrm{d}t\\ &=\left|f^{(n)}(a)\right|^p\int_0^1(2t+n-2)t^\alpha(1-t)^{p(n-1)}\,\mathrm{d}t+m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p\int_0^1(2t+n-2)(1-t^\alpha)(1-t)^{p(n-1)}\,\mathrm{d}t\\ &=[(n-2)\mathrm{B}(\alpha+1,p(n-1)+1)+2\mathrm{B}(\alpha+2,p(n-1)+1)]\left|f^{(n)}(a)\right|^p+m[(n-2)\mathrm{B}(1,p(n-1)+1)\\ &\quad+2\mathrm{B}(2,p(n-1)+1)-(n-2)\mathrm{B}(\alpha+1,p(n-1)+1)-2\mathrm{B}(\alpha+2,p(n-1)+1)]\left|f^{(n)}\left(\frac{b}{m}\right)\right|^p.\end{aligned} \quad (3.16)$$

Employing the identities (3.2) and (3.6) give

$$(n-2)\mathrm{B}(1,p(n-1)+1)+2\mathrm{B}(2,p(n-1)+1)=\frac{(n-1)[p(n-2)+2]}{[p(n-1)+1][p(n-1)+2]} \quad (3.17)$$

and

$$(n-2)\mathrm{B}(\alpha+1,p(n-1)+1)+2\mathrm{B}(\alpha+2,p(n-1)+1) \\ =\frac{p(n-2)(n-1)+n(\alpha+2)-2}{p(n-1)+\alpha+2}\mathrm{B}(\alpha+1,p(n-1)+1). \quad (3.18)$$

Substituting (3.17) and (3.18) into (3.16), and then combining (3.16) with (3.15) and (3.14) in sequence, yield the inequality (3.11). This completes the proof of Theorem 3.3. □

Corollary 3.3. *Under the conditions of Theorem 3.3,*

1. *when* $n=2$, *we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ \le\frac{(b-a)^2}{4}\left\{2\mathrm{B}(\alpha+2,p+1)\left|f''(a)\right|^p+m\left[\frac{2}{(p+1)(p+2)}-2\mathrm{B}(\alpha+2,p+1)\right]\left|f''\left(\frac{b}{m}\right)\right|^p\right\}^{1/p};$$

2. *when* $\alpha = 1$ *and* $n = 2$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^2}{4}\left[\frac{4|f''(a)|^p}{(p+1)(p+2)(p+3)} + \frac{2m|f''(b/m)|^p}{(p+2)(p+3)}\right]^{1/p};$$

3. *when* $m = \alpha = p = 1$ *and* $n = 2$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)^2}{24}\left[|f''(a)| + |f''(b)|\right].$$

Acknowledgements

The authors thank the anonymous referee and Professors Bo-Yan Xi and Shu-Hong Wang for their helpful corrections to and valuable comments on the original version of this manuscript.

References

[1] R.-F. Bai, F. Qi, B.-Y. Xi, Hermite-Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions, Filomat 27 (1) (2013) 1–7; Available online at http://dx.doi.org/10.2298/FIL1301001B.

[2] S.-P. Bai, S.-H. Wang, F. Qi, Some Hermite-Hadamard type inequalities for n-time differentiable (α, m)-convex functions, J. Inequal. Appl. 2012, 2012:267, 11 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2012-267.

[3] M. K. Bakula, M. E. Özdemir, J. Pečarić, Hadamard type inequalities for m-convex and (α, m)-convex functions, J. Inequal. Pure Appl. Math. 9 (4) (2008) Art. 96, 12 pages; Available online at http://www.emis.de/journals/JIPAM/article1032.html.

[4] S. S. Dragomir, C. E. M. Pearce, Selected Topics on Hermite-Hadamard Type Inequalities and Applications, RGMIA Monographs, Victoria University, 2000; Available online at http://rgmia.org/monographs/hermite_hadamard.html.

[5] A.-P. Ji, T.-Y. Zhang, F. Qi, Integral inequalities of Hermite-Hadamard type for (α, m)-GA-convex functions, available online at http://arxiv.org/abs/1306.0852.

[6] W.-H. Li, F. Qi, *Hermite-Hadamard type inequalities of functions whose derivatives of n-th order are (α, m)-convex*, available online at http://arxiv.org/abs/1308.2948.

[7] V. G. Miheşan, A generalization of the convexity, Seminar on Functional Equations, Approx. Convex, Cluj-Napoca, 1993. (Romania)

[8] C. P. Niculescu, L.-E. Persson, Convex Functions and their Applications, CMS Books in Mathematics, Springer-Verlag, 2005.

[9] M. E. Özdemir, M. Avci, E. Set, On some inequalities of Hermite-Hadamard type via m-convexity, Appl. Math. Lett. 23 (9) (2010) 1065–1070; Available online at http://dx.doi.org/10.1016/j.aml.2010.04.037.

[10] M. E. Özdemir, M. Avci, H. Kavurmaci, Hermite-Hadamard-type ineuqlities via (α, m)-convexity, Comput. Math. Appl. 61 (9) (2011) 2614–2620; Available online at http://dx.doi.org/10.1016/j.camwa.2011.02.053.

[11] F. Qi, Z.-L. Wei, Q. Yang, Generalizations and refinements of Hermite-Hadamard's inequality, Rocky Mountain J. Math. 35 (1) (2005) 235–251; Available online at http://dx.doi.org/10.1216/rmjm/1181069779.

[12] M. Z. Sarikaya, N. Aktan, On the generalization of some integral inequalities and their applications, Math. Comput. Model. 54 (9-10) (2011) 2174–2182; Available online at http://dx.doi.org/10.1016/j.mcm.2011.05.026.

[13] Y. Shuang, H.-P. Yin, F. Qi, Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions, Analysis (Munich) 33 (1) (2013) 197–208; Available online at http://dx.doi.org/10.1524/anly.2013.1192.

[14] G. Toader, *Some generalizations of the convexity*, Univ. Cluj-Napoca, Cluj-Napoc. 1985, 329–338.

[15] S.-H. Wang and F. Qi, Inequalities of Hermite-Hadamard type for convex functions which are n-times differentiable, Math. Inequal. Appl. 17 (2014), in press.

[16] S.-H. Wang, B.-Y. Xi, F. Qi, Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, Analysis (Munich) 32 (3) (2012) 247–262; Available online at http://dx.doi.org/10.1524/anly.2012.1167.

[17] B.-Y. Xi, R.-F. Bai, F. Qi, Hermite-Hadamard type inequalities for the m- and (α, m)-geometrically convex functions, Aequationes Math. 84 (3) (2012) 261–269; Available online at http://dx.doi.org/10.1007/s00010-011-0114-x.

[18] B.-Y. Xi, J. Hua, F. Qi, Hermite-Hadamard type inequalities for extended s-convex functions on the co-ordinates in a rectangle, J. Appl. Anal. 20 (1) (2014) in press.

[19] B.-Y. Xi, F. Qi, Hermite-Hadamard type inequalities for functions whose derivatives are of convexities, Nonlinear Funct. Anal. Appl. 18 (2) (2013) 163–176.

[20] B.-Y. Xi, F. Qi, Some Hermite-Hadamard type inequalities for differentiable convex functions and applications, Hacet. J. Math. Stat. 42 (3) (2013) 243–257.

[21] B.-Y. Xi, F. Qi, Some inequalities of Hermite-Hadamard type for h-convex functions, Adv. Inequal. Appl. 2 (1) (2013) 1–15.

[22] B.-Y. Xi, F. Qi, Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means, J. Funct. Spaces Appl. 2012 (2012) Article ID 980438, 14 pages; Available online at http://dx.doi.org/10.1155/2012/980438.

[23] T.-Y. Zhang, A.-P. Ji, F. Qi, Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means, Matematiche (Catania) 68 (1) (2013) 229–239; Available online at http://dx.doi.org/10.4418/2013.68.1.17.