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Some identities of degenerate special polynomials

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Abstract: In this paper, by considering higher-order degenerate Bernoulli and Euler polynomials which were introduced by Carlitz, we investigate some properties of mixed-type of those polynomials. In particular, we give some identities of mixed-type degenerate special polynomials which are derived from the fermionic integrals on \mathbb{Z}_p and the bosonic integrals on \mathbb{Z}_p .

Keywords: Mixed-type degenerate special polynomial, Fermionic integral, Bosonic integral

MSC: 11B68, 11S80

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p , respectively. The p -adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the fermionic p -adic integral on \mathbb{Z}_p is defined by Kim as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^N. \quad (1)$$

Then, by (1), we get

$$I_{-1}(f_1) = -I(f) + 2f(0), \quad \text{where } f_1(x) = f(x+1), \quad (\text{see [9–13]}). \quad (2)$$

From (2), we can derive the following integral equation:

$$I_{-1}(f_n) = (-1)^n I(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (3)$$

where $f_n(x) = f(x+n)$, ($n \in \mathbb{N}$).

The bosonic integral on \mathbb{Z}_p (or p -adic invariant integral on \mathbb{Z}_p) is defined by Volkenborn as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) \mu_0(x + p^N \mathbb{Z}_p) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (4)$$

(see [11, 22, 23]). Then, by (4), we get

$$I_0(f_1) - I_0(f) = f'(0), \quad \text{where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}, \quad (5)$$

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and

$$I_0(f_n) - I_0(f) = \sum_{l=0}^{n-1} f'(l), \quad \text{where } f_n(x) = f(x+n), \quad (n \in \mathbb{N}). \tag{6}$$

For $r \in \mathbb{N}$, the higher-order Bernoulli polynomials are given by the generating function as

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [1-24]}). \tag{7}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the higher-order Bernoulli numbers. For $r = 1$, $B_n(x) = B_n^{(1)}(x)$ are called the ordinary Bernoulli polynomials and $B_n = B_n^{(1)}(0)$ are called the ordinary Bernoulli numbers. As is well known, the higher-order Euler polynomials are also defined by the generating function

$$\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [9, 10, 13]}). \tag{8}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the higher-order Euler numbers. For $r = 1$, $E_n(x) = E_n^{(1)}(x)$ are called the ordinary Euler polynomials and $E_n = E_n^{(1)}(0)$ are called the ordinary Euler numbers. By (7) and (8), we get

$$B_n^{(r)}(x) = \sum_{l=0}^n \binom{n}{l} B_l^{(r)} x^{n-l}, \quad E_n^{(r)}(x) = \sum_{l=0}^n \binom{n}{l} E_{n-l}^{(r)} x^l, \quad (\text{see [18]}). \tag{9}$$

The Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^n S_2(n, l) (x)_l, \quad (n \geq 0), \tag{10}$$

and the Stirling numbers of the first kind are given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0), \quad (\text{see [18]}). \tag{11}$$

Now, we consider the analogue of $(x)_n$ as follows:

$$(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda) = \sum_{l=0}^n S_1(n, l | \lambda) x^l, \quad \text{where } \lambda \neq 0. \tag{12}$$

Note that $\lim_{\lambda \rightarrow 1} S_1(n, l | \lambda) = S_1(n, l)$, $(n \geq 0)$.

For $\lambda \neq 0$, Carlitz considered the degenerate Bernoulli polynomials given by the generating function

$$\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}, \quad (\text{see [3, 9]}). \tag{13}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are called the degenerate Bernoulli numbers. Note that $\lim_{\lambda \rightarrow 0} \beta_n(\lambda, x) = B_n(x)$ and $\lim_{\lambda \rightarrow \infty} \lambda^{-n} \beta_n(\lambda, \lambda x) = b_n(x)$, where $b_n(x)$ are the Bernoulli polynomials of the second kind given by the generating function

$$\frac{t}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}, \quad (\text{see [18]}). \tag{14}$$

The degenerate Euler polynomials are also defined by the generating function

$$\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda, x) \frac{t^n}{n!}, \quad (\text{see [3, 9]}). \tag{15}$$

When $x = 0$, $\mathcal{E}_n(\lambda) = \mathcal{E}_n(\lambda, 0)$ are called the degenerate Euler numbers. Note that $\lim_{\lambda \rightarrow 0} \mathcal{E}_n(\lambda, x) = E_n(x)$.

From (13) and (15), we note that

$$\beta_n(\lambda, x) = \sum_{l=0}^n \binom{n}{l} \beta_l(\lambda)(x)_{n-l, \lambda}, \quad (n \geq 0), \tag{16}$$

and

$$\mathcal{E}_n(\lambda, x) = \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l(\lambda)(x)_{n-l, \lambda}, \quad (n \geq 0). \tag{17}$$

For $r \in \mathbb{N}$, the higher-order degenerate Bernoulli polynomials and the higher-order degenerate Euler polynomials are introduced by Carlitz and are respectively given by

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(r)}(\lambda, x) \frac{t^n}{n!}, \tag{18}$$

and

$$\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(\lambda, x) \frac{t^n}{n!}, \quad (\text{see [3, 9]}). \tag{19}$$

When $x = 0$, $\beta_n^{(r)}(\lambda) = \beta_n^{(r)}(\lambda, 0)$ and $\mathcal{E}_n^{(r)}(\lambda) = \mathcal{E}_n^{(r)}(\lambda, 0)$ are respectively called the higher-order degenerate Bernoulli numbers and the higher-order degenerate Euler numbers.

From (18) and (19), we have

$$\beta_n^{(r)}(\lambda, x) = \sum_{l=0}^n \binom{n}{l} \beta_l^{(r)}(\lambda)(x)_{n-l, \lambda}, \quad \mathcal{E}_n^{(r)}(\lambda, x) = \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l^{(r)}(\lambda)(x)_{n-l, \lambda}. \tag{20}$$

The Bernoulli polynomials of the second kind with order r and the Daehee polynomials of order r are respectively defined by the generating functions

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [18]}). \tag{21}$$

and

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [10, 18]}). \tag{22}$$

When $x = 0$, $b_n^{(r)} = b_n^{(r)}(0)$ and $D_n^{(r)} = D_n^{(r)}(0)$ are called the higher-order Bernoulli numbers of the second kind and the higher-order Daehee numbers.

In this paper, by considering higher-order degenerate Bernoulli and Euler polynomials which were introduced by Carlitz, we investigate some properties of mixed-type of those polynomials. In particular, we give some identities of mixed-type degenerate special polynomials which are derived from the fermionic integrals on \mathbb{Z}_p and the bosonic integrals on \mathbb{Z}_p (also called p -adic invariant integrals on \mathbb{Z}_p).

2 Some identities of mixed-type degenerate polynomials

For $\lambda \neq 0$ and $t \in \mathbb{C}_p$ with $|t|_p < |\lambda|_p^{-1} p^{-\frac{1}{p-1}}$, we observe that

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}} d\mu_0(x_1) \cdots d\mu_0(x_r) = \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r \left(\frac{\log(1+\lambda t)}{\lambda t}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} \\ & = \left(\sum_{l=0}^{\infty} \beta_l^{(r)}(\lambda, x) \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} D_m^{(r)} \lambda^m \frac{t^m}{m!}\right) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \beta_l^{(r)}(\lambda, x) D_{n-l}^{(r)} \lambda^{n-l} \frac{n!}{l!(n-l)!}\right) \frac{t^n}{n!} \\ & = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} \lambda^{n-l} \beta_l^{(r)}(\lambda, x) D_{n-l}^{(r)}\right) \frac{t^n}{n!}. \end{aligned} \tag{23}$$

It is not difficult to show that

$$\left(\frac{\log(1 + \lambda t)}{\lambda t}\right)^r = r! \sum_{l=0}^{\infty} \frac{S_1(l + r, r) l! \lambda^l t^l}{(l + r)! l!} = \sum_{l=0}^{\infty} \frac{S_1(l + r, r)}{\binom{l+r}{l}} \lambda^l \frac{t^l}{l!}. \tag{24}$$

Thus, by (24), we get

$$\left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}\right)^r \left(\frac{\log(1 + \lambda t)}{\lambda t}\right)^r (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \frac{S_1(l + r, r)}{\binom{l+r}{r}} \binom{n}{l}\right) \lambda^l \beta_{n-l}^{(r)}(\lambda, x) \frac{t^n}{n!}. \tag{25}$$

The left hand side of (23) is given by

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}} d\mu_0(x_1) \cdots d\mu_0(x_r) \\ &= \sum_{n=0}^{\infty} \lambda^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x + x_1 + \cdots + x_r}{\lambda}_n d\mu_0(x_1) \cdots d\mu_0(x_r) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_{n,\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r) \frac{t^n}{n!}. \end{aligned} \tag{26}$$

Therefore, by (23), (25) and (26), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_{n,\lambda} d\mu_0(x_1) \cdots d\mu_0(x_r) &= \sum_{l=0}^n \binom{n}{l} \lambda^{n-l} \beta_l^{(r)}(\lambda, x) D_{n-l}^{(r)} \\ &= \sum_{l=0}^n \binom{n}{l} \frac{S_1(l + r, r)}{\binom{l+r}{r}} \lambda^l \beta_{n-l}^{(r)}(\lambda, x). \end{aligned}$$

From (12), we note that

$$(x_1 + \cdots + x_r + x)_{n,\lambda} = \sum_{l=0}^n S_1(n, l | \lambda) (x_1 + \cdots + x_r + x)^l. \tag{27}$$

By (2), we easily get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t} d\mu_0(x_1) \cdots d\mu_0(x_r) = \left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}. \tag{28}$$

Thus, by (28), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_0(x_1) \cdots d\mu_0(x_r) = B_n^{(r)}(x), \quad (n \geq 0). \tag{29}$$

Therefore, by (29), we obtain the following corollary.

Corollary 2.2. For $n \geq 0$, we have

$$\sum_{l=0}^n S_1(n, l | \lambda) B_l^{(r)}(x) = \sum_{l=0}^n \binom{n}{l} \frac{S_1(l + r, r)}{\binom{l+r}{r}} \lambda^l \beta_{n-l}^{(r)}(\lambda, x).$$

From (1), (2) and (3), we can derive the following equation:

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x_1+\cdots+x_r+x}{\lambda}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \tag{30}$$

$$= \left(\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \right)^r (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(\lambda, x) \frac{t^n}{n!}.$$

Thus, by (30), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + x_1 + \cdots + x_r)_{n,\lambda} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \mathcal{E}_n^{(r)}(\lambda, x), \quad (n \geq 0). \tag{31}$$

Now, we observe that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1 + \cdots + x_r + x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{32}$$

Thus, by (32), we see

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x), \quad (n \geq 0). \tag{33}$$

From (12), we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_{n,\lambda} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{l=0}^n S_1(n, l | \lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + x_1 + \cdots + x_r)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \sum_{l=0}^n S_1(n, l | \lambda) E_l^{(r)}(x). \end{aligned} \tag{34}$$

Therefore, by (31) and (34), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$\mathcal{E}_n^{(r)}(\lambda, x) = \sum_{l=0}^n S_1(n, l | \lambda) E_l^{(r)}(x).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (13), we get

$$\begin{aligned} & \sum_{m=0}^{\infty} \beta_m^{(r)}(\lambda, x) \frac{1}{\lambda^m} (e^{\lambda t} - 1)^m \frac{1}{m!} = \left(\frac{\frac{1}{\lambda}(e^{\lambda t} - 1)}{e^t - 1} \right)^r e^{xt} = \left(\frac{e^{\lambda t} - 1}{\lambda t} \right)^r \left(\frac{t}{e^t - 1} \right)^r e^{xt} \\ &= \left(\sum_{l=0}^{\infty} \frac{S_2(l+r, r)}{\binom{l+r}{r}} \lambda^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{t^m}{m!} \right) = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \frac{S_2(l+r, r)}{\binom{l+r}{r}} \binom{n}{l} \lambda^l B_{n-l}^{(r)}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{35}$$

On the other hand,

$$\begin{aligned} \sum_{m=0}^{\infty} \beta_m^{(r)}(\lambda, x) \frac{1}{\lambda^m} (e^{\lambda t} - 1)^m \frac{1}{m!} &= \sum_{m=0}^{\infty} \beta_m^{(r)}(\lambda, x) \frac{1}{\lambda^m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \beta_m^{(r)}(\lambda, x) S_2(n, m) \lambda^{n-m} \right) \frac{t^n}{n!}. \end{aligned} \tag{36}$$

Therefore, by (35) and (36), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$\sum_{l=0}^n \frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r, r) \lambda^l B_{n-l}^{(r)}(x) = \sum_{l=0}^n \beta_l^{(r)}(\lambda, x) S_2(n, l) \lambda^{n-l}.$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (19), we get

$$\sum_{m=0}^{\infty} \mathcal{E}_m^{(r)}(\lambda, x) \lambda^{-m} (e^{\lambda t} - 1)^m \frac{1}{m!} = \left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \tag{37}$$

We observe that

$$\begin{aligned} \sum_{m=0}^{\infty} \mathcal{E}_m^{(r)}(\lambda, x) \lambda^{-m} (e^{\lambda t} - 1)^m \frac{1}{m!} &= \sum_{m=0}^{\infty} \mathcal{E}_m^{(r)}(\lambda, x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \mathcal{E}_m^{(r)}(\lambda, x) \lambda^{n-m} S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \tag{38}$$

Thus, by (37) and (38), we get

$$E_n^{(r)}(x) = \sum_{m=0}^n \mathcal{E}_m^{(r)}(\lambda, x) \lambda^{n-m} S_2(n, m), \quad (n, \geq 0). \tag{39}$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)}(\lambda, x) \frac{t^n}{n!} &= \left(\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}\right)^r (1 + \lambda t)^{\frac{x}{\lambda}} = \left(\frac{2}{e^{\frac{1}{\lambda} \log(1 + \lambda t)} + 1}\right)^r e^{\frac{x}{\lambda} \log(1 + \lambda t)} \\ &= \sum_{m=0}^{\infty} E_m^{(r)}(x) \frac{\lambda^{-m}}{m!} (\log(1 + \lambda t))^m = \sum_{m=0}^{\infty} E_m^{(r)}(x) \frac{1}{\lambda^m} \sum_{n=m}^{\infty} S_1(n, m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n E_m^{(r)}(x) \lambda^{n-m} S_1(n, m) \right) \frac{t^n}{n!}. \end{aligned} \tag{40}$$

By comparing the coefficients on both sides of (40), we get

$$\mathcal{E}_n^{(r)}(\lambda, x) = \sum_{m=0}^n E_m^{(r)}(x) \lambda^{n-m} S_1(n, m). \tag{41}$$

Therefore, by (39) and (41), we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$E_n^{(r)}(x) = \sum_{m=0}^n \mathcal{E}_m^{(r)}(\lambda, x) \lambda^{n-m} S_2(n, m),$$

and

$$\mathcal{E}_n^{(r)}(\lambda, x) = \sum_{m=0}^n E_m^{(r)}(x) \lambda^{n-m} S_1(n, m).$$

From (18), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n^{(r)}(\lambda, x) \frac{t^n}{n!} &= \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1 + \lambda t)^{\frac{x}{\lambda}} \\ &= \left(\frac{t}{e^{\frac{1}{\lambda} \log(1 + \lambda t)} - 1}\right)^r e^{\frac{x}{\lambda} \log(1 + \lambda t)} \\ &= \left(\frac{\lambda t}{\log(1 + \lambda t)}\right)^r \left(\frac{\frac{1}{\lambda} \log(1 + \lambda t)}{e^{\frac{1}{\lambda} \log(1 + \lambda t)} - 1}\right)^r e^{\frac{x}{\lambda} \log(1 + \lambda t)} \\ &= \left(\sum_{l=0}^{\infty} b_l^{(r)} \lambda^l \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{1}{\lambda^m} \frac{(\log(1 + \lambda t))^m}{m!}\right) \end{aligned} \tag{42}$$

$$\begin{aligned}
 &= \left(\sum_{l=0}^{\infty} b_l^{(r)} \lambda^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{1}{\lambda^m} \sum_{j=m}^{\infty} S_1(j, m) \lambda^j \frac{t^j}{j!} \right) \\
 &= \left(\sum_{l=0}^{\infty} b_l^{(r)} \lambda^l \frac{t^l}{l!} \right) \left(\sum_{j=0}^{\infty} \left(\sum_{m=0}^j B_m^{(r)}(x) \lambda^{j-m} S_1(j, m) \right) \frac{t^j}{j!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \left(\sum_{m=0}^j B_m^{(r)}(x) \lambda^{j-m} S_1(j, m) \right) \frac{b_{n-j}^{(r)} \lambda^{n-j} n!}{j!(n-j)!} \right) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \sum_{m=0}^j B_m^{(r)} \lambda^{n-m} S_1(j, m) b_{n-j}^{(r)} \right) \frac{t^n}{n!}.
 \end{aligned}$$

By comparing the coefficients on the both sides of (42), we obtain the following theorem.

Theorem 2.6. For $n \geq 0$, we have

$$\beta_n^{(r)}(\lambda, x) = \sum_{j=0}^n \binom{n}{j} \sum_{m=0}^j B_m^{(r)}(x) \lambda^{n-m} S_1(j, m) b_{n-j}^{(r)}.$$

For $r, s \in \mathbb{N}$, let us consider the degenerate Bernoulli-Euler mixed-type polynomials of order (r, s) as follows:

$$\beta \mathcal{E}_n^{(r,s)}(\lambda, x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \mathcal{E}_n^{(s)}(\lambda, x + y_1 + \cdots + y_r) d\mu_0(y_1) \cdots d\mu_0(y_r), \quad (n \geq 0). \tag{43}$$

From (43), (20) and Theorem 2.1, we have

$$\begin{aligned}
 \beta \mathcal{E}_n^{(r,s)}(\lambda, x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \mathcal{E}_n^{(s)}(\lambda, x + y_1 + \cdots + y_r) d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + y_1 + \cdots + y_r)_{n-l, \lambda} d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l} \binom{n-l}{m} \frac{S_1(m+r, r)}{\binom{m+r}{r}} \lambda^m \beta_{n-l-m}^{(r)}(\lambda, x).
 \end{aligned} \tag{44}$$

Therefore, by (44), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$\beta \mathcal{E}_n^{(r,s)}(\lambda, x) = \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l} \binom{n-l}{m} \frac{S_1(m+r, r)}{\binom{m+r}{r}} \lambda^m \beta_{n-l-m}^{(r)}(\lambda, x).$$

Remark. The generating function of $\beta \mathcal{E}_n^{(r,s)}(\lambda, x)$ is given by

$$\begin{aligned}
 \sum_{n=0}^{\infty} \beta \mathcal{E}_n^{(r,s)}(\lambda, x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \mathcal{E}_n^{(s)}(\lambda, x + y_1, \dots, y_r) \frac{t^n}{n!} d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \left(\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x+y_1+\cdots+y_r}{\lambda}} d\mu_0(y_1) \cdots d\mu_0(y_r) \\
 &= \left(\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} \right)^s \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \left(\frac{\log(1 + \lambda t)}{\lambda t} \right)^r (1 + \lambda t)^{\frac{x}{\lambda}}.
 \end{aligned} \tag{45}$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (45), we get

$$\sum_{n=0}^{\infty} \beta \mathcal{E}_n^{(r,s)}(\lambda, x) \frac{\lambda^{-n}}{n!} (e^{\lambda t} - 1)^n = \left(\frac{2}{e^t + 1}\right)^s \left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} E_l^{(s)} B_{n-l}^{(r)}(x)\right) \frac{t^n}{n!}. \tag{46}$$

On the other hand,

$$\begin{aligned} \sum_{m=0}^{\infty} \beta \mathcal{E}_m^{(r,s)}(\lambda, x) \frac{\lambda^{-m}}{m!} (e^{\lambda t} - 1)^m &= \sum_{m=0}^{\infty} \beta \mathcal{E}_m^{(r,s)}(\lambda, x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \beta \mathcal{E}_m^{(r,s)}(\lambda, x) S_2(n, m) \lambda^{n-m}\right) \frac{t^n}{n!}. \end{aligned} \tag{47}$$

Therefore, by (46) and (47), we obtain the following theorem.

Theorem 2.8. For $n \geq 0$, we have

$$\sum_{m=0}^n \beta \mathcal{E}_m^{(r,s)}(\lambda, x) S_2(n, m) \lambda^{n-m} = \sum_{m=0}^n \binom{n}{m} E_m^{(s)} B_{n-m}^{(r)}(x).$$

For $r, s \in \mathbb{N}$, let us define the higher-order degenerate Euler-Bernoulli mixed-type polynomials as follows:

$$\mathcal{E}\beta_n^{(r,s)}(\lambda, x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \beta_n^{(s)}(\lambda, y_1 + \cdots + y_r + x) d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r). \tag{48}$$

Thus, by (20) and (48), we get

$$\begin{aligned} \mathcal{E}\beta_n^{(r,s)}(\lambda, x) &= \sum_{l=0}^n \binom{n}{l} \beta_l^{(s)}(\lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y_1 + \cdots + y_r + x)_{n-l, \lambda} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \sum_{l=0}^n \binom{n}{l} \beta_l^{(s)}(\lambda) \mathcal{E}_{n-l}^{(r)}(\lambda, x). \end{aligned} \tag{49}$$

The generating function of $\mathcal{E}\beta_n^{(r,s)}(\lambda, x)$ is given by

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}\beta_n^{(r,s)}(\lambda, x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \beta_n^{(s)}(\lambda, y_1 + \cdots + y_r + x) \frac{t^n}{n!} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}\right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{y_1 + \cdots + y_r + x}{\lambda}} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}\right)^s \left(\frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1}\right)^r (1 + \lambda t)^{\frac{x}{\lambda}}. \end{aligned} \tag{50}$$

From (12), we note that

$$\begin{aligned} &\mathcal{E}\beta_n^{(r,s)}(\lambda, x) \\ &= \sum_{l=0}^n \binom{n}{l} \beta_l^{(s)}(\lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y_1 + \cdots + y_r + x)_{n-l, \lambda} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \sum_{l=0}^n \binom{n}{l} \beta_l^{(s)}(\lambda) \sum_{m=0}^{n-l} S_1(n-l, m | \lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (y_1 + \cdots + y_r + x)^m d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \sum_{l=0}^n \binom{n}{l} \beta_l^{(s)}(\lambda) \sum_{m=0}^{n-l} S_1(n-l, m | \lambda) E_m^{(r)}(x). \end{aligned} \tag{51}$$

Therefore, by (49) and (51), we obtain the following theorem.

Theorem 2.9. For $n \geq 0$, we have

$$\mathcal{E}_n^{(r,s)}(\lambda, x) = \sum_{l=0}^n \binom{n}{l} \beta_l^{(s)}(\lambda) \mathcal{E}_{n-l}^{(r)}(\lambda, x) = \sum_{l=0}^n \sum_{m=0}^{n-l} \binom{n}{l} \beta_l^{(s)}(\lambda) S_1(n-l, m | \lambda) E_m^{(r)}(x).$$

Remark. From (43), we have

$$\begin{aligned} \beta \mathcal{E}_n^{(r,s)}(\lambda, x) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \mathcal{E}_n^{(s)}(\lambda, x + y_1 + \cdots + y_r) d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + y_1 + \cdots + y_r)_{n-l, \lambda} d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l} S_1(n-l, m | \lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + y_1 + \cdots + y_r)^m d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \sum_{l=0}^n \binom{n}{l} \mathcal{E}_l^{(s)}(\lambda) \sum_{m=0}^{n-l} S_1(n-l, m | \lambda) B_m^{(r)}(x). \end{aligned}$$

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