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SOME IMMERSIONS IN PSEUDO-RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE

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In [2], Obata considered immersions of Riemannian manifolds in spaces of constant curvature and obtained a relationship among the Ricci form of the immersed manifold and the second and third fundamental forms of the immersion. A geometric interpretation of the third fundamental form was given by using the notion of the Gauss map and several applications as well. He expected one can generalize his method to pseudo-riemannian manifolds with arbitrary signature of metric. The purpose of the present paper is to consider certain immersions in pseudo-riemannian manifolds of constant curvature for which his method can be generalized.

1. Preliminaries.

By a differentiable manifold we will always mean a connected, paracompact, C^∞-differentiable manifold. Moreover, all our functions, forms and mappings will be understood to be C^∞.

A pseudo-riemannian metric b on an n-dimensional differentiable manifold M is a differentiable field of nondegenerate symmetric bilinear forms b_p on the tangent spaces M_p of M. A pseudo-riemannian manifold is a differentiabe manifold with a pseudo-riemannian metric. Since M is connected, the signature of b is constant.

A basis $\{v_1, \cdots, v_n\}$ of M_p is called orthonormal if $b_p(v_i, v_j)=\pm\delta_{ij}$. If U_p is a subset of M_p then $U_p^\perp$ denotes $\{v \in M_p \mid b_p(v, U_p)=0\}$, a linear subspace of M_p. If U_p is a linear subspace, then $\dim U_p + \dim U_p^\perp = n$. A linear subspace U_p of M_p is said to be nondegenerate, if b_p restricts to a nondegenerate form on U_p. This means $U_p \cap U_p^\perp = 0$, so we have $M_p = U_p \oplus U_p^\perp$ for nondegenerate U_p. M_p has a basis $\{v_1, \cdots, v_n\}$ such that $U_p=\{v_1, \cdots, v_s\}$, $s \leqq n$, if U_p is nondegenerate. Furthermore there is an orthonormal basis $\{u_1, \cdots, u_n\}$ such that $\{u_1, \cdots, u_i\}=\{v_1, \cdots, v_i\}$ for $i=1, 2, \cdots, n$. (cf. Wolf [3], p. 50).

Let R_s^n, $0 \leqq s \leqq n$, denote the vector space of real n-tuples $x=(x^1, \cdots, x^n)$ with the bilinear form b defined by

$$b_s^n(x, y) = -\sum_{i=1}^{s} x^i y^i + \sum_{j=s+1}^{n} x^j y^j.$$

Then b is a pseudo-riemannian metric on R_s^n with signature $(s, n-s)$. Let Σ be

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the quadratic in R_s^n defined by

$$\Sigma=\{x\in R_s^n \mid b_s^n(x, x)=\varepsilon r^2\}, \qquad n\geqq 3, \quad \varepsilon=\pm 1.$$

If $x\in\Sigma$ then Σ_x is a nondegenerate subspace of $(R_s^n)_x$, so R_s^n induces a pseudo-riemannian metric on Σ. With this metric, Σ is a complete pseudo-riemannian manifold of constant curvature $K=\varepsilon r^{-2}$ and signature $(s, n-s-1)$ if $\varepsilon=1$, $(s-1, n-s)$ if $\varepsilon=-1$. Let $O^s(n)$ denote the group of all linear isometries of the vector space R_s^n onto itself, then $O^s(n)$ is known to be the group of all isometries of Σ (cf. Wolf [3], p. 66).

Given integers s, n such that $0\leqq s\leqq n$, we define

$$S_s^n=\{x\in R_s^{n+1} \mid b_s^{n+1}(x, x)=r^2\},$$

$$H_s^n=\{x\in R_{s+1}^{n+1} \mid b_{s+1}^{n+1}(x, x)=-r^2\}.$$

Let V denote one of the following simply connected complete pseudo-riemannian manifold of dimension N and of signature $(s, N-s)$:

(i) S_s^N, a pseudo-riemannian sphere,

(ii) R_s^N, a pseudo-euclidean space,

(iii) H_s^N, a pseudo-riemannian hyperbolic space.

The bundle $F(V)$ of the orthonomal frames on V can be identified with the group $G(N)$ which is one of the following according as the type of V:

(i) $O^s(N+1)$,

(ii) $E^s(N)$, the euclidean group of R_s^N consisting of all transformations $y\to\alpha(y)+x$, $\alpha\in O^s(N)$, $x\in R_s^N$,

(iii) $O^{s+1}(N+1)$.

In any case, the isotropy subgroup at a given point is $O^s(N)$ and hence V is the homogeneous space $G(N)/O^s(N)$.

2. Nondegerate isometric immersions.

Definition. Let M, M' be pseudo-riemannian manifolds; b, b' be their pseudo-riemannian metrics respectively and x: $M\to M'$ be an immersion. Then M is called a *nondegenerate isometrically immersed submanifold of* M', if

(i) $b'(dx(u), dx(v))=b(u, v)$,

(ii) $dx(M_p)$ and $(dx(M_p))^{\perp}$ are both nondegenerate linear subspaces of $M'_{x(p)}$.

Let M be a pseudo-riemannian n-dimensional manifold with signature $(t, n-t)$, $t\leqq s$, which is an isometrically immersed nondegenerate submanifold of the space V by a mapping x: $M\to V$. $b_{x(p)}$ restricts to a nondegenerate form on $dx(M_p)$. We denote $dx(M_p)$ by $M_{x(p)}$. Then $V_{x(p)}=M_{x(p)}\oplus M_{x(p)}^{\perp}$. Let $F(M)$ denote the bundle of orthonormal frames on M, $F(V)$ denote the bundle of orthonormal frames on V, and B denote the set of elements $(p, e_1, \cdots, e_N)$ such that $(p, e_1, \cdots, e_n)\in F(M)$, $\{e_{n+1}, \cdots, e_N\}\in M_{x(p)}^{\perp}$ and $(x(p), e_1, \cdots, e_N)\in F(V)$, where e_i, $1\leqq i\leqq n$, are identified with $dx(e_i)$, Then ψ: $B\to M$, $\psi(p, e_1, \cdots, e_N)=p$, can be viewed as a principal bundle with the fiber $O^t(n)\times O^{s-t}(N-n)$. Let $\tilde{x}$: $B\to F(V)=G(N)$ be the natural immersion defined by $\tilde{x}(p, e_1, \cdots, e_N)=(x(p), e_1, \cdots, e_N)$.

We will agree on the following range of indices:

$$1\leqq A, B, C, \cdots\leqq N; \qquad 1\leqq i, j, k, \cdots\leqq n; \qquad n+1\leqq\alpha, \beta, \gamma, \cdots\leqq N.$$

$b(u, v)$ will simply be denoted by (u, v). Now

(1) $$x\colon M\to V\to R_S^{N+1} \qquad (S=s \text{ or } s+1).$$

We have

(2) $$(x, x)=\varepsilon r^2, \qquad (x, e_A)=0, \qquad (e_A, e_B)=\varepsilon_A\delta_{AB} \qquad (\varepsilon_A=\pm 1).$$

From (2), we have linear forms ω_A, ω_{AB} so that

(3) $$dx=\sum \varepsilon_A\omega_A\otimes e_A, \qquad de_A=\sum \varepsilon_B\omega_{AB}\otimes e_B-\frac{\varepsilon}{r^2}\omega_A\otimes x,$$

where

(4) $$\omega_{AB}+\omega_{BA}=0.$$

Exterior differentiation of (3) gives

(5) $$d\omega_A=\sum \varepsilon_B\omega_{AB}\wedge\omega_B, \qquad d\omega_{AB}-\sum \varepsilon_C\omega_{AC}\wedge\omega_{CB}=-\frac{\varepsilon}{r^2}\omega_A\wedge\omega_B.$$

By the theorem of structure equations (Wolf [3], p. 50), we have that $\{\varepsilon_A\omega_{AB}\}$ are the connection forms of V relative to $\{e_A\}$ and that $\{\omega_A\}$ is the dual coframe of $\{e_A\}$ in the sense that $\langle\omega_A, e_A\rangle=\varepsilon_A$. The expression at the right-hand side of the second equation of (5) gives the curvature form of the pseudo-riemannian metric on V.

When submanifold (1) is given, we choose a frame field e_A in a neighborhood of R_S^{N+1} at x such that e_i are tangent vectors to $dx(M)$ at x and e_α span $(dx(M))^\perp$ (w. r. t. V_x) at x. Equations (3), when restricted to this frame field, become

(3′) $$dx=\sum \varepsilon_A\theta_A\otimes e_A, \qquad de_A=\sum \varepsilon_B\theta_{AB}\otimes e_B-\frac{\varepsilon}{r^2}\theta_A\otimes x$$

with

(6) $$\theta_\alpha=0.$$

The pseudo-riemannian metric ds^2 on $x(M)$ is given by

(7) $$\mathrm{I}=ds^2=\sum \varepsilon_i\theta_i\otimes\theta_i.$$

The $\varepsilon_i\theta_{ij}$ $(1\leqq i, j\leqq n)$ are connection forms of the induced metric on M and its curvature forms are

(8) $$d\theta_{ij}-\sum \varepsilon_k\theta_{ik}\wedge\theta_{kj}=\sum \varepsilon_\alpha\theta_{i\alpha}\wedge\theta_{\alpha j}-\frac{\varepsilon}{r^2}\theta_i\wedge\theta_j.$$

Taking the exterior differentiation of (6) and making use of (5), we get

$$\sum_i \varepsilon_i\theta_i\wedge\theta_{i\alpha}=0.$$

By Cartan's lemma, we have

$$\theta_{i\alpha}=\sum_j h_{i\alpha j}\varepsilon_j\theta_j, \qquad h_{i\alpha j}=h_{j\alpha i}.$$

The form

$$\mathrm{II}=\sum_\alpha \varepsilon_\alpha\Theta_\alpha\otimes e, \qquad \text{where} \qquad \Theta_\alpha=\sum_{i,j}\varepsilon_i\varepsilon_j h_{i\alpha j}\theta_i\otimes\theta_j$$

is called the *second fundamental form* of M in V.

The *mean curvature vector* of M in V is defined by

$$N=\frac{1}{n}\sum_{j,\alpha}\varepsilon_j\varepsilon_\alpha h_{j\alpha j}e_\alpha=\frac{1}{n}\sum \varepsilon_\alpha h_\alpha e_\alpha, \qquad \text{where} \qquad h_\alpha=\sum_i \varepsilon_i h_{i\alpha i}.$$

The curvature form (8), denoted by Ω_{ij}, is written as

$$\begin{aligned}\Omega_{ij}&=d\theta_{ij}-\sum_k \varepsilon_k\theta_{ik}\wedge\theta_{kj}\\ &=-\sum_{\alpha,l,m}\varepsilon_\alpha\varepsilon_l\varepsilon_m h_{i\alpha l}h_{j\alpha m}\theta_l\wedge\theta_m-\frac{\varepsilon}{r^2}\theta_i\wedge\theta_j.\end{aligned}$$

Let us put

$$\Omega_{ij}=\frac{1}{2}\sum_{k,l}K_{ijkl}\varepsilon_k\varepsilon_l\theta_k\wedge\theta_l.$$

Then

(9) $$K_{ijlm}=-\frac{\varepsilon}{r^2}\varepsilon_l\varepsilon_m(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})-\sum_\alpha \varepsilon_\alpha(h_{i\alpha l}h_{j\alpha m}-h_{i\alpha m}h_{j\alpha l}).$$

The immersion satisfying that $\mathrm{II}=0$ is called *totally geodesic immersion.* If the mean curvature vector vanishes identically, then the immersion is said to be *minimal.*

Let a be a fixed vector in R_S^{N+1}. We consider the height function (a, x) as a function on $x(M)$. By (3′) and (6) we get

$$\begin{aligned}d(x, a)&=\sum_i \varepsilon_i(a, e_i)\theta_i,\\ D(a, e_i)&=(a, De_i)\\ &=\left(a, \sum_B \varepsilon_B\theta_{iB}\otimes e_B-\frac{\varepsilon}{r^2}\theta_i\otimes x+\sum_j \varepsilon_j\theta_{ji}\otimes e_j\right)\end{aligned}$$

$$=\left(a, \sum_{\alpha} \varepsilon_\alpha \theta_{i\alpha} \otimes e_\alpha - \frac{\varepsilon}{r^2} \theta_i \otimes x\right)$$

$$=\left(a, \sum_{\alpha,j} \varepsilon_\alpha \varepsilon_j h_{i\alpha j} \theta_j \otimes e_\alpha - \frac{\varepsilon}{r^2} \theta_i \otimes x\right)$$

$$=\sum_{\alpha,j} \varepsilon_\alpha \varepsilon_j h_{i\alpha j}(a, e_\alpha)\theta_j - \frac{\varepsilon}{r^2}(x, a)\theta_i.$$

Thus we have

$$(a, e_i),_i = (D(a, e_i), e_i) = \sum_{\alpha} \varepsilon_\alpha h_{i\alpha i}(a, e_\alpha) - \frac{\varepsilon\varepsilon_i}{r^2}(x, a).$$

On the other hand, we have

$$\Delta(x, a) = \sum_i (x, a),_{i,i} = \sum_i (x,_i, a),_i$$

$$= \sum_i \varepsilon_i (a, e_i),_i = \sum_{\alpha,i} \varepsilon_\alpha \varepsilon_i h_{i\alpha i}(a, e_\alpha) - \frac{n\varepsilon}{r^2}(x, a)$$

$$= n(a, N) - \frac{n\varepsilon}{r^2}(x, a).$$

Thus we have the following theorem: (cf. Chern [1], § 5A))

THEOREM 1. *Let x be a nondegenerate isometric immersion of M in V. Then x is a minimal immersion if and only if there is a fixed vector a in R_S^{N+1} so that the functions (a, x) satisfy the differential equation*

$$\Delta(x, a) + \frac{n\varepsilon}{r^2}(x, a) = 0.$$

Let $X = \sum_\alpha \varepsilon_\alpha X_\alpha e_\alpha$ be a normal vector of $x(M)$ at x. Then the quadratic differential form defined by

$$\mathrm{II}_X = \langle \mathrm{II}, X\rangle = \sum_{\alpha,i,k} \varepsilon_\alpha \varepsilon_i \varepsilon_k h_{i\alpha k} X_\alpha \theta_i \otimes \theta_k$$

is called the second fundamental form of the immersion x in the direction X. For the mean curvature vector N we have

$$\mathrm{II}_N = \frac{1}{n} \sum_{\alpha,i,k} \varepsilon_\alpha \varepsilon_i \varepsilon_k h_{i\alpha k} h_\alpha \theta_i \otimes \theta_k. \tag{10}$$

It is clear that $\mathrm{II}_N = 0$ if and only if $N = 0$.

By (8), the Ricci tensor $K_{jl} = \sum_i \varepsilon_i K_{ijli}$ is written as

$$K_{jl} = \frac{\varepsilon}{r^2}(n-1)\varepsilon_l \delta_{jl} + \sum_\alpha \varepsilon_\alpha h_{j\alpha l} h_\alpha - \sum_{\alpha,i} \varepsilon_\alpha \varepsilon_i h_{i\alpha l} h_{j\alpha i}. \tag{11}$$

The Ricci form ψ is defined by

(12) $$\psi=\sum_{j,k}\varepsilon_j\varepsilon_k K_{jk}\theta_j\otimes\theta_k.$$

By (7), (10), (11) and (12), we have

$$\psi=\frac{\varepsilon(n-1)}{r^2}\sum_i\varepsilon_i\theta_i\otimes\theta_i+n\mathrm{II}_N-\sum_\alpha\varepsilon_\alpha\varepsilon_i\theta_{i\alpha}\otimes\theta_{i\alpha}.$$

If we write $\mathrm{III}=\sum_{\alpha,i}\varepsilon_\alpha\varepsilon_i(\theta_{i\alpha})^2$, then we have finally

$$\psi=\frac{\varepsilon(n-1)}{r^2}\mathrm{I}+n\mathrm{II}_N-\mathrm{III}.$$

THEOREM 2. *Suppose that a pseudo-riemannian manifold M is nondegenerate isometrically immersed into a simply-connected complete pseudo-riemannian space of constant curvature ε/r^2. Then*

$$\psi-n\mathrm{II}_N+\mathrm{III}=\frac{\varepsilon(n-1)}{r^2}\mathrm{I}.$$

holds.

3. The generalized Gauss map.

Let Q be the set of all totally geodesic n-space with signature $(t,n-t)$ in V. The group $G(N)$ acts on Q transitively, and Q is identified with a homogeneous space

$$Q=G(N)/G(n)\times O^{s-t}(N-n).$$

We introduce a quadratic differential form $d\Sigma^2$ on Q:

$$d\Sigma^2=\sum_\alpha\varepsilon_\alpha\theta_\alpha\otimes\theta_\alpha+\sum_{\alpha,i}\varepsilon_\alpha\varepsilon_i\theta_{i\alpha}\otimes\theta_{i\alpha}.$$

With the immersion x: $M\to V$ we associate the generalized Gauss map f: $M\to Q$ where $f(p)$, $p\in M$, is the totally geodesic n-space tangent to $x(M)$ at $x(p)$. Consider the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{F} & F(V)=G(N) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & Q(N)/G(n)\times O^{s-t}(N-n) \end{array}$$

where π is the natural projection and F is the natural identification. It is clear that

$$\mathrm{III}=f^*(d\Sigma^2).$$

Thus we have

THEOREM 3. *The generalized Gauss mas f is a constant map if and only if the immersion x is totally geodesic.*

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