

SOME IMPROVEMENTS FOR A CLASS OF THE CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

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ABSTRACT. In this paper, we concern a weighted version of the Hardy inequality, which is a special case of the more general Caffarelli-Kohn-Nirenberg inequalities. We improve the inequality on the whole space or on a bounded domain by adding various remainder terms. On the whole space, we show the existence of a remainder term which has the form of ratio of two weighted integrals. Also we give a simple derivation of the remainder term involving a distance from the manifold of the “virtual extremals”. Finally on a bounded domain, we prove the existence of remainder terms involving the gradient of functions.

1. INTRODUCTION

In this paper, we are concerned with the weighted version of the Hardy inequality:

$$(1.1) \quad \int_{\Omega} |\nabla u|^p |x|^{-pa} dx \geq \left(\frac{N - p - pa}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^{p(a+1)}} dx$$

for all $u \in C_0^\infty(\Omega)$, where Ω is a smooth bounded domain in \mathbb{R}^N ($N \geq 3$) with $0 \in \Omega$, or $\Omega = \mathbb{R}^N$, $1 < p < N$ and $-\infty < a < \frac{N-p}{p}$. Actually, much more general weighted type inequalities are shown by Caffarelli, Kohn and Nirenberg [4] and (1.1) is one of the special cases. Let $D_{0,a}^{1,p}(\Omega)$ and $W_{0,a}^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to each norm

$$\|u\|_{D_{0,a}^{1,p}(\Omega)} = \left(\int_{\Omega} |\nabla u|^p |x|^{-pa} dx \right)^{1/p},$$
$$\|u\|_{W_{0,a}^{1,p}(\Omega)} = \left(\int_{\Omega} (|\nabla u|^p + |u|^p) |x|^{-pa} dx \right)^{1/p},$$

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respectively. Then (1.1) holds true for all $u \in D_{0,a}^{1,p}(\Omega)$. If Ω is bounded, the Poincaré type inequality implies that $D_{0,a}^{1,p}(\Omega) = W_{0,a}^{1,p}(\Omega)$. Also the constant $\left(\frac{N-p-pa}{p}\right)^p$ in (1.1) is known optimal and never attained in $D_{0,a}^{1,p}(\Omega)$.

When $a = 0$, (1.1) becomes the classical Hardy inequality

$$(1.2) \quad \int_{\Omega} |\nabla u|^p dx \geq \left(\frac{N-p}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} dx,$$

again the equality in (1.2) is never achieved by any function in $D_{0,0}^{1,p}(\Omega)$. There are many papers up to now that treat the improvement of (1.2) when Ω is a smooth bounded domain (see [2], [3], [6], [8], [10], [11], [12], [17], [18], and references therein). On the other hand, when $\Omega = \mathbb{R}^N$, Ghoussoub and Moradifam [11] show that there is no strictly positive $V \in C^1((0, \infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V(|x|)|u|^2 dx$$

holds for all $u \in W^{1,2}(\mathbb{R}^N)$. Therefore we cannot expect the same type of remainder terms as in the bounded domain case, one of the reasons of what is a lack of the Poincaré inequality in the whole space. Instead, Cianchi and Ferone [7] provided the following “non-standard” remainder term: Let $p^* = \frac{Np}{N-p}$ be the critical Sobolev exponent, $u_0(x) = |x|^{-\frac{N-p}{p}}$ for $x \in \mathbb{R}^N$, and define

$$d_p(u) = \inf_{c \in \mathbb{R}} \frac{\|u - cu_0\|_{L^{p^*, \infty}(\mathbb{R}^N)}}{\|u\|_{L^{p^*, p}(\mathbb{R}^N)}} \quad (1 < p < N).$$

Here $L^{\tau, \sigma}(\mathbb{R}^N)$ ($0 < \tau \leq \infty, 1 \leq \sigma \leq \infty$) is the Lorentz space with the norm

$$\|u\|_{L^{\tau, \sigma}(\mathbb{R}^N)} = \|s^{\frac{1}{\tau} - \frac{1}{\sigma}} u^*(\cdot)\|_{L^{\sigma}(0, \infty)},$$

where u^* denotes the (one-dimensional) decreasing rearrangement of u . Then in [7] it is shown that for any $1 < p < N$ there exists a constant $C = C(p, N)$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx (1 + Cd_p(u)^{2p^*})$$

holds for every real-valued weakly differentiable function u in \mathbb{R}^N decaying to zero at infinity with $|\nabla u| \in L^p(\mathbb{R}^N)$. Recently, the authors of this paper have succeeded to obtain a remainder term for the classical Hardy inequality (1.2) on the whole space [16]. Our method consists

of the well-known Brezis-Vázquez transformation [3] and the use of the Gagliardo-Nirenberg inequality.

Concerning the inequality (1.1), on the other hand, Wang and Willem [19] obtained the following improved version of (1.1) on a bounded domain $\Omega \subset \mathbb{R}^N$: Assume $\Omega \subset \subset B_R(0)$ for some $R > 0$. Then

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx \\ & \geq C \int_{\Omega} \left(\log \left(\frac{R}{|x|} \right) \right)^{-2} |x|^{-2a} |\nabla u|^2 dx \end{aligned}$$

holds for all $u \in D_{0,a}^{1,2}(\Omega)$, where $C = C(\Omega, a)$ is a positive constant. Their method consists of the use of a conformal transformation introduced by Catrina and Wang [5], which transforms the problem on a bounded domain of \mathbb{R}^N to that of a cylinder $\mathcal{C} = \mathbb{R} \times S^{N-1}$; see §4. Later, Abdellaoui, Colorado and Peral [1] obtained the following improvement of (1.1): Let $|\Omega|$ denote the volume of Ω . Then for all $1 < q < p$, there exists a positive constant $C = C(N, p, q, a, |\Omega|)$ such that the inequality

$$\begin{aligned} & \int_{\Omega} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ & \geq C \left(\int_{\Omega} |x|^{-ra} |\nabla u|^q dx \right)^{p/q} \end{aligned}$$

holds true for all $u \in D_{0,a}^{1,p}(\Omega)$, where r is any number such that $q \leq r < \infty$ if $a \leq 0$ and $1 \leq r < p + \rho$ for some positive constant ρ when $a > 0$. Their method is based on the Picone type inequality for the operator $\operatorname{div}(|x|^{-pa} |\nabla u|^{p-2} \nabla u)$. Note that from the proof of [1], the constant $C(N, p, q, a, |\Omega|) \rightarrow 0$ as $|\Omega| \rightarrow \infty$.

In this paper, firstly we improve the inequality (1.1) when $\Omega = \mathbb{R}^N$ by adding a remainder term of the form of ratio of two weighted integrals to (1.1).

Theorem 1. *(A remainder term of the form of ratio on the whole space) Let $N \geq 3$, $2 \leq p < N$ and $-\infty < a < \frac{N-p}{p}$. For given $n \in \mathbb{N}$, $t \in (0, 1)$ and $\gamma < \min\{1-t, \frac{p-n}{p}\}$, set $\delta = n - N + \frac{n}{1-t-\gamma} \left(\gamma + \frac{N-p-pa}{p} \right)$.*

Then there exists a constant $C > 0$ such that the inequality

$$(1.3) \quad \begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ & \geq C \frac{\left(\int_{\mathbb{R}^N} |x|^\delta |u|^{\frac{n}{1-t-\gamma}} dx \right)^{\frac{p(1-t-\gamma)}{nt}}}{\left(\int_{\mathbb{R}^N} |x|^{-pa} |u|^p dx \right)^{\frac{1-t}{t}}} \end{aligned}$$

holds for all radially symmetric function $u \in W_{0,a}^{1,p}(\mathbb{R}^N)$, $u \not\equiv 0$.

Remark 1. If we put $F_{A,B}(u) = \int_{\mathbb{R}^N} |u|^A |x|^B dx$ for constants A, B , and put $u_\lambda(x) = \lambda^C u(\lambda x)$ for $x \in \mathbb{R}^N$ and $C \in \mathbb{R}$, then a simple computation shows that $F_{A,B}(u_\lambda) = \lambda^{AC-B-N} F_{A,B}(u)$. The remainder term in Theorem 1 can be written as

$$R(u) = \frac{\left(\int_{\mathbb{R}^N} |u|^{\frac{n}{1-t-\gamma}} |x|^\delta dx \right)^{\frac{p(1-t-\gamma)}{nt}}}{\left(\int_{\mathbb{R}^N} |u|^p |x|^{-pa} dx \right)^{\frac{1-t}{t}}} = \frac{\left\{ F_{\frac{n}{1-t-\gamma}, \delta}(u) \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ F_{p, -pa}(u) \right\}^{\frac{1-t}{t}}},$$

therefore it satisfies

$$R(u_\lambda) = \frac{\left\{ F_{\frac{n}{1-t-\gamma}, \delta}(u_\lambda) \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ F_{p, -pa}(u_\lambda) \right\}^{\frac{1-t}{t}}} = \frac{\left\{ \lambda^{(\frac{n}{1-t-\gamma})C - \delta - N} F_{\frac{n}{1-t-\gamma}, \delta}(u) \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ \lambda^{pC + pa - N} F_{p, -pa}(u) \right\}^{\frac{1-t}{t}}}.$$

If we put $C = \frac{N-p-pa}{p}$, we see

$$\begin{aligned} R(u_\lambda) &= \frac{\left\{ \lambda^{(\frac{n}{1-t-\gamma})(\frac{N-p-pa}{p}) - \delta - N} \right\}^{\frac{p(1-t-\gamma)}{nt}}}{\left\{ \lambda^{p(\frac{N-p-pa}{p}) + pa - N} \right\}^{\frac{1-t}{t}}} R(u) = \left(\frac{\lambda^{-p(\frac{1-t}{t})}}{\lambda^{-p(\frac{1-t}{t})}} \right) R(u) \\ &= R(u). \end{aligned}$$

Thus the remainder term is invariant under the scaling $u(x) \mapsto u_\lambda(x) = \lambda^{\frac{N-p-pa}{p}} u(\lambda x)$.

Remark 2. If we choose n, t, γ satisfying $\frac{n}{1-t-\gamma} = p$, then (1.3) in Theorem 1 also holds even for non-radial functions. Indeed, for a non-radial function u , let us consider the radial function

$$U(r) = \left(\omega_N^{-1} \int_{S^{N-1}} |u(r\omega)|^p dS_\omega \right)^{\frac{1}{p}}.$$

Then Hölder's inequality implies that

$$U'(r) \leq \left(\omega_N^{-1} \int_{S^{N-1}} |\partial_r u(r\omega)|^p dS_\omega \right)^{\frac{1}{p}},$$

so we obtain

$$(1.4) \quad \omega_N \int_0^\infty |U'(r)|^p r^{N-1-pa} dr \leq \int_{\mathbb{R}^N} \left| \nabla u \cdot \frac{x}{|x|} \right|^p |x|^{-pa} dx.$$

Also we have

$$(1.5) \quad \int_{\mathbb{R}^N} |U(|x|)|^p |x|^A dx = \int_{\mathbb{R}^N} |u|^p |x|^A dx$$

for any $A \in \mathbb{R}$. Thus when $\frac{n}{1-t-\gamma} = p$, (1.3) for U , (1.4) and (1.5) imply that the same inequality holds for all non-radial functions.

Note that the standard rearrangement argument, see for example [13], is not applicable because of the presence of weights, since a may be negative and $|x|^{-p(a+1)}$ may be an increasing function. Thus, differently from [16], here we use another type of pointwise estimate for the expansion of $|a - b|^p$, see Lemma 1. The use of full version of the Caffarelli-Kohn-Nirenberg inequality, see Proposition 1, is another key point. Since the technique used here is different from that of [16], the remainder term obtained in Theorem 1 is also different from that in [16] even when $a = 0$.

In §3, we improve (1.1) on the whole space by adding a remainder term which involves a distance from “the manifold of the virtual extremals” $\{cu_a \mid c \in \mathbb{R}\}$, where $u_a(x) = |x|^{-\frac{N-p-pa}{p}}$. For the proof, differently from that of Cianchi-Ferone [7], we use a new inequality recently obtained by Machihara, Ozawa and Wadade [14]. It is surprising for the authors that a direct use of the Machihara-Ozawa-Wadade inequality leads to the existence of “non-standard” type remainder terms for (1.1) very simply, at least in the radially symmetric case.

In §4, we will show some improvements of (1.1) when $p = 2$:

$$\int_{\Omega} |\nabla u|^2 |x|^{-2a} dx \geq \left(\frac{N-2-2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx,$$

where Ω is a bounded domain in \mathbb{R}^N ($N \geq 3$). In this part, the method of the proof is to combine two ideas: One is to transform the problem to the one on a cylinder, which was initiated by Catrina and Wang [5] in this context, and the another is to improve the one-dimensional Hardy (weighted Poincaré) inequality by the method of Picone’s identity see [11], [12] for $p = 2$ case and [1] for $p \neq 2$ case.

2. PROOF OF THEOREM 1.

In this section, we prove Theorem 1. For the proof, we need the following lemma.

Lemma 1 ([9]). *Let $p \geq 2$ and a, b be real numbers. Then there exists $c_p > 0$ such that*

$$|a - b|^p \geq |a|^p - p|a|^{p-2}ab + c_p|b|^p$$

holds true. c_p is explicitly given as $c_p = \min_{0 < t \leq 1/2} ((1-t)^p - t^p + pt^{p-1})$ and sharp in this inequality.

We recall here the following inequality obtained by Caffarelli, Kohn and Nirenberg [4] in its full version:

Proposition 1. (Caffarelli-Kohn-Nirenberg [4]) *Let $n \in \mathbb{N}$ and let $p, q, r, \alpha, \beta, \sigma$ and t be real constants such that $p, q \geq 1$, $r > 0$, $0 \leq t \leq 1$, and*

$$\frac{1}{p} + \frac{\alpha}{n}, \quad \frac{1}{q} + \frac{\beta}{n}, \quad \frac{1}{r} + \frac{\gamma}{n} > 0$$

where $\gamma = t\sigma + (1-t)\beta$. Then there exists a positive constant C such that

$$(2.1) \quad \left\| |x|^\gamma u \right\|_{L^r(\mathbb{R}^n)} \leq C \left\| |x|^\alpha |\nabla u| \right\|_{L^p(\mathbb{R}^n)}^t \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^n)}^{1-t}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

(1) *(balance of dimension)*

$$\frac{1}{r} + \frac{\gamma}{n} = t \left(\frac{1}{p} + \frac{\alpha - 1}{n} \right) + (1-t) \left(\frac{1}{q} + \frac{\beta}{n} \right),$$

(2) $0 \leq \alpha - \sigma$ *if* $t > 0$,

(3) $1 \geq \alpha - \sigma$ *if* $t > 0$ *and* $\frac{1}{r} + \frac{\gamma}{n} = \frac{1}{p} + \frac{\alpha-1}{n}$.

Proof of Theorem 1. We show Theorem 1 for a radial function $u \in C_0^\infty(\mathbb{R}^N)$. Then a density argument implies the desired result. Since u is radial, u can be written as $u(x) = \tilde{u}(|x|)$ for some function $\tilde{u} \in C_0^\infty([0, +\infty))$. Appealing to Brezis-Vázquez's idea, we put

$$(2.2) \quad \tilde{v}(r) = r^{\frac{N-p-p\alpha}{p}} \tilde{u}(r).$$

We see $\tilde{v}(0) = 0$ since $a < \frac{N-p}{p}$ and $\tilde{v}(+\infty) = 0$, since $\tilde{u} \equiv 0$ near $r = +\infty$. Put $v(y) = \tilde{v}(|y|)$ for $y \in \mathbb{R}^n$, $n \in \mathbb{N}$. Calculation shows that

$$\begin{aligned} J &:= \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ &= \omega_N \int_0^\infty |\tilde{u}'(r)|^p r^{-pa} r^{N-1} dr - \omega_N \left(\frac{N-p-pa}{p} \right)^p \int_0^\infty |\tilde{u}(r)|^p r^{-p(a+1)} r^{N-1} dr \\ &= \omega_N \int_0^\infty \left| \left(\frac{N-p-pa}{p} \right) r^{-\frac{N-pa}{p}} \tilde{v}(r) - r^{-\frac{N-p-pa}{p}} \tilde{v}'(r) \right|^p r^{N-1-pa} dr \\ &\quad - \omega_N \left(\frac{N-p-pa}{p} \right)^p \int_0^\infty |\tilde{v}(r)|^p r^{-1} dr. \end{aligned}$$

We apply Lemma 1 to the integrand of the first term:

$$\begin{aligned} &\left| \left(\frac{N-p-pa}{p} \right) r^{-\frac{N-pa}{p}} \tilde{v}(r) - r^{-\frac{N-p-pa}{p}} \tilde{v}'(r) \right|^p r^{N-1-pa} \\ &\geq \left[\left(\frac{N-p-pa}{p} \right)^p r^{-N+pa} |\tilde{v}(r)|^p \right. \\ &\quad \left. - p \left(\frac{N-p-pa}{p} \right)^{p-1} |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r) r^{-\left(\frac{N-pa}{p}\right)(p-1)} r^{-\left(\frac{N-p-pa}{p}\right)} \right. \\ &\quad \left. + c_p |\tilde{v}'(r)|^p r^{-N+p+pa} \right] r^{N-1-pa} \\ &= \left(\frac{N-p-pa}{p} \right)^p r^{-1} |\tilde{v}(r)|^p - p \left(\frac{N-p-pa}{p} \right)^{p-1} |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r) \\ &\quad + c_p |\tilde{v}'(r)|^p r^{p-1}. \end{aligned}$$

By using the fact $\tilde{v}(0) = \tilde{v}(+\infty) = 0$ and $p \geq 2$, we see

$$p \int_0^\infty |\tilde{v}(r)|^{p-2} \tilde{v}(r) \tilde{v}'(r) dr = \int_0^\infty \frac{d}{dr} (|\tilde{v}(r)|^p) dr = 0.$$

Finally, we note that the terms involving $\int_0^\infty |\tilde{v}(r)|^p r^{-1} dr$ cancel out by subtracting each other. Thus we obtain

$$(2.3) \quad J \geq c_p \omega_N \int_0^\infty |\tilde{v}'(r)|^p r^{p-1} dr = c_p \frac{\omega_N}{\omega_n} \int_{\mathbb{R}^n} |\nabla v(y)|^p |y|^{p-n} dy.$$

From now on, we estimate the right hand side of (2.3) by using the Caffarelli-Kohn-Nirenberg inequality (2.1) (Proposition 1) for v on \mathbb{R}^n .

We take

$$q = p \quad \text{and} \quad \alpha = \beta = \frac{p-n}{p}.$$

By these choices, we see

$$\frac{1}{p} + \frac{\alpha}{n} = \frac{1}{q} + \frac{\beta}{n} = \frac{1}{n} > 0.$$

The first condition of the Proposition 1 (balance of dimension) reduces to

$$\frac{1}{r} + \frac{\gamma}{n} = \frac{1-t}{n}, \quad \text{i.e.,} \quad r = \frac{n}{1-t-\gamma}.$$

Thus $\frac{1}{r} + \frac{\gamma}{n} > 0$ if $t < 1$, and by the assumption $\gamma < \min\{1-t, \frac{p-n}{p}\}$, r is positive and the condition $\alpha \geq \sigma = \frac{\gamma-(1-t)\beta}{t}$ when $t > 0$ is fulfilled. Also under the choice $0 < t < 1$, we see $0 = \frac{1}{p} + \frac{\alpha-1}{n} \neq \frac{1}{r} + \frac{\gamma}{n} = \frac{1-t}{n}$, thus we do not need to consider the third condition. In conclusion, we assure that the following inequality holds true for v :

$$\left\| |y|^\gamma v \right\|_{L^{\frac{n}{1-t-\gamma}}(\mathbb{R}^n)} \leq C \left\| |y|^{\frac{p-n}{p}} |\nabla v| \right\|_{L^p(\mathbb{R}^n)}^t \left\| |y|^{\frac{p-n}{p}} v \right\|_{L^p(\mathbb{R}^n)}^{1-t},$$

that is,

$$\left(\int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{1-t-\gamma}} |v|^{\frac{n}{1-t-\gamma}} dy \right)^{\frac{p(1-t-\gamma)}{nt}} \leq C \left(\int_{\mathbb{R}^n} |y|^{p-n} |\nabla v|^p dy \right) \left(\int_{\mathbb{R}^n} |y|^{p-n} |v|^p dy \right)^{\frac{1-t}{t}}.$$

Combining this to (2.3), we have

$$(2.4) \quad J \geq C' \frac{\left(\int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{1-t-\gamma}} |v|^{\frac{n}{1-t-\gamma}} dy \right)^{\frac{p(1-t-\gamma)}{nt}}}{\left(\int_{\mathbb{R}^n} |y|^{p-n} |v|^p dy \right)^{\frac{1-t}{t}}}$$

where $C' = C^{-1} c_p \frac{\omega_N}{\omega_n}$. By the definition $v(y) = \tilde{v}(|y|)$, $y \in \mathbb{R}^n$ and the assumption that $u \in W_{0,a}^{1,p}(\mathbb{R}^N)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |y|^{p-n} |v|^p dy &= \omega_n \int_0^\infty r^{N-p-pa} |\tilde{u}(r)|^p r^{p-n} r^{n-1} dr \\ &= \frac{\omega_n}{\omega_N} \int_{\mathbb{R}^N} |u|^p |x|^{-pa} dx < \infty. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^n} |y|^{\frac{n\gamma}{1-t-\gamma}} |v|^{\frac{n}{1-t-\gamma}} dy &= \omega_n \int_0^\infty \left(r^{\frac{N-p-pa}{p}} |\tilde{u}(r)| \right)^{\frac{n}{1-t-\gamma}} r^{\frac{n\gamma}{1-t-\gamma}} r^{n-1} dr \\ &= \omega_n \int_0^\infty |\tilde{u}(r)|^{\frac{n}{1-t-\gamma}} r^{\frac{n}{1-t-\gamma}(\gamma + \frac{N-p-pa}{p}) + n - N} r^{n-1} dr \\ &= \frac{\omega_n}{\omega_N} \int_{\mathbb{R}^N} |u|^{\frac{n}{1-t-\gamma}} |x|^\delta dx. \end{aligned}$$

Inserting these into (2.4), we obtain the desired conclusion. \square

Remark 3. If we assume that the function $u \in W_{0,a}^{1,p}(\mathbb{R}^N)$ is nonnegative, radially symmetric and radially decreasing, then we can apply the same argument in [16] also in our situation. In this case we obtain the following theorem, the proof of it is exactly the same as in [16].

Theorem 2. *For given $N \geq 3$, $2 \leq p < N$ and $q > 2$, set $\alpha = \alpha(p, q, N) = 2 - N + \frac{q(N-p-pa)}{2}$. Then there exists $D = D(p, q, N) > 0$ such that the inequality*

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ & \geq D \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-p-pa} dx} \right)^{\frac{2}{q-2}} \end{aligned}$$

holds for all nonnegative, radially symmetric and radially decreasing function $u \in W_{0,a}^{1,p}(\mathbb{R}^N)$, $u \not\equiv 0$.

3. A REMAINDER TERM INVOLVING A DISTANCE FROM THE VIRTUAL EXTREMALS

For $-\infty < a < \frac{N-p}{p}$, let $u_a(x) = |x|^{-\frac{N-p-pa}{p}}$. Note that u_a is a solution to the Euler-Lagrange equation associated with the best constant of the inequality (1.1)

$$\begin{aligned} -\operatorname{div}(|x|^{-pa} |\nabla u|^{p-2} \nabla u) &= \left(\frac{N-p-pa}{p} \right)^p |x|^{-p(a+1)} u^{p-1}, \\ u &\geq 0 \quad \text{in } \mathbb{R}^N, \end{aligned}$$

however, $u_a \notin D_{0,a}^{1,p}(B_R(0))$ for any $R > 0$. Thus u_a is not a genuine minimizer for the best constant of (1.1) in the admissible class $D_{0,a}^{1,p}(\mathbb{R}^N)$, but just approximates the non-existing extremals on the whole space.

In this section, we prove an improved version of (1.1), which involves a sort of the “distance” of the associated function u from the one-dimensional space of “virtual extremals” $\{cu_a \mid c \in \mathbb{R}\}$. For the (sub-critical, also the critical) Hardy, or higher order Hardy-Rellich inequalities, see [15].

For $R > 0$, let us define

$$(3.1) \quad d_R(f, g) = \left(\int_{\mathbb{R}^N} \frac{|f(x) - g(x)|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{p(a+1)}} dx \right)^{1/p}$$

for functions f, g , for which the right hand side is finite.

Theorem 3. *Let $N \geq 3$, $2 \leq p < N$ and assume $-\infty < a < \frac{N-p}{p}$. Then*

$$(3.2) \quad \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ \geq c_p \left(\frac{p-1}{p} \right)^p \sup_{R>0} \inf_{c \in \mathbb{R}} d_R(u, cu_a)^p$$

holds for any radially symmetric function $u \in D_{0,a}^{1,p}(\mathbb{R}^N)$. Here c_p is a constant in Lemma 1 and $d_R(\cdot, \cdot)$ is defined in (3.1).

For the proof, we need the following result.

Proposition 2. (Machihara-Ozawa-Wadade [14]:Theorem 1.1) *Let $N \in \mathbb{N}$, $1 < \alpha < \infty$ and $\max\{1, \alpha - 1\} < \beta < \infty$. Then for any $R > 0$, the inequality*

$$(3.3) \quad \int_{\mathbb{R}^N} \frac{|f(x) - f(R\frac{x}{|x|})|^\beta}{\left| \log \frac{R}{|x|} \right|^\alpha |x|^N} dx \leq \left(\frac{\beta}{\alpha - 1} \right)^\beta \int_{\mathbb{R}^N} \frac{\left| \frac{x}{|x|} \cdot \nabla f(x) \right|^\beta}{|x|^{N-\beta} \left| \log \frac{R}{|x|} \right|^{\alpha-\beta}} dx$$

holds for all $f \in W^1 L_{N,\beta, \frac{\beta-\alpha}{\beta}}(\mathbb{R}^N)$. Also the constant $\left(\frac{\beta}{\alpha-1} \right)^\beta$ is best possible in (3.3).

Remark 4. Here, $W^1 L_{p,q,\lambda}(\mathbb{R}^N)$ denotes the Sobolev-Lorentz-Zygmund spaces. For the precise definition of these spaces, we refer to [14]. However, we note that if $\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) |x|^{p-N} dx < \infty$, then the function $u \in W^1 L_{N,p,0}(\mathbb{R}^N)$.

Proof of Theorem 3. First, we prove Theorem for a radial function $u \in C_0^\infty(\mathbb{R}^N)$. Let $u \in C_0^\infty(\mathbb{R}^N)$, $u(x) = \tilde{u}(r)$, $r = |x|$ be a radial function. Define $v(x) = \tilde{v}(|x|)$ for $x \in \mathbb{R}^N$ where $\tilde{v}(r)$ is defined in (2.2). As in the proof of Theorem 1, we obtain

$$(3.4) \quad J(u) = \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-pa} dx - \left(\frac{N-p-pa}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{p(a+1)}} dx \\ \geq c_p \omega_N \int_0^\infty |\tilde{v}'_r|^p r^{p-1} dr = c_p \int_{\mathbb{R}^N} |\nabla v|^p |x|^{p-N} dx,$$

since we assume $p \geq 2$, see (2.3). Here we claim that if $u \in C_0^\infty(\mathbb{R}^N) \subset W_{0,a}^{1,p}(\mathbb{R}^N)$, v satisfies

$$\int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p) |x|^{p-N} dx < \infty.$$

In particular, $v \in W^1 L_{N,p,0}(\mathbb{R}^N)$ by the above remark. Indeed, from (3.4), we have $\int_{\mathbb{R}^N} |\nabla v|^p |x|^{p-N} dx \leq J(u)/c_p < \infty$. Also by the definition of v , we see $\int_{\mathbb{R}^N} |x|^{p-N} |v|^p dx = \int_{\mathbb{R}^N} |x|^{-pa} |u|^p dx < \infty$. Thus we have obtained the claim.

By the claim, we can apply Proposition 2 to $v \in W^1 L_{N,p,0}(\mathbb{R}^N)$. Then we derive

$$\begin{aligned}
(3.5) \quad J(u) &\geq c_p \int_{\mathbb{R}^N} |\nabla v|^p |x|^{p-N} dx \geq c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{R}^N} \frac{\left| v(x) - v\left(\frac{Rx}{|x|}\right) \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^N} dx \\
&= c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{N-p-pa}{p}} u(x) - R^{\frac{N-p-pa}{p}} u\left(\frac{Rx}{|x|}\right) \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^N} dx \\
&= c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{R}^N} \frac{\left| u(x) - R^{\frac{N-p-pa}{p}} \tilde{u}(R) |x|^{-\frac{N-p-pa}{p}} \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{p(a+1)}} dx \\
&\geq c_p \left(\frac{p-1}{p} \right)^p \inf_{c \in \mathbb{R}} \int_{\mathbb{R}^N} \frac{\left| u(x) - c |x|^{-\frac{N-p-pa}{p}} \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{p(a+1)}} dx
\end{aligned}$$

for any $R > 0$, here we have used (3.4) in the first inequality, (3.3) with $f = v$ and $\alpha = \beta = p$ in the second inequality, the definition of v in the first equality, and the fact that u is radially symmetric in the second equality. This proves Theorem for a radial function $u \in C_0^\infty(\mathbb{R}^N)$.

Next, we prove Theorem for a radial function $u \in D_{0,a}^{1,p}(\mathbb{R}^N)$. Here we follow an argument by Machihara, Ozawa and Wadade [14]. Let $\{u_m\}_{m=1}^\infty \subset C_0^\infty(\mathbb{R}^N)$ be a sequence of radially symmetric functions such that $u_m \rightarrow u$ in $D_{0,a}^{1,p}(\mathbb{R}^N)$ as $m \rightarrow \infty$. Then there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ such that

$$\begin{aligned}
\frac{u_{m_j}}{|x|^{a+1}} &\rightarrow \frac{u}{|x|^{a+1}} \quad \text{in } L^p(\mathbb{R}^N), \\
u_{m_j} &\rightarrow u \quad \text{a.e. in } \mathbb{R}^N
\end{aligned}$$

by (1.1). Now, define

$$w_R(x) = \frac{w(x) - R^{\frac{N-p-pa}{p}} \tilde{w}(R) |x|^{-\frac{N-p-pa}{p}}}{|x|^{a+1} \left| \log \frac{R}{|x|} \right|}$$

for a radial function $w \in L_{\text{loc}}^1(\mathbb{R}^N)$, $w(x) = \tilde{w}(|x|)$, and $R > 0$. Since the inequality (3.5) holds for $u_{m_j} - u_{m_k} \in C_0^\infty(\mathbb{R}^N)$, we observe that $\{(u_{m_j})_R\}_{j=1}^\infty$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$. Since $(u_{m_j})_R \rightarrow u_R$ a.e.

in \mathbb{R}^N , we easily see that $(u_{m_j})_R \rightarrow u_R$ in $L^p(\mathbb{R}^N)$ as $j \rightarrow \infty$. Therefore, for any $R > 0$, we have

$$\begin{aligned} J(u) &= \lim_{j \rightarrow \infty} J(u_{m_j}) \\ &\geq c_p \left(\frac{p-1}{p} \right)^p \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_{m_j}(x) - R^{\frac{N-p-pa}{p}} \widetilde{u_{m_j}}(R)|x|^{-\frac{N-p-pa}{p}}|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{p(a+1)}} dx \\ &= c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u(x) - R^{\frac{N-p-pa}{p}} \tilde{u}(R)|x|^{-\frac{N-p-pa}{p}}|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{p(a+1)}} dx \end{aligned}$$

for all radial functions $u \in \mathcal{D}_{0,a}^{1,p}(\mathbb{R}^N)$, here we have used (3.5) for $u_{m_j} \in C_0^\infty(\mathbb{R}^N)$. As before, this ends the proof. \square

4. IMPROVED CAFFARELLI-KOHN-NIRENBERG TYPE INEQUALITIES ON A BOUNDED DOMAIN

In this section, we revisit the idea by Catrina and Wang [5], Wang-Willem [19] to improve the inequality

$$(4.1) \quad \int_{\Omega} |\nabla u|^2 |x|^{-2a} dx \geq \left(\frac{N-2-2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx$$

on a smooth bounded domain Ω in \mathbb{R}^N ($N \geq 3$) with $0 \in \Omega$. Here u is a function in $C_0^\infty(\Omega \setminus \{0\})$ and $-\infty < a < \frac{N-2}{2}$ is always assumed. We can check that under the assumption $a < \frac{N-2}{2}$, $D_{0,a}^{1,2}(\Omega)$ is identical to the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to the norm $\|\cdot\|_{D_{0,a}^{1,2}(\Omega)}$. The idea of Catrina and Wang consists of the use of a conformal transformation which converts a problem to an equivalent one in a domain on a cylinder $\mathcal{C} = \mathbb{R} \times S^{N-1}$. More precisely, for a function $u \in C_0^\infty(\Omega \setminus \{0\})$, let us associate $v \in C_0^\infty(\tilde{\Omega})$ by the transformation

$$(4.2) \quad u(x) = |x|^{-\frac{N-2-2a}{2}} v \left(-\log |x|, \frac{x}{|x|} \right),$$

where $\tilde{\Omega}$ is a domain on the cylinder \mathcal{C} defined as

$$\tilde{\Omega} = \left\{ (t, \theta) \in \mathbb{R} \times S^{N-1} : t = -\log |x|, \theta = \frac{x}{|x|}, x \in \Omega \right\}.$$

Catrina and Wang proved in [5] that when $\Omega = \mathbb{R}^N$ and $a < \frac{N-2}{2}$, the transformation (4.2) provides an isomorphism between two Hilbert

spaces $D_{0,a}^{1,2}(\mathbb{R}^N)$ and $H^1(\mathcal{C})$, the inner product of the latter is given by

$$(v, w)_{H^1(\mathcal{C})} = \int_{\mathcal{C}} \left\{ \nabla v \cdot \nabla w + \left(\frac{N-2-2a}{2} \right)^2 vw \right\} d\mu.$$

Here $|\nabla v|^2 = v_t^2 + |\nabla_{\theta} v|^2$ and $d\mu = dt dS_{\theta} = |x|^{-N} dx$ are the length of the gradient vector and the volume element on \mathcal{C} . By a direct computation, we see

$$\nabla u(x) = -|x|^{-\frac{N-2a}{2}} \left[\left\{ v_t(t, \theta) + \left(\frac{N-2-2a}{2} \right) v \right\} \theta + \nabla_{\theta} v(t, \theta) \right],$$

and since $\langle \theta, \nabla_{\theta} v \rangle \equiv 0$, it holds

$$(4.3) \quad |\nabla u(x)|^2 = |x|^{-N+2a} \left[\left\{ v_t(t, \theta) + \left(\frac{N-2-2a}{2} \right) v \right\}^2 + |\nabla_{\theta} v(t, \theta)|^2 \right].$$

Furthermore, we have

$$(4.4) \quad \int_{\tilde{\Omega}} |\nabla v|^2 d\mu = \int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^{2(a+1)}} dx,$$

$$(4.5) \quad \int_{\tilde{\Omega}} v^2 d\mu = \int_{\Omega} \frac{u^2}{|x|^{2(a+1)}} dx.$$

On the other hand, Wang and Willem [19] proved the weighted Poincaré inequality of the form

$$\int_{\tilde{\Omega}} |\nabla v|^2 d\mu \geq \frac{1}{4} \int_{\tilde{\Omega}} \frac{v^2}{t^2} d\mu$$

for $v \in C_0^{\infty}(\tilde{\Omega})$. Using this inequality and the transformation (4.2), they obtained the improved inequality mentioned in the Introduction, See [5], [19].

Following their arguments, we prove the next theorem.

Theorem 4. *Let $V = V(t)$ is nonnegative, monotone decreasing function on $t \in (0, +\infty)$ and assume that there exists a strictly positive function $\phi = \phi(t) \in C^2((0, +\infty))$ such that $-\phi''(t) \geq V(t)\phi(t)$ holds on $(0, +\infty)$. Let Ω be a domain in \mathbb{R}^N such that $\Omega \subset\subset B_R(0)$ for some*

$R > 0$. Then there exists a constant $C = C(a, \Omega, V) > 0$ such that

$$(4.6) \quad \begin{aligned} & \int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx \\ & \geq C \int_{\Omega} V \left(\log \left(\frac{R}{|x|} \right) \right) |x|^{-2a} |\nabla u|^2 dx \end{aligned}$$

holds for all $u \in D_{0,a}^{1,2}(\Omega)$.

For example, $V(t) = \frac{1}{4t^2}$ satisfies the assumption of Theorem 4 with $\phi(t) = t^{1/2}$, $t \in (0, +\infty)$.

As a corollary, we obtain the following improved inequalities.

Corollary 1. *Let Ω be a domain in \mathbb{R}^N such that $\Omega \subset\subset B_R(0)$ for some $R > 0$. Assume $-\infty < a < \frac{N-2}{2}$. Then for any integer $k \in \mathbb{N}$, there exists a constant $C = C(a, \Omega, k) > 0$ such that the followings hold.*

(i) *Define the functions $X_i = X_i(s)$ for $s \in (0, 1]$ iteratively as $X_1(s) = (1 - \log s)^{-1}$, $X_i(s) = X_1(X_{i-1}(s))$ for $i \geq 2$. Then*

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx \\ & \geq C \sum_{i=1}^k \int_{\Omega} \frac{1}{|x|^{2a}} X_1^2 \left(\frac{|x|}{R} \right) X_2^2 \left(\frac{|x|}{R} \right) \cdots X_i^2 \left(\frac{|x|}{R} \right) |\nabla u|^2 dx \end{aligned}$$

holds for all $u \in D_{0,a}^{1,2}(\Omega)$.

(ii) *Let $e_{-1} = 0$, $e_0 = 1$, $e_i = e^{e_{i-1}}$ for $i \geq 1$, and define functions iteratively as $\log^{(1)} s = \log s$, $\log^{(i)}(s) = \log^{(1)}(\log^{(i-1)} s)$ for $i \geq 2$, those are well defined when $s > e_{i-2}$ and positive when $s > e_{i-1}$. For any integer $k \in \mathbb{N}$, take $\rho \geq Re_{k-1}$. Then there exists a constant $C = C(a, \Omega, k) > 0$ such that*

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 |x|^{-2a} dx - \left(\frac{N-2-2a}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^{2(a+1)}} dx \\ & \geq C \sum_{i=1}^k \int_{\Omega} \frac{1}{|x|^{2a}} \left(\log^{(1)} \frac{\rho}{|x|} \right)^{-2} \left(\log^{(2)} \frac{\rho}{|x|} \right)^{-2} \cdots \left(\log^{(i)} \frac{\rho}{|x|} \right)^{-2} |\nabla u|^2 dx \end{aligned}$$

holds for all $u \in D_{0,a}^{1,2}(\Omega)$.

For the proof of Theorem 4, first we show a lemma.

Lemma 2. *(The weighted Poincaré inequality on a cylinder) Let $\tilde{\Omega} \subset \mathcal{C}_+$ be a bounded domain where $\mathcal{C}_+ = \{(t, \theta) \in \mathbb{R} \times S^{N-1} : t > 0\}$. Let*

$V = V(t)$ satisfy the assumptions in Theorem 4. Then it holds

$$(4.7) \quad \int_{\tilde{\Omega}} |\nabla v|^2 d\mu \geq \int_{\tilde{\Omega}} V(t)v^2 d\mu$$

for all $v \in C_0^\infty(\tilde{\Omega})$.

Proof. Put $\psi(t, \theta) = v(t, \theta)/\phi(t)$ where $\phi(t) > 0$, $\phi = \phi(t) \in C^2((0, +\infty))$ such that $-\phi''(t) \geq V(t)\phi(t)$ on $(0, +\infty)$. Then $\psi \equiv 0$ near $t = 0$ and $t = \infty$, since $v \in C_0^\infty(\tilde{\Omega})$. Thus

$$\begin{aligned} \int_0^\infty v_t^2(t, \theta) dt &= \int_0^\infty (\phi_t \psi + \phi \psi_t)^2 dt = \int_0^\infty (\phi_t^2 \psi^2 + \phi^2 \psi_t^2 + \phi \phi_t (\psi^2)_t) dt \\ &= \int_0^\infty (\phi_t^2 \psi^2 + \phi^2 \psi_t^2) dt + [\psi^2 \phi \phi_t]_0^\infty - \int_0^\infty \psi^2 (\phi \phi_t)_t dt \\ &= \int_0^\infty \phi^2 \psi_t^2 dt - \int_0^\infty \psi^2 \phi \phi_{tt} dt \geq - \int_0^\infty \psi^2 \phi \phi_{tt} dt \\ &= \int_0^\infty v^2 \left(-\frac{\phi''(t)}{\phi(t)} \right) dt \geq \int_0^\infty V(t)v^2 dt. \end{aligned}$$

Integrating both sides on S^{N-1} with respect to dS_θ and adding $\int_{\tilde{\Omega}} |\nabla_\theta v|^2 dt dS_\theta$ to the right hand side, we obtain (4.7). \square

The inequality (4.7) also holds for $v \in H_0^1(\tilde{\Omega})$ by a density argument.

Proof of Theorem 4. By using a scaling $x \mapsto x/R$, it is enough to prove Theorem when $R = 1$. In this case, $\tilde{\Omega} \subset \mathcal{C}_+$ where $\mathcal{C}_+ = \{(t, \theta) \in \mathbb{R} \times S^{N-1} : t > 0\}$. Now, we will prove

$$(4.8) \quad \int_{\tilde{\Omega}} |\nabla v|^2 d\mu \geq C \int_{\tilde{\Omega}} V(t) \left[|\nabla_\theta v|^2 + \left\{ v_t(t, \theta) + \left(\frac{N-2-2a}{2} \right) v \right\}^2 \right] d\mu$$

for some $C > 0$ independent of v . Indeed, since $r_0 = \sup_{x \in \Omega} |x| < 1$, we have $t = -\log |x| \geq -\log r_0 > 0$ on $\tilde{\Omega}$. Then the positivity and the

decreasing property of V imply that

$$\begin{aligned}
& \int_{\tilde{\Omega}} V(t) \left[|\nabla_{\theta} v|^2 + \left\{ v_t(t, \theta) + \left(\frac{N-2-2a}{2} \right) v \right\}^2 \right] d\mu \\
& \leq 2 \int_{\tilde{\Omega}} V(t) |\nabla v|^2 d\mu + 2 \left(\frac{N-2-2a}{2} \right)^2 \int_{\tilde{\Omega}} V(t) v^2 d\mu \\
& \leq 2V(-\log r_0) \int_{\tilde{\Omega}} |\nabla v|^2 d\mu + 2 \left(\frac{N-2-2a}{2} \right)^2 \int_{\tilde{\Omega}} V(t) v^2 d\mu \\
& \leq \left(2V(-\log r_0) + 2 \left(\frac{N-2-2a}{2} \right)^2 \right) \int_{\tilde{\Omega}} |\nabla v|^2 d\mu,
\end{aligned}$$

where we have used the weighted Poincaré inequality (4.7) in the last inequality. Thus we get (4.8). Simple computation using (4.3), (4.4), (4.5) in (4.8) yields the desired inequality (4.6). \square

Remark 5. The constant C in Theorem 4 can be chosen that

$$C = \left(2V(-\log r_0) + 2 \left(\frac{N-2-2a}{2} \right)^2 \right)^{-1}.$$

Proof of Corollary 1. We follow the computation in [17].

(i) We may assume $R = 1$. Note that X_i is well-defined and $X_i(0) = 0$, $X_i(1) = 1$, $0 < X_i(s) < 1$ for $s \in (0, 1)$. We compute

$$\begin{aligned}
X_1'(s) &= \frac{1}{s} X_1(s)^2, \quad X_i'(s) = \frac{X_1(s) \cdots X_{i-1}(s)}{s} X_i(s)^2, \\
(X_1 \cdots X_k)'(s) &= \frac{(X_1 \cdots X_k)(s)}{s} \{X_1(s) + (X_1 X_2)(s) + \cdots + (X_1 \cdots X_k)(s)\}.
\end{aligned}$$

Define $Y_i(t) = X_i(e^{-t})$ for $t \in [0, +\infty)$. Then we obtain

$$\begin{aligned}
Y_1'(t) &= -Y_1(t)^2, \quad Y_i'(t) = -(Y_1 \cdots Y_i)(t) Y_i(t), \\
(Y_1 \cdots Y_k)'(t) &= (Y_1 \cdots Y_k)(t) \{Y_1(t) + (Y_1 Y_2)(t) + \cdots + (Y_1 \cdots Y_k)(t)\}.
\end{aligned}$$

Now, define

$$\phi_k(t) = (Y_1 \cdots Y_k)^{-1/2}(t) > 0.$$

Then by differentiating $\log \phi_k(t) = -\frac{1}{2} \sum_{i=1}^k \log Y_i(t)$, we check that

$$\phi_k'(t) = \frac{\phi_k(t)}{2} \sum_{i=1}^k (Y_1 \cdots Y_i)(t), \quad \phi_k''(t) = -\frac{\phi_k(t)}{4} \sum_{i=1}^k (Y_1 \cdots Y_i)^2(t),$$

thus ϕ_k is a solution of $-\frac{\phi_k''(t)}{\phi_k(t)} = \frac{1}{4} \sum_{i=1}^k (Y_1 \cdots Y_i)^2(t)$. It is easy to check that $V_k(t) = \frac{1}{4} \sum_{i=1}^k (Y_1 \cdots Y_i)^2(t)$ is nonnegative and decreasing on $(0, +\infty)$. Applying Theorem 4, we obtain the result.

(ii) We compute

$$\left(\log^{(i)} s\right)' = s^{-1} \left(\log^{(1)} s\right)^{-1} \left(\log^{(2)} s\right)^{-1} \cdots \left(\log^{(i-1)} s\right)^{-1}$$

for $i = 1, 2, \dots, k$. Put $L_i(t) = \log^{(i)}(\rho e^t)$ and $W_i(t) = (L_1 \cdots L_i)^{-1}(t)$ for $i = 1, 2, \dots, k$. We see $L_i(t) > 0$ if $\rho > e^{-t} e_{i-1}$ and obtain

$$L_1'(t) = 1, \quad L_i'(t) = (L_1 \cdots L_{i-1})^{-1}(t) \quad \text{for } i = 2, 3, \dots, k,$$

$$W_i'(t) = -W_i(t) \sum_{j=1}^i (L_1 \cdots L_j)^{-1}(t).$$

Define

$$\psi_k(t) = (L_1 \cdots L_k)^{1/2}(t) > 0.$$

Then we check that

$$\begin{aligned} \psi_k'(t) &= \frac{\psi_k(t)}{2} \sum_{i=1}^k (L_1 \cdots L_i)^{-1}(t) = \frac{\psi_k(t)}{2} \sum_{i=1}^k W_i(t), \\ \psi_k''(t) &= \frac{\psi_k(t)}{4} \left(\sum_{i=1}^k W_i(t) \right)^2 - \frac{\psi_k(t)}{2} \sum_{i=1}^k W_i(t) \sum_{j=1}^i W_j(t), \\ -\frac{\psi_k''(t)}{\psi_k(t)} &= \frac{1}{4} \sum_{i=1}^k W_i(t)^2. \end{aligned}$$

On the way of computation, we have used the identity

$$2 \sum_{i=1}^k a_i \sum_{j=1}^i a_j - \left(\sum_{i=1}^k a_i \right) \left(\sum_{j=1}^k a_j \right) = \sum_{i=1}^k a_i^2$$

for any $a_1, \dots, a_k \in \mathbb{R}$. Again $V_k(t) = \frac{1}{4} \sum_{i=1}^k (L_1 \cdots L_i)^{-2}(t)$ is nonnegative and decreasing on $(0, +\infty)$, so we may apply Theorem 4 to obtain the result. \square

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