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# SOME INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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ABSTRACT. Some inequalities for convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

## 1. INTRODUCTION

Let  $A$  be a selfadjoint linear operator on a complex Hilbert space  $(H; \langle \cdot, \cdot \rangle)$ . The *Gelfand map* establishes a  $*$ -isometrically isomorphism  $\Phi$  between the set  $C(Sp(A))$  of all *continuous functions* defined on the *spectrum* of  $A$ , denoted  $Sp(A)$ , an the  $C^*$ -algebra  $C^*(A)$  generated by  $A$  and the identity operator  $1_H$  on  $H$  as follows (see for instance [6, p. 3]):

For any  $f, g \in C(Sp(A))$  and any  $\alpha, \beta \in \mathbb{C}$  we have

- (i)  $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$ ;
- (ii)  $\Phi(fg) = \Phi(f)\Phi(g)$  and  $\Phi(\bar{f}) = \Phi(f)^*$ ;
- (iii)  $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$ ;
- (iv)  $\Phi(f_0) = 1_H$  and  $\Phi(f_1) = A$ , where  $f_0(t) = 1$  and  $f_1(t) = t$ , for  $t \in Sp(A)$ .

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator  $A$ .

If  $A$  is a selfadjoint operator and  $f$  is a real valued continuous function on  $Sp(A)$ , then  $f(t) \geq 0$  for any  $t \in Sp(A)$  implies that  $f(A) \geq 0$ , *i.e.*  $f(A)$  is a positive operator on  $H$ . Moreover, if both  $f$  and  $g$  are real valued functions on  $Sp(A)$  then the following important property holds:

- (P)  $f(t) \geq g(t)$  for any  $t \in Sp(A)$  implies that  $f(A) \geq g(A)$

in the operator order of  $B(H)$ .

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [6] and the references therein. For other results, see [13], [7] and [9].

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [11] (see also [6, p. 5]):

**Theorem 1** (Mond-Pečarić, 1993, [11]). *Let  $A$  be a selfadjoint operator on the Hilbert space  $H$  and assume that  $Sp(A) \subseteq [m, M]$  for some scalars  $m, M$  with  $m < M$ . If  $f$  is a convex function on  $[m, M]$ , then*

$$(MP) \quad f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle$$

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for each  $x \in H$  with  $\|x\| = 1$ .

The following result that provides a reverse of the Mond & Pečarić has been obtained in [3]:

**Theorem 2** (Dragomir, 2008, [3]). *Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a convex and differentiable function on  $\overset{\circ}{I}$  (the interior of  $I$ ) whose derivative  $f'$  is continuous on  $\overset{\circ}{I}$ . If  $A$  is a selfadjoint operators on the Hilbert space  $H$  with  $Sp(A) \subseteq [m, M] \subset \overset{\circ}{I}$ , then*

$$(1.1) \quad (0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \langle f'(A)Ax, x \rangle - \langle Ax, x \rangle \cdot \langle f'(A)x, x \rangle$$

for any  $x \in H$  with  $\|x\| = 1$ .

Perhaps more convenient reverses of the Mond & Pečarić result are the following inequalities that have been obtained in the same paper [3]:

**Theorem 3** (Dragomir, 2008, [3]). *Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a convex and differentiable function on  $\overset{\circ}{I}$  (the interior of  $I$ ) whose derivative  $f'$  is continuous on  $\overset{\circ}{I}$ . If  $A$  is a selfadjoint operators on the Hilbert space  $H$  with  $Sp(A) \subseteq [m, M] \subset \overset{\circ}{I}$ , then*

$$(1.2) \quad (0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \begin{cases} \frac{1}{2} \cdot (M - m) \left[ \|f'(A)x\|^2 - \langle f'(A)x, x \rangle^2 \right]^{1/2} \\ \frac{1}{2} \cdot (f'(M) - f'(m)) \left[ \|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2} \end{cases} \leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$

for any  $x \in H$  with  $\|x\| = 1$ .

We also have the inequality

$$(1.3) \quad (0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \frac{1}{4} (M - m) (f'(M) - f'(m)) - \begin{cases} [\langle Mx - Ax, Ax - mx \rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x \rangle]^{1/2}, \\ \left| \langle Ax, x \rangle - \frac{M+m}{2} \right| \left| \langle f'(A)x, x \rangle - \frac{f'(M)+f'(m)}{2} \right| \end{cases} \leq \frac{1}{4} (M - m) (f'(M) - f'(m))$$

for any  $x \in H$  with  $\|x\| = 1$ .

Moreover, if  $m > 0$  and  $f'(m) > 0$ , then we also have

$$(1.4) \quad (0 \leq) \langle f(A)x, x \rangle - f(\langle Ax, x \rangle) \leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x \rangle \langle f'(A)x, x \rangle, \\ \left( \sqrt{M} - \sqrt{m} \right) \left( \sqrt{f'(M)} - \sqrt{f'(m)} \right) [\langle Ax, x \rangle \langle f'(A)x, x \rangle]^{1/2}, \end{cases}$$

for any  $x \in H$  with  $\|x\| = 1$ .

For generalisations to  $n$ -tuples of operators as well as for some particular cases of interest, see [3].

The main aim of the present paper is to provide more general vector inequalities for convex functions whose derivatives are continuous.

## 2. SOME INEQUALITIES FOR TWO OPERATORS

The following result holds:

**Theorem 4.** *Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a convex and differentiable function on  $\overset{\circ}{I}$  (the interior of  $I$ ) whose derivative  $f'$  is continuous on  $\overset{\circ}{I}$ . If  $A$  and  $B$  are selfadjoint operators on the Hilbert space  $H$  with  $Sp(A), Sp(B) \subseteq [m, M] \subset \overset{\circ}{I}$ , then*

$$(2.1) \quad \langle f'(A)x, x \rangle \langle By, y \rangle - \langle f'(A)Ax, x \rangle \\ \leq \langle f(B)y, y \rangle - \langle f(A)x, x \rangle \leq \langle f'(B)By, y \rangle - \langle Ax, x \rangle \langle f'(B)y, y \rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

In particular, we have

$$(2.2) \quad \langle f'(A)x, x \rangle \langle Ay, y \rangle - \langle f'(A)Ax, x \rangle \\ \leq \langle f(A)y, y \rangle - \langle f(A)x, x \rangle \leq \langle f'(A)Ay, y \rangle - \langle Ax, x \rangle \langle f'(A)y, y \rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and

$$(2.3) \quad \langle f'(A)x, x \rangle \langle Bx, x \rangle - \langle f'(A)Ax, x \rangle \\ \leq \langle f(B)x, x \rangle - \langle f(A)x, x \rangle \leq \langle f'(B)Bx, x \rangle - \langle Ax, x \rangle \langle f'(B)x, x \rangle$$

for any  $x \in H$  with  $\|x\| = 1$ .

*Proof.* Since  $f$  is convex and differentiable on  $\overset{\circ}{I}$ , then we have that

$$(2.4) \quad f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s)$$

for any  $t, s \in [m, M]$ .

Now, if we fix  $t \in [m, M]$  and apply the property (P) for the operator  $A$ , then for any  $x \in H$  with  $\|x\| = 1$  we have

$$(2.5) \quad \langle f'(A) \cdot (t \cdot 1_H - A)x, x \rangle \\ \leq \langle [f(t) \cdot 1_H - f(A)]x, x \rangle \leq \langle f'(t) \cdot (t \cdot 1_H - A)x, x \rangle$$

for any  $t \in [m, M]$  and any  $x \in H$  with  $\|x\| = 1$ .

The inequality (2.5) is equivalent with

$$(2.6) \quad t \langle f'(A)x, x \rangle - \langle f'(A)Ax, x \rangle \leq f(t) - \langle f(A)x, x \rangle \leq f'(t)t - f'(t) \langle Ax, x \rangle$$

for any  $t \in [m, M]$  and any  $x \in H$  with  $\|x\| = 1$ .

If we fix  $x \in H$  with  $\|x\| = 1$  in (2.6) and apply the property (P) for the operator  $B$ , then we get

$$(2.7) \quad \langle [f'(A)x, x]B - \langle f'(A)Ax, x \rangle 1_H y, y \rangle \\ \leq \langle [f(B) - \langle f(A)x, x \rangle 1_H]y, y \rangle \leq \langle [f'(B)B - \langle Ax, x \rangle f'(B)]y, y \rangle$$

for each  $y \in H$  with  $\|y\| = 1$ , which is clearly equivalent to the desired inequality (2.1). ■

**Remark 1.** If we fix  $x \in H$  with  $\|x\| = 1$  and choose  $B = \langle Ax, x \rangle \cdot 1_H$ , then we obtain from the first inequality in (2.1) the reverse of the Mond-Pečarić inequality obtained by the author in [3]. The second inequality will provide the inequality (MP) for convex functions whose derivatives are continuous.

The following corollary is of interest:

**Corollary 1.** Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a convex and differentiable function on  $\overset{\circ}{I}$  whose derivative  $f'$  is continuous on  $\overset{\circ}{I}$ . Also, suppose that  $A$  is a selfadjoint operator on the Hilbert space  $H$  with  $Sp(A) \subseteq [m, M] \subset \overset{\circ}{I}$ . If  $g$  is nonincreasing and continuous on  $[m, M]$  and

$$(2.8) \quad f'(A)[g(A) - A] \geq 0$$

in the operator order of  $B(H)$ , then

$$(2.9) \quad (f \circ g)(A) \geq f(A)$$

in the operator order of  $B(H)$ .

*Proof.* If we apply the first inequality from (2.3) for  $B = g(A)$  we have

$$(2.10) \quad \langle f'(A)x, x \rangle \langle g(A)x, x \rangle - \langle f'(A)Ax, x \rangle \leq \langle f(g(A))x, x \rangle - \langle f(A)x, x \rangle$$

any  $x \in H$  with  $\|x\| = 1$ .

We use the following Čebyšev type inequality for functions of operators established by the author in [4]:

Let  $A$  be a selfadjoint operator with  $Sp(A) \subseteq [m, M]$  for some real numbers  $m < M$ . If  $h, g : [m, M] \rightarrow \mathbb{R}$  are continuous and *synchronous (asynchronous)* on  $[m, M]$ , then

$$(2.11) \quad \langle h(A)g(A)x, x \rangle \geq (\leq) \langle h(A)x, x \rangle \cdot \langle g(A)x, x \rangle$$

for any  $x \in H$  with  $\|x\| = 1$ .

Now, since  $f'$  and  $g$  are continuous and are asynchronous on  $[m, M]$ , then by (2.11) we have the inequality

$$(2.12) \quad \langle f'(A)g(A)x, x \rangle \leq \langle f'(A)x, x \rangle \cdot \langle g(A)x, x \rangle$$

for any  $x \in H$  with  $\|x\| = 1$ .

Subtracting in both sides of (2.12) the quantity  $\langle f'(A)Ax, x \rangle$  and taking into account, by (2.8), that  $\langle f'(A)[g(A) - A]x, x \rangle \geq 0$  for any  $x \in H$  with  $\|x\| = 1$ , we then have

$$\begin{aligned} 0 &\leq \langle f'(A)[g(A) - A]x, x \rangle = \langle f'(A)g(A)x, x \rangle - \langle f'(A)Ax, x \rangle \\ &\leq \langle f'(A)x, x \rangle \cdot \langle g(A)x, x \rangle - \langle f'(A)Ax, x \rangle \end{aligned}$$

which together with (2.10) will produce the desired result (2.9). ■

We provide now some particular inequalities of interest that can be derived from Theorem 4:

**Example 1. a.** Let  $A, B$  two positive definite operators on  $H$ . Then we have the inequalities

$$(2.13) \quad 1 - \langle A^{-1}x, x \rangle \langle By, y \rangle \leq \langle \ln Ax, x \rangle - \langle \ln By, y \rangle \leq \langle Ax, x \rangle \langle B^{-1}y, y \rangle - 1$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

In particular, we have

$$(2.14) \quad 1 - \langle A^{-1}x, x \rangle \langle Ay, y \rangle \leq \langle \ln Ax, x \rangle - \langle \ln Ay, y \rangle \leq \langle Ax, x \rangle \langle A^{-1}y, y \rangle - 1$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and

$$(2.15) \quad 1 - \langle A^{-1}x, x \rangle \langle Bx, x \rangle \leq \langle \ln Ax, x \rangle - \langle \ln Bx, x \rangle \leq \langle Ax, x \rangle \langle B^{-1}x, x \rangle - 1$$

for any  $x \in H$  with  $\|x\| = 1$ .

**b.** With the same assumption for  $A$  and  $B$  we have the inequalities

$$(2.16) \quad \langle By, y \rangle - \langle Ax, x \rangle \leq \langle B \ln By, y \rangle - \langle \ln Ax, x \rangle \langle By, y \rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

In particular, we have

$$(2.17) \quad \langle Ay, y \rangle - \langle Ax, x \rangle \leq \langle A \ln Ay, y \rangle - \langle \ln Ax, x \rangle \langle Ay, y \rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and

$$(2.18) \quad \langle Bx, x \rangle - \langle Ax, x \rangle \leq \langle B \ln Bx, x \rangle - \langle \ln Ax, x \rangle \langle Bx, x \rangle$$

for any  $x \in H$  with  $\|x\| = 1$ .

The proof of Example **a** follows from Theorem 4 for the convex function  $f(x) = -\ln x$  while the proof of the second example follows by the same theorem applied for the convex function  $f(x) = x \ln x$  and performing the required calculations. The details are omitted.

The following result may be stated as well:

**Theorem 5.** Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a convex and differentiable function on  $\overset{\circ}{I}$  (the interior of  $I$ ) whose derivative  $f'$  is continuous on  $\overset{\circ}{I}$ . If  $A$  and  $B$  are selfadjoint operators on the Hilbert space  $H$  with  $Sp(A), Sp(B) \subseteq [m, M] \subset \overset{\circ}{I}$ , then

$$(2.19) \quad f'(\langle Ax, x \rangle) (\langle By, y \rangle - \langle Ax, x \rangle) \leq \langle f(B)y, y \rangle - f(\langle Ax, x \rangle) \\ \leq \langle f'(B)By, y \rangle - \langle Ax, x \rangle \langle f'(B)y, y \rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

In particular, we have

$$(2.20) \quad f'(\langle Ax, x \rangle) (\langle Ay, y \rangle - \langle Ax, x \rangle) \leq \langle f(A)y, y \rangle - f(\langle Ax, x \rangle) \\ \leq \langle f'(A)Ay, y \rangle - \langle Ax, x \rangle \langle f'(A)y, y \rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and

$$(2.21) \quad f'(\langle Ax, x \rangle) (\langle Bx, x \rangle - \langle Ax, x \rangle) \leq \langle f(B)x, x \rangle - f(\langle Ax, x \rangle) \\ \leq \langle f'(B)Bx, x \rangle - \langle Ax, x \rangle \langle f'(B)x, x \rangle$$

for any  $x \in H$  with  $\|x\| = 1$ .

*Proof.* Since  $f$  is convex and differentiable on  $\overset{\circ}{I}$ , then we have that

$$(2.22) \quad f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s)$$

for any  $t, s \in [m, M]$ .

If we choose  $s = \langle Ax, x \rangle \in [m, M]$ , with a fix  $x \in H$  with  $\|x\| = 1$ , then we have

$$(2.23) \quad f'(\langle Ax, x \rangle) \cdot (t - \langle Ax, x \rangle) \leq f(t) - f(\langle Ax, x \rangle) \leq f'(t) \cdot (t - \langle Ax, x \rangle)$$

for any  $t \in [m, M]$ .

Now, if we apply the property (P) to the inequality (2.23) and the operator  $B$ , then we get

$$\begin{aligned} & \langle f'(\langle Ax, x \rangle) \cdot (B - \langle Ax, x \rangle \cdot 1_H) y, y \rangle \\ & \leq \langle [f(B) - f(\langle Ax, x \rangle) \cdot 1_H] y, y \rangle \leq \langle f'(B) \cdot (B - \langle Ax, x \rangle \cdot 1_H) y, y \rangle \end{aligned}$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ , which is equivalent with the desired result (2.19). ■

**Remark 2.** We observe that if we choose  $B = A$  in (2.21) or  $y = x$  in (2.20) then we recapture the Mond-Pečarić inequality and its reverse from (1.1).

The following particular case of interest follows from Theorem 5

**Corollary 2.** Assume that  $f, A$  and  $B$  are as in Theorem 5. If, either  $f$  is increasing on  $[m, M]$  and  $B \geq A$  in the operator order of  $B(H)$  or  $f$  is decreasing and  $B \leq A$ , then we have the Jensen's type inequality

$$(2.24) \quad \langle f(B)x, x \rangle \geq f(\langle Ax, x \rangle)$$

for any  $x \in H$  with  $\|x\| = 1$ .

The proof is obvious by the first inequality in (2.21) and the details are omitted.

We provide now some particular inequalities of interest that can be derived from Theorem 5:

**Example 2. a.** Let  $A, B$  be two positive definite operators on  $H$ . Then we have the inequalities

$$(2.25) \quad 1 - \langle Ax, x \rangle^{-1} \langle By, y \rangle \leq \ln(\langle Ax, x \rangle) - \langle \ln By, y \rangle \leq \langle Ax, x \rangle \langle B^{-1}y, y \rangle - 1$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

In particular, we have

$$(2.26) \quad 1 - \langle Ax, x \rangle^{-1} \langle Ay, y \rangle \leq \ln(\langle Ax, x \rangle) - \langle \ln Ay, y \rangle \leq \langle Ax, x \rangle \langle A^{-1}y, y \rangle - 1$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and

$$(2.27) \quad 1 - \langle Ax, x \rangle^{-1} \langle Bx, x \rangle \leq \ln(\langle Ax, x \rangle) - \langle \ln Bx, x \rangle \leq \langle Ax, x \rangle \langle B^{-1}x, x \rangle - 1$$

for any  $x \in H$  with  $\|x\| = 1$ .

**b.** With the same assumption for  $A$  and  $B$ , we have the inequalities

$$(2.28) \quad \langle By, y \rangle - \langle Ax, x \rangle \leq \langle B \ln By, y \rangle - \langle By, y \rangle \ln(\langle Ax, x \rangle)$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

In particular, we have

$$(2.29) \quad \langle Ay, y \rangle - \langle Ax, x \rangle \leq \langle A \ln Ay, y \rangle - \langle Ay, y \rangle \ln(\langle Ax, x \rangle)$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and

$$(2.30) \quad \langle Bx, x \rangle - \langle Ax, x \rangle \leq \langle B \ln Bx, x \rangle - \langle Bx, x \rangle \ln(\langle Ax, x \rangle)$$

for any  $x \in H$  with  $\|x\| = 1$ .

## 3. INEQUALITIES FOR TWO SEQUENCES OF OPERATORS

The following result may be stated:

**Theorem 6.** *Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a convex and differentiable function on  $\overset{\circ}{I}$  (the interior of  $I$ ) whose derivative  $f'$  is continuous on  $\overset{\circ}{I}$ . If  $A_j$  and  $B_j$  are selfadjoint operators on the Hilbert space  $H$  with  $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \overset{\circ}{I}$  for any  $j \in \{1, \dots, n\}$ , then*

$$\begin{aligned}
(3.1) \quad & \sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle B_j y_j, y_j \rangle - \sum_{j=1}^n \langle f'(A_j) A_j x_j, x_j \rangle \\
& \leq \sum_{j=1}^n \langle f(B_j) y_j, y_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \\
& \leq \sum_{j=1}^n \langle f'(B_j) B_j y_j, y_j \rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \sum_{j=1}^n \langle f'(B_j) y_j, y_j \rangle
\end{aligned}$$

for any  $x_j, y_j \in H, j \in \{1, \dots, n\}$  with  $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$ .

In particular, we have

$$\begin{aligned}
(3.2) \quad & \sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle A_j y_j, y_j \rangle - \sum_{j=1}^n \langle f'(A_j) A_j x_j, x_j \rangle \\
& \leq \sum_{j=1}^n \langle f(A_j) y_j, y_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \\
& \leq \sum_{j=1}^n \langle f'(A_j) A_j y_j, y_j \rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \sum_{j=1}^n \langle f'(A_j) y_j, y_j \rangle
\end{aligned}$$

for any  $x_j, y_j \in H, j \in \{1, \dots, n\}$  with  $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$  and

$$\begin{aligned}
(3.3) \quad & \sum_{j=1}^n \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^n \langle B_j x_j, x_j \rangle - \sum_{j=1}^n \langle f'(A_j) A_j x_j, x_j \rangle \\
& \leq \sum_{j=1}^n \langle f(B_j) x_j, x_j \rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j \rangle \\
& \leq \sum_{j=1}^n \langle f'(B_j) B_j x_j, x_j \rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \sum_{j=1}^n \langle f'(B_j) x_j, x_j \rangle
\end{aligned}$$

for any  $x_j \in H, j \in \{1, \dots, n\}$  with  $\sum_{j=1}^n \|x_j\|^2 = 1$ .

*Proof.* As in [6, p. 6], if we put

$$\tilde{A} := \begin{pmatrix} A_1 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & A_n \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} B_1 & \cdot & \cdot & \cdot & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & B_n \end{pmatrix}$$



and

$$\tilde{x} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \tilde{y} = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

then we have  $Sp(\tilde{A}), Sp(\tilde{B}) \subseteq [m, M]$ ,  $\|\tilde{x}\| = \|\tilde{y}\| = 1$ ,

$$\langle f'(\tilde{A})\tilde{x}, \tilde{x} \rangle = \sum_{j=1}^n \langle f'(A_j)x_j, x_j \rangle, \langle B\tilde{y}, \tilde{y} \rangle = \sum_{j=1}^n \langle B y_j, y_j \rangle$$

and so on.

Applying Theorem 4 for  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{x}$  and  $\tilde{y}$  we deduce the desired result (3.1). ■

The following particular case may be of interest:

**Corollary 3.** *Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  be a convex and differentiable function on  $\mathring{I}$  (the interior of  $I$ ) whose derivative  $f'$  is continuous on  $\mathring{I}$ . If  $A_j$  and  $B_j$  are selfadjoint operators on the Hilbert space  $H$  with  $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$  for any  $j \in \{1, \dots, n\}$ , then for any  $p_j, q_j \geq 0$  with  $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$ , we have the inequalities*

$$(3.4) \quad \left\langle \sum_{j=1}^n p_j f'(A_j)x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j f'(A_j)A_j x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n q_j f(B_j)y, y \right\rangle - \left\langle \sum_{j=1}^n p_j f(A_j)x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n q_j f'(B_j)B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j f'(B_j)y, y \right\rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

In particular, we have

$$(3.5) \quad \left\langle \sum_{j=1}^n p_j f'(A_j)x, x \right\rangle \left\langle \sum_{j=1}^n q_j A_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j f'(A_j)A_j x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n q_j f(A_j)y, y \right\rangle - \left\langle \sum_{j=1}^n p_j f(A_j)x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n q_j f'(A_j)B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j f'(A_j)y, y \right\rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and

$$(3.6) \quad \left\langle \sum_{j=1}^n p_j f'(A_j) x, x \right\rangle \left\langle \sum_{j=1}^n p_j B_j x, x \right\rangle - \left\langle \sum_{j=1}^n p_j f'(A_j) A_j x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n p_j f(B_j) x, x \right\rangle - \left\langle \sum_{j=1}^n p_j f(A_j) x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n p_j f'(B_j) B_j x, x \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n p_j f'(B_j) x, x \right\rangle$$

for any  $x \in H$  with  $\|x\| = 1$ .

*Proof.* Follows from Theorem 6 on choosing  $x_j = \sqrt{p_j} \cdot x$ ,  $y_j = \sqrt{q_j} \cdot y$ ,  $j \in \{1, \dots, n\}$ , where  $p_j, q_j \geq 0$ ,  $j \in \{1, \dots, n\}$ ,  $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$  and  $x, y \in H$ , with  $\|x\| = \|y\| = 1$ . The details are omitted. ■

**Example 3. a.** Let  $A_j, B_j$ ,  $j \in \{1, \dots, n\}$ , be two sequences of positive definite operators on  $H$ . Then we have the inequalities

$$(3.7) \quad 1 - \sum_{j=1}^n \langle A_j^{-1} x_j, x_j \rangle \sum_{j=1}^n \langle B_j y_j, y_j \rangle \\ \leq \sum_{j=1}^n \langle \ln A_j x_j, x_j \rangle - \sum_{j=1}^n \langle \ln B_j y_j, y_j \rangle \leq \sum_{j=1}^n \langle A_j x_j, x_j \rangle \sum_{j=1}^n \langle B_j^{-1} y_j, y_j \rangle - 1$$

for any  $x_j, y_j \in H$ ,  $j \in \{1, \dots, n\}$  with  $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$ .

**b.** With the same assumption for  $A_j$  and  $B_j$  we have the inequalities

$$(3.8) \quad \sum_{j=1}^n \langle B_j y_j, y_j \rangle - \sum_{j=1}^n \langle A_j x_j, x_j \rangle \\ \leq \sum_{j=1}^n \langle B_j \ln B_j y_j, y_j \rangle - \sum_{j=1}^n \langle \ln A_j x_j, x_j \rangle \sum_{j=1}^n \langle B_j y_j, y_j \rangle$$

for any  $x_j, y_j \in H$ ,  $j \in \{1, \dots, n\}$  with  $\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|y_j\|^2 = 1$ .

Finally, we have

**Example 4. a.** Let  $A_j, B_j$ ,  $j \in \{1, \dots, n\}$ , be two sequences of positive definite operators on  $H$ . Then for any  $p_j, q_j \geq 0$  with  $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$ , we have the inequalities

$$(3.9) \quad 1 - \left\langle \sum_{j=1}^n p_j A_j^{-1} x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle \\ \leq \left\langle \sum_{j=1}^n p_j \ln A_j x, x \right\rangle - \left\langle \sum_{j=1}^n q_j \ln B_j y, y \right\rangle \\ \leq \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j^{-1} y, y \right\rangle - 1$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

**b.** With the same assumption for  $A_j, B_j, p_j$  and  $q_j$ , we have the inequalities

$$(3.10) \quad \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \\ \leq \left\langle \sum_{j=1}^n q_j B_j \ln B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j \ln A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle$$

for any  $x, y \in H$  with  $\|x\| = \|y\| = 1$ .

**Remark 3.** We observe that all the other inequalities for two operators obtained in Section 2 can be extended for two sequences of operators in a similar way. However, the details are left to the interested reader.

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