SOME INEQUALITIES FOR HYPERGEOMETRIC FUNCTIONS[1]

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1. **Introduction.** The hypergeometric R-function, $R(a; b_1, \cdots, b_n; x_1, \cdots, x_n)$, has recently been shown [**4**] to have a close connection with the theory of elementary mean values. By exploiting this connection we shall arrive at inequalities which compare R with (combinations of) mean values of the form $(\sum w_i x_i^t)^{1/t}$, where the weights (w) are related to the b-parameters by $w_i = b_i / \sum b_i$, $(i=1, \cdots, n)$. Both the b-parameters and the arguments (x) are required to be positive, but the a-parameter may be any real number.

The inequalities in their general form (Theorem 2) subsume a wide variety of special results, including some inequalities for elementary transcendental functions and elliptic integrals, inequalities due to Watson [**10**] and Szegö [**9**] for a certain integral, and new inequalities for the Gaussian and confluent hypergeometric functions as well as Appell's function F_1. New inequalities for the surface area and capacity of an ellipsoid in n dimensions are also included.

This work began from a conversation with Dr. G. D. Chakerian at a time when I was privileged to use the facilities of the Mathematics Department of the California Institute of Technology. I wish to thank also Mr. Malcolm D. Tobey for discussions of Theorem 1, which he first proved for restricted values of t by a method different from the one used here.

2. **The inequalities.** The R-function will be defined for present purposes by an integral representation,

$$R(a, b, x) = \int_E \left(\sum_{i=1}^n u_i x_i \right)^{-a} P(b, u') du', \tag{2.1}$$

where (b) and (x) stand for n-tuples of positive numbers, $(u') = (u_1, \cdots, u_{n-1})$, and $du' = du_1 \cdots du_{n-1}$. The domain of integration E is the set of points satisfying $u_i > 0$ $(i=1, \cdots, n-1)$ and $\sum_{i=1}^{n-1} u_i < 1$. Because we define $u_n \equiv 1 - \sum_{i=1}^{n-1} u_i$, the last condition is equivalently $u_n > 0$. The positive weight function

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$$P(b, u') = \frac{\Gamma(b_1 + \cdots + b_n)}{\Gamma(b_1) \cdots \Gamma(b_n)} \prod_{i=1}^{n} u_i^{b_i - 1}, \qquad (u') \in E,$$

satisfies

$$\int_E P(b, u')\, du' = 1.$$

A homogeneous mean of the values (x) is constructed from the R-function in the following way. We define the parameter c, the positive weights (w), and the hypergeometric mean $M(t, c)$ by

$$c \equiv \sum_{i=1}^{n} b_i, \tag{2.2}$$

$$w_i \equiv b_i/c, \quad (i = 1, \cdots, n), \qquad \left(\sum_{i=1}^{n} w_i = 1\right), \tag{2.3}$$

$$M(t, c; x, w) \equiv [R(-t, cw, x)]^{1/t}, \qquad (t \neq 0, c > 0). \tag{2.4}$$

If $t=0$ or $c=0$, the mean value is defined to be the limiting value of $M(t, c)$ as $t \to 0$ or $c \to 0$. It is shown in [4] that

$$\lim_{c \to 0} M(t, c; x, w) = M_t(x, w), \tag{2.5}$$

where M_t is the mean of order t:

$$\begin{aligned} M_t(x, w) &\equiv \left(\sum_{i=1}^{n} w_i x_i^t\right)^{1/t}, \qquad (t \neq 0), \\ M_0(x, w) &\equiv \prod_{i=1}^{n} x_i^{w_i}. \end{aligned} \tag{2.6}$$

We shall want several results taken from [4] which are valid for $c \geqq 0$:

$$\begin{aligned} &M(s, c; x, w) < M(t, c; x, w) \quad \text{if } s < t \text{ and } \min(x) < \max(x); \\ &M(1, c; x, w) = M_1(x, w), \qquad M(-c, c; x, w) = M_0(x, w). \end{aligned} \tag{2.7}$$

By $\max(x)$ we denote the largest of the positive numbers $x_1, \cdots, x_n$. In addition to (2.7) we shall want a new result:

THEOREM 1. *If* $\min(x) < \max(x)$, *the hypergeometric mean of the positive values* (x) *with positive weights* (w) *satisfies*

$$\begin{aligned} M(t, c; x, w) &< M_t(x, w), \qquad (t > 1, c > 0), \\ M(t, c; x, w) &> M_t(x, w), \qquad (t < 1, c > 0). \end{aligned} \tag{2.8}$$

PROOF. By (2.1) and (2.4) we have

$$[M(t,c;x,w)]^t = \int_E \left(\sum_{i=1}^{n} u_i x_i\right)^t P(cw,u')\,du', \qquad (c>0). \tag{2.9}$$

If $\min(x) < \max(x)$, then $(\sum u_i x_i)^t < \sum u_i x_i^t$ for $t > 1$ or $t < 0$, with reversed inequality for $0 < t < 1$; this follows directly from the fact that $M_t(x, u)$ is a strictly increasing function of t. Using (2.9) and (2.7), we have

$$\int_E (\sum u_i x_i^t) P(cw,u')\,du' = M(1,c;x^t,w) = M_1(x^t,w)$$
$$= \sum w_i x_i^t = [M_t(x,w)]^t.$$

The inequalities of Theorem 1 follow immediately provided that $t \neq 0$. If $t = 0$ and $c > 0$, the integral representation [**4**] of $M(0, c)$ and the concavity of the logarithmic function show that

$$\begin{aligned}\log[M(0,c;x,w)] &= \int_E \log\left(\sum_{i=1}^{n} u_i x_i\right) P(cw,u')\,du' \\ &> \int_E \left(\sum_{i=1}^{n} u_i \log x_i\right) P(cw,u')\,du' \\ &= \sum w_i \log x_i = \log M_0(x,w).\end{aligned} \tag{2.10}$$

In preparation for the statement of Theorem 2, we define several quantities in terms of elementary mean values, assuming always that (b), c, and (w) are related by (2.2) and (2.3):

$$\begin{aligned}H(a,b,x) &\equiv [M_1(x,w)]^{-a} = (\sum b_i x_i/c)^{-a}, \\ J(a,b,x) &\equiv [M_{-a}(x,w)]^{-a} = \sum b_i x_i^{-a}/c, \\ K(a,b,x) &\equiv [M_0(x,w)]^{-a} = \prod x_i^{-ab_i/c}, \\ J'(a,b,x) &\equiv [M_0(x,w)]^{-c}[M_{c-a}(x,w)]^{c-a} = (\prod x_i^{-b_i}) \sum b_i x_i^{c-a}/c, \\ H'(a,b,x) &\equiv [M_0(x,w)]^{-c}[M_{-1}(x,w)]^{c-a} \\ &= (\prod x_i^{-b_i})(\sum b_i x_i^{-1}/c)^{a-c}.\end{aligned} \tag{2.11}$$

As implied by (2.7) and (2.20) below, there are several values of a for which the hypergeometric function $R(a, b, x)$ takes a simple form:

$$\begin{aligned} &R = M_1 = H = J, && (a = -1), \\ &R = 1 = H = J = K, && (a = 0), \\ &R = M_0^{-c} = H' = J' = K, && (a = c), \\ &R = M_0^{-c} M_{-1}^{-1} = H' = J', && (a = c+1). \end{aligned} \tag{2.12}$$

Having disposed of these special values, we now state

THEOREM 2. *If the arguments* (x) *and parameters* (b) *are all positive and if* $\min(x) < \max(x)$, *the hypergeometric function* $R(a, b, x)$ *and the elementary functions defined by* (2.11) *satisfy*

$$H' < K < H < R < J < J', \qquad (-\infty < a < -1), \tag{2.13}$$

$$H' < K < J < R < \min\{H, J'\}, \qquad (-1 < a < 0), \tag{2.14}$$

$$\max\{H, H'\} < R < K < \min\{J, J'\}, \qquad (0 < a < c), \tag{2.15}$$

$$H < K < J' < R < \min\{H', J\}, \qquad (c < a < c + 1), \tag{2.16}$$

$$H < K < H' < R < J' < J, \qquad (c + 1 < a < \infty). \tag{2.17}$$

If $a \leqq c-1$, *then* $\min\{H, J'\} = H < J'$. *If* $a \geqq 1$, *then* $\min\{H', J\} = H' < J$.

PROOF. The main inequalities are those relating to R; all other parts of the theorem follow easily from the inequality $M_s < M_t$ if $s < t$. It is evident from (2.7) and (2.8) that

$$\begin{aligned} &M_1 < M(t, c) < M_t, && (1 < t < \infty),\\ &M_t < M(t, c) < M_1, && (0 < t < 1),\\ &M_0 < M(t, c) < M_1, && (-c < t < 0),\\ &M_t < M(t, c), && (-c - 1 < t < -c). \end{aligned} \tag{2.18}$$

Several additions to this list could be made (for instance, the last inequality is valid also for $-\infty < t < -c-1$) without sharpening the final results. We now raise each of the inequalities (2.18) to the power t, use (2.4), and substitute $t = -a$, thereby proving those parts of Theorem 2 which compare R with H, J, or K. In the inequalities

$$H(a, b, x) < R(a, b, x) < J(a, b, x), \qquad (-\infty < a < -1),$$

we now replace a by $c-a$ and (x) by $(x^{-1}) \equiv (x_1^{-1}, \cdots, x_n^{-1})$. The elementary relation $M_t(x^{-1}, w) = [M_{-t}(x, w)]^{-1}$ implies that

$$H(c - a, b, x^{-1}) = [M_0(x, w)]^c H'(a, b, x) \tag{2.19}$$

and that (2.19) still holds if H and H' are replaced by J and J' (or by K and $K' = K$). The Euler transformation of R is a similar equation [**2**]:

$$R(c - a, b, x^{-1}) = \left(\prod_{i=1}^{n} x_i^{b_i} \right) R(a, b, x) = [M_0(x, w)]^c R(a, b, x). \tag{2.20}$$

Thus (2.13) implies (2.17), and similarly all those parts of Theorem 2 which compare R with H' or J' are implied by those parts which compare R with H or J.

3. **Some special cases.** Assume x and y to be positive and unequal. From (2.1) and (2.15) we find that

$$R(1; 1, 1; x, y) = (\log x - \log y)/(x - y)$$

satisfies $H<R<K$; hence

$$(xy)^{1/2} < \frac{x - y}{\log x - \log y} < \frac{x + y}{2}. \tag{3.1}$$

If $r\neq 0$, then $R(1-r; 1, 1; x, y)=(x^r-y^r)/r(x-y)$ lies always between $H=[\frac{1}{2}(x+y)]^{r-1}$ and $J=\frac{1}{2}(x^{r-1}+y^{r-1})$ according to Theorem 2. This statement is sharper than the very useful result [**5**, Theorem 41] that it lies always between x^{r-1} and y^{r-1}.

If $0<a<c$ the R-function has the integral representation [**2**, Equation (7.1)]

$$B(a, c - a)R(a, b, x) = \int_0^\infty t^{c-a-1} \prod_{i=1}^n (t + x_i)^{-b_i}\, dt, \tag{3.2}$$

where B is the beta function. The integral was considered by Watson [**10**] and Szegö [**9**], who proved the right and left parts, respectively, of the inequality $H<R<K$ for the case $0<a=c-1$. This is a special case of (2.15).

Gauss' hypergeometric function is expressible in terms of the R-function by

$${}_2F_1(a, b; c; x) = R(a; b, c - b; 1 - x, 1). \tag{3.3}$$

Assume $c>b>0$ and $-\infty<x<1$, $x\neq 0$. Then the inequalities of Theorem 2 remain valid when the following substitutions are made:

$$\begin{aligned} R &= {}_2F_1(a, b; c; x), \\ K &= (1 - x)^{-ab/c}, \\ H &= (1 - bx/c)^{-a}, \\ H' &= (1 - x)^{c-a-b}[1 - x + bx/c]^{a-c}, \\ J &= 1 + (b/c)[(1 - x)^{-a} - 1], \\ J' &= (b/c)(1 - x)^{c-a-b} + (1 - b/c)(1 - x)^{-b}. \end{aligned} \tag{3.4}$$

If $c>a>0$ it may be useful to recall that both members of (3.3) are symmetric in a and b.

Inequalities for confluent hypergeometric functions can be obtained by taking a limit of Equations (3.4) in either of two ways. In the first limiting process (replacing x by x/a and letting $a \to \infty$), (2.13) or (2.17) yields only a weaker form (with $<$ replaced by $\leqq$) of the inequalities

$$(3.5) \qquad e^{bx/c} < {}_1F_1(b; c; x) < 1 + (b/c)(e^x - 1), \qquad (c > b > 0,\ x \neq 0).$$

However, strict inequality is easy to prove if $x > 0$ by comparing the power series of the three members, and if $x < 0$ by Kummer's transformation of ${}_1F_1$. In the other limiting process (replacing x by cx and letting $c \to \infty$), (2.13)–(2.15) become the following inequalities, valid for $b \geqq 0$ and $x \leqq 0$:

$$(3.6) \qquad (1 - bx)^{-a} \leqq {}_2F_0(a, b; x), \qquad (-\infty < a \leqq -1),$$

$$(3.7) \qquad 1 \leqq {}_2F_0(a, b; x) \leqq (1 - bx)^{-a}, \qquad (-1 \leqq a \leqq 0),$$

$$(3.8) \qquad (1 - bx)^{-a} \leqq {}_2F_0(a, b; x) \leqq 1, \qquad (0 \leqq a < \infty).$$

Results analogous to those for Gauss' hypergeometric function are easily found for Appell's function F_1 by using the relation

$$(3.9) \qquad F_1(a, b, b'; c; x, y) = R(a; b, b', c - b - b'; 1 - x, 1 - y, 1).$$

4. **Area and capacity of an ellipsoid.** The ellipsoid $\sum_{i=1}^{n} x_i^2/a_i^2 = 1$ has volume $V = a_1 a_2 \cdots a_n \pi^{n/2}/\Gamma(\frac{1}{2}n + 1)$; its surface area S and capacity C are given by

$$(4.1) \qquad S/nV = R(-\tfrac{1}{2}; \tfrac{1}{2}, \cdots, \tfrac{1}{2}; a_1^{-2}, \cdots, a_n^{-2}), \qquad (n \geqq 2),$$

$$(4.2) \qquad (n - 2)/C = R(\tfrac{1}{2}n - 1; \tfrac{1}{2}, \cdots, \tfrac{1}{2}; a_1^2, \cdots, a_n^2), \qquad (n \geqq 3).$$

The expression for S is derived by writing the surface area as an $(n-1)$-fold integral [**6**, Equation (10)] and comparing with (2.1); the expression for C comes from separating Laplace's equation in confocal coordinates [**1**] and using (3.2).

Assume that $a_1, \cdots, a_n$ are positive, finite, and not all equal. Then Theorem 2 gives inequalities for S and C, in particular

$$(4.3) \qquad \prod a_i^{-1/n} < \sum a_i^{-1}/n < S/nV < \left(\sum a_i^{-2}/n\right)^{1/2}.$$

For $n = 3$ these inequalities were found by Peano and Pólya [**7**]; for general n the members of (4.3) were discussed by Lehmer [**6**, p. 230] as approximations (but not as bounds) for S in the case of nearly spherical ellipsoids.

If $n = 3$, inequalities for C and S are equivalent to inequalities for the symmetric normal elliptic integrals [**3**]

$$(4.4)\qquad \begin{aligned} R_F(x, y, z) &\equiv R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x, y, z), \\ R_G(x, y, z) &\equiv R(-\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x, y, z). \end{aligned}$$

If x, y, z are positive and not all equal, Theorem 2 shows that

$$(4.5)\qquad \left(\frac{3}{x+y+z}\right)^{1/2} < R_F(x, y, z) < (xyz)^{-1/6},$$

$$(4.6)\qquad (xyz)^{1/6} < \frac{x^{1/2}+y^{1/2}+z^{1/2}}{3} < R_G(x, y, z) < \left(\frac{x+y+z}{3}\right)^{1/2}.$$

Inequalities for C due to Pólya and Szegö [8, Equation (1.7)] are equivalent to inequalities for R_F that are sharper than (4.5):

$$(4.7)\qquad \frac{3}{x^{1/2}+y^{1/2}+z^{1/2}} < R_F(x, y, z) < \frac{3}{(yz)^{1/4}+(zx)^{1/4}+(xy)^{1/4}}.$$

The left side of (4.7) is generalized to any $n \geqq 3$ by

THEOREM 3. *If $x_1, \cdots, x_n$ are nonnegative and not all equal, then*

$$(4.8)\qquad \left(n \Big/ \sum_{i=1}^{n} x_i^{1/2}\right)^{n-2} < R(\tfrac{1}{2}n-1; \tfrac{1}{2}, \tfrac{1}{2}, \cdots, \tfrac{1}{2}; x_1, x_2, \cdots, x_n), \qquad (n \geqq 3).$$

PROOF. Assume $y_1, \cdots, y_n$ are positive and not all equal. The inequality

$$\left(\sum y_i/n\right)^{n-1} > \left(\prod y_i\right) \sum y_i^{-1}/n, \qquad (n \geqq 3),$$

is part of Maclaurin's theorem for elementary symmetric functions [5, Theorem 52]. Replacing y_i by $(t+x_i)^{1/2}$, where $t>0$, we have

$$\begin{aligned} \prod (t+x_i)^{-1/2} &> \left[\sum (t+x_i)^{1/2}/n\right]^{1-n} \sum (t+x_i)^{-1/2}/n \\ &= (1-\tfrac{1}{2}n)^{-1}(d/dt)\left[\sum (t+x_i)^{1/2}/n\right]^{2-n}. \end{aligned}$$

We now integrate with respect to t and use (3.2) with $a = \frac{1}{2}n - 1 = c - 1$.

Theorems 2 and 3 show that the capacity of an ellipsoid in n dimensions ($n \geqq 3$) with positive semiaxes satisfies

$$(4.9)\qquad \prod a_i^{1-2/n} < C/(n-2) < \left(\sum a_i/n\right)^{n-2}$$

unless the ellipsoid is a sphere.

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