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SOME INEQUALITIES FOR RADIAL BLASCHKE-MINKOWSKI HOMOMORPHISMS

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Abstract. We establish some Brunn-Minkowski type inequalities for radial Blaschke-Minkowski homomorphisms with respect to Orlicz radial sums and differences of dual quermassintegrals.

Keywords: radial Blaschke-Minkowski homomorphism; Orlicz radial sum

MSC 2010: 52A20, 52A40

1. INTRODUCTION

During the last three decades, convex geometric analysis has achieved important developments. The classical Brunn-Minkowski theory has been extended to the L_p Brunn-Minkowski theory (see e.g. [6], [21]), and more recently to the more general Orlicz-Brunn-Minkowski theory, see [7], [8], [28], [29], [34].

Projection bodies and intersection bodies play a critical role in the solution of Shephard's problem, respectively the Busemann-Petty problem. We refer the reader to [16], [17], [33], [5], [4], [9], [10], [12], [13], [14], [18], [19], [20], [23], [22]. The projection body operator and intersection body operator are continuous and $GL(n)$ contravariant valuations, see [16], [17]. Schuster [25], [26] introduced the notion of Blaschke-Minkowski homomorphisms, respectively radial Blaschke-Minkowski homomorphisms, which are more general than the well known projection body operator, respectively the intersection body operator.

For $n \geq 3$, let \mathcal{K}^n be the space of compact convex sets with nonempty interior in \mathbb{R}^n endowed with the Hausdorff topology. A map $\Phi: \mathcal{K}^n \rightarrow \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism, see [25], if it satisfies the following conditions:

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- (a) Φ is continuous.
- (b) For $K_1, K_2 \in \mathcal{K}^n$,

$$\Phi(K_1 \# K_2) = \Phi K_1 + \Phi K_2,$$

where $K_1 \# K_2$ denotes the Blaschke sum of K_1 and K_2 , and $\Phi K_1 + \Phi K_2$ is the Minkowski sum of ΦK_1 and ΦK_2 .

- (c) For all $K \in \mathcal{K}^n$ and every $v \in SO(n)$,

$$\Phi(vK) = v\Phi K,$$

where $SO(n)$ is the group of rotations of \mathbb{R}^n .

A star body is a compact subset of \mathbb{R}^n that is star-shaped with respect to the origin and has a positive continuous radial function (see Section 2). For $n \geq 3$, we denote by \mathcal{S}^n the set of all star bodies of \mathbb{R}^n endowed with the Hausdorff topology. A map $\Psi: \mathcal{S}^n \rightarrow \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism, see [25], if it satisfies the following conditions:

- (a) Ψ is continuous.
- (b) For $K_1, K_2 \in \mathcal{S}^n$,

$$\Psi(K_1 \tilde{\#} K_2) = \Psi K_1 \tilde{+} \Psi K_2,$$

where $K_1 \tilde{\#} K_2$ denotes the radial Blaschke sum of K_1 and K_2 , and $\Psi K_1 \tilde{+} \Psi K_2$ is the radial Minkowski sum of ΨK_1 and ΨK_2 .

- (c) For all $K \in \mathcal{S}^n$ and every $v \in SO(n)$,

$$\Psi(vK) = v\Psi K,$$

where $SO(n)$ is the group of rotations of \mathbb{R}^n .

Volume inequalities for convex body and star body valued valuations are an active field of research (see [1], [2], [25], [26], [27], [30], [32], [11]). In particular, Schuster in [25] established the following Brunn-Minkowski type inequalities.

Theorem A. *Let $\Psi: \mathcal{S}^n \rightarrow \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{S}^n$, then*

$$V(\Psi(K_1 \tilde{+} K_2))^{1/n(n-1)} \leq V(\Psi K_1)^{1/n(n-1)} + V(\Psi K_2)^{1/n(n-1)},$$

with equality if and only if K_1 and K_2 are dilates.

In fact, a more general form of the Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms Ψ and dual quermassintegrals \widetilde{W}_j holds (see [25], Theorem 7.6): If $K_1, L_1 \in \mathcal{S}^n$, $0 \leq i \leq n-1$ and $0 \leq j < n-2$, then

$$(1.1) \quad \begin{aligned} \widetilde{W}_i(\Psi_j(K_1 \widetilde{+} K_2))^{1/(n-i)(n-j-1)} \\ \leq \widetilde{W}_i(\Psi_j K_1)^{1/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j K_2)^{1/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K_1 and K_2 are dilates.

Wang in [27] established the following L_p Brunn-Minkowski type inequalities for mixed radial Blaschke-Minkowski homomorphisms.

Theorem B. Let $\Psi: \underbrace{\mathcal{S}^n \times \dots \times \mathcal{S}^n}_{n-1} \rightarrow \mathcal{S}^n$ be a mixed radial Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{S}^n$, then for $0 \leq i \leq n-2$, $0 \leq j \leq n-2$, $0 < p < n-j-1$,

$$(1.2) \quad \begin{aligned} \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p K_2))^{p/(n-i)(n-j-1)} \\ \leq \widetilde{W}_i(\Psi_j K_1)^{p/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j K_2)^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K_1 and K_2 are dilates.

Leng in [15] established the following Brunn-Minkowski type inequality for the volume difference function.

Theorem C. If K, L, K_1 and L_1 are compact domains, $K_1 \subseteq K, L_1 \subseteq L$ and K_1 is a homothetic copy of L_1 , then

$$(V(K+L) - V(K_1+L_1))^{1/n} \geq (V(K) - V(K_1))^{1/n} + (V(L) - V(L_1))^{1/n}.$$

Equality holds if and only if K and L are homothetic and $(V(K), V(K_1)) = R(V(L), V(L_1))$, where R is a constant.

The aim of this paper is to establish Brunn-Minkowski type inequalities for radial Blaschke-Minkowski homomorphisms. Let \mathcal{C}^+ be the set of all Orlicz functions, that is, strictly decreasing convex functions $\phi: (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{t \rightarrow 0^+} \phi(t) = \infty$ and $\lim_{t \rightarrow \infty} \phi(t) = 0$. For $\phi \in \mathcal{C}^+$ the associated radial Orlicz addition is denoted by $\widetilde{+}_\phi$, see (2.1). First we show the following Orlicz-Brunn-Minkowski type inequality for radial Blaschke-Minkowski homomorphisms:

Theorem 1.1. Let $\Psi: \underbrace{S^n \times \dots \times S^n}_{n-1} \rightarrow S^n$ be a mixed radial Blaschke-Minkowski homomorphism. If $K_1, K_2 \in S^n$, $0 \leq i \leq n-1$, and $0 \leq j < n-2$, while $\phi \in \mathcal{C}^+$, then

$$\begin{aligned} \phi(1) \geq & \phi \left(\left(\frac{\widetilde{W}_i(\Psi_j K_1)}{\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_\phi K_2))} \right)^{1/(n-i)(n-j-1)} \right) \\ & + \phi \left(\left(\frac{\widetilde{W}_i(\Psi_j K_2)}{\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_\phi K_2))} \right)^{1/(n-i)(n-j-1)} \right), \end{aligned}$$

with equality if and only if K_1 and K_2 are dilates.

In particular, for $\phi(t) = t^p$, $p < 0$, Theorem 1.1 implies

$$\begin{aligned} & \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p K_2))^{p/(n-i)(n-j-1)} \\ & \geq \widetilde{W}_i(\Psi_j K_1)^{p/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j K_2)^{p/(n-i)(n-j-1)}, \end{aligned}$$

which complements Theorem B obtained by Wang.

Theorem 1.2. Let $K, L, K_1, L_1 \in S^n$, $K \subseteq K_1$, $L \subseteq L_1$ and K_1 is a dilate of L_1 . Then for $0 \leq i < n$, $0 \leq j < n-2$, $0 < p < n-j-1$,

$$\begin{aligned} & [\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1)) - \widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))]^{p/(n-i)(n-j-1)} \\ & \geq (\widetilde{W}_i(\Psi_j K_1) - \widetilde{W}_i(\Psi_j K))^{p/(n-i)(n-j-1)} \\ & \quad + (\widetilde{W}_i(\Psi_j L_1) - \widetilde{W}_i(\Psi_j L))^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K and L are dilates and

$$\widetilde{W}_i(\Psi_j K) / \widetilde{W}_i(\Psi_j K_1) = \widetilde{W}_i(\Psi_j L) / \widetilde{W}_i(\Psi_j L_1).$$

2. NOTATION AND BACKGROUND MATERIAL

Let \mathcal{K}^n denote the class of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space \mathbb{R}^n . For the class of convex bodies containing the origin in their interior, we write \mathcal{K}_0^n . $V(K)$ and $\widetilde{W}_i(K)$, $0 \leq i < n$, denote the n -dimensional volume and the quermassintegrals of a convex body K . Let S^{n-1} denote the unit sphere in \mathbb{R}^n .

For $K \in \mathcal{K}^n$, its support function $h_K(\cdot) := h(K, \cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ is defined for $u \in \mathbb{R}^n$ by $h(K, u) = \max\{x \cdot u: x \in K\}$, where $x \cdot u$ denotes the standard inner product

of u and x in \mathbb{R}^n . The support function determines a convex body uniquely. The Minkowski sum $\alpha K + \beta L$ of $K, L \in \mathcal{K}^n$ with $\alpha, \beta > 0$ is determined by $h(\alpha K + \beta L, u) = \alpha h(K, u) + \beta h(L, u)$.

For a set $K \subset \mathbb{R}^n$ that is star-shaped with respect to the origin, the radial function $\varrho_K(\cdot) := \varrho(K, \cdot): \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ is defined for $u \in \mathbb{R}^n \setminus \{0\}$ by $\varrho(K, u) = \max\{\lambda \geq 0: x \in \lambda K\}$. If ϱ_K is positive and continuous, then K is called a star body about the origin. We write \mathcal{S}^n for the space of all star bodies in \mathbb{R}^n . A star body is uniquely determined by its radial function, and the radial sum $\alpha K \tilde{+} \beta L$ of $K, L \in \mathcal{S}^n$ with $\alpha, \beta > 0$ is determined by $\varrho(\alpha K \tilde{+} \beta L, u) = \alpha \varrho(K, u) + \beta \varrho(L, u)$. If $\varrho(K, u)/\varrho(L, u)$ is independent of $u \in S^{n-1}$, then K and L are dilates.

The Orlicz radial sum is defined by Zhu, Zhou and Xu in [34]: Let $\alpha, \beta > 0$ and $\phi \in \mathcal{C}^+$. The Orlicz radial sum $\alpha \cdot K \tilde{+}_\phi \beta \cdot L$ is given for all $u \in \mathbb{R}^n \setminus \{0\}$ by

$$(2.1) \quad \varrho(\alpha \cdot K \tilde{+}_\phi \beta \cdot L, u) = \sup \left\{ t > 0: \alpha \phi\left(\frac{\varrho(K, u)}{t}\right) + \beta \phi\left(\frac{\varrho(L, u)}{t}\right) \leq \phi(1) \right\}.$$

If $\phi(t) = t^p$, $p < 0$, then the Orlicz radial sum reduces to the L_p radial sum $\tilde{+}_p$, which is given by

$$\varrho(\alpha \cdot K \tilde{+}_p \beta \cdot L, u)^p = \alpha \varrho(K, u)^p + \beta \varrho(L, u)^p$$

for all $u \in \mathbb{R}^n \setminus \{0\}$.

2.1. Mixed volumes and mixed quermassintegrals. Zhao in [31] defines dual Orlicz mixed quermassintegrals $\overline{W}_{\psi, i}(K, L)$ of K, L with respect to a strictly increasing convex function $\psi: (0, \infty) \rightarrow (0, \infty)$ with $\lim_{t \rightarrow 0^+} = 0$ and $\lim_{t \rightarrow \infty} = \infty$, by

$$\overline{W}_{\psi, i}(K, L) := \frac{1}{n} \int_{S^{n-1}} \psi\left(\frac{\varrho(K, u)}{\varrho(L, u)}\right) \varrho(K, u)^{n-i} dS(u).$$

We use a slightly different definition: For $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}^+$ we define $\widetilde{W}_{\phi, i}(K, L)$ by

$$\widetilde{W}_{\phi, i}(K, L) := \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\varrho(L, u)}{\varrho(K, u)}\right) \varrho(K, u)^{n-i} dS(u).$$

If $\phi \in \mathcal{C}^+$ is such that $\psi(t) = \phi(1/t)$ is convex, then $\overline{W}_{\psi, i}(K, L) = \widetilde{W}_{\phi, i}(K, L)$.

For $K, M, N \in \mathcal{S}^n$, $0 \leq i \leq n-2$ and $\phi \in \mathcal{C}^+$, the dual mixed Orlicz quermassintegral $\widetilde{W}_{\phi, i, 1}(K, M, N)$ is defined by

$$(2.2) \quad \frac{n-i-1}{\phi'_r(1)} \widetilde{W}_{\phi, i, 1}(K, M, N) = \lim_{\varepsilon \rightarrow 0^+} \frac{\widetilde{W}_i(K \tilde{+}_\phi \varepsilon \cdot M, N) - \widetilde{W}_i(K, N)}{\varepsilon},$$

where $\phi'_r(1)$ is the right-derivative of ϕ at 1, which exists, since ϕ is convex (see, e.g., [24], Theorem 1.5.4), and $\phi'_r(1) < 0$, since ϕ is strictly decreasing. For the unit ball B in \mathbb{R}^n , by Lemma 3.3 (see Section 3) we have $\widetilde{W}_{\phi,i,1}(K, M, B) = \widetilde{W}_{\phi,i+1}(K, M)$.

Lutwak in [19] showed that the volume of a radial Minkowski combination $\lambda_1 L_1 \widetilde{+} \dots \widetilde{+} \lambda_m L_m$ of star bodies L_1, \dots, L_m can be expressed as a homogeneous polynomial of degree n :

$$V(\lambda_1 L_1 \widetilde{+} \dots \widetilde{+} \lambda_m L_m) = \sum_{i_1, \dots, i_n=1}^m \widetilde{V}(L_{i_1}, \dots, L_{i_n}) \lambda_{i_1} \dots \lambda_{i_n}.$$

The coefficients $\widetilde{V}(L_{i_1}, \dots, L_{i_n})$ are called the dual mixed volumes of L_{i_1}, \dots, L_{i_n} . The definition of dual mixed volumes $\widetilde{W}_i(L, N)$ and $\widetilde{W}_i(L)$ are analogous to those for the mixed volume. The dual mixed volume $\widetilde{W}_i(L, N)$ has the integral representation

$$(2.3) \quad \widetilde{W}_i(L, N) = \frac{1}{n} \int_{S^{n-1}} \varrho(L, u)^{n-i-1} \varrho(N, u) dS(u),$$

where $dS(u)$ is the spherical Lebesgue measure of S^{n-1} .

The Minkowski inequality for dual mixed quermassintegrals is stated by Wang in [27] as follows: For $L_1, L_2 \in \mathcal{S}^n$ and $0 \leq i < n - 1$,

$$(2.4) \quad \widetilde{W}_i(L_1, L_2)^{n-i} \leq \widetilde{W}_i(L_1)^{n-i-1} \widetilde{W}_i(L_2),$$

with equality if and only if L_1 and L_2 are dilates. It is a direct consequence of Hölder's inequality for integrals (see, e.g., [24], Lemma 9.3.1). A variant, which also follows from Hölder's inequality, is the following: For $L_1, L_2, Q \in \mathcal{S}^n$ and $0 \leq i < n - 2$, set

$$\widetilde{W}_{i,1}(L_1, L_2, Q) = \frac{1}{n} \int_{S^{n-1}} \varrho(L_1, u)^{n-i-2} \varrho(L_2, u) \varrho(Q, u) dS(u).$$

Then

$$(2.5) \quad \widetilde{W}_{i,1}(L_1, L_2, Q)^{n-i-1} \leq \widetilde{W}_i(L_1, Q)^{n-i-2} \widetilde{W}_i(L_2, Q).$$

2.2. Mixed Blaschke-Minkowski homomorphisms and mixed radial Blaschke-Minkowski homomorphisms. Let $\Phi: \mathcal{K}^n \rightarrow \mathcal{K}^n$ be a Blaschke-Minkowski homomorphism. Schuster in [25], Theorem 1.2, showed that if $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$, then there is a continuous operator

$$\Phi: \underbrace{\mathcal{K}^n \times \dots \times \mathcal{K}^n}_{n-1} \rightarrow \mathcal{K}^n,$$

symmetric in its arguments, such that

$$\Phi(\lambda_1 K_1 + \dots + \lambda_m K_m) = \sum_{i_1, \dots, i_{n-1}=1}^m \Phi(K_{i_1}, \dots, K_{i_{n-1}}) \lambda_{i_1} \dots \lambda_{i_{n-1}},$$

where the $\Phi(K_{i_1}, \dots, K_{i_{n-1}})$ are called mixed Blaschke-Minkowski homomorphisms, and the sum is with respect to Minkowski addition.

If $K_1 = \dots = K_{n-i-1} = K$ and $K_{n-i} = \dots = K_{n-1} = M$, then $\Phi(K_1, \dots, K_{n-1})$ will be written as $\Phi_i(K, M)$. In particular, $\Phi_i K = \Phi_i(K, B)$ and $\Phi K = \Phi_0 K$.

Let $\Psi: \mathcal{S}^n \rightarrow \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. Schuster in [25], Theorem 1.2_d, showed that if $K_1, \dots, K_m \in \mathcal{S}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$, then there is a continuous operator

$$\Psi: \underbrace{\mathcal{S}^n \times \dots \times \mathcal{S}^n}_{n-1} \rightarrow \mathcal{S}^n,$$

symmetric in its arguments, such that

$$\Psi(\lambda_1 K_1 \tilde{+} \dots \tilde{+} \lambda_m K_m) = \sum_{i_1, \dots, i_{n-1}=1}^m \Psi(K_{i_1}, \dots, K_{i_{n-1}}) \lambda_{i_1} \dots \lambda_{i_{n-1}},$$

where the $\Psi(K_{i_1}, \dots, K_{i_{n-1}})$ are called mixed radial Blaschke-Minkowski homomorphisms, and the sum is with respect to the radial sum.

If $K_1 = \dots = K_{n-i-1} = K$ and $K_{n-i} = \dots = K_{n-1} = M$, then $\Psi(K_1, \dots, K_{n-1})$ will be written as $\Psi_i(K, M)$. In particular, $\Psi_i K = \Psi_i(K, B)$ and $\Psi K = \Psi_0 K$.

3. LEMMAS AND MAIN RESULTS

Lemma 3.1 (Jensen's inequality [8]). *Let μ be a probability measure in a space X , let U be an open convex set in \mathbb{R}^n , and let φ be a convex real-valued function on U . Assume that $g: X \rightarrow U$ is measurable and component-wise μ -integrable, and that $\varphi \circ g$ is μ -integrable. Let $z_0 = \int_X g(x) d\mu(x)$. Then $z_0 \in U$ and*

$$\int_X \varphi(g(x)) d\mu(x) \geq \varphi\left(\int_X g(x) d\mu(x)\right).$$

Lemma 3.2 ([34]). *If $\phi \in \mathcal{C}^+$ and $K, L \in \mathcal{S}^n$, then*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\varrho(K \tilde{+}_{\phi} \varepsilon \cdot L, u) - \varrho(K, u)}{\varepsilon} = \frac{\varrho(K, u)}{\phi'_r(1)} \phi\left(\frac{\varrho(L, u)}{\varrho(K, u)}\right), \quad \text{uniformly on } S^{n-1}.$$

Lemma 3.3. If $\phi \in \mathcal{C}^+$ and $K, L, Q \in S^n$, then for each $i = 0, 1, \dots, n - 2$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_\phi \varepsilon \cdot L, Q) - \widetilde{W}_i(K, Q)}{\varepsilon} \\ &= \frac{n - i - 1}{\phi'_r(1)n} \int_{S^{n-1}} \phi\left(\frac{\varrho(L, u)}{\varrho(K, u)}\right) \varrho(K, u)^{n-i-1} \varrho(Q, u) \, dS(u). \end{aligned}$$

Proof. From Lemma 3.2 we derive

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_\phi \varepsilon \cdot L, Q) - \widetilde{W}_i(K, Q)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \left[\frac{1}{n\varepsilon} \int_{S^{n-1}} \varrho(K \widetilde{+}_\phi \varepsilon \cdot L, u)^{n-i-1} \varrho(Q, u) \right. \\ & \quad \left. - \frac{1}{n\varepsilon} \int_{S^{n-1}} \varrho(K, u)^{n-i-1} \varrho(Q, u) \, dS(u) \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{\varrho(K \widetilde{+}_\phi \varepsilon \cdot L, u)^{n-i-1} - \varrho(K, u)^{n-i-1}}{\varepsilon} \varrho(Q, u) \, dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{\varrho(K \widetilde{+}_\phi \varepsilon \cdot L, u)^{n-i-1} - \varrho(K, u)^{n-i-1}}{\varepsilon} \varrho(Q, u) \, dS(u) \\ &= \frac{n - i - 1}{\phi'_r(1)n} \int_{S^{n-1}} \phi\left(\frac{\varrho(L, u)}{\varrho(K, u)}\right) \varrho(K, u)^{n-i-1} \varrho(Q, u) \, dS(u). \end{aligned}$$

□

Lemma 3.4. If $\phi \in \mathcal{C}^+$ and $K, L, Q \in S^n$, while $0 \leq i \leq n - 2$, then

$$(3.1) \quad \frac{\widetilde{W}_{\phi, i, 1}(K, L, Q)}{\widetilde{W}_i(K, Q)} \geq \phi\left(\left(\frac{\widetilde{W}_i(L, Q)}{\widetilde{W}_i(K, Q)}\right)^{1/(n-i-1)}\right).$$

Proof. Since

$$\widetilde{W}_i(K, Q) = \frac{1}{n} \int_{S^{n-1}} \varrho(K, u)^{n-i-1} \varrho(Q, u) \, dS(u),$$

and since $\varrho(K, \cdot)$ and $\varrho(Q, \cdot)$ are positive functions, we have $\widetilde{W}_i(K, Q) > 0$. Hence, $\varrho(K, u)^{n-i-1} \varrho(Q, u) \, dS(u) / n \widetilde{W}_i(K, Q)$ is a probability measure on S^{n-1} . It follows

from Lemma 3.1, Lemma 3.3 and (2.5) and the fact that ϕ is decreasing, that

$$\begin{aligned}
 \frac{\widetilde{W}_{\phi,i,1}(K, L, Q)}{\widetilde{W}_i(K, Q)} &= \frac{1}{\widetilde{W}_i(K, Q)n} \int_{S^{n-1}} \phi\left(\frac{\varrho(L, u)}{\varrho(K, u)}\right) \varrho(K, u)^{n-i-1} \varrho(Q, u) \, dS(u) \\
 &\geq \phi\left(\frac{1}{n\widetilde{W}_i(K, Q)} \int_{S^{n-1}} \varrho(L, u) \varrho(K, u)^{n-i-2} \varrho(Q, u) \, dS(u)\right) \\
 &= \phi\left(\frac{\widetilde{W}_{i,1}(K, L, Q)}{\widetilde{W}_i(K, Q)}\right) \\
 &\geq \phi\left(\frac{\widetilde{W}_i(K, Q)^{(n-i-2)/(n-i-1)} \widetilde{W}_i(L, Q)^{1/(n-i-1)}}{\widetilde{W}_i(K, Q)}\right) \\
 &= \phi\left(\left(\frac{\widetilde{W}_i(L, Q)}{\widetilde{W}_i(K, Q)}\right)^{1/(n-i-1)}\right).
 \end{aligned}$$

□

Lemma 3.5. *If $\phi \in C^+$ and $K_1, K_2, Q \in S^n$, while $0 \leq i \leq n-2$, then*

$$(3.2) \quad \phi(1) \geq \phi\left(\left(\frac{\widetilde{W}_i(K_1, Q)}{\widetilde{W}_i(K_1 \tilde{+}_\phi K_2, Q)}\right)^{1/(n-i-1)}\right) + \phi\left(\left(\frac{\widetilde{W}_i(K_2, Q)}{\widetilde{W}_i(K_1 \tilde{+}_\phi K_2, Q)}\right)^{1/(n-i-1)}\right).$$

Proof. For brevity, let $\widetilde{K} = K_1 \tilde{+}_\phi K_2$. The definition of \widetilde{K} necessarily implies for all $u \in S^{n-1}$ that

$$\phi(1) = \phi\left(\frac{\varrho(K_1, u)}{\varrho(\widetilde{K}, u)}\right) + \phi\left(\frac{\varrho(K_2, u)}{\varrho(\widetilde{K}, u)}\right).$$

From

$$\widetilde{W}_i(\widetilde{K}, Q) = \frac{1}{n} \int_{S^{n-1}} \varrho(\widetilde{K}, u)^{n-i-1} \varrho(Q, u) \, dS(u),$$

and by Lemma 3.4, we derive

$$\begin{aligned}
 \phi(1)\widetilde{W}_i(\widetilde{K}, Q) &= \frac{1}{n} \int_{S^{n-1}} \left(\phi\left(\frac{\varrho(K_1, u)}{\varrho(\widetilde{K}, u)}\right) + \phi\left(\frac{\varrho(K_2, u)}{\varrho(\widetilde{K}, u)}\right)\right) \varrho(\widetilde{K}, u)^{n-i-1} \varrho(Q, u) \, dS(u) \\
 &= \widetilde{W}_{\phi,i,1}(\widetilde{K}, K_1, Q) + \widetilde{W}_{\phi,i,1}(\widetilde{K}, K_2, Q) \\
 &\geq \widetilde{W}_i(\widetilde{K}, Q) \left[\phi\left(\left(\frac{\widetilde{W}_i(K_1, Q)}{\widetilde{W}_i(\widetilde{K}, Q)}\right)^{1/(n-i-1)}\right) + \phi\left(\left(\frac{\widetilde{W}_i(K_2, Q)}{\widetilde{W}_i(\widetilde{K}, Q)}\right)^{1/(n-i-1)}\right)\right].
 \end{aligned}$$

Thus, (3.2) is established. □

Lemma 3.6 ([25]). Let $\Psi: \underbrace{\mathcal{S}^n \times \dots \times \mathcal{S}^n}_{n-1} \rightarrow \mathcal{S}^n$ be a mixed radial Blaschke-Minkowski homomorphism. If $K, M \in \mathcal{S}^n$, while $0 \leq i \leq n-2$ and $0 \leq j \leq n-2$, then

$$(3.3) \quad \widetilde{W}_i(K, \Psi_j M) = \widetilde{W}_j(M, \Psi_i K).$$

Furthermore, the image of a ball B under a radial Blaschke-Minkowski homomorphism is again a ball. Let r_Ψ be such that $\Psi B = r_\Psi B$. Then

$$(3.4) \quad \widetilde{W}_{n-1}(\Psi_i K) = r_\Psi \widetilde{W}_{i+1}(K).$$

We are now ready to prove Theorem 1.1:

Theorem 3.7. Let $\Psi: \underbrace{\mathcal{S}^n \times \dots \times \mathcal{S}^n}_{n-1} \rightarrow \mathcal{S}^n$ be a mixed radial Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{S}^n$, $0 \leq i \leq n-2$ and $0 \leq j < n-2$, while $\phi \in \mathcal{C}^+$, then

$$(3.5) \quad \begin{aligned} \phi(1) \geq & \phi \left(\left(\frac{\widetilde{W}_i(\Psi_j K_1)}{\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_\phi K_2))} \right)^{1/(n-i)(n-j-1)} \right) \\ & + \phi \left(\left(\frac{\widetilde{W}_i(\Psi_j K_2)}{\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_\phi K_2))} \right)^{1/(n-i)(n-j-1)} \right), \end{aligned}$$

with equality if and only if K_1 and K_2 are dilates.

Proof. By (2.4), (3.2), (3.3) and the fact that ϕ is decreasing, we have for $M \in \mathcal{S}^n$ that

$$\begin{aligned} & \phi(1) \\ \geq & \phi \left(\left(\frac{\widetilde{W}_j(K_1, \Psi_i M)}{\widetilde{W}_j(K_1 \widetilde{+}_\phi K_2, \Psi_i M)} \right)^{1/(n-j-1)} \right) + \phi \left(\left(\frac{\widetilde{W}_j(K_2, \Psi_i M)}{\widetilde{W}_j(K_1 \widetilde{+}_\phi K_2, \Psi_i M)} \right)^{1/(n-j-1)} \right) \\ = & \phi \left(\left(\frac{\widetilde{W}_i(M, \Psi_j K_1)}{\widetilde{W}_i(M, \Psi_j(K_1 \widetilde{+}_\phi K_2))} \right)^{1/(n-j-1)} \right) + \phi \left(\left(\frac{\widetilde{W}_i(M, \Psi_j K_2)}{\widetilde{W}_i(M, \Psi_j(K_2 \widetilde{+}_\phi K_2))} \right)^{1/(n-j-1)} \right) \\ \geq & \phi \left(\left(\frac{\widetilde{W}_i(M)^{(n-i-1)/(n-i)} \widetilde{W}_i(\Psi_j K_1)^{1/(n-i)}}{\widetilde{W}_i(M, \Psi_j(K_1 \widetilde{+}_\phi K_2))} \right)^{1/(n-j-1)} \right) \\ & + \phi \left(\left(\frac{\widetilde{W}_i(M)^{(n-i-1)/(n-i)} \widetilde{W}_i(\Psi_j K_2)^{1/(n-i)}}{\widetilde{W}_i(M, \Psi_j(K_1 \widetilde{+}_\phi K_2))} \right)^{1/(n-j-1)} \right). \end{aligned}$$

By the equality conditions of (2.4), equality holds if and only if M , $\Psi_j K_1$, and $\Psi_j K_2$ are dilates.

Set $M = \Psi_j(K_1 \tilde{+}_\phi K_2)$ and note that $\widetilde{W}_i(M, M) = \widetilde{W}_i(M)$ to obtain (3.5). If there is equality in (3.5), then there exist $\lambda_1, \lambda_2 > 0$ such that

$$(3.6) \quad \Psi_j K_1 = \lambda_1 \Psi_j(K_1 \tilde{+}_\phi K_2) \quad \text{and} \quad \Psi_j K_2 = \lambda_2 \Psi_j(K_1 \tilde{+}_\phi K_2).$$

From equality (3.5), it follows that

$$\phi(\lambda_1^{1/(n-j-1)}) + \phi(\lambda_2^{1/(n-j-1)}) = \phi(1).$$

Moreover, (3.4) and (3.6) imply

$$(3.7) \quad \lambda_1 = \frac{\widetilde{W}_{j+1}(K_1)}{\widetilde{W}_{j+1}(K_1 \tilde{+}_\phi K_2)} \quad \text{and} \quad \lambda_2 = \frac{\widetilde{W}_{j+1}(K_2)}{\widetilde{W}_{j+1}(K_1 \tilde{+}_\phi K_2)}.$$

Hence, we have

$$\phi(1) = \phi\left(\left(\frac{\widetilde{W}_{j+1}(K_1)}{\widetilde{W}_{j+1}(K_1 \tilde{+}_\phi K_2)}\right)^{1/(n-j-1)}\right) + \phi\left(\left(\frac{\widetilde{W}_{j+1}(K_2)}{\widetilde{W}_{j+1}(K_1 \tilde{+}_\phi K_2)}\right)^{1/(n-j-1)}\right),$$

and using the equality conditions of an Orlicz-Brunn-Minkowski inequality for the dual mixed quermassintegral, see [31], we obtain K_1 and K_2 are dilates. \square

The intersection body operator $I: \mathcal{S}^n \rightarrow \mathcal{S}^n$ is a radial Blaschke-Minkowski homomorphism and we derive the following corollary from Theorem 3.7:

Corollary 3.8. *If $K_1, K_2 \in \mathcal{S}^n$ and $\phi \in \mathcal{C}^+$, then*

$$(3.8) \quad \phi(1) \geq \phi\left(\left(\frac{V(IK_1)}{V(I(K_1 \tilde{+}_\phi K_2))}\right)^{1/n(n-1)}\right) + \phi\left(\left(\frac{V(IK_2)}{V(I(K_1 \tilde{+}_\phi K_2))}\right)^{1/n(n-1)}\right),$$

with equality if and only if K_1 and K_2 are dilates.

Lemma 3.9 ([3]). *Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be two series of positive real numbers. If $p > 1$, $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$, then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} \leq \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{1/p},$$

with equality if and only if $a_1/b_1 = a_2/b_2 = \dots = a_n/b_n$.

Lemma 3.10 ([27]). Let $\Phi: \underbrace{\mathcal{K}^n \times \dots \times \mathcal{K}^n}_{n-1} \rightarrow \mathcal{K}^n$ be a mixed Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{K}^n$ and $p > 1$, then for $0 \leq i \leq n-2$, $0 \leq j \leq n-2$,

$$(3.9) \quad \begin{aligned} W_i(\Phi_j(K_1 +_p K_2))^{p/(n-i)(n-j-1)} \\ \geq W_i(\Phi_j K_1)^{p/(n-i)(n-j-1)} + W_i(\Phi_j K_2)^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K_1 and K_2 are dilates.

Theorem 3.11. Let $K, L \in \mathcal{K}^n$, $K_1, L_1 \in \mathcal{S}^n$, $K_1 \subseteq K$, $L_1 \subseteq L$ and let K_1, L_1 be dilates of each other. Then for $0 \leq i < n$, $0 \leq j < n-2$, $1 < p < n-j-1$,

$$\begin{aligned} [W_i(\Phi_j(K +_p L)) - \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))]^{p/(n-i)(n-j-1)} \\ \geq (W_i(\Phi_j K) - \widetilde{W}_i(\Psi_j K_1))^{p/(n-i)(n-j-1)} \\ + (W_i(\Phi_j L) - \widetilde{W}_i(\Psi_j L_1))^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K and L are dilates and

$$W_i(\Phi_j K) / \widetilde{W}_i(\Psi_j K_1) = W_i(\Phi_j L) / \widetilde{W}_i(\Psi_j L_1).$$

Proof. By Theorem B and since K_1 and L_1 are dilates, we have

$$\begin{aligned} \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))^{p/(n-i)(n-j-1)} \\ = \widetilde{W}_i(\Psi_j K_1)^{p/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j L_1)^{p/(n-i)(n-j-1)}. \end{aligned}$$

It follows from Lemma 3.9 and Lemma 3.10 that

$$\begin{aligned} [W_i(\Phi_j(K +_p L)) - \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))]^{p/(n-i)(n-j-1)} \\ \geq [(W_i(\Phi_j K)^{p/(n-i)(n-j-1)} + W_i(\Phi_j L)^{p/(n-i)(n-j-1)})^{(n-i)(n-j-1)/p} \\ - (\widetilde{W}_i(\Psi_j K_1))^{p/(n-i)(n-j-1)} \\ + \widetilde{W}_i(\Psi_j L_1)^{p/(n-i)(n-j-1)})^{(n-i)(n-j-1)/p}]^{p/(n-i)(n-j-1)} \\ \geq (W_i(\Phi_j K) - \widetilde{W}_i(\Psi_j L))^{p/(n-i)(n-j-1)} \\ + (W_i(\Phi_j K_1) - \widetilde{W}_i(\Psi_j L_1))^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K and L are dilates and

$$W_i(\Phi_j K) / \widetilde{W}_i(\Psi_j K_1) = W_i(\Phi_j L) / \widetilde{W}_i(\Psi_j L_1).$$

□

Taking the mixed projection body Π_i and the mixed intersection operator I_i as the mixed Blaschke-Minkowski and the mixed radial Blaschke-Minkowski homomorphisms, we derive:

Corollary 3.12. *Let $K, L \in \mathcal{K}^n$, $K_1, L_1 \in \mathcal{S}^n$, $K_1 \subseteq K$, $L_1 \subseteq L$ and let K_1, L_1 be dilates of each other. Then for $0 \leq i < n$, $0 \leq j < n - 2$, $1 < p < n - j - 1$,*

$$\begin{aligned} & [W_i(\Pi_j(K \dot{+}_p L)) - \widetilde{W}_i(I_j(K_1 \widetilde{+}_p L_1))]^{p/(n-i)(n-j-1)} \\ & \geq (W_i(\Pi_j K) - \widetilde{W}_i(I_j K_1))^{p/(n-i)(n-j-1)} \\ & \quad + (W_i(\Pi_j L) - \widetilde{W}_i(I_j L_1))^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K and L are dilates and

$$W_i(\Pi_j K) / \widetilde{W}_i(I_j K_1) = W_i(\Pi_j L) / \widetilde{W}_i(I_j L_1).$$

Taking $i = 0$, $j = 0$ in Corollary 3.12 yields

Corollary 3.13. *Let $K, L \in \mathcal{K}^n$, $K_1, L_1 \in \mathcal{S}^n$, $K_1 \subseteq K$, $L_1 \subseteq L$ and let K_1, L_1 be dilates of each other. Then for $1 < p < n - 1$,*

$$\begin{aligned} & [V(\Pi(K \dot{+}_p L)) - V(I(K_1 \widetilde{+}_p L_1))]^{p/n(n-1)} \\ & \geq (V(\Pi K) - V(IK_1))^{p/n(n-1)} + (V(\Pi L) - V(IL_1))^{p/n(n-1)}, \end{aligned}$$

with equality if and only if K and L are dilates and

$$V(\Pi K) / V(IK_1) = V(\Pi L) / V(IL_1).$$

Theorem 3.14. *Let $K, L, K_1, L_1 \in \mathcal{S}^n$, $K \subseteq K_1$, $L \subseteq L_1$ and let K_1 be a dilate of L_1 . Then for $0 \leq i < n$, $0 \leq j < n - 2$, $0 < p < n - j - 1$,*

$$\begin{aligned} & [\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1)) - \widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))]^{p/(n-i)(n-j-1)} \\ & \geq (\widetilde{W}_i(\Psi_j K_1) - \widetilde{W}_i(\Psi_j K))^{p/(n-i)(n-j-1)} \\ & \quad + (\widetilde{W}_i(\Psi_j L_1) - \widetilde{W}_i(\Psi_j L))^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K and L are dilates and

$$\widetilde{W}_i(\Psi_j K) / \widetilde{W}_i(\Psi_j K_1) = \widetilde{W}_i(\Psi_j L) / \widetilde{W}_i(\Psi_j L_1).$$

Proof. We set $q = (n - i)(n - j - 1)$. Applying Theorem B, we obtain

$$\begin{aligned} & \widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))^{p/q} \leq \widetilde{W}_i(\Psi_j K)^{p/q} + \widetilde{W}_i(\Psi_j L)^{p/q}, \\ & \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))^{p/q} = \widetilde{W}_i(\Psi_j K_1)^{p/q} + \widetilde{W}_i(\Psi_j L_1)^{p/q}. \end{aligned}$$

It follows from Lemma 3.9, which we can use since $q/p > 1$, that

$$\begin{aligned} & [\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1)) - \widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))]^{p/q} \\ & \geq [(\widetilde{W}_i(\Psi_j K_1)^{p/q} + \widetilde{W}_i(\Psi_j L_1)^{p/q})^{q/p} - (\widetilde{W}_i(\Psi_j K)^{p/q} + \widetilde{W}_i(\Psi_j L)^{p/q})^{q/p}]^{p/q} \\ & \geq (\widetilde{W}_i(\Psi_j K_1) - \widetilde{W}_i(\Psi_j K))^{p/q} + (W_i(\Psi_j L_1) - \widetilde{W}_i(\Psi_j L))^{p/q}. \end{aligned}$$

Equality holds if and only if equality holds in Theorem B and Lemma 3.9, hence equality holds if and only if K and L are dilates and

$$\widetilde{W}_i(\Psi_j K) / \widetilde{W}_i(\Psi_j K_1) = \widetilde{W}_i(\Psi_j L) / \widetilde{W}_i(\Psi_j L_1).$$

□

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