

SOME INFINITE FIBONACCI GROUPS

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The Fibonacci groups are a special case of the following class of groups first studied by G. A. Miller (7). Given a natural number n , let θ be the automorphism of the free group $F = \langle x_1, \dots, x_n \mid \rangle$ of rank n which permutes the subscripts of the generators in accordance with the cycle $(1, 2, \dots, n)$. Given a word w in F , let R be the smallest normal subgroup of F which contains w and is closed under θ . Then define $G_n(w) = F/R$ and write $A_n(w)$ for the derived factor group of $G_n(w)$. Putting, for $r \geq 2, k \geq 1$,

$$w = x_1 \dots x_r x_{r+k}^{-1},$$

with subscripts reduced modulo n , we obtain the groups $F(r, n, k)$ studied in (1) and (2), while the $F(r, n, 1)$ are the ordinary Fibonacci groups $F(r, n)$ of (3), (5) and (6). To conform with earlier notation, we write $A(r, n, k)$ and $A(r, n)$ for the derived factor groups of $F(r, n, k)$, and $F(r, n)$ respectively.

Our purpose in the present note is threefold: firstly to explain the connection between the two apparently different formulae for $|A(r, n)|$ given in (5) (numbered (3) and (6)), secondly to derive necessary and sufficient conditions for $A_n(w)$ to be infinite, and finally to apply this to the groups $A(r, n, k)$ by way of example. The second item is the analogue of a result of Dunwoody (4), while the third extends Theorems 6 and 7 of (1) and at the same time complements Theorem 1 of (2).

The symbol $| \cdot |$ will stand indiscriminately for the order of an element of a group, the order of a group or the absolute value of a complex number, it being clear from the context which is intended. $\mathbf{Z}[x]$ denotes the ring of polynomials over the integers \mathbf{Z} , while $\mathbf{Z}\langle x \mid x^n \rangle$ is the integral group ring of a cyclic group of order n , so that

$$\mathbf{Z}\langle x \mid x^n \rangle \cong \mathbf{Z}[x]/(x^n - 1),$$

as rings. Finally, we observe the usual conventions with regard to empty sums (for example, formula (1) below when $s = 0$) and empty products (for example, the definition of $f * g$ when f is a constant polynomial).

The n -generator, n -relation presentation given above for $F(r, n)$ shows that a relation matrix for $A(r, n)$ is the circulant matrix C whose first row is

$$\underbrace{k+1, \dots, k+1}_s, \quad k-1, \quad \underbrace{k, \dots, k}_{n-s-1}$$

where,

$$r = kn + s, \quad 0 \leq s < n.$$

Thus we obtain formula (6) of (5), viz.

$$|A(r, n)| = \pm \det C = \pm \prod_{i=1}^n \left(\sum_{j=1}^s (k+1)\omega_i^{j-1} + (k-1)\omega_i^s + \sum_{j=s+2}^n k\omega_i^{j-1} \right), \tag{1}$$

where $\omega_1, \dots, \omega_n$ are distinct n th roots of unity. We can express this more conveniently as follows: let b_{i-1} be the exponent-sum of x_i in $w = x_1 \dots x_r x_{r+1}^{-1}$ and put

$$f(x) = b_0 + b_1x + \dots + b_{n-1}x^{n-1}.$$

Writing

$$g(x) = x^n - 1,$$

formula (1) reduces to

$$|A(r, n)| = \prod_{g(\xi) = 0} |f(\xi)|. \tag{2}$$

The presentation for $F(r, n)$ on r generators (x_1, \dots, x_r) obtained using Tietze transformations in the obvious way, leads to the relation matrix

$$M^n - I$$

for $A(r, n)$, where M is the companion matrix of the polynomial

$$f'(x) = x^r - x^{r-1} - \dots - x - 1.$$

Thus we have another expression for $|A(r, n)|$:

$$|A(r, n)| = \prod_{i=1}^n |\xi_i^n - 1|,$$

where ξ_1, \dots, ξ_r are the zeros of $f'(x)$, in other words

$$|A(r, n)| = \prod_{f'(\xi) = 0} |g(\xi)|. \tag{3}$$

Now we have

$$-f'(x) = (1 + x^n + \dots + x^{(k-1)n})(1 + x + \dots + x^{n-1}) + x^{kn}(1 + x + \dots + x^{s-1} - x^s),$$

so that

$$f + f' \equiv 0 \pmod{(g)}. \tag{4}$$

Definition. Let $f, g \in \mathbf{Z}[x]$ and define $f * g \in \mathbf{Z}$ to be the product of the values of g on the complex zeros of f .

The following easily-proved theorem now shows that (2) and (3) are in fact the same formula.

Theorem 1. For polynomials $f, g, f' \in \mathbf{Z}[x]$, we have

- (i) $f * g = \pm g * f$, if f and g are monic,
- (ii) $g * f = g * f'$, if $f \equiv f' \pmod{(g)}$,
- (iii) $g * ff' = (g * f)(g * f')$.

Passing now to the $G_n(w)$, let b_{i-1} be the exponent sum of x_i in w and define

$$f(x) = b_0 + b_1x + \dots + b_{n-1}x^{n-1},$$

as above, a member of $\mathbf{Z}[x]$. Call $f(x)$ the polynomial associated with w .

Dunwoody's Theorem (4) asserts that $G_n(w)$ is perfect if and only if the polynomial associated with w is a unit in $\mathbf{Z}\langle x \mid x^n \rangle$. Our next result is an analogue of this.

Theorem 2. *Let $f(x)$ be the polynomial associated with the word w in the free group of rank n , and let $g(x) = x^n - 1$. The following three assertions are then equivalent:*

- (a) $G_n(w)$ has an infinite abelian factor group,
- (b) $g * f = 0$,
- (c) $f(x)$ is a zero-divisor in $\mathbf{Z}\langle x \mid x^n \rangle$.

Proof. We first prove the equivalence of (a) and (b). (a) is equivalent to the assertion that $A_n(w)$ is infinite. Now a relation matrix for $A_n(w)$ is the circulant matrix C with first row $(b_0, b_1, \dots, b_{n-1})$, and so $A_n(w)$ is infinite if and only if $\det C = 0$. But $\det C = \pm g * f$, so the result follows. To show that (b) and (c) are equivalent, first note that $g * f = 0$ if and only if g and f have a common zero, that is $f(\xi) = 0$ for some n th root ξ of unity. Let $\phi_k(x)$ be the cyclotomic polynomial of order k ; (b) is equivalent to the assertion: $\phi_k(x)$ is a divisor of $f(x)$ for some k dividing n . Since $\phi_k(x)\phi'_k(x) = x^n - 1$ for some $\phi'_k(x) \in \mathbf{Z}[x]$, this condition is clearly equivalent to (c).

Thus we see that $A_n(w)$ is infinite if and only if the polynomial associated with w vanishes on some n th root of unity. We apply this to the classification of the infinite members of the set $A(r, n, k)$, thereby generalising Theorems 6 and 7 of (1), as well as Corollary 2 of (5). Theorem 1 of (2) asserts that $F(r, n, k)$ is metacyclic of order $r^n - 1$ provided that

$$r \equiv 1 \pmod{n} \quad \text{and} \quad (n, k) = 1.$$

Our theorem shows that $F(r, n, k)$ is infinite if

$$r \equiv 1 \pmod{n} \quad \text{and} \quad (n, k) \neq 1.$$

Theorem 3. *$A(r, n, k)$ is infinite if and only if at least one of the following two conditions holds:*

- (i) $(r - 1, k, n) \neq 1$,
- (ii) $v_2(r + 1)$ and $v_2(n)$ are each greater than $v_2(k - 1)$,

where v_2 denotes the 2-part of a positive integer and $v_2(0) = \infty$.

Proof. By Theorem 2, $A(r, n, k)$ is infinite if and only if the polynomial

$$1 + x + \dots + x^{r-1} - x^{r+k-1}$$

vanishes on an n th root of unity. Since $r \geq 2$, this is equivalent to the vanishing of

$$(x^{r+k} - x^{r+k-1}) - (x^r - 1) \tag{5}$$

on some non-trivial n th root of unity. We first assume that some n th root ξ of unity is a root of (5), so that

$$\xi^{r+k} - \xi^{r+k-1} = \xi^r - 1. \tag{6}$$

It follows that

$$|\xi - 1| = |\xi^r - 1|,$$

so that either

$$\xi^r = \xi \quad \text{or} \quad \xi^r = \xi^{-1}.$$

In the first case, substitution in (6) yields

$$\xi^k(\xi - 1) = (\xi - 1),$$

so that

$$\xi \neq 1, \quad \xi^k = 1, \quad \xi^{r-1} = 1, \quad \xi^n = 1,$$

from which (i) follows. In the second case, substitution in (6) yields

$$\xi^{k-1} - \xi^{k-2} = \xi^{-1} - 1,$$

and multiplication of this by $\xi/(\xi - 1)$ gives

$$\xi^{k-1} = -1.$$

The conditions

$$\xi \neq 1, \quad \xi^{k-1} = -1, \quad \xi^{r+1} = 1, \quad \xi^n = 1$$

now give condition (ii).

For the converse, let $h \neq 1$ be a common factor of $r-1$, k and n . If ξ is a primitive h th root of unity, then $\xi \neq 1$, $\xi^n = 1$ and (5) vanishes on ξ . If, on the other hand, (ii) holds, let $h = 2.v_2(k-1)$, so that h is a divisor both of $r+1$ and n , and $h \neq 1$. Again let ξ be a primitive h th root of unity, so that again $\xi \neq 1$, $\xi^n = 1$. Since $\xi^{2(k-1)} = 1$ and $\xi^{k-1} \neq 1$, we must have $\xi^{k-1} = -1$ and ξ is again a zero of (5), which completes the proof.

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