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SOME INTEGRAL FORMULAS OF THE GAUSS-KRONECKER CURVATURE

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Let M^{2k} be a compact, oriented $2k$-dimensional manifold and f: $M^{2k} \to E^{2k+1}$ be an immersion of M^{2k} into a $(2k+1)$-dimensional Euclidean space, and let B_ν be the bundle of unit normal vectors of $f(M^{2k})$, and the mapping $\tilde{\nu}$: $B_\nu \to S_0^{2k}$ of B_ν into the unit sphere S_0^{2k} of E^{2k+1}, be defined by $\nu(p, e)=e$ for (p, e) in B_ν.

Let dV be the volume element of M^{2k}, $d\Sigma_{2k}$ be the volume element of S_0^{2k}. Then the integral:

$$T(f)=\int_{M^{2k}} |\tilde{\nu}^* d\Sigma_{2k}|$$

is called the *total curvature* of the immersion

(1) $$f:\ M^{2k} \to E^{2k+1}.$$

Chern and Lashof [2] proves that $T(f)$ satisfies the following inequality:

(2) $$T(f) \geqq \left(\sum_{i=0}^{2k} \beta_i \right) c_{2k}$$

where c_{2k} denotes the volume of the unit sphere S_0^{2k}, and β_i is the i-th Betti number of M^{2k}. If the equality in (2) holds, then the immersion (1) is called a *minimal imbedding*.

Now, let N be the outer unit normal vector field on $f(M^{2k})$, and let

$$\eta:\ M^{2k} \to S_0^{2k}$$

be the sphere mapping defined by $\eta(p)=N(p)$, then the function $G(p)$ defined by

(3) $$\eta^* d\Sigma_{2k}=G(p)dV,$$

where η^* is the dual mapping of η, is called the Gauss-Kronecker curvature of f.

The object of this note is to find some integral formulas for the Gauss-Kronecker curvature, and to prove that these integral formulas play a main role in the minimal imbedding of even-dimensional hypersurfaces in Euclidean spaces.

1. Some integral formulas for Gauss-Kronecker curvature.

THEOREM 1. *Let f: $M^{2k} \to E^{2k+1}$ be an immersion of an compact, oriented $2k$-*

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dimensional manifold in Euclidean space of dimension $(2k+1)$, *and let* $U=\{p\in M^{2k}: G(p)>0\}$, *and* $V=\{p\in M^{2k}: G(p)<0\}$, *then*

$$\int_U G(p)dV \geqq \left(\sum_{i=0}^{k}\beta_{2i}\right)\frac{c_{2k}}{2}, \tag{4}$$

and

$$\int_V G(p)dV \leqq -\left(\sum_{i=1}^{k}\beta_{2i-1}\right)\frac{c_{2k}}{2}. \tag{5}$$

Proof. Let $\mathcal{J}(N)$ denote the index of the unit normal vector field N of M^{2k} then by the Hopf index theorem, we have

$$\mathcal{J}(N)=\frac{1}{c_{2k}}\int_{M^{2k}}\eta^* d\Sigma_{2k}=\frac{1}{2}\chi(M^{2k}), \tag{6}$$

where $\chi(M^{2k})$ denotes the Euler characteristic of M^{2k}. Therefore, we have

$$\int_{M^{2k}} G(p)dV=\frac{c_{2k}}{2}\chi(M^{2k}), \tag{7}$$

so that

$$\int_U G(p)dV+\int_V G(p)dV=\left(\sum_{i=0}^{n}(-1)^i\beta_i\right)\frac{c_{2k}}{2}. \tag{8}$$

On the other hand, by the definition of $G(p)$ and, we can easily prove that

$$|\tilde{\nu}^* d\Sigma_{2k}|=2|G(p)|dV.$$

Therefore, by (2), we get

$$\int_{M^{2k}}|\tilde{\nu}^* d\Sigma_{2k}|=2\int_{M^{2k}}|G(p)|dV=\left(\sum_{i=0}^{2k}\beta_i\right)c_{2k}, \tag{9}$$

hence, we have

$$\int_U G(p)dV-\int_V G(p)dV \geqq \left(\sum_{i=0}^{2k}\beta_i\right)\frac{c_{2k}}{2}. \tag{10}$$

Combining (8) and (10), we can easily get the inequalities (4) and (5). This completes the proof of the theorem.

THEOREM 2. *Let* $f: M^{2k}\to E^{2k+1}$ *be given as in Theorem* 1, *then the immersion* $f: M^{2k}\to E^{2k+1}$ *is a minimal imbedding if and only if*

$$\int_U G(p)dV=\left(\sum_{i=0}^{k}\beta_{2i}\right)\frac{c_{2k}}{2}, \tag{11}$$

and

$$\int_V G(p)dV=-\left(\sum_{i=1}^{k}\beta_{2i-1}\right)\frac{c_{2k}}{2}. \tag{12}$$

Proof. If equalities (11) and (12) hold, then by (9), we have

$$T(f)=\int_{M^{2k}}|\tilde{\nu}^*d\Sigma_{2k}|=2\int_U G(p)dV-2\int_V G(p)dV=\left(\sum_{i=0}^{k}\beta_i\right)c_{2k}.$$

Therefore, the immersion f: $M^{2k}\to E^{2k+1}$ is a minimal imbedding.

Conversely, if the immersion f: $M^{2k}\to E^{2k+1}$ is a minimal imbedding, then we have

$$T(f)=\int_{M^{2k}}|\tilde{\nu}^*d\Sigma_{2k}|=\left(\sum_{i=0}^{2k}\beta_i\right)c_{2k},$$

that is

$$\int_U G(p)dV-\int_V G(p)dV=\left(\sum_{i=0}^{2k}\beta_i\right)\frac{c_{2k}}{2}. \tag{13}$$

Therefore, by (8) and (13), we can easily get equalities (11) and (12). This completes the proof of the theorem.

2. Some Applications.

In this section, I want to use Theorem 1 and 2 to prove the following theorem on the minimal imbedding of a even dimensional manifold with the odd dimensional Betti numbers vanishing.

THEOREM 3. *Let f: $M^{2k}\to E^{2k+1}$ be an immersion of a compact, oriented $2k$-dimensional manifold M^{2k} in E^{2k+1} with the Gauss-Kronecker curvature $G(p)\geqq 0$, then the odd dimensional Betti number of M^{2k} are zero, and the immersion f is a minimal imbedding.*

Conversely, if the immersiou f: $M^{2k}\to E^{2k+1}$ is a minimal imbedding of a compact, oriented $2k$-dimensional manifold in E^{2k+1}, with the odd dimensional Betti numbers vanishing, then the Gauss-Kronecker curvature $G(p)\geqq 0$ for all p in M^{2k}.

Proof. By the hypothesis, if $G(p)\geqq 0$ for all p in M^{2k}, then we have $V=\phi$, and therefore by inequality (5), we have

$$\sum_{i=1}^{k}\beta_{2i-1}=0,$$

so that

$$\beta_{2i-1}=0 \qquad \text{for} \quad i=1,2,\cdots,k.$$

Furthermore, by the inequality (7), we have

$$T(f)=\int_{M^{2k}}|\tilde{\nu}^*d\Sigma_{2k}|=2\int_U G(p)dV=2\int_{M^{2k}}G(p)dV$$
$$=\chi(M^{2k})c_{2k}=\left(\sum_{i=0}^{2k}\beta_i\right)c_{2k},$$

so that the immersion f: $M^{2k}\to E^{2k+1}$ is a minimal imbedding.

Conversely, if the immersion f: $M^{2k}\to E^{2k+1}$ is a minimal imbedding and $\beta_{2i-1}=0$ for all $i=1,2,\cdots,k$. Now, let us suppose that there exist some points in M^{2k} with the Gauss-Kronecker curvature $G(p)$ negative, then by the continuity of $G(p)$ on M^{2k}, the set $V=\{p\in M^{2k}: G(p)<0\}$ is a positive measure set on M^{2k}, therefore the integral

$$\int_V G(p)dV$$

is negative, hence by (9) and (11), we have

$$T(f)=\int_{M^{2k}}|\tilde{\nu}^* d\Sigma_{2k}|=2\int_U G(p)dV-2\int_V G(p)dV$$

$$>2_U G(p)dV=\left(\sum_{i=0}^{k}\beta_{2i}\right)c_{2k}=\left(\sum_{i=0}^{k}\beta_i\right)c_{2k}.$$

This is a contradiction, hence $G(p)\geqq 0$ for all p in M^{2k}. This completes the proof of the theorem.

In the special case, when $k=1$, then the Gauss-Kronecker curvature $G(p)$ is just the Gaussian curvature $K(p)$, hence by the fact of $c_2=4\pi$, we have the following corollary:

COROLLARY. *If f: $M^2\to E^3$ is an immersion of a compact, oriented surface in the* 3-*dimensional Eculidean space, then we have*

$$\int_U K(p)dV\geqq 4\pi, \tag{14}$$

and

$$\int_V K(p)dV\leqq -4g\pi, \tag{15}$$

where $U=\{p\in M^2: K(p)>0\}$ and $V=[p\in M^2: K(p)<0\}$.

The immersion f: $M^2\to E^3$ is a minimal imbedding if and only if the equalities in (14) *and* (15) *hold. In particular if M^2 is a sphere, then the immersion f is a minimal imbedding if and only if the Gaussian curvature of f is non-negative everywhere.*

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