SOME INTERESTING FEATURES OF FREQUENCY CURVES

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Introduction

It is well known that in the normal error curve the points of inflection are equidistant from the mode. However it has never been pointed out that this is also a characteristic of all of the bell-shaped Pearson Frequency Curves. This fact can be most easily shown by placing the mode at the abscissa $x = 0$.

Many rough checks have been developed for use in applying the Theory of Least Squares. The second part of this paper develops a rough check on the computation for use when fitting a Pearson Frequency Curve to a set of observations. No rough checks on computation are given in textbooks on Pearson's Frequency Curves.

At present it is customary to follow a separate procedure for each Type of curve when computing the constants of a Pearson Frequency Curve. The third part of this paper shows how a single system may be followed for all Types. A single procedure is very desirable in order that the rough check of Part 2 may be quickly applied.

Part 1. Points of Inflection

Perhaps nothing brings out the limitations of the bell-shaped Pearson Curves in a more striking manner than a discussion of their points of inflection. In dealing with frequency curves it is well known that any curve can be fitted to a given distribution and that the real problem in curve fitting is the selection of a curve. Figures 1, 2, and 3 illustrate three hypothetical histograms. All three of these histograms are bell-shaped yet none of them will be closely fitted by any of the Pearson Curves. The reasons will be pointed out presently.

The differential equation from which Pearson derived his system of frequency curves is

$$\frac{dy}{dx} = \frac{y(x - P)}{b_2x^2 + b_1x + b_0}.$$

By putting $x - P = X$, i.e. by placing the mode at the abscissa $X = 0$, this differential equation may be written:

$$\frac{dy}{dX} = \frac{yX}{\pm B_2X \pm B_1X + B_0}$$

where the $+$ or $-$ sign is taken according to the type of the curve. (It will be shown later that the constant term of the denominator must be less than zero.)

Since in the Type III curve B_2 is 0 and in the "Normal Curve" both B_2 and B_1 are 0 it will be advantageous to consider the general case of

$$\frac{dy}{dX} = \frac{yX}{F(X)},$$

where $F(X)$ is an integral rational function of the n^{th} degree, at once rather than considering special cases first.

If

$$\frac{dy}{dX} = \frac{yX}{F(X)};$$

then

$$\frac{d^2y}{dX^2} = \frac{y}{[F(X)]^2}\{X^2 + F(X) - XF'(X)\}.$$

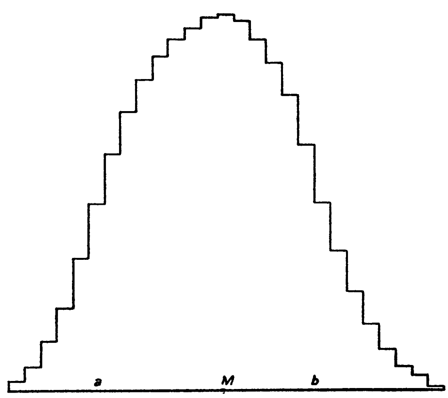

FIG. 1

In order to locate the points of inflection, $\frac{d^2y}{dX^2}$ is equated to zero. Then we have:

$$X^2 + F(X) - XF'(X) = 0. \tag{1}$$

This equation is always of the same degree as $F(X)$ except when $F(X)$ is linear or constant. Hence we have proved the Theorem: If $y = G(X)$ be the solution of the differential equation

$$\frac{dy}{dX} = \frac{yX}{F(X)},$$

then the number of points of inflection of y cannot exceed the degree of $F(X)$ when $F(X)$ is of degree greater than one.

Now $F(X) = B_nX^n + B_{n-1}X^{n-1} + \cdots + B_2X^2 + B_1X + B_0$. Whence equation (1) can be written in the form:

$$(1 - n)B_nX^n + (2-n)B_{n-1}X^{n-1} + (3 - n)B_{n-2}X^{n-2} + \cdots$$
$$+ (r - n)B_{n-r+1}X^{n-r+1} + \cdots - 3B_4X^4 - 2B_3X^3 + (1 - B_2)X^2 + B_0 = 0.$$

Hence we have established the Theorem: The coefficient of the linear term of X in the equation of the points of inflection is zero.

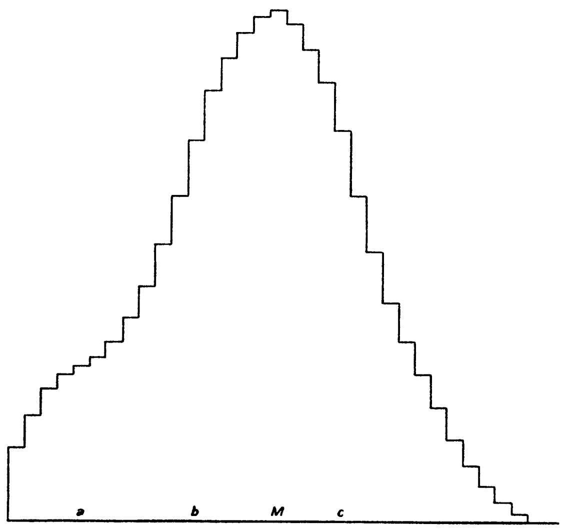

FIG. 2

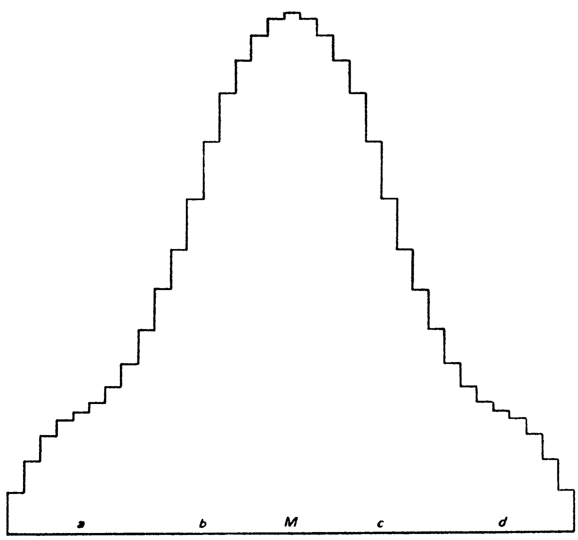

FIG. 3

For the "Normal Curve" and also for Type III,

$$B_2 = B_3 = B_4 = \cdots = B_n = 0.$$

Hence the points of inflection of these two Types are given by $X = \pm\sqrt{-B_0}$.

For Types I and II, B_2 is positive and $B_3 = B_4 = \cdots = B_n = 0$, and the points of inflection are $X = \pm\sqrt{\frac{-B_0}{1 - B_2}}$. Hence the points of inflection are undefined if $B_2 = 1$, are pure imaginary if $B_2 > 1$, and real if $B_2 < 1$.

For Types IV, V, VI and VII, B_2 is negative and $B_3 = \cdots = B_n = 0$, and the points of inflection are at $X = \pm\sqrt{\frac{-B_0}{1 + |B_2|}}$.

In some of these Types it may happen that the abscissae of the points of inflection though real will lie beyond the range of the curve. Thus Types III and VI may have 1 or 2 points of inflection, the single point of inflection occurring when $\left|\sqrt{\frac{-B_0}{1 + B_2}}\right| >$ the range of the curve in the direction that the range is limited. Type II may have 0 or 2 points of inflection, as there will be no real points of inflection when $B_2 \geqq 1$. Type I may have 0, 1 or 2 points of inflection. Types IV, V and VII as well as the "Normal Curve" always have 2 and only two points of inflection.

Now it should be noted that when one of the eight bell-shaped Pearson curves has two points of inflection then the abscissae of these 2 points of inflection are equidistant from the abscissa of the mode. In figure 1 a point of inflection will be at abscissa b and another at abscissa a. (M is the abscissa of the mode.) Since $b - M \neq M - a$ none of the Pearson curves will fit this histogram closely. In figure 2, points of inflection occur at abscissae a, b, and c. Since a Pearson curve can have at most two points of inflection no Pearson curve will fit this histogram closely. In figure 3 there are four points of inflection and no Pearson curve will fit this histogram closely.

Part 2. Range

Definition: A bell-shaped curve is a continuous curve which starts at zero (or zero as a limit), rises to a single maximum, at which maximum point the first derivative is zero, and then falls to zero (or zero as a limit).

Or, more formally, $y = G(x)$ is a bell-shaped curve if $G(x_1) = G(x_2) = 0$ and if $G'(P) = 0$ and $G''(P) < 0$ where $G(x)$ is continuous and does not vanish in the interval from x_1 to x_2 and P is a unique point in this interval.

If a bell-shaped curve has the value of zero at two finite points, one on each side of the maximum (mode), it is said to be of limited range in both directions, or briefly, of limited range.

If a bell-shaped curve has the value of zero at only one finite point it is said to be of limited range in one direction, or also of unlimited range in one direction.

If a bell-shaped curve has the value of zero only at $\pm\infty$, i.e. at no finite points, it is said to be of unlimited range in both directions, or briefly, of unlimited range.

Theorem I: If $F(x)$ can be separated into a finite number of factors each either of the form $(x - r_i)$ or $(x^2 + 2r_j\, x + r_j^2 + r_{0j}^2)$ where no real root is repeated and $y = G(x)$ is a *bell-shaped* curve which is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y(x - P)}{F(x)},$$

then if $F(x)$ has no real roots, y is of unlimited range in both directions; if all of the real roots of $F(x)$ lie on the same side of P, y is of limited range in one (that) direction; if at least one real root of $F(x)$ lies on one side of P and at least one on the other side, y is of limited range in both directions.

PROOF: If $F(x) = 0$ when $x = P$, we have

$$\frac{dy}{dx} = \frac{y}{g(x)}$$

where $g(x) = F(x) \div (x - P)$. This derivative is zero only when $y = 0$ or $g(x) = \pm\infty$. Hence the solution does not have a finite maximum and therefore is not a bell-shaped curve. If $F(x) > 0$ when $x = P$, we have

$$\frac{d^2y}{dx^2} = \frac{y}{[F(x)]^2}\left[(x - P)^2 + F(x) - (x - P)\frac{d}{dx}F(x)\right]$$

and

$$\left.\frac{d^2y}{dx^2}\right|_{x=P} = \frac{y}{[F(x)]^2}[F(x)]$$

which is greater than zero and, since at a maximum the second derivative must not be greater than zero, in this case the solution would have a minimum at $x = P$ and therefore would not be a bell-shaped curve. As the theorem concerns only those solutions which are bell-shaped curves, $F(x) < 0$ when $x = P$. If $F(x) = 0$ when $x \neq P$ then $\frac{dy}{dx} = \pm\infty$ unless y is also zero. Assume $y \neq 0$. Since $F(x)$ is negative, if $y \neq 0$ when $F(x) = 0$ then $\frac{dy}{dx} \to -\infty$ as $F(x) \to 0$, for an $x > P$, and changes to $+\infty$ as $F(x)$ changes sign on passing through the value 0. Hence the curve would contain another maximum before falling to zero and therefore the solution is not a bell-shaped curve. Similar reasoning holds for an $x < P$. Therefore if $y \neq 0$ when $F(x) = 0$, the curve is not bell-shaped. If $y = 0$ when $F(x) = 0$, the curve has its range limited at this point. That is, any real number which makes $F(x)$ vanish will also make y vanish if y represents a bell-shaped curve. Hence if all of the real roots lie on the same side of P the curve is of limited range in that direction only, while if at least one of the real roots lies on each side of P the curve is of limited range in both directions. If $F(x)$ contains no real roots it does not vanish for any real value of x. In this case, by partial fractions the differential equation becomes:

$$\frac{dy}{y} = \frac{k_{11}\,dx}{(x + r_1)^2 + r_{01}^2} + \frac{k_{21}\,dx}{(x + r_2)^2 + r_{02}^2} + \cdots + \frac{2k_{21}(x + r_1)\,dx}{(x + r_1)^2 + r_{01}^2} + \frac{2k_{22}(x + r_2)\,dx}{(x + r_2)^2 + r_{02}^2} + \cdots$$

On integrating,

$$y = C\left[(x + r_1)^2 + r_{0_1}\right]^{k_{21}}\left[(x + r_2)^2 + r_{0_2}^2\right]^{k_{22}} \cdots e^{k_{11} \arctan \frac{x+r_1}{r_{0_1}} + \cdots}.$$

Hence y does not vanish for a finite real value of x and the Theorem is fully established.

THEOREM II: If $F(x)$ can be separated into a finite number of factors each either of the from $(x - r_i)$ or $(x^2 + 2r_jx + r_j^2 + r_{0_j}^2)$ where no real root is repeated and $y = G(x)$ is a *bell-shaped* curve which is a solution of the differential equation $\frac{dy}{dx} = \frac{y(x - P)}{F(x)}$, then if y is of unlimited range, $F(x)$ contains no real roots; if y is of limited range in one direction, all of the real roots of $F(x)$ lie on the same (that) side of P; if y is of limited range in both directions, at least one of the real roots of $F(x)$ lies on one side of P and at least one on the other.

PROOF: By partial fractions the differential equation may be written:

$$\frac{dy}{y} = \frac{k_{11}\,dx}{x - r_{11}} + \frac{k_{21}\,dx}{x - r_{12}} + \cdots + \frac{k_{21}\,dx}{(x + r_{21})^2 + r_{0_1}^2}$$
$$+ \frac{k_{22}\,dx}{(x + r_{22})^2 + r_{0_2}^2} + \cdots \frac{2k_{31}(x + r_{21})\,dx}{(x + r_{21})^2 + r_{0_1}^2} + \frac{2k_{32}(x + r_{23})\,dx}{(x + r_{22})^2 + r_{0_2}^2} + \cdots$$

and on integrating:

$$y = C(x - r_{11})^{k_{11}}(x - r_{12})^{k_{12}} \cdots \left[(x + r_{21})^2 + r_{0_1}^2\right]^{k_{31}} \cdots e^{k_{21} \arctan \frac{x+r_{21}}{r_{0_1}} + \cdots}.$$

Hence $y = 0$ for $x = r_{11}, r_{12}, \cdots$ and for no other finite values of x provided k_{11}, $k_{12}, \cdots$ are positive. If one or more of the k_{ij} are negative, $y = \infty$ at such points and unless some r_{ij} closer to P has previously made y vanish, the curve is not bell-shaped. Therefore, for bell-shaped curves, the exponent of the factor containing the real root of smallest absolute value on each side of P is positive. Therefore: if y is of limited range in both directions, at least one real root lies on each side of P; if y is of unlimited range in one direction, all of the real roots lie on the same side of P; if y is of unlimited range it contains no real roots. Hence the Theorem is established.

The effect of repeated real roots will now be considered. If a real root is repeated an odd number of times at $x = r$, then $F(x)$ changes sign at $x = r$ and the first theorem is true. If a real root is repeated an even number of times at $x = r$, then $F(x)$ does not change sign at $x = r$ and we know that either (a) $y = 0$ at $x = r$; or (b) y is finite and $\neq 0$ and $\frac{dy}{dx} = \pm\infty$ at $x = r$, i.e. there is a point of inflection at $x = r$. It will now be shown that (b) cannot occur. If case (b) is possible, y is continuous at $x = r$, $\frac{dy}{dx} = \pm\infty$ according as $(r - P) \lessgtr 0$

moreover $\frac{dy}{dx}$ does not change sign in the neighborhood of the point $x = r$, and $\frac{d^2y}{dx^2}$ changes sign from $+\infty$ to $-\infty$ or vice versa according as $(r - P) \lessgtr 0$. Now

$$\frac{d^2y}{dx^2} = \frac{y}{[F(x)]^2}\left[(x - P)^2 + F(x) - (x - P)\frac{d}{dx}F(x)\right].$$

Whence if y is finite and $\neq 0$, $\frac{d^2y}{dx^2}$ does not change sign at $x = r$ because it is possible to select a neighborhood such that

$$|(x - P)^2| > \left|F(x) - (x - P)\frac{d}{dx}F(x)\right|$$

for an x differing from r by ϵ where ϵ is a small positive quantity. Therefore case (b) is not possible and $y = 0$ when a real root is repeated an even number of times. That is to say the range of the curve is limited at a point where a real root is repeated an even number of times. Thus Theorem I always holds for repeated roots.

For Theorem II it is clear that this Theorem holds for repeated roots when a non-repeated root lies closer to P, and on the same side, than the repeated root. Suppose that the repeated root is the nearest root to P (on a given side of P). Then by partial fractions:

$$\frac{dy}{y} = \frac{k_{11}\,dx}{(x - r_{11})} + \frac{k_{12}\,dx}{(x - r_{11})^2} + \frac{k_{13}\,dx}{(x - r_{11})^3} + \cdots + \frac{k_{41}\,dx}{(x - r_{41})} + \frac{k_{42}\,dx}{(x - r_{42})}$$
$$+ \cdots + \frac{k_{21}\,dx}{(x + r_{21})^2 + r_{0_1}^2} + \frac{k_{22}\,dx}{(x + r_{22})^2 + r_{0_2}^2} + \cdots + \frac{2k_{31}(x + r_{21})\,dx}{(x + r_{21})^2 + r_{0_1}^2} + \cdots$$

and on integrating:

$$y = C(x - r_{11})^{k_{11}}(x - r_{41})^{k_{41}}(x - r_{42})^{k_{42}} \cdots \left[(x + r_{21})^2 + r_{0_1}^2\right]^{k_{31}}$$
$$\cdots e^{k_{21} \arctan \frac{x + r_{21}}{r_{0_1}} + \cdots - \frac{k_{12}}{(x - r_{11})} - \frac{k_{13}}{2(x - r_{11})^2} - \cdots}$$

Hence y can $= 0$ only for $x = r_{11}$ or for $x = r_{41}, r_{42}, \cdots$ and for no other finite values of x. Since by hypothesis y is bell-shaped, then the proper k_{ij} must be positive and Theorem II always holds for repeated roots.

Theorems I and II can now be combined and generalized in the form:

THEOREM: If $F(x)$ is a polynomial with real coefficients and $y = G(x)$ is a bell-shaped curve which is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y(x - P)}{F(x)},$$

then the necessary and sufficient condition: that y be of unlimited range in both directions is that $F(x)$ have no real roots; that y be of limited range in one direction is that all of the real roots of $F(x)$ lie on the same side of P; that y be of limited range in both directions is that at least one real root of $F(x)$ lie on one side of P and one on the other.

COROLLARY: $F(x)$ must be negative throughout the range of y.

Suppose now that we have some statistics which we wish to graduate and the statistics are of such nature that we would expect a bell-shaped curve, rather than a J- or U-shaped curve, and we desire the best fit: If we use a curve which is a solution of the differential equation

$$\frac{dy}{dx} = \frac{y(x - P)}{F(x)}$$

(the Pearson Curves being special cases) to fit the statistics and if in computing the constants for the curve one of the following cases arise:

(a) $b_0' \nless 0$ when this constant is computed,

or (b) $B_0 \nless 0$ when the origin is moved to the mode,

or (c) a root is located within the range of the statistics then it means that:

1. A mistake may have been made in the computation; thus the Theorem just established provides a rough check on the work of computation,

2. If no mistake has been made in the computation it may indicate that the bell-shaped Pearson Curves will not closely fit the statistics and that some other graduation curves be used, e.g. the Gram-Charlier Types A or B might be tried,

3. If no mistake has been made in the computation it may happen that one of the bell-shaped Pearson Curves will give an excellent fit but a different method than or a modification of the Method or Moments should be used in order to compute the constants.

Part 3. Computing the Constants

At present, the constants of a frequency curve are computed as follows: First the moments are computed about an arbitrary origin, then the moments about the A.M. are determined, then β_1 and β_2 and the criterion are computed, after which the type of curve can be selected. From this point a separate procedure is followed for each curve. Now in the above method one will not know whether a root has been located in the range of statistics or not.

Take Pearson's differential equation

$$\frac{dy}{dx} = \frac{y(x - P)}{b_2x^2 + b_1x + b_0}.$$

Put $X = x - P$. Then $dX = dx$ and $x = X + P$, and

$$\frac{dy}{dx} = \frac{yX}{b_2(X + P)^2 + b_1(X + P) + b_0} = \frac{yX}{b_2X^2 + 2Pb_2X + b_1X + P^2b_2 + Pb_1 + b_0}.$$

Now put

$$b_2 = B_2$$
$$2Pb_2 + b_1 = B_1$$
$$P^2b_2 + Pb_1 + b_0 = B_0 .$$

Then we have

$$\frac{dy}{dX} = \frac{yX}{B_2X^2 + B_1X + B_0} \quad \text{or} \quad \frac{dy}{dx} = \frac{y(x - P)}{B_2(x - P)^2 + B_1(x - P) + B_0} . \tag{1}$$

It should be noted that for a particular curve, B_2, B_1 and B_0 are constants; i.e., their values do not change with a change of the origin. The values of b_1 and b_0 do change with a change in the origin.

If we clear equation (1) of fractions, multiply by $e^{\eta x}$ and integrate with respect to x over the range from x_1 to x_2, where

$$e^{\lambda_1\eta + \frac{\lambda_2\eta^2}{2!} + \frac{\lambda_3\eta^3}{3!} + \cdots} \equiv \int_{x_1}^{x_2} e^{\eta x}\, y dx ,$$

then successively differentiate with respect to η, and equate coefficients of like powers of η, we finally obtain:

$$\left.\begin{aligned}
&\lambda_1 - P + B_1 - 2PB_2 + 2B_2\lambda_1 = 0 ,\\
&\lambda_2 + B_0 - PB_1 + P^2B_2 + B_1\lambda_1 - 2PB_2\lambda_1 + 3B_2\lambda_2 + B_2\lambda_1^2 = 0 ,\\
&\lambda_3 + 2\lambda_2B_1 - 4PB_2\lambda_2 + 4B_2\lambda_3 + 4B_2\lambda_1\lambda_2 = 0 ,\\
&\lambda_4 + 3B_1\lambda_3 - 6PB_2\lambda_3 + 5B_2\lambda_4 + 6B_2\lambda_2^2 + 6B_2\lambda_1\lambda_3 = 0 .
\end{aligned}\right\} \tag{2}$$

Since we can compute the moments from the raw statistics and the semi-invariants from the moments, we may regard λ_2, λ_3 and λ_4 in these equations as knowns and the B_0, B_1, B_2, P and λ_1 as unknowns. But the origin has not yet been specified. Let the origin be placed at the A.M. where $\mu_1 = \lambda_1 = 0$. As λ_2, λ_3, λ_4, B_0, B_1 and B_2 are unchanged by a change of origin, we have:

$$\left.\begin{aligned}
&B_1 - P_0 - 2P_0B_2 = 0 .\\
&\lambda_2 + B_0 - P_0B_1 + P_0^2B_2 + 3B_2\lambda_2 = 0 ,\\
&\lambda_3 + 2B_1\lambda_2 - 4P_0B_2\lambda_2 + 4B_2\lambda_3 = 0 ,\\
&\lambda_4 + 3B_1\lambda_3 - 6P_0B_2\lambda_3 + 5B_2\lambda_4 + 6B_2\lambda_2^2 = 0 .
\end{aligned}\right\} \tag{3}$$

Now put

$$\left.\begin{aligned}
b_0' &= B_0 - P_0B_1 + P_0^2B_2 ,\\
b_1' &= B_1 - 2P_0B_2 ,\\
b_2' &= B_2 ;
\end{aligned}\right\} \tag{4}$$

then

$$\left.\begin{aligned} b_1' - P_0 &= 0\,, \\ \lambda_2 + b_0' + 3b_2'\lambda_2 &= 0\,, \\ \lambda_3 + 2b_1'\lambda_2 + 4b_2'\lambda_3 &= 0\,, \\ \lambda_4 + 3b_1'\lambda_3 + 5b_2'\lambda_4 + 6b_2'\lambda_2^2 &= 0\,. \end{aligned}\right\} \qquad (5)$$

By reversing the transformation (4) we get:

$$\left.\begin{aligned} B_2 &= b_2'\,, \\ B_1 &= b_1' + 2P_0b_2' \\ B_0 &= b_0' + P_0(b_1' + P_0b_2')\,. \end{aligned}\right\} \qquad (6)$$

Now the above theory suggests the following procedure for computing the constants of a frequency curve: First the moments are computed about an arbitrary origin, then the semi-invariants are computed (or alternatively the moments about the A.M., either step involves about the same amount of work), then the equations (5) are solved and then by means of equations (6) the B_2, B_1 and B_0 are computed. Next solve the quadratic equation

$$B_2X^2 + B_1X + B_0 = 0.$$

The character of the roots of this equation indicates which type to use and it is unnecessary to compute the criterion. The constants of the frequency curve are simple functions of the roots of the above quadratic equation and can be readily found by integrating the diff. eq. (1) being careful to write the solution as a function of $X = x - P$. The rough checks mentioned in Part 2 can be quickly and conveniently applied when this procedure is followed.

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