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Some Interesting Results on $\mathcal{F}$-metric Spaces

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Abstract. In this manuscript, we prove that the newly introduced $\mathcal{F}$-metric spaces are Hausdorff and first countable. We investigate some interrelations among the Lindelöfness, separability and second countability axiom in the setting of $\mathcal{F}$-metric spaces. Moreover, we acquire some interesting fixed point results concerning altering distance functions for contractive type mappings and Kannan type contractive mappings in this exciting context. In addition, most of the findings are well-furnished by several non-trivial examples. Finally, we raise an open problem regarding the structure of an open set in this setting.

1. Introduction

As a generalization of Euclidean geometry and a common setting for continuous functions, topology of normed spaces is one of the most fascinating and instrumental branches of research in contemporary mathematics. This fact has prompted many mathematicians to deal with the topology induced by a norm defined on a linear space on a non-empty set in plenty of research articles. Therefore the topology of b-metric spaces, dislocated metric spaces and several other abstract spaces are thoroughly investigated and also improved by several authors (see [3, 7, 12] and references therein).

On the other hand, metric fixed point theory appears as one of the most salient means to work out various research ventures in non-linear functional analysis and a variety of other fields in science and technology. It all emerged with the illustrious Banach contraction principle in the setting of a complete metric space, due to Banach [1], in 1922 and subsequently, plenty of results appeared which complement, extend and obviously improve the pioneer one [2, 5–7, 9, 10].

Right through the years, mathematicians got involved with improving the underlying metric structure of the previous result and in a very recent article, Jleli and Samet [8] proposed the notion of an $\mathcal{F}$-metric space which is another interesting framework to work with. The authors made use of a certain class of auxiliary functions to coin the idea of such abstract spaces. We begin with the collection of such functions.

Let $\mathcal{F}$ be the set of functions $f : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(\mathcal{F}_1)$ f is non-decreasing, i.e., $0 < s < t \Rightarrow f(s) \leq f(t)$.

$(\mathcal{F}_2)$ For every sequence $(t_n) \subseteq (0, +\infty)$, we have

$$\lim_{n\to+\infty} t_n = 0 \Longleftrightarrow \lim_{n\to+\infty} f(t_n) = -\infty.$$

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Example 1.1. *The following are some examples of the previously discussed kind of auxiliary functions:*

(*i*) $-\frac{1}{t}$, *where* $t \in (0, \infty)$;

(*ii*) $-e^{\frac{1}{t}}$, *for all* $t \in (0, \infty)$.

Utilizing such functions, the authors generalized the concept of usual metric spaces and originated the notion of $\mathcal{F}$-metric spaces as follows:

Definition 1.2. *[8] Let* X *be a non-empty set, and let* $d : X \times X \to [0, \infty)$ *be a mapping. Suppose that there exists* $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ *such that*

(D1) $(x, y) \in X \times X, \ d(x, y) = 0 \Longleftrightarrow x = y$.

(D2) $d(x, y) = d(y, x)$, *for all* $(x, y) \in X \times X$.

(D3) For every $(x, y) \in X \times X$, *for each* $N \in \mathbb{N}, N \geq 2$, *and for every* $(u_i)_{i=1}^N \subseteq X$ *with* $(u_1, u_N) = (x, y)$, *we have*

$$d(x, y) > 0 \Longrightarrow f(d(x, y)) \leq f\Big(\sum_{i=1}^{N-1} d(u_i, u_{i+1})\Big) + \alpha.$$

Then d *is said to be an* $\mathcal{F}$*-metric on* X, *and the pair* (X, d) *is said to be an* $\mathcal{F}$*-metric space.*

We observe that any metric on X is an $\mathcal{F}$-metric, but the converse is not true, which is given in [8, Example 2.1]. The succeeding example is an example of one such.

Example 1.3. *We consider the set* $X = [1, 4]$ *and define a mapping* $d : X \times X \to \mathbb{R}$ *by*

$$d(x, y) = \begin{cases} 0, & \text{when } x = y; \\ x + y, & \text{when } x \neq y \text{ and } x, y \in [1, 2]; \\ 2(x + y), & \text{when } x \neq y \text{ and } x, y \in [2, 3]; \\ 3(x + y), & \text{when } x \neq y \text{ and } x, y \in [3, 4]; \\ 1, & \text{elsewhere}. \end{cases}$$

Then it is easy to note that (X, d) *is an* $\mathcal{F}$*-metric space with* $f(t) = -\frac{1}{t}$ *and* $\alpha = 2$.

Further, the notions of completeness, convergence and Cauchy sequences in this framework along with some other terminologies can be found in [8].

In this literature, we assert a couple of topological observations concerning the newly introduced $\mathcal{F}$-metric spaces. In fact, being a vast generalization of usual metric spaces, such kind of spaces still hold some beautiful topological properties like Hausdorff and also first countability. Further, we prove that a Lindelöf $\mathcal{F}$-metric space is separable as well as second countable. On the other hand, we also enquire into a few exciting fixed point results involving altering distance functions in the later half of this article. Finally, we construct several non-trivial examples to validate the obtained theorems.

2. Results on topology of $\mathcal{F}$-metric spaces

In this section, we deal with the topological developments of $\mathcal{F}$-metric spaces which is equipped with the $\mathcal{F}$-metric topology $\tau_{\mathcal{F}}$. First of all, we attest that such metric spaces are Hausdorff.

Theorem 2.1. *Every* $\mathcal{F}$*-metric space is Hausdorff.*

Proof. Let (X, d) be an $\mathcal{F}$-metric space, so there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ satisfying the conditions ($D1$-$D3$) of Definition 1.2. Let x, y be two arbitrary points in X with $x \neq y$, and take $a_n = \frac{d(x,y)}{n}$. Then (a_n) is a sequence in $(0, \infty)$ and $a_n \to 0$ as $n \to \infty$. So by $(\mathcal{F}_2)$, we have,

$$f(a_n) \to -\infty \ \text{ as } \ n \to \infty.$$

We claim that $B(x, \frac{a_n}{2}) \cap B(y, \frac{a_n}{2}) = \emptyset$ for some $n \in \mathbb{N}$. Suppose to the contrary that $B(x, \frac{a_n}{2}) \cap B(y, \frac{a_n}{2}) \neq \emptyset$ for each $n \in \mathbb{N}$. Then we can find a sequence (z_n) in X such that $z_n \in B(x, \frac{a_n}{2}) \cap B(y, \frac{a_n}{2})$ for all $n \in \mathbb{N}$. Therefore,

$$d(x, z_n) < \frac{a_n}{2} \text{ and } d(y, z_n) < \frac{a_n}{2}.$$

Since, $d(x, y) > 0$, so using (D3), we get,

$$\begin{aligned} f(d(x,y)) &\leq f(d(x,z_n) + d(z_n, y)) + \alpha \\ &\leq f\Big(\frac{a_n}{2} + \frac{a_n}{2}\Big) + \alpha \\ &= f(a_n) + \alpha. \end{aligned}$$

Taking limit as $n \to \infty$ in both sides of above equation we get

$$f(d(x,y)) \leq -\infty,$$

which is a contradiction.

Therefore, $B(x, \frac{a_n}{2}) \cap B(y, \frac{a_n}{2}) = \emptyset$ for some $n \in \mathbb{N}$. Now, we consider $A = \text{int}\Big(B(x, \frac{a_n}{2})\Big)$ and $B = \text{int}\Big(B(y, \frac{a_n}{2})\Big)$, where $\text{int}(Y)$ stands for the interior of Y, which, by definition, are open. Clearly, $x \in A$ and $y \in B$ and

$$A \cap B = \emptyset.$$

Hence we are done. □

Remark 2.2. *It is worthy to mention that the Hausdorff property is a sufficient condition to claim the uniqueness of a limit for a convergent sequence. Therefore, this property holds good for every $\mathcal{F}$-metric space also.*

Now we study the first countability axiom in the following result.

Theorem 2.3. *Every $\mathcal{F}$-metric space (X, d) is first countable.*

Proof. Let $x \in X$ be arbitrary. Then the family

$$\beta = \left\{\text{int}\left(B\left(x, \frac{1}{n}\right)\right) : n \in \mathbb{N}\right\}$$

is a countable set of open neighborhoods of x. Let U be an open neighborhood of x. Then by the definition of an $\mathcal{F}$-open set, $B(x, r) \subseteq U$, for some $r > 0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < r$ and therefore we have

$$\begin{aligned} & B\Big(x, \frac{1}{n}\Big) \subseteq B(x, r) \subseteq U \\ \Rightarrow \text{int}\left(B\left(x, \frac{1}{n}\right)\right) \subseteq\ & B\Big(x, \frac{1}{n}\Big) \subseteq B(x, r) \subseteq U. \end{aligned}$$

Hence β is a local basis at x. Hence any $\mathcal{F}$-metric space is first countable. □

In the next theorem, we relate the Lindelöfness and separability of $\mathcal{F}$-metric spaces.

Theorem 2.4. *Every Lindelöf $\mathcal{F}$-metric space is separable.*

Proof. Let (X, d) be a Lindelöf $\mathcal{F}$-metric space and take $\mathcal{A}_n = \{\text{int}\Big(B(x, \frac{1}{n})\Big) : x \in X\}$. Then $\mathcal{A}_n$ is an open cover of X for each $n \in \mathbb{N}$. Since (X, d) is Lindelöf, $\mathcal{A}_n$ has a countable subcover, say $\mathcal{A}'_n = \{\text{int}\Big(B(x_{n_i}, \frac{1}{n})\Big) : i \in \mathbb{N}\}$ for all $n \in \mathbb{N}$.

Let $D = \{x_{n_i} : i, n \in \mathbb{N}\}$. Then D is a countable subset of X. Next, we show that $\overline{D} = X$. Let $x \in X$ be arbitrary and U be an open set in X containing x. Then there exists $r > 0$ such that $x \in B(x, r) \subseteq U$. Let us choose $m \in \mathbb{N}$ such that $\frac{1}{m} < r$. Since, $\mathcal{A}'_m = \{\text{int}\left(B(x_{m_i}, \frac{1}{m})\right) : i \in \mathbb{N}\}$ is an open cover of X, $x \in \left\{\text{int}\left(B(x_{m_k}, \frac{1}{m})\right)\right\}$ for some $k \in \mathbb{N}$. Therefore,

$$d(x_{m_k}, x) < \frac{1}{m} < r \Rightarrow x_{m_k} \in B(x, r) \subseteq U.$$

Also, $x_{m_k} \in D$. Thus $D \cap U \neq \phi$, and hence $x \in \overline{D}$. So $\overline{D} = X$ and consequently X is separable. □

We now establish a relationship between Lindelöfness and second countability axiom of $\mathcal{F}$-metric spaces in the following theorem.

Theorem 2.5. *Every Lindelöf $\mathcal{F}$-metric space is second countable.*

Proof. Let (X, d) be a Lindelöf $\mathcal{F}$-metric space and take $\mathcal{A}_n = \{\text{int}\left(B(x, \frac{1}{n})\right) : x \in X\}$ for all $n \in \mathbb{N}$. Then $\mathcal{A}_n$ is an open cover of X for all $n \in \mathbb{N}$. Since (X, d) is Lindelöf, $\mathcal{A}_n$ has a countable subcover, say $\mathcal{A}'_n = \{\text{int}\left(B(x_{n_i}, \frac{1}{n})\right) : i \in \mathbb{N}\}$ for each $n \in \mathbb{N}$.

Let $\mathcal{A} = \{\text{int}\left(B(x_{n_i}, \frac{1}{n})\right) : i, n \in \mathbb{N}\}$. Then $\mathcal{A}$ is a countable collection of open sets in X. Now, we show that $\mathcal{A}$ is a base for the topology on X. To prove this, let U be an open set in X and $x \in U$. So there exists $r > 0$ such that $x \in B(x, r) \subseteq U$. Let us choose $n \in \mathbb{N}$ such that $\frac{1}{n} < r$. Then $B(x, \frac{1}{n}) \subseteq B(x, r) \subseteq U$.

Then by $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f\left(\frac{1}{n}\right) - \alpha. \tag{1}$$

Again, choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$. Since, $\mathcal{A}'_{2m}$ is an open cover of X, $x \in \text{int}\left(B(x_{2m_i}, \frac{1}{2m})\right)$ for some $i \in \mathbb{N}$.

Let, $y \in B(x_{2m_i}, \frac{1}{2m})$. Then we have,

$$\begin{aligned} d(y, x_{2m_i}) + d(x_{2m_i}, x) &< \frac{1}{2m} + \frac{1}{2m} \\ &= \frac{1}{m} < \delta. \end{aligned}$$

So, by Equation (1) we have,

$$f(d(y, x_{2m_i}) + d(x_{2m_i}, x)) < f\left(\frac{1}{n}\right) - \alpha.$$

If $y = x$, then clearly $y \in B(x, \frac{1}{n}) \subseteq U$. If not, then by $(D3)$ we get,

$$\begin{aligned} f(d(y, x)) \leq f(d(y, x_{2m_i}) + d(x_{2m_i}, x)) + \alpha &< f\left(\frac{1}{n}\right) - \alpha + \alpha \\ &= f\left(\frac{1}{n}\right) \\ \Rightarrow d(y, x) &< \frac{1}{n} \\ \Rightarrow y &\in B\left(x, \frac{1}{n}\right) \subseteq U. \end{aligned}$$

Therefore,

$$\begin{aligned} & B\left(x_{2m_i}, \frac{1}{2m}\right) \subseteq U \\ & \Rightarrow x \in \text{int}\left(B\left(x_{2m_i}, \frac{1}{2m}\right)\right) \subseteq B\left(x_{2m_i}, \frac{1}{2m}\right) \subseteq U \\ & \Rightarrow x \in \text{int}\left(B\left(x_{2m_i}, \frac{1}{2m}\right)\right) \subseteq U, \text{ where } \text{int}\left(B\left(x_{2m_i}, \frac{1}{2m}\right)\right) \in \mathcal{A}. \end{aligned}$$

Therefore, $\mathcal{A}$ is a countable base for the topology on X and hence (X, d) is second countable. □

Therefore it is interesting to note from the above theorems that some of the important topological properties of usual metric spaces hold good in this structure. However, we are not sure about certain other important properties hold good or not in it. In this context, we pose the following open problem:

Open Problem 2.6. *Is every open ball an open set in $\mathcal{F}$-metric spaces?*

3. Fixed point results via altering distance functions

In this section, we present a few fixed point results concerning some special kinds of self-maps via altering distance functions. To begin with, we recall a crucial notion of an altering distance function which was originally coined by Khan et al. [11].

Definition 3.1. *A function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if*

(i) *φ is continuous,*
(ii) *φ is non-decreasing,*
(iii) $\varphi(t) = 0 \iff t = 0$.

In 1962, Edelstein [4] obtained the following version of the Banach contraction principle [1] relevant to the contractive mappings.

Theorem 3.2. *[4] Let (X, d) be a compact metric space and $T : X \to X$ be a self-map. Assume that*

$$d(Tx, Ty) < d(x, y)$$

holds for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point in X.

Now, employing the idea of altering distance functions, we generalize Theorem 3.2 in $\mathcal{F}$-metric setting. The following result assures the existence and uniqueness of a fixed point arising of contractive type mappings in this framework.

Theorem 3.3. *Let (X, d) be a sequentially compact $\mathcal{F}$-metric space and T be a self-map on X such that*

$$\varphi(d(Tx, Ty)) < \varphi(d(x, y))$$

for all $x, y \in X$ with $x \neq y$, where φ is an altering distance function. Also, assume that the $\mathcal{F}$-metric d is continuous. Then T has a unique fixed point and for any $x \in X$, the sequence $(T^n x)$ is $\mathcal{F}$-convergent to that fixed point.

Proof. Let $x_0 \in X$ and we define a sequence (x_n) by $x_n = T^n x_0$ for $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n \in X$ is a fixed point of T. So, we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since (X, d) is a sequentially compact $\mathcal{F}$-metric space, there exists a convergent subsequence (x_{n_k}) of (x_n) that converges to z. Since T is continuous, it follows that the subsequence (x_{n_k+1}) converges to Tz. Take $s_n = \varphi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$. Then we have,

$$\begin{aligned} s_{n+1} &= \varphi(d(x_{n+1}, x_{n+2})) \\ &< \varphi(d(x_n, x_{n+1})) \\ &= s_n. \end{aligned}$$

This shows that the sequence of non-negative real numbers (s_n) is a decreasing sequence and hence convergent to some $a \geq 0$. Next, if $a > 0$, then we have

$$0 < a = \lim_{k \to \infty} \varphi(d(x_{n_k}, x_{n_k+1})) = \varphi(d(z, Tz)).$$

Also we have

$$0 < a = \lim_{k\to\infty} \varphi(d(x_{n_k+1}, x_{n_k+2})) = \varphi(d(Tz, T^2z)) < \varphi(d(z, Tz)) = a,$$

which leads to a contradiction. So we must have $a = 0$. Thus the sequence (s_n) converges to zero. Therefore we obtain

$$\begin{aligned} &\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0 \\ \implies &\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \end{aligned}$$

Now we show that z is a fixed point of T. On the contrary, let $z \neq Tz$. Then $d(z, Tz) > 0$ and so by $(D3)$, we have

$$f(d(z, Tz)) \leq f\left(d(z, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, Tz)\right) \to -\infty \text{ as } k \to \infty,$$

which is a contradiction. So we must have $z = Tz$, i.e., z is a fixed point of T.

Now, to prove the uniqueness of the fixed point, if possible, we assume T has two fixed points $z_1, z_2 \in X$ with $z_1 \neq z_2$, i.e., $Tz_1 = z_1$ and $Tz_2 = z_2$. Then

$$\begin{aligned} \varphi(d(z_1, z_2)) &= \varphi(d(Tz_1, Tz_2)) \\ &< \varphi(d(z_1, z_2)), \end{aligned}$$

which is impossible. Therefore, we obtain $z_1 = z_2$. Hence T has a unique fixed point. Hence, we prove that the sequence (x_n) converges to z. If $x_n = z$ for finitely many $n \in \mathbb{N}$, then we can exclude those x_n from (x_n) and assume that, $x_n \neq z$ for all $n \in \mathbb{N}$. Then from the sequentially compactness, we have

$$\lim_{k\to\infty} d(x_{n_k}, z) = 0,$$

that is, z is the accumulation point of the sequence (x_n). Again, if z_1 be another accumulation point of (x_n), then there exists a subsequence of (x_n) which converges to z_1. Then continuing as above discussion, we can show that z_1 is a fixed point of T, which implies that $z = z_1$. So, z is the unique accumulation point of (x_n).

Now, we consider a sequence (α_n) of real numbers where $\alpha_n = \varphi(d(x_n, z))$ for all $n \in \mathbb{N}$. Therefore, $\alpha_{n_k} = \varphi(d(x_{n_k}, z)) \to 0$ as $k \to \infty$, which implies (α_n) has a subsequence (α_{n_k}) that converges to 0. So 0 is an accumulation point of (α_n). Now, we have,

$$\begin{aligned} \alpha_{n+1} &= \varphi(d(x_{n+1}, z)) \\ &= \varphi(d(Tx_n, Tz)) \\ &< \varphi(d(x_n, z)) \\ &= \alpha_n \end{aligned}$$

for all $n \in \mathbb{N}$. Hence (α_n) is a monotone decreasing sequence of non-negative real numbers and 0 is an accumulation point of (α_n). Then (α_n) must converge to 0. Therefore, letting $n \to \infty$, we obtain $\varphi(d(x_n, z)) \to 0$. This implies

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Hence (x_n) converges to z and so $(T^n x_0)$ converges to z. Since $x_0 \in X$ is arbitrary, $(T^n x)$ converges to z for each $x \in X$. □

From the above theorem, we can establish the subsequent corollary by taking $\varphi(t) = t$ for all $t \in [0, \infty)$.

Corollary 3.4. *Let (X, d) be a sequentially compact $\mathcal{F}$-metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$, where φ is an altering distance function. Then T has a unique fixed point in X and for any $x \in X$, the sequence $(T^n x)$ converges to that fixed point.

The succeeding example authenticates previously discussed Theorem 3.3.

Example 3.5. *Let $X = [0,1]$ and define the metric $d : X \times X \to [0,\infty)$ by*

$$d(x,y) = |x-y|$$

for all $x, y \in X$. Also consider a self-map T on X by

$$T(x) = 1 - \frac{x}{2},$$

for all $x \in X$. Then (X,d) is a $\mathcal{F}$-metric space with $f(t) = \ln t$ and $\alpha = 0$ and also $\mathcal{F}$-compact. Now, consider $\varphi(t) = t^2, \ \ t \in [0,\infty)$. Then we have,

$$\begin{aligned} \varphi(d(Tx,Ty)) &= \varphi(|Tx-Ty|) \\ &= |Tx-Ty|^2 \\ &= \left(1 - \frac{x}{2} - 1 + \frac{y}{2}\right)^2 \\ &= \frac{1}{4}(x-y)^2 \\ &< (x-y)^2 \\ &= \varphi(d(x,y)), \end{aligned}$$

for all $x, y \in X$ with $x \neq y$. Therefore,

$$\varphi(d(Tx,Ty)) < \varphi(d(x,y))$$

holds for all $x, y \in X$ with $x \neq y$. Thus T satisfies all the hypotheses of Theorem 3.3 and hence possesses a unique fixed point $x = \frac{2}{3} \in X$.

In the next theorem, we consider Kannan type contractive mappings defined on an $\mathcal{F}$-metric space.

Theorem 3.6. *Let T be a self-mapping on an $\mathcal{F}$-metric space (X,d) and assume that the $\mathcal{F}$-metric d is continuous. Suppose there exists $x_0 \in X$ such that the orbit $\phi(x_0) = \{T^n x_0 : n \in \mathbb{N}\}$ has an accumulation point $z \in X$. If T is orbitally continuous at z and there exists an altering distance function φ such that*

$$\varphi(d(Tx,Ty)) < \frac{1}{2}\{\varphi(d(x,Tx)) + \varphi(d(y,Ty))\}$$

holds for all $x, y = Tx \in \overline{\phi(x_0)}$ with $x \neq y$, then z is the unique fixed point of T.

Proof. Let us define a sequence (x_n) by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}_0$. If $x_n = x_{n+1}$ for some n, then T has a fixed point. So, we assume $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}_0$. Let $\alpha_n = \varphi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}_0$. Then by the given condition it follows that

$$\begin{aligned} \alpha_{n+1} &= \varphi(d(x_{n+1}, x_{n+2})) \\ &= \varphi(d(Tx_n, Tx_{n+1})) \\ &< \frac{1}{2}\{\varphi(d(x_n, Tx_n)) + \varphi(d(x_{n+1}, Tx_{n+1}))\} \\ &= \frac{1}{2}\{\varphi(d(x_n, x_{n+1})) + \varphi(d(x_{n+1}, x_{n+2}))\} \\ &= \frac{1}{2}\{\alpha_n + \alpha_{n+1}\} \\ \Rightarrow \frac{1}{2}\alpha_{n+1} &< \frac{1}{2}\alpha_n \\ \Rightarrow \alpha_{n+1} &< \alpha_n. \end{aligned}$$

This shows that (α_n) is a strictly decreasing sequence of positive reals and hence convergent to some non-negative real number a. Since z is an accumulation point of $\phi(x_0)$, there exists a sequence of positive integers (n_k) such that (x_{n_k}) converges to $z \in X$. Therefore, by orbitally continuity of T, we get (x_{n_k+1}) converges to Tz.

Now, we claim that $a = 0$. If $a \neq 0$, then we have

$$0 < a = \lim_{k\to\infty} \varphi(d(x_{n_k}, x_{n_k+1})) = \varphi(d(z, Tz)).$$

Also we have

$$0 < a = \lim_{k\to\infty} \varphi(d(x_{n_k+1}, x_{n_k+2})) = \varphi(d(Tz, T^2z)) < \varphi(d(z, Tz)) = a,$$

which leads to a contradiction. So we must have $a = 0$. Thus the sequence (α_n) converges to zero. Let there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ satisfying the conditions $(D1$-$D3)$ of Definition 1.2. Then by $(\mathcal{F}_2)$, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < t < \delta \ \Rightarrow f(t) < f(\varphi(\epsilon)) - \alpha. \tag{2}$$

Now,

$$\begin{aligned}
\varphi(d(x_n, x_{n+1})) &< \frac{1}{2}\{\varphi(d(x_{n-1}, Tx_{n-1})) + \varphi(d(x_n, Tx_n))\} \\
&= \frac{1}{2}\{\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_n, x_{n+1}))\} \\
&= \frac{1}{2}\{\varphi(d(T^{n-1}x_0, T^n x_0)) + \varphi(d(T^n x_0, T^{n+1}x_0))\}.
\end{aligned}$$

Finally we obtain,

$$\sum_{i=n}^{m-1} \varphi(d(x_i, x_{i+1})) < \sum_{i=n}^{m-1} \frac{1}{2}\{\varphi(d(T^{i-1}x_0, T^i x_0)) + \varphi(d(T^i x_0, T^{i+1}x_0))\}.$$

Since

$$\lim_{n\to\infty}\{\varphi(d(T^{n-1}x_0, T^n x_0)) + \varphi(d(T^n x_0, T^{n+1}x_0))\} = 0,$$

there exists some $N \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{m-1} \varphi(d(x_i, x_{i+1})) < \delta,$$

holds for all $n \geq N$. Hence by (2) and $(\mathcal{F}_1)$, we have

$$f\Big(\sum_{i=n}^{m-1} \varphi(d(x_i, x_{i+1}))\Big) < f(\varphi(\epsilon)) - \alpha. \tag{3}$$

Now, we show that

$$d(x_n, x_m) < \epsilon$$

for all $m > n \geq N$. Let $m, n \in \mathbb{N}$ be fixed but arbitrary such that $m > n \geq N$. If $d(x_n, x_m) = 0$, then clearly $d(x_n, x_m) < \epsilon$ and if $d(x_n, x_m) > 0$, then using (D3) and (3), we get

$$\begin{aligned}
&\varphi(d(x_n, x_m)) > 0 \\
\Rightarrow f(\varphi(d(x_n, x_m))) &\leq f\Big(\sum_{i=n}^{m-1} \varphi(d(x_i, x_{i+1}))\Big) + \alpha \\
&< f(\varphi(\epsilon)),
\end{aligned}$$

which gives by $(\mathcal{F}_1)$ that

$$\varphi(d(x_n, x_m)) < \varphi(\epsilon)$$
$$\Rightarrow d(x_n, x_m) < \epsilon.$$

This proves that (x_n) is $\mathcal{F}$-Cauchy. Since (x_n) has a subsequence (x_{n_k}) which converges to z, the limit of (x_n) will be z. This means that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{4}$$

This implies that (x_n) converges to z and by orbitally continuity of T, we get (Tx_n) converges to Tz. Since $Tx_n = x_{n+1}$, we have, by the uniqueness of limit of sequence, $z = Tz$. Hence z is a fixed point of T. For uniqueness, let z^* be another fixed point of T. Then

$$\begin{aligned}\varphi(d(z, z^*)) &= \varphi(d(Tz, Tz^*))\\ &< \frac{1}{2}\{\varphi(d(z, Tz)) + \varphi(d(z^*, Tz^*))\}\\ &< 0,\end{aligned}$$

a contradiction. Therefore z is the unique fixed point of T. □

Remark 3.7. *The following example shows that, if we take any two points $x, y = Tx \in \phi(x_0)$ with $x \neq y$ satisfying the inequality*

$$\varphi(d(Tx, Ty)) < \frac{1}{2}\{\varphi(d(x, Tx)) + \varphi(d(y, Ty))\},$$

then the sequence of iterates $(T^n x_0)$ may not converge to the accumulation point of $\phi(x_0)$.

The following example validates the previous remark.

Example 3.8. *Let $X = \{2, -2, 2 + \frac{1}{3n}, -2 - \frac{1}{3n+1} : n \in \mathbb{N}\}$ and define the metric $d : X \times X \to [0, \infty)$ by*

$$d(x_1, y_1) = |x_1 - y_1|$$

for all $x_1, y_1 \in X$. Then (X, d) is an $\mathcal{F}$-metric space with $f(t) = \ln t$, for $t > 0$ and $\alpha = 0$. Now, we define T on X by

$$\begin{aligned}&T(2) = -2,\ \ T(-2) = 2,\ \ T\Big(2 + \frac{1}{3n}\Big) = -2 - \frac{1}{3n+1}\\ &\qquad \text{and}\ \ T\Big(-2 - \frac{1}{3n+1}\Big) = 2 + \frac{1}{3(n+1)}.\end{aligned}$$

Now, for $x_0 = 2 + \frac{1}{3}$, we have

$$\phi(x_0) = \Big\{2 + \frac{1}{3}, -2 - \frac{1}{4}, 2 + \frac{1}{6}, -2 - \frac{1}{7}, 2 + \frac{1}{9}, -2 - \frac{1}{10}, \cdots\Big\}.$$

Then it can be easily verified that the inequality

$$\varphi(d(Tx, Ty)) < \frac{1}{2}\{\varphi(d(x, Tx)) + \varphi(d(y, Ty))\}$$

is satisfied for all $x, y = Tx \in \phi(x_0)$ with $x \neq y$ and $\varphi(t) = t,\ t \geq 0$. Whenever $z = 2$ in $\overline{\phi(x_0)}$, we have $Tz = -2$. Moreover, T is orbitally continuous at z. But still z is not a fixed point of T.

Example 3.9. *Let $(e^i_n)_{n\in\mathbb{N}}$ be the sequence of real numbers whose i-th term is i and all other terms are 0. Take $X = \{(e^i_n)_{n\in\mathbb{N}} : i \geq 1\}$. Define $d : X \times X \to \mathbb{R}$ by*

$$d(x,y) = \begin{cases} 1 + \left|\frac{1}{\sup_n |e^i_n|} - \frac{1}{\sup_n |e^j_n|}\right|, & \text{if } x = (e^i_n)_{n\in\mathbb{N}} \text{ and } y = (e^j_n)_{n\in\mathbb{N}} \text{ with } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Then it is easy to note that (X,d) is an $\mathcal{F}$-metric space with $f(t) = \ln t$ and $\alpha = 0$. Also, it is clear that (X,d) is $\mathcal{F}$-complete but not $\mathcal{F}$-compact.

Now, define a mapping $T : X \to X$ by

$$T((e^i_n)_{n\in\mathbb{N}}) = (e^{3i}_n)_{n\in\mathbb{N}}$$

for all $(e^i_n) \in X$. Therefore, for $i < j$, we have

$$\begin{aligned} d(T(e^i_n), T(e^j_n)) &= 1 + \left|\frac{1}{3i} - \frac{1}{3j}\right| \\ &= 1 + \frac{1}{3i} - \frac{1}{3j} < 1 + \frac{1}{3i} \end{aligned}$$

whereas,

$$\begin{aligned} \frac{1}{2}\{d((e^i_n), T(e^i_n)) + d((e^j_n), T(e^j_n))\} &= \frac{1}{2}\left\{1 + \left|\frac{1}{i} - \frac{1}{3i}\right| + 1 + \left|\frac{1}{j} - \frac{1}{3j}\right|\right\} \\ &= 1 + \frac{1}{3i} + \frac{1}{3j} > 1 + \frac{1}{3i}. \end{aligned}$$

So,

$$d(T(e^i_n), T(e^j_n)) < \frac{1}{2}\{d((e^i_n), T(e^i_n)) + d((e^j_n), T(e^j_n))\}.$$

In a similar manner, we can show that

$$d(T(e^i_n), T(e^j_n)) < \frac{1}{2}\{d((e^i_n), T(e^i_n)) + d((e^j_n), T(e^j_n))\}$$

if $i > j$. Therefore,

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in X$ with $x \neq y$, but T does not possess any fixed point.

From the above example, we observe that the completeness of X can not alone guarantee the existence of a fixed point for the Kannan type contractive mappings.

Theorem 3.10. *Let (X,d) be an $\mathcal{F}$-complete metric space and T be a continuous self-map on T such that*

$$\varphi(d(Tx, Ty)) < \frac{1}{2}\{\varphi(d(x, Tx)) + \varphi(d(y, Ty))\} \tag{5}$$

for all $x, y \in X$ with $x \neq y$, where φ is an altering distance function. Also assume that for any $x \in X$ and for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\varphi(d(T^i x, T^j x)) < \epsilon + \delta \Rightarrow \varphi(d(T^{i+1}x, T^{j+1}x)) \leq \epsilon$$

for any $i, j \in \mathbb{N}$. Then T has a unique fixed point, and for any $x \in X$, the sequence of iterates $(T^n x)$ converges to that fixed point.

Proof. Let $x_0 \in X$ be arbitrary but fixed. We consider the sequence (x_n) in X, where $x_n = T^n x_0$ for all natural numbers n. Also, we take the sequence of real numbers (s_n) defined by $s_n = \varphi(d(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$.

If $x_n = x_{n+1}$ for some n, then it is easily noticeable that x_n is a fixed point of T. So now we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Putting $x = x_n,\ y = x_{n+1}$ in (5), we get,

$$\begin{aligned}
\varphi(d(Tx_n, Tx_{n+1})) &< \frac{1}{2}\{\varphi(d(x_n, Tx_n)) + \varphi(d(x_{n+1}, Tx_{n+1}))\}\\
\Rightarrow \varphi(d(x_{n+1}, x_{n+2})) &< \frac{1}{2}\{\varphi(d(x_n, x_{n+1})) + \varphi(d(x_{n+1}, x_{n+2}))\}\\
\Rightarrow s_{n+1} &< \frac{1}{2}\{s_n + s_{n+1}\}\\
\Rightarrow s_{n+1} &< s_n.
\end{aligned}$$

Therefore, (s_n) is a decreasing sequence of non-negative real numbers and hence convergent to some $a \geq 0$. We claim that $a = 0$. If $a > 0$, then by the given condition there exists $\delta > 0$ such that

$$\varphi(d(T^i x, T^j x)) < a + \delta \Rightarrow \varphi(d(T^{i+1}x, T^{j+1}x)) \leq a \tag{6}$$

for any $i, j \in \mathbb{N}$. But since (s_n) converges to a, there exists $n \in \mathbb{N}$ such that

$$s_n < a + \delta.$$

Then using (6) we get,

$$s_{n+1} \leq a,$$

which contradicts the fact that (s_n) converges to a. Therefore, (s_n) converges to 0.

Now, we show that (x_n) is a Cauchy sequence. To show this, we put $x = x_n, y = x_m$ in (5) and get,

$$\begin{aligned}
\varphi(d(Tx_n, Tx_m)) &< \frac{1}{2}\{\varphi(d(x_n, Tx_n)) + \varphi(d(x_m, Tx_m))\}\\
\Rightarrow \varphi(d(x_{n+1}, x_{m+1})) &< \frac{1}{2}\{s_n + s_m\} \to 0 \text{ when } n, m \to \infty.
\end{aligned}$$

Therefore, the double sequence $(\varphi(d(x_n, x_m)))$ of real numbers converges to 0. So for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\varphi(d(x_n, x_m)) < \varphi(\epsilon)$$

for all $n, m \geq N$, which gives

$$d(x_n, x_m) < \epsilon$$

for all $n, m \geq N$. Therefore (x_n) is a Cauchy sequence in X. Again, as X is $\mathcal{F}$-complete, (x_n) converges to some $z \in X$. Since, T is continuous, (Tx_n) converges to Tz, i.e., (x_n) converges to Tz. So we have $z = Tz$, i.e., z is a fixed point of T. The uniqueness of the fixed point can be similarly derived as in Theorem 3.6. □

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