Annali di Matematica pura ed applicata
(IV), Vol. CLXXV (1998), pp. 59-91

Some Interpolation Inequalities Involving Stokes Operator and First Order Derivatives (*).

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1. – Introduction.

As is well known, an interpolation inequality states *a priori* properties of a function (or of its derivatives) beloging to some Sobolev spaces [2, 4, 10-12, 17, 19, 23-25, 33-34, 37-38, 42]. Usually, on the right hand side of such inequalities in a multiplicative form the L^p*-norm* of the function and of its derivatives of maximum order appear. However, in the study of some partial differential equations it is required for an interpolation inequality to involve on the right hand side a suitable differential operator, instead of the maximum order of the derivatives. This is the case of Navier-Stokes equations and Stokes operator, owing to the nonlinear character of the equations and the fact that the maximum order of derivatives which appears in the equations is due to the Stokes operator ($P\Delta$).

One of the aims of the paper is to prove some interpolation inequalities involving Stokes operator in a multiplicative form. Our chief requirement is to state the results in exterior domains Ω of $\mathbb{R}^n$. If Ω denotes a bounded and sufficiently smooth domain, $\boldsymbol{w}(x) \in W^{2,p}(\Omega) \cap J^{1,p}(\Omega)$ (see section 2 for notations), then the following inequality holds [5, 13, 15-16, 41, 43-44]

$$|\boldsymbol{w}|_{2,p} \leq C|P\Delta \boldsymbol{w}|_p, \tag{1.1}$$

where C, independent of $\boldsymbol{w}(x)$, among other parameters, takes into account of the Poincaré inequality. After which by general interpolation inequalities of second order by Gagliardo and Nirenberg type, one is able to prove properties for $\boldsymbol{w}(x)$ and $\nabla \boldsymbol{w}(x)$ in

(*) Entrata in Redazione il 10 giugno 1996.

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a suitable L^r*-space* by means of an inequality of the type (1):

$$|D^j w|_r \leq C|PΔw|_p^a |w|_q^{1-a}, \qquad j = 0, 1,$$

where j, r, a, p, q satisfy a suitable relation. When Ω is unbounded in any directions, in particular if it is an exterior domain, the above considerations fail to hold. In this connection, it is sufficient to think that in (1.1) C depends on Poincaré inequality. Actually the situation is more involved, since an inequality of the type $|D^2 w|_p \leq C|PΔw|_p$ for any $p \geq n/2$ does not already hold [13, 15, 30]. Here we employ a technique which does not make use of the above inequality and it is able to prove the desidered interpolation inequalities when Ω is an exterior domain. Of course, as a consequence we have that inequality $|D^2 w|_p \leq C|PΔw|_p$ for any $p \geq n/2$ is not necessary to obtain our estimates on function $w(x)$, in particular we can say that $(p > n/2)$ $w(x) \in L^\infty(\Omega)$. So it is completely avoid the use of standard Sobolev imbedding theorems.

Another aim of this paper is to prove interpolation inequalities between $H^{1,p}(\mathbb{R}^n)$ and $H^{-1,p}(\mathbb{R}^n)$ either of the first order of derivatives. Since to explain the results it is necessary to introduce some notations, here we want only to point out that we can prove some interpolation Sobolev inequalities without require that the functions have traces equal zero on the boundary. Moreover, we try to give a numerical value to the constants which appear in interpolation inequalities of the first order (for functions beloging to the completeness of $C_0^\infty(\mathbb{R}^n)$ in suitable norm). Of course, we consider exponents that are not connected with the cases $q = ns/(n-s)$, $s \in [1, n)$ for which are known the results of [1, 45]. The constants are not sharp, however they are deduced in such a simple way that it seems interesting to communicate.

We conclude stressing that our results also come if we substute the Stokes operator with some elliptic operators. As well as in the inequalities of the first order, we can substute the operator «∇» with «$A(x)\cdot\nabla$», where $A(x)$ is a matrix which defines an elliptic operator. However, the extension is not so trivial with exception of the Laplace operator. Our technique consents to consider the case of $\nabla\cdot(A(x)\cdot\nabla w(x)) \in L^1(\Omega)$ and in the view of the results of [39] this appears of particular interest also for the domain Ω not exterior.

Finally we quote the paper [32, 47]. In [32] MASUDA for the first time considered our question for the case of Laplace operator. He proved an inequality of the type $\sup_\Omega |u(x)| \leq C|\Delta u|_2^{1/2} |\nabla u|_2^{1/2}$, $u(x)_{|\partial\Omega} = 0$, where $\Omega \subset \mathbb{R}^3$ is an exterior domain. Subsequently, in [47] XIE proves the same inequality considering an arbitrary domain $\Omega \subset \mathbb{R}^3$. Moreover he gives a precise value of the constant, in fact $C = (2\pi)^{-1/2}$ is the best constant for the inequality.

Further comments on the results of the paper we refer the reader to the remarks of section 2.

(1) Such an inequality, in an ambit of Navier-Stokes equations, was proved and used in [8] for questions connected to the existence of solutions. More precisaly they proved ($p = q = 2$ and $r = \infty$) $|w|_\infty \leq C|PΔw|_2^{3/4} |w|_2^{1/4}$. However in [8] the domain $\Omega \subset \mathbb{R}^3$ is bounded. In exterior domains this inequality becomes interesting not only for questions connected to the existence, but also for problems of stability of solutions (see [18, 32]).

2. – Notations and statement of the results.

Throughout the paper Ω will denote an exterior domain of $\mathbb{R}^n$ ($n \geq 2$). As far as the regularity of $\partial\Omega$ is concerned, it is specified in the statements of the below theorems.

$L^p(\Omega)$ ($p \geq 1$) denotes the set of all fields $\varphi(x)$ on Ω such that

$$|\varphi|_p^p = \int_\Omega |\varphi(x)|^p\,dx < \infty\ ;$$

$$L^\infty(\Omega) = \{\textit{Lebesgue measurable } \varphi(x)\text{: ess}\sup_\Omega |\varphi(x)| < \infty\}\,;$$

$$W^{m,p}(\Omega) = \{\varphi(x)\colon\ |\varphi|_{m,p}^p = \sum_{|\alpha|=0}^{m} |D^\alpha\varphi|_p^p < \infty\}\,,$$

where $D^\alpha\varphi(x)$ denotes weak derivatives of $\varphi(x)$ of order $|\alpha|$. $\mathring{W}^{m,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $|\cdot|_{m,p}$ norm. $\mathcal{C}_0(\Omega)$ is the whole set of $C^\infty(\Omega)$ vector field $\varphi(x)$ with compact support on Ω such that $\nabla\cdot\varphi(x) = 0$; $J^p(\Omega)$ and $J^{m,p}(\Omega)$ denote the completion of $\mathcal{C}_0(\Omega)$ in $L^p(\Omega)$ and $W^{m,p}(\Omega)$ respectively. It is well known [13] that $L^p(\Omega) = J^p(\Omega) \oplus G^p(\Omega)$, where $G^p(\Omega) = \{\psi(x)\colon \psi(x) = \nabla h(x),\ h(x) \in L^p_{\text{loc}}(\Omega)\}$ (here $\partial\Omega$ is smooth of C^2-*class*). As is standard, we set $(\boldsymbol{f}, \boldsymbol{g}) = \int_\Omega \boldsymbol{f}(x)\cdot\boldsymbol{g}(x)\,dx$, for any $\boldsymbol{f}(x), \boldsymbol{g}(x)$ such that $\boldsymbol{f}(x)\cdot\boldsymbol{g}(x)$ is integrable over Ω. If $1/p + 1/q = 1$, we have for any $\boldsymbol{f}(x) \in J^p(\Omega)$ and $\psi(x) \in G^p(\Omega)$, $(\boldsymbol{f}, \psi) = (\boldsymbol{f}, \nabla h) = 0$. The operator $P_q\cdot$ is the projector from $L^q(\Omega)$ into $J^q(\Omega)$; if in the context there is no ambiguity then we omit index q. We indicate by $H_0^{-1,p}(\Omega)$ the completion of $L^p(\Omega)$ with respect the norm $|\varphi|_{-1,p} = \sup\limits_{\psi(x)\in \mathring{W}^{1,q}(\Omega),\ |\psi|_{1,q}=1} \left|\int_\Omega \varphi(x)\cdot\psi(x)\,dx\right|$ with $1/p + 1/q = 1$. $\widehat{W}^{1,p}(\Omega) = \{\varphi(x)\colon |\nabla\varphi|_p < \infty\}$ and $D_0^{1,p}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect $|\nabla\cdot|_p$. Following [13, 15] we indicate by $D_0^{-1,p}(\Omega)$ the completion of $C_0(\Omega)$ with respect the norm $\|\varphi\|_{-1,p} = \sup\limits_{\psi(x)\in D_0^{1,q}(\Omega),\ |\nabla\psi|_q=1} \left|\int_\Omega \varphi(x)\cdot\psi(x)\,dx\right|$ with $1/p + 1/q = 1$. We set

$$H_{q,p}^{P\Delta}(\Omega) = \{\boldsymbol{w}(x) \in J^q(\Omega)\colon\ \nabla\boldsymbol{w}(x) \in L^p_{\text{loc}}(\Omega) \text{ and } P\Delta\boldsymbol{w}(x) \in J^p(\Omega)\};$$

$$\mathring{H}_{q,p}^{P\Delta}(\Omega) = \{\boldsymbol{w}(x) \in H_{q,p}^{P\Delta}(\Omega) \text{ and } \boldsymbol{w}(x)_{|\partial\Omega} = 0\}\,;$$

$$\mathring{H}_{q,p}^{P\Delta,1}(\Omega) = \{\boldsymbol{w}(x) \in \mathring{H}_{q,p}^{P\Delta}(\Omega) \text{ and } \nabla\boldsymbol{w}(x)_{|\partial\Omega} = 0\}\,.$$

In the definition of the above sets of functions it is assumed that $\partial\Omega$ is smooth of C^2-class. We denote by $\boldsymbol{E}(x)$, $P(x)$ the Stokes fundamental solution, $\boldsymbol{E}(x)$ is a tensor and $P(x)$ is a vector field, they have components:

$$E_{ij}(x) = \frac{1}{4\pi}\left(\delta_{ij}\ln\frac{1}{|x|} + \frac{x_i x_j}{|x|^2}\right), \qquad P_j(x) = \frac{1}{2\pi}\frac{x_j}{|x|^2}, \qquad n = 2,$$

$$E_{ij}(x) = \frac{1}{2\omega_n}\left(\frac{\delta_{ij}}{n-2}\frac{1}{|x|} + \frac{x_i x_j}{|x|^n}\right), \qquad P_j(x) = \frac{1}{\omega_n}\frac{x_j}{|x|^n}, \qquad n \geq 3,$$

where ω_n is the measure of unit sphere.

We define for $w(x) \in \mathring{H}^{P\Delta}_{q,p}(\Omega)$ the boundary integral quantities:

i) $$\int_{\partial\Omega} \boldsymbol{E}(x-x_0)\cdot \boldsymbol{T}(\boldsymbol{w}(x), \pi(x))\cdot \vec{n}\, d\sigma = B_n, \qquad p \in \left[\frac{n}{2}, n\right), \qquad n > 2;$$

$$\int_{\partial\Omega} \boldsymbol{T}(\boldsymbol{w}(x), \pi(x))\cdot \vec{n}\, d\sigma = B_2, \qquad p \in (1, 2), \qquad n = 2;$$

in the case of $p \geq n \geq 2$, with further condition

ii) $$\int_{\partial\Omega} \nabla \boldsymbol{E}(x-x_0)\cdot \boldsymbol{T}(\boldsymbol{w}(x), \pi(x))\cdot \vec{n}\, d\sigma = B^1_n;$$

where $\pi(x)$ is the function such that $P\Delta \boldsymbol{w}(x) = \Delta \boldsymbol{w}(x) - \nabla\pi(x)$ in Ω and $\boldsymbol{T}(\boldsymbol{w}(x), \pi(x))$ is the stress tensor of components $T_{ij}(x) = -\delta_{ij}\pi(x) + (\partial w_i(x)/\partial x_j + \partial w_j(x)/\partial x_i)$. The symbol $S\mathring{H}^{P\Delta}_{q,p}(\Omega)$ denotes the set of functions $\boldsymbol{w}(x) \in \mathring{H}^{P\Delta}_{q,p}(\Omega)$ for which $B_n = 0$, or $B_n = B^1_n = 0$, $n \geq 2$. Of course it is $\mathring{H}^{P\Delta,1}_{q,p}(\Omega) \subset S\mathring{H}^{P\Delta}_{q,p}(\Omega)$, while $S\mathring{H}^{P\Delta}_{q,p}(\Omega)$ is a subset of the class of solutions to Stokes system with boundary conditions $\boldsymbol{w}(x)_{|\partial\Omega} = 0$ and $B_n = 0$, or $B_n = B^1_n = 0$, $n \geq 2$, whose existence was stated in [30], for the case of Laplace equation see [29].

Now we are in a position to stand our results:

THEOREM 2.1. – *Let $\Omega \subseteq \mathbb{R}^n$ be, $n \geq 2$, and assume $\partial\Omega$ of C^m class with $2m > n$, m an even positive integer. Assume $\boldsymbol{w}(x) \in \mathring{H}^{P\Delta}_{q,p}(\Omega)$, with $p, q \in (1, \infty)$. Then there exists a constant C indipendent of $\boldsymbol{w}(x)$ such that for $a \in [0, 1]$ and $p \in (1, n/2)$ $(n \geq 3)$*

$$|\boldsymbol{w}|_r \leq \begin{cases} (C|P\Delta \boldsymbol{w}|_p)^a\, |\boldsymbol{w}|_q^{1-a} & \text{if } q \leq \dfrac{np}{n-2p}, \\[2ex] (C|P\Delta \boldsymbol{w}|_p)^{1-a}\, |\boldsymbol{w}|_q^{a} & \text{if } q \geq \dfrac{np}{n-2p}, \end{cases}$$

where

$$r \in \begin{cases} \left[q, \dfrac{np}{n-2p}\right) & \text{and} \quad \dfrac{1}{r} = a\left(\dfrac{1}{p} - \dfrac{2}{n}\right) + (1-a)\dfrac{1}{q}, \quad \text{if } q \leqslant \dfrac{np}{n-2p}, \\ \left[\dfrac{np}{n-2p}, q\right] & \text{and} \quad \dfrac{1}{r} = (1-a)\left(\dfrac{1}{p} - \dfrac{2}{n}\right) + a\dfrac{1}{q}, \quad \text{if } q \geqslant \dfrac{np}{n-2p}. \end{cases}$$

Moreover for $p \geqslant n/2$ *(for* $n = 2, p > 1$*), there exists a constant* C *indipendent of* $\boldsymbol{w}(x)$ *such that*

$$|\boldsymbol{w}|_r \leqslant C|P\Delta \boldsymbol{w}|_p^a |\boldsymbol{w}|_q^{1-a}, \qquad r \in \begin{cases} [q, \infty] & \text{if } n \leqslant 3, \\ [q, \infty) & \text{if } n = 2; \end{cases} \tag{2.1}$$

provided that for $a \in [0, 1)$

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{2}{n}\right) + (1-a)\frac{1}{q}, \tag{2.2}$$

where C *depends on* p, q, r, a *and* $\partial\Omega$.

THEOREM 2.2. – *Let* $w(x) \in \widehat{W}^{1,s}(\mathbb{R})^n \cap L^q(\mathbb{R}^n)$, $n \geqslant 1$, $s \in [1, \infty]$, $q \geqslant 1$. *Then there exists a constant* C *such that for* $a \in [0, 1]$ *and* $s \in [1, n)$ $(n \geqslant 2)$

$$|w|_r \leqslant \begin{cases} (C|\nabla w|_s)^a |w|_q^{1-a} & \text{if } q \leqslant \dfrac{ns}{n-s}, \\ (C|\nabla w|_s)^{1-a} |w|_q^a & \text{if } q \geqslant \dfrac{ns}{n-s}, \end{cases} \tag{2.3}$$

where

$$r \in \begin{cases} \left[q, \dfrac{ns}{n-s}\right) & \text{and} \quad \dfrac{1}{r} = a\left(\dfrac{1}{s} - \dfrac{1}{n}\right) + (1-a)\dfrac{1}{q}, \quad \text{if } q \leqslant \dfrac{ns}{n-s}, \\ \left[\dfrac{ns}{n-s}, q\right] & \text{and} \quad \dfrac{1}{r} = (1-a)\left(\dfrac{1}{s} - \dfrac{1}{n}\right) + a\dfrac{1}{q}, \quad \text{if } q \geqslant \dfrac{ns}{n-s}; \end{cases}$$

constant C *is the following*

$$\pi^{-1/2} n^{-1/p} \left(\frac{p-1}{n-p}\right)^{1-1/p} \left\{\frac{\Gamma(1+n/2)\,\Gamma(n)}{\Gamma(n/p)\,\Gamma(1+n-n/p)}\right\}.$$

Moreover, for $s \in [n, \infty]$ *and* $r \geqslant q$ *we have* $(n \geqslant 1)$

$$|w|_r \leqslant M|\nabla w|_s^a |w|_q^{1-a}, \tag{2.4}$$

with $a \in [0, 1)$

$$\frac{1}{r} = a\left(\frac{1}{s} - \frac{1}{n}\right) + (1-a)\frac{1}{q}. \tag{2.5}$$

If $r \geq s$, *then in* (2.4) *there is*

$$M(r, s, q, n) = \alpha^{-\alpha/(\alpha+\beta)} M_1^{\beta/(\alpha+\beta)} M_2^{\alpha/(\alpha+\beta)} + \beta^{-1} \alpha^{\beta/(\alpha+\beta)} M_1^{\alpha/(\alpha+\beta)} M_2^{\beta/(\alpha+\beta)};$$

otherwise for $s \geq r \geq q$, *there is* $M(r, s, q, n) = M(s, s, q, n)^b$ *with* $b = s(r-q)/ /r(s-q)$, *where*

$$\alpha = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right),$$

$$\beta = \frac{1}{2} - \frac{n}{2}\left(\frac{1}{s} - \frac{1}{r}\right),$$

$$M_1 = (4\pi)^{-(n/2)(1/q - 1/r)} \beta_1^{-n/2\beta_1}, \qquad \frac{1}{\beta_1} = 1 + \frac{1}{r} - \frac{1}{q};$$

$$M_2 = (4\pi)^{-(n/2)(1/s - 1/r)} \beta_2^{-1/2 - n/2\beta_2} \frac{\Gamma^{1/\beta_2}(n/2 + \beta_2/2)}{\Gamma^{1/\beta_2}(n/2)}, \qquad \frac{1}{\beta_2} = 1 + \frac{1}{r} - \frac{1}{s}.$$

THEOREM 2.3. – *Let* $\Omega \subset \mathbb{R}^n$ *be,* $n \geq 2$, $\partial\Omega$ *locally lipschitzian. Let* $w(x) \in \widehat{W}^{1,s}(\Omega) \cap \cap L^q(\Omega)$. *Then there exists a constant* C *indipendent of* $w(x)$ *such that*

$$|w|_r \leq C |\nabla w|_s^a \, |w|_q^{1-a}, \tag{2.6}$$

with the following restrictions:

(2.7) *if* $s \in [1, n)$,

$$\text{then } r \in \begin{cases} \left[q, \dfrac{ns}{n-s}\right] & \text{and } \dfrac{1}{r} = a\left(\dfrac{1}{s} - \dfrac{1}{n}\right) + (1-a)\dfrac{1}{q} \text{ if } q \leq \dfrac{ns}{n-s}, \\[2ex] \left[\dfrac{ns}{n-s}, q\right] & \text{and } \dfrac{1}{r} = (1-a)\left(\dfrac{1}{s} - \dfrac{1}{n}\right) + a\dfrac{1}{q} \text{ if } q \geq \dfrac{ns}{n-s}, \end{cases}$$

with $a \in [0, 1]$; *if* $s \in [n, \infty)$, *then* $r \in [q, \infty)$ *and*

$$\frac{1}{r} = a\left(\frac{1}{s} - \frac{1}{n}\right) + (1-a)\frac{1}{q}. \tag{2.8}$$

with $a \in [0, 1)$. *Constant* C *depends on* r, s, q *and* a.

THEOREM 2.4. – *Let $\Omega \subseteq \mathbb{R}^n$ be, $n \geq 2$, $\partial\Omega$ of C^2-class. Let $w(x) \in D_0^{1,s}(\Omega) \cap H_0^{-1,q}(\Omega)$, $s \in [n/(n-1), \infty]$ and $q \in [n/(n-1), \infty)$. Then there exist constants C_0 and C_1 indipendent of $w(x)$ such that*

$$|w|_r \leq C_1 |\nabla w|_s^a |w|_{-1,q}^{1-a} + C_0 |\nabla w|_s^b |w|_{-1,q}^{1-b}, \tag{2.9}$$

with $a \in [1/2, 1)$ and $b \in [0, 1)$, provided that $r \geq \bar{r} = \max\{s, q\}$ and

$$\begin{cases} \dfrac{1}{r} = \dfrac{1}{n} + a\left(\dfrac{1}{s} - \dfrac{2}{n}\right) + (1-a)\dfrac{1}{q}, \\ \dfrac{1}{r} = b\left(\dfrac{1}{s} - \dfrac{2}{n}\right) + (1-b)\dfrac{1}{q}, \end{cases} \qquad r \in \begin{cases} \left[\bar{r}, \dfrac{ns}{n-s}\right) & \text{if } s \in [1, n), \\ [\bar{r}, \infty] & \text{if } s \geq n. \end{cases} \tag{2.10}$$

In particular, if $w(x) = D_j \widehat{w}(x)$ with $\widehat{w}(x) \in L^q(\Omega)$, then inequality (2.9) *becomes*

$$|D_j \widehat{w}|_r \leq C_1 |D^2 \widehat{w}|_s^a |\widehat{w}|_q^{1-a}, \qquad a \in [1/2, 1), \tag{2.11}$$

provided that relation $(2.10)_1$ *holds. If $\Omega = \mathbb{R}^n$, $n \geq 2$, then inequality* (2.9) *and* (2.11) *hold for $s \in [1, \infty]$ and $q \in [1, \infty)$.*

If $w(x) \in D_0^{1,s}(\Omega) \cap D_0^{-1,q}(\Omega)$, there exists a constant C_2 indipendent of $w(x)$ such that

$$|w|_r \leq C_2 |\nabla w|_s^a \| w \|_{-1,q}^{1-a}, \tag{2.12}$$

provided that $(2.10)_1$ *holds.*

Finally, if $w(x) \in L^s(\mathbb{R}^n) \cap H^{-1,q}(\mathbb{R}^n)$, then $w(x) \in H^{-1,r}(\mathbb{R}^n)$ and there exists a constant C_3 indipendent of $w(x)$ such that

$$|w|_{-1,r} \leq C_3 |w|_s^a |w|_{-1,q}^{1-a}, \tag{2.13}$$

provided that for $a \in [0, 1)$ $(2.10)_1$ *holds. If $w(x) \in L^s(\mathbb{R}^n) \cap D_0^{-1,q}(\mathbb{R}^n)$, then $w(x) \in D_0^{-1,r}(\mathbb{R}^n)$ and there exists a constant C_4 indipendent of $w(x)$ such that*

$$\| w \|_{-1,r} \leq C_4 |w|_s^a \| w \|_{-1,q}^{1-a}, \tag{2.14}$$

provided that for $a \in [0, 1)$ $(2.10)_1$ *holds.*

For $r = ns/(n-s)$ inequalities (2.9)-(2.12) *hold with $a = 1$ and $C_0 = 0$. Constants C_i $(i = 0, 1, \ldots, 4)$ in* (2.9), (2.11)-(2.14) *depend on r, s, q, a and in* (2.9), (2.11)-(2.12) *on $\partial\Omega$ also*

THEOREM 2.5. – *Let $\Omega \subseteq \mathbb{R}^n$ be as in Theorem 2.1. Let $\boldsymbol{w}(x) \in \mathring{H}^{P\Delta}_{q,p}(\Omega)$, $q, p \in (1, \infty)$. Then the following inequalities hold:*

$$|\nabla \boldsymbol{w}|_r \leq C |P\Delta \boldsymbol{w}|_p^a |\boldsymbol{w}|_q^{1-a}, \tag{2.15}$$

provided that $p \in (1, n/2)$, $a \in [1/2, 1)$, $r \geq \bar{r} = \max\{p, q\}$, *and*

$$\frac{1}{r} = \frac{1}{n} + a\left(\frac{1}{p} - \frac{2}{n}\right) + (1-a)\frac{1}{q}, \tag{2.16}$$

with C *indipendent of* $\boldsymbol{w}(x)$; *moreover, for* $p \geq n/2$

$$|\nabla \boldsymbol{w}|_r \leq C_5 |P\Delta \boldsymbol{w}|_p^a |\boldsymbol{w}|_q^{1-a} + C_6 |P\Delta \boldsymbol{w}|_p^{ab} |\boldsymbol{w}|_q^{1-ab} + C_7 |P\Delta \boldsymbol{w}|_p^b |\boldsymbol{w}|_q^{1-b}, \tag{2.17}$$

with C_i $i = 5, 6, 7$ *indipendent of* $\boldsymbol{w}(x)$, *provided that* $a \in [1/2, 1)$, $r \in [\bar{r}, \infty]$ ($r \in [\bar{r}, \infty)$ *for* $n = 2$), *relation* (2.16) *holds and*

$$b = \begin{cases} \dfrac{np}{2pq + n(p-q)} & \text{if } n \geq 3 \text{ and } p > \dfrac{n}{2}, \\ \dfrac{np}{2pq + n(p-q)} - \varepsilon, \quad \forall \varepsilon \in \left(0, \dfrac{np}{\bar{r}(2pq + n(p-q))}\right) & \\ \qquad \text{if } n = 2 \text{ and } p > 1, \ n \geq 3 \text{ and } p = \dfrac{n}{2}. \end{cases} \tag{2.18}$$

If r *is less than* n, *then inequality* (2.17) *holds with* $C_7 = 0$.

If p *belongs to* $(1, n/2)$, *then inequality* (2.15) *holds with* $a = 1$.

Finally, inequality (2.15) *fails to hold in the following cases: for* $r > n$ *and* $p \geq q \geq n/2$; *for* $r = np/(n-p)$, $p \in [n/2, n)$ *and* $a = 1$. *Constant* C, C_i $(i = 5, \ldots, 7)$ *depend on* r, p, q *and* a.

THEOREM 2.6. – *Let* $\Omega \subseteq \mathbb{R}^n$ *be as in Theorem 2.1. Let* $\boldsymbol{w}(x) \in \mathring{H}_{q,p}^{P\Delta,1}(\Omega)$ *with* $p \geq n/2$ ($p > 1$ *for* $n = 2$) *and* $q \in (1, \infty)$. *Then there exists a constant* C *indipendent of* $\boldsymbol{w}(x)$ *such that,* $\bar{r} = \max\{q, p\}$,

$$|\nabla \boldsymbol{w}|_r \leq C |P\Delta \boldsymbol{w}|_p^a |\boldsymbol{w}|_q^{1-a}, \qquad r \in \begin{cases} [\bar{r}, \infty], & n \geq 3, \\ [\bar{r}, \infty), & n = 2, \end{cases} \tag{2.19}$$

provided that for $a \in [(1/2), 1)$

$$\frac{1}{r} = \frac{1}{n} + a\left(\frac{1}{p} - \frac{2}{n}\right) + (1-a)\frac{1}{q}, \tag{2.20}$$

where C *depends on* p, q, r *and* a.

Moreover, if $\boldsymbol{w}(x) \in S\mathring{H}_{q,p}^{P\Delta}(\Omega)$, *then inequality* (2.19) *holds with dimensional balance* (2.20) *and with the following restrictions:* $r \geq \bar{r}$ *if* $p \in [n/2, n)$; $r \geq p \geq q$ *if* $p \geq n$.

COROLLARY 2.1. – *Let $\Omega \subseteq \mathbb{R}^n$ be as in Theorem 2.1. Let $w(x) \in \overset{\circ}{H}{}^{PA}_{q,p}(\Omega) \cap \widehat{W}^{1,s}(\Omega)$ for some $p, q \in (1, \infty)$ and $s \in [1, \infty)$. If $s \in [1, n)$, $ns/(n-s) > q$, $r \in [q, ns/(n-s)]$, then*

$$|w|_r \leqslant C |PAw|_p^{a\chi} |\nabla w|_s^{b\chi'} |w|_q^{1-a\chi-b\chi'}, \tag{2.21}$$

for some constant C only dependent on r, p, s, q, a, b, χ and $\partial\Omega$, provided that for $a, b \in [0, 1]$,

$$\begin{cases} \chi, \chi' \in [0, 1] \ \text{and} \ \chi + \chi' = 1, \\ \dfrac{1}{r} = a\left(\dfrac{1}{p} - \dfrac{2}{n}\right) + (1-a)\dfrac{1}{q} = b\left(\dfrac{1}{s} - \dfrac{1}{n}\right) + (1-b)\dfrac{1}{q}. \end{cases} \tag{2.22}$$

If $s \in [1, n)$, $ns/(n-s) > q$, $r > ns/(n-s)$, then for any $\chi \in [0, 1]$

$$|w|_r \leqslant C |PAw|_p^{a} |\nabla w|_s^{\chi(1-a)} |w|_q^{(1-\chi)(1-a)}, \tag{2.23}$$

for some constant C only dependent on r, p, s, q, a, χ and $\partial\Omega$, provided that for $a \in [0, 1]$,

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{2}{n}\right) + (1-a)\left[\chi\left(\frac{1}{s} - \frac{1}{n}\right) + (1-\chi)\frac{1}{q}\right]. \tag{2.24}$$

If $s \in [1, n)$, $q > ns/(n-s)$, $r \in [ns/(n-s), q]$, then inequality (2.6) *holds again (in other words* (2.21) *with $a = 0$). Finally, if $s > n$ and $r \geqslant q$, then we have again inequality* (2.21) *with restrictions* (2.23). *We stress that $r = \infty$ may be for $n \geqslant 3$, while $r < \infty$ for $n = 2$, in accord with Theorem* 2.1.

Some words of comments about the theorems.

REMARK 2.1. – The theory of the interpolation spaces has application in several fields of the analisys. Apart from the pioner and fundamental results by Riesz and Thorin, or of their generalizations, with papers [10-11, 23-25, 37-38] by Gagliardo, Lions and Nirenberg the theory had a meaning devolopment and application in the field of the partial differential equations, as well different methods of interpolation were introduced (real and complex interpolation methods by Peetre and Calderon respectively). Here we cannot be exhaustive for the whole research field, thus we refer the reader to [2] for a general theory of the interpolation and to the books by Lions-Magenes [26], where the interpolation theory (of functional spaces and linear operators) is used as systematic tool to solve questions related to some partial differential equations. Nevertheless, we refer to [27] who wants a clear and complete (at least up to 1963) matter on the theory of the interpolation and about its application in partial differential equations.

REMARK 2.2. – We are essentially interested to interpolation inequalities which are not connected with ones of *Sobolev exponents:* $s \in [1, n)$ and $r = ns/(n-s)$. This is made since for the above exponents the theory becomes a repetition; in Theorem 2.2 they are considered uniquely for the sake of completeness in the statement.

The study devoloped in this paper is based on the properties of the resolving operator associated to (unsteady) Stokes system and for Theorem 2.2 to the heat equation. In the papers [6, 9, 46, 48] (2) properties of symmetric Markov semigroups are connected with inequalities of Hardy-Littlewood-Sobolev type. More precisely, in [6] symmetric Markov semigroups on $L^2(\Omega)$ are considered and an equivalence is given between $L^\infty - L^2$ estimates of semigroups and the following inequality:

$$|f|_2^{2+4/\mu} \leqslant CQ(f)\,|f|_1^{4/\mu},$$

for some constant C, $\forall f \in D(Q) \cap L^1$, where $Q(f) = \int a_{ij}(x)(\partial/\partial x_i)f(\partial/\partial x_j)f dx$, μ is a suitable parameter connected to the $L^\infty - L^2$ semigroup properties (concerning the above estimate also see the papers [7, 36]). In [46] $L^\infty - L^p$ estimates of semigroups are considered and an equivalence is given with the Hardy-Littlewood-Sobolev inequalities. In a class of operator containing the negative of Laplace operator, the necessary condition of the above result is due to [48]. Although we employ properties of the resolving operator associated to the Stokes problem, stated recently in [31], the approach to the result is completely different from [6, 46, 48]. As well as, the order of interpolation inequalities (we consider second derivatives, Sobolev spaces of negative order) and exponents of summability of the L^p-Lebesgue space interpolated (we can consider $p = 1$) are different from ones deduced in [6, 46, 48].

To obtain our Sobolev inequalities we employ an argument by duality. We consider the solutions of a suitable initial boundary value problem (heat equation and Stokes problem) as *test functions* for an integral variational formulation of Laplace equation and (steady) Stokes system. Thus the solutions are test functions with a parameter, that is the time variable t. After which, making use of the *evolution equation* we attain a sort of Green formula. The arbitrarity of t consent to us to obtain the inequality in a suitable form. Of course the choosen of the evolution equation is close connected with the interpolation inequality consedered.

In this approach there is a sort of equivalence between *dimensional balance* of the inequalities (see for example (2.2), (2.5)) and properties of resolving operator of the solutions to equations (more precisely we refer to properties of Theorem 3.1-3.2). In [31] results of optimality have been obtained for solutions to Stokes problem in exterior domains. Among these it is proved that the exponents μ and μ' in (3.7) are sharp for $p \geqslant n/2$. These results are not connected with Stokes operator, since also for the heat equation it is possible to obtain the same results, but to the fact that the domain Ω is exterior. For a Cauchy problem (3.7) is substuted by (3.11). Now in the light of our technique the optmality of (3.7) and (3.11) can be seen as a consequence of the dimensional belance stated for the inequalities of interpolation. Indeed it is not difficult to prove that if we modify (3.7) or (3.11), then *a posteriori* we can violate the dimensional be-

(2) When this paper was completed Professor Y. GIGA informed the author of the existence of papers [6, 9, 46, 48]. The author wishes to express his thanks to Professor Y. GIGA for drawing his attention to the quoted papers.

lance of the interpolation inequalities, which gives an *absurdum*. This last assertion prove the optmality. Conversely assume that it is possible to modify the dimensional belance of the interpolation inequalities, then it is possible to violate the optimality of (3.7), which is an *absurdum* by virtue of the results of [31].

REMARK 2.3. – Theorem 2.1 is our chief result. In this theorem the assumption $\partial\Omega \in C^m$ with $2m > n$ seems too much requirement (in effect, as it will be clear from the proof, for $p \in (1, n/2)$ it is sufficient to require $\partial\Omega \in C^2$). Moreover, for $n = 2$ it is required $r < \infty$. Actually both the assumptions are consequence of the results obtained in [31] (see Theorem 3.1 of the present paper), which are employed for the proof of the theorem. However both the assumptions can be removed if we consider the Laplace operator instead of Stokes one. In fact the equivalent of Theorem 3.1 for heat equation can be proved with $\partial\Omega \in C^2$ and $r = \infty$ for $n = 2$ (see sect. 2 Theorem 3.2). It is not so immediate, but more in general, following for example the arguments employed in [31] and for some estimates in [28], it is possible to extend the results of Theorem 3.1 and Theorem 3.2 to parabolic operator: $\mathcal{L}(u) = \nabla\cdot(\boldsymbol{A}(x)\cdot\nabla u(x, t)) - (\partial/\partial t)\, u(x, t) = 0$. As a consequence we have (2.1) with $\nabla\cdot(\boldsymbol{A}(x)\cdot\nabla w(x))$ instead of $P\Delta \boldsymbol{w}(x)$. Moreover for such elliptic operator it is possible consider (2.1) also $p = 1$. Taking into account the result of [39], our technique assume a particular interest for elliptic operator, indipendently of the case of exterior domain Ω.

For the validity of (2.1) in $H^{P\Delta}_{q,p}(\Omega)$ the requirement $\boldsymbol{w}(x)_{|\partial\Omega} = 0$ is a necessary and sufficient condition. In fact to prove that the condition is necessary, it is sufficient to consider the set of function $\mathcal{H} = \{\boldsymbol{w}(x)$ such that $P\Delta\boldsymbol{w}(x) = 0$, $\boldsymbol{w}(x)_{|\partial\Omega} = \boldsymbol{a}(x) \neq 0$, $\boldsymbol{a}(x)\cdot\vec{n}_{|\partial\Omega} = 0$ and $\boldsymbol{w}(x) \to 0$ for $|x| \to \infty\}$ (harmonic function in the case of Laplace operator). It is well known that $\mathcal{H}$ is a non void set for $n \geq 3$, with $\mathcal{H} \subset H^{P\Delta}_{q,p}(\Omega)$, $q > n/(n-2)$. Therefore if inequality (2.1) holds in $H^{P\Delta}_{q,p}(\Omega)$, it implies that $\mathcal{H} = \emptyset$, which is an *absurdum*. Neverthless it is of some interest to stress that for $\boldsymbol{w}(x) \in H^{P\Delta}_{q,p}(\Omega)$, $q > n/n-2$, we can modify (2.1) as it follows. Consider the function $\boldsymbol{W}(x) \in \mathcal{H}$ with $\boldsymbol{W}(x)_{|\partial\Omega} = \boldsymbol{w}(x)$. Then, setting $\boldsymbol{U}(x) = \boldsymbol{w}(x) - \boldsymbol{W}(x)$ we have $\boldsymbol{U}(x) \in \overset{\circ}{H}{}^{P\Delta}_{q,p}(\Omega)$, inequality (2.1) implies

$$|\boldsymbol{U}|_r \leq C|P\Delta\boldsymbol{w}|_p^a\,|\boldsymbol{U}|_q^{1-a},$$

or in particular

$$|\boldsymbol{w}|_r \leq C|P\Delta\boldsymbol{w}|_p^a\,|\boldsymbol{U}|_q^{1-a} + |\boldsymbol{W}|_r.$$

REMARK 2.4. – Here we like to recall that, contemporaneously with the quoted papers in the introduction, as far as interpolation inequalties of the first order is concerned, in [21] Ladyzhenskaya proved the following fundamental inequality for the theory of two dimensional Navier-Stokes system: $\forall w(x) \in \overset{\circ}{W}{}^{1,2}(\Omega)$ $\Omega \subseteq \mathbb{R}^2$, $|w|_4 \leq 2^{1/4}\,|\nabla w|_2^{1/2}\,|w|_2^{1/2}$. Theorem 2.2 and Theorem 2.3 are connected to such family of inequalities, they state inequalities which have the same *dimensional balance*. However the theorems are completely different in the aims. Theorem 2.2 tries to give a value of constants sufficiently precise. This is trivial consequence of the results of [1, 45] on the best constant for exponent $s \in [1, n)$ connected to the well known Sobolev inequality. While for exponents of summability which are not connected to Sobolev inequality (see

(2.4) and (2.5)) the results are obtained making use of properties concerning the solutions of the heat equation. The employed technique makes very simple the computation of the constants.

In Remark 2.3 we have already observed that if we perform a suitable study for the parabolic equation $\mathcal{L}(\cdot) = 0$ (see Remark 2.3), then operator $P\Delta \cdot$ in (2.1) can be substuted by $\nabla\cdot(A(x)\cdot\nabla\cdot)$. Also in Theorem 2.2 we have, by the same considerations, the possibility to substitute in inequality (2.4) the «∇» with «$A(x)\cdot\nabla$» (for any $s \geq n$, $q \geq 1$ and r given in (2.5)). In this way we discover a result near to ones of [6] proved for $s = 2$, $q = 1$ and $r = 2$ (the case of $L^\infty - L^2$ estimates of semigroups already quoted in Remark 2.2.).

In Theorem 2.3 a quite different purpose is considered. In [14] (we do not known similar results in literature former ones proved in [14]) for the first time was taken in consideration the possibility to prove a Sobolev inequality of type (2.3) with $r = ns/(n - s)$, $s \in [1, n)$, without requiring that the function belongs to the completion of $C_0^\infty(\Omega)$ in norm $|\nabla\cdot|_s$, $s \in [1, n)$. The idea is to consider only the *condition* at infinity of the function, in other words we employ the fact that $w(x) \to 0$ for $|x| \to \infty$ in a generalized sense. When this last condition is not satysfied, then it is proved in [14] that for any function $w(x)$ such that $|\nabla w|_s < \infty$, $s \in [1, n)$, there exists a constant w_0 such that $|w(x) - w_0| \to 0$ for $|x| \to \infty$ in a suitable sense. An exaustive devolopment of these ideas it is given in [13], moreover it is given the value of w_0 (for analogous results see also papers [20] and [3]). A natural extension of the above results is to prove inequalities interpolating function $w(x) \in \widehat{W}^{1,s}(\Omega) \cap L^q(\Omega)$ for $s \geq n$, without requaring $w(x)_{|\partial\Omega} = 0$. So that the results of Theorem 2.3 cannot be seen as a particular case of Theorem 2.2. We explicity point out that unfortunately our technique only proves Theorem 2.3 for $r < \infty$, while Theorem 2.2 holds for $r \leq \infty$. However we note that from (3.4) of Lemma 3.3 section 3 and (2.6), we can deduce the following inequality:

$$|w(x)|_\infty \leq C|\nabla w|_s^a |w|_q^{1-a} + C(\varepsilon)|\nabla w|_s^{a-\varepsilon} |w|_q^{1+\varepsilon-a}, \qquad \forall\varepsilon \in (0, a].$$

REMARK 2.5. – In Theorem 2.4 is proved an interpolation inequality for function $w(x) \in W_0^{1,s}(\Omega) \cap H^{-1,q}(\Omega)$. The meaning of the inequalities is immediate. We stress that if Ω is bounded, by virtue of Poincaré inequality ($w(x)_{|\partial\Omega} = 0$), constant C_0 in (2.9) is equal zero. It is interesting to note that the constant C_0 is equal zero also in the case of $w(x) \in D_0^{1,s}(\Omega) \cap D^{-1,q}(\Omega)$ as stated in the theorem.

REMARK 2.6. – It is well known that for any $\boldsymbol{w}(x) \in \overset{\circ}{H}{}^{P\Delta}_{2,2}(\Omega)$ the following inequality holds: $|\nabla \boldsymbol{w}|_2 \leq |P\Delta \boldsymbol{w}|_2^{1/2} |\boldsymbol{w}|_2^{1/2}$. This inequality is true for any $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, whose boundary $\partial\Omega$ is locally lipschitzian. The aim of Theorem 2.5 is to generalize the above inequality compatibly to any exponents of summability. The theorem proves that the inequality can not be generalized to the cases of $r > n$ and $r = np/(n - p)$, $p \in [n/2, n)$. For $r \in (1, n]$ the result is not complete. Indeed inequality (2.15) holds for $p \in (1, n/2)$. For $p \geq n/2$, we have estimate (2.17). We gess that estimate (2.17) can be improved with constant $C_6 = 0$ in the case of $r > n$ and, of course, with constants $C_6 = C_7 = 0$ in the case of $r \in [n/2, n]$.

Inequality (2.19) of Theorem 2.6 has the same *dimensional balance* of ones proved in Theorem 2.5. In Theorem 2.6 we consider the cases $p \geq n/2$ ($n \geq 3$), since the cases

$p \in (1, n/2)$ are discussed in Theorem 2.5 in the weakner hypothesis $w(x) \in \overset{\circ}{H}{}^{P\Delta}_{q,p}(\Omega)$. In fact, the aim of the theorem is just to give no restriction on the exponents of summability obtaining (2.19). However, it is achieved by requaring that $w(x) \in \overset{\circ}{H}{}^{P\Delta,1}_{q,p}(\Omega)$ or $w(x) \in S\overset{\circ}{H}{}^{P\Delta}_{q,p}(\Omega)$.

REMARK 2.7. – Corollary 2.1 gives results by suitable coupling of the above interpolation inequalities. Apart of regularity of $\partial\Omega$ (in the case of Laplace opearator $\partial\Omega$ can be choosen of C^2-*class*) and the value of the constant C as particular case Corollary has the result of [47]. In fact in (2.23) for $n = 3$, $p = s = 2$, we can choose $\chi = 1$.

3. – Some preliminary lemmas.

Let us consider steady Stokes system in Ω:

$$\Delta u(x) + \nabla\pi(x) = f(x), \qquad \nabla\cdot u(x) = 0 \quad \text{in } \Omega .$$

For Stokes problem we mean the Stokes system with Dirichlet boundary condition:

$$u(x)_{|\partial\Omega} = 0 , \qquad u(x) \to 0 \quad \text{for } |x| \to \infty .$$

The following lemma holds

LEMMA 3.1. – *Let* $\Omega \subset \mathbb{R}^n$ *be,* $n \geq 2$, $\partial\Omega$ *of* C^2-*class. Let* $f(x) \in J^p(\Omega)$, $p \in (1, n/2)$ $(n \geq 3)$. *Then, there exists a unique solution* $(u(x), \pi(x)) \in W^{1,p}_{\text{loc}}(\Omega)$ *to steady Stokes problem such that*

$$|D^2 u|_p + |\nabla\pi|_p \leq C|f|_p . \tag{3.1}$$

Let $u(x) \in \overset{\circ}{H}{}^{P\Delta}_{q,p}(\Omega)$, $q, p \in (1, \infty)$, $n \geq 2$. *Then, for some* $\pi(x)$, $(u(x), \pi(x)) \in W^{1,p}_{\text{loc}}(\Omega) \cap J^q(\Omega) \times L^p_{\text{loc}}(\Omega)$ *is a solution to steady Stokes system with* $f(x) = P\Delta u(x)$; *moreover there exists a constant C independent of* $u(x)$ *such that*

$$|D^2 u|_p + |\nabla\pi|_p \leq C(|f|_p + |u|_{L^{p_0}(\Omega^*)}) , \qquad p_0 \geq 1 , \tag{3.2}$$

where $\Omega^* \subset \Omega$ *is an arbitrary bounded domain such that* $\partial\Omega \cap \partial(\Omega - \Omega^*) = \emptyset$.

Inequality (3.1) *is sharp, in the sense that it fails to hold for* $p \geq n/2$.

Finally, if $u(x) \in S\overset{\circ}{H}{}^{P\Delta}_{q,p}(\Omega)$, $q, p \in (1, \infty)$, $n \geq 2$. *Then* $(u(x), \pi(x)) \in W^{1,p}_{\text{loc}}(\Omega) \cap J^q(\Omega) \times L^p_{\text{loc}}(\Omega)$ *is a solution to Stokes system in* Ω *with* $f(x) = P\Delta u(x)$ *and with* $u(x)_{|\partial\Omega} = 0$ *and boundary integral condition* $B_n = 0$ *for* $p \in [n/2, n)$ $(p > 1, n = 2)$ *and* $B_n = B^1_n = 0$ *for* $p \geq n$. *Also, there exists a constant C such that* $(u(x), \pi(x))$ *satisfies inequality* (3.1).

PROOF. – Inequality (3.1) is proved in [30] and see [44] for the case of $n = 3$. As far as inequality (3.2) is concerned, the proof is possible to deduce from the results of [30]. The optimality of inequality (3.1) is proved in [30]. However inequalities (3.1)-(3.2) have been object of study of several authors [3, 15, 16, 20]; in [13] these results are quoted and, also for a more general Stokes problem, are devoloped and discussed the same questions. Finally the case of $\boldsymbol{u}(x) \in S\overset{\circ}{H}{}^{P\Delta}_{q,p}(\Omega)$ is a particular case of the results for Stokes system proved in [30].

LEMMA 3.2. – *Let $\Omega \subseteq \mathbb{R}^n$ be, $n \geq 2$, $\partial\Omega$ locally lipschitzian. Let $\nabla\psi(x) \in L^p(\Omega)$, $p \in [1, n)$. Then, there exist constants ψ_0 and C such that*

$$\begin{cases} \lim\limits_{r \to \infty} \int\limits_{\omega_n} |\psi(r, \sigma) - \psi_0|^p d\sigma = 0 , \\ |\psi - \psi_0|_q \leq C |\nabla\psi|_\pi , \qquad 1/q = 1/p - 1/n , \end{cases} \tag{3.3}$$

where ω_n is the sphere of radius 1.

PROOF. – The above lemma is proved by Galdi in [13], Chapt. II, Theorem 5.1. See [14] for the particular case of $\Omega \subset \mathbb{R}^3$ and $\partial\Omega$ of C^2-class.

LEMMA 3.3. – *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the cone property. Let $\psi(x) \in W^{m,p}(\Omega) \cap L^{p_0}(\Omega)$, $p \geq 1$ and $p_0 \in [1, \infty]$. If $m - n/p \notin \mathbb{N} \cup \{0\}$, then there exist a constant $C(\Omega, n, m, p)$ indipendent of $\psi(x)$ such that*

$$|D^j \psi|_r \leq C(|D^m \psi|_p^a |\psi|_{p_0}^{1-a} + |\psi|_{p_0}), \tag{3.4}$$

provided that $1/r = 1/n + a(1/p - m/n) + (1-a)(1/q)$ and $a \in [j/m, 1]$. If $m - n/p \in \mathbb{N} \cup \{0\}$, then (3.4) *holds with $a \in [j/m, 1)$.*

PROOF. – The lemma is due to Gagliardo [11] and Nirenberg [37]. See also [35] Theorem 58.X.

Let X be a Banach space. By $L^p((0, T); X)$ we denote the Banach space of function $\varphi(\tau)$ from $(0, T)$ in X normed by $\left(\int\limits_0^T |\varphi(\tau)|_X^p d\tau \right)^{1/p}$.

Consider the nonstationary Stokes system:

$$\begin{cases} \varphi_t(x, t) - \Delta\varphi(x, t) = \nabla p(x, t), \qquad \nabla \cdot \varphi(x, t) = 0 , \quad \text{on } \Omega \times (0, T), \\ \varphi(x, t)_{|\partial\Omega} = 0 , \qquad \varphi(x, t) \to 0 \quad \text{for } |x| \to \infty , \qquad \forall t \in (0, T), \\ \varphi(x, 0) = \varphi_0(x) . \end{cases} \tag{3.5}$$

For solutions to system (3.5) the following theorem holds:

THEOREM 3.1. – *Let $\Omega \subseteq \mathbb{R}^n$ be, $n \geq 2$, $\partial\Omega$ of C^m class with $2m > n$. For any $\varphi_0(x)$ belonging to $\mathcal{C}_0(\Omega)$, there exists a unique solution $(\varphi(x, t), p(x, t))$ corresponding to*

$\varphi_0(x)$ such that

$$(3.6)\qquad \begin{cases} \varphi(x,t) \in \bigcap_{p>1} L^p((0,T); J^{1,p}(\Omega)\cap W^{2,p}(\Omega)), \\ \nabla p(x,t),\ \varphi_t(x,t) \in \bigcap_{p>1} L^p((0,T); L^p(\Omega)). \end{cases}$$

Moreover, there exists a constant M such that

$$(3.7)\qquad \begin{cases} |\varphi(t)|_q \leq M|\varphi_0|_p t^{-\mu}, \quad \mu = \dfrac{n}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right), \\ \qquad\qquad q \in \begin{cases} (1,\infty], & \text{if } p=1,\ n\geq 3; \\ [p,\infty], & \text{if } p>1,\ n\geq 3; \\ [p,\infty), & \text{if } p>1,\ n=2; \end{cases} \\ |\nabla\varphi(t)|_q \leq M|\varphi_0|_p t^{-\mu'}, \quad \mu' = \dfrac{1}{2} + \mu, \\ \qquad\qquad q \in \begin{cases} (1,n], & \text{if } p=1,\ n\geq 3; \\ [p,n], & \text{if } p>1,\ n\geq 2; \\ [p,\infty), & \text{if } p>1,\ n\geq 2,\ t\in(0,1]; \end{cases} \\ |\nabla\varphi(t)|_q \leq M|\varphi_0|_p t^{-\mu''}, \quad \mu'' = \dfrac{n}{2p}, \quad q\geq n,\ t\geq 1; \\ |\varphi_t(t)|_q \leq M|\varphi_0|_p t^{-\mu'''}, \quad \mu''' = 1 + \dfrac{n}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right), \quad n\geq 3, \\ \qquad\qquad q \in \begin{cases} (1,\infty], & \text{if } p=1, \\ [p,\infty], & \text{if } p>1. \end{cases} \end{cases}$$

In estimate (3.7) *M is a constant independent of $\varphi_0(x)$ and if $q>p$ it is also indipendent of $p\geq 1$. For $p\geq n/2$ exponent μ'' is sharp, in the sense that it is not possible to improve it in $\mu''+\varepsilon$, $\forall\varepsilon>0$, with M indipendent of t and $\varphi_0(x)$.*

PROOF. – Theorem 3.1 is part of results proved in [31] Theorems 1.1-1.2, Lemma 3.1.

REMARK 3.1. – In the present paper, in some proofs is crucial the property of constant M stated in Theorem 3.1. That is: constant M is indipendent of p for $q>p\geq 1$. Since this result is enclosed in paper [31], for the sake of completeness, we like here to repeat the proof of the property as it follows.

Assume that $(3.7)_1$ *holds for some constant* C, *with* C *a priori depending on* p, q, *then we can establish the existence of a constant* M *such that* $(3.7)_1$ *holds and* M *is indipendent of* p *for* $q > p \geq 1$.

To this end, consider two solutions of system (3.5), say $(\boldsymbol{v}(x, t), \pi(x, t))$ with $\boldsymbol{v}_0(x) \in L^p(\Omega), p \geq 1$, and $(\boldsymbol{h}(x, s), \varpi(x, s))$ with $\boldsymbol{h}(x, 0) = \boldsymbol{h}_0(x) \in \mathcal{C}_0(\Omega)$. For a fixed $t > 0$, we set $\widehat{\boldsymbol{h}}(x, \tau) = \boldsymbol{h}(x, t - \tau)$, $\forall \tau \in [0, t]$. Multiplying $(3.5)_1$ by $\widehat{\boldsymbol{h}}(x, \tau)$, taking into account (3.6), an integration by parts gives:

$$(\boldsymbol{v}(t), \boldsymbol{h}_0(x)) = (\boldsymbol{v}_0, \boldsymbol{h}(t)) .$$

Applying the Hölder inequality to the right hand side of the last relation, subsequently the L^p-convexity theorem, we arrive at

$$|(\boldsymbol{v}(t), \boldsymbol{h}_0)| \leq |\boldsymbol{v}_0|_p |\boldsymbol{h}(t)|_{p'} \leq |\boldsymbol{v}_0|_p |\boldsymbol{h}(t)|_\infty^\theta |\boldsymbol{h}(t)|_{q'}^{1-\theta},$$

$$\text{with } q' = \frac{q}{q-1}, \; \theta = \frac{1}{p}\frac{q-p}{q-1}, \quad \forall q > p,$$

Employing $(3.7)_1$ for the solution $\boldsymbol{h}(x, t)$ with exponents $q = p = q'$ and $q = \infty$, $p = q'$, we obtain

$$|(\boldsymbol{v}(t), \boldsymbol{h}_0)| \leq (C(q', \infty))^\theta (C(q', q'))^{1-\theta} |\boldsymbol{v}_0|_p |\boldsymbol{h}_0|_{q'} t^{-(n/2)(1/p - 1/q)}, \quad \forall \boldsymbol{h}_0(x) \in \mathcal{C}_0(\Omega),$$

which implies

$$|\boldsymbol{v}(t)|_q \leq (C(q', \infty))^\theta (C(q', q'))^{1-\theta} |\boldsymbol{v}_0|_p t^{-(n/2)(1/p - 1/q)}, \quad \text{for } q > p \geq 1, \quad t > 0 .$$

Therefore, setting $M = \max_{\theta \in [0, 1]} (C(q', \infty))^\theta (C(q', q'))^{1-\theta}$, M is indipendent of p. We conclude observing that, from properties of resolving operator associated to system (3.5), it is easy to prove the result also for M in $(3.7)_2$-$(3.7)_4$.

Consider in Ω the initial value problem for heat equation:

$$(3.8) \qquad \begin{cases} \Delta\varphi(x, t) - \varphi_t(x, t) = 0, & \text{on } \Omega \times (0, T), \\ \varphi(x, t) \to 0 \quad \text{for } |x| \to \infty, & \varphi(x, t)_{|\partial\Omega} = 0, \quad \varphi(x, 0) = \varphi_0(x). \end{cases}$$

The below theorem holds for solutions to system (3.8). It is more complete then Theorem 3.1.

THEOREM. – 3.2. – *Let* $\Omega \subseteq \mathbb{R}^n$ *be,* $n \geq 2$, $\partial\Omega$ *of* C^2*-class. Let* $\varphi(x) \in C_0^\infty(\Omega)$. *Then corresponding to* $\varphi(x)$ *there exists a unique solution* $\varphi(x, t)$ *such that*

$$\varphi(x, t) \in \bigcap_{p > 1} L^p((0, T); J^{1,p}(\Omega) \cap W^{2,p}(\Omega)), \qquad \varphi_t(x, t) \in \bigcap_{p > 1} L^p((0, T); L^p(\Omega)) .$$

Moreover, there exists a constant C such that

$$(3.9)\quad \begin{cases} |\varphi(t)|_{t^q} \leq C|\varphi_0|_p t^{-\mu}, \quad \mu = \dfrac{n}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right), \quad q \in [p, \infty], \ p \geq 1; \\ |\nabla\varphi(t)|_q \leq C|\varphi_0|_p t^{-\mu'} \mu' = \dfrac{1}{2} + \mu, \\ \qquad q \in \begin{cases} (1, n], & \text{if } p = 1, \\ [p, n], & \text{if } p > 1, \\ [p, \infty), & \text{if } p > 1, \ n \geq 2, \ t \in (0, 1]; \end{cases} \\ |\varphi(t)|_q \leq C|\varphi_0|_p t^{-\mu''}, \quad \mu'' = \dfrac{n}{2p}, \quad q \geq n, \ t \geq 1; \\ |\varphi_t(t)|_q \leq M|\varphi_0|_p t^{-\mu'''}, \quad \mu''' = 1 + \dfrac{n}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right), \quad n \geq 3, \\ \qquad q \in \begin{cases} (1, \infty], & \text{if } p = 1, \\ [p, \infty], & \text{if } p > 1. \end{cases} \end{cases}$$

In inequality (3.9) *C is a constant indipendent of* $p \geq 1$ *and* $\varphi_0(x)$. *Estimate* $(3.9)_3$ *is sharp for* $p \geq n/2$.

PROOF. – The proof of this theorem can be performed following the ideas of [31] for Stokes problem. However, here we want to point out some further references. As far as the existence is concerned we refer [22] and for some aspects on the asymptotic decay we recall [28]. In these theorems it is sufficient to require $\partial\Omega$ of C^2 class since we follow [28] to obtain $(3.9)_1$. In [28] a different technique with respect the one of [31] is employed. The difference of technique is essentialy due to the presence of the pressure term. Since we are considering the heat equation, apart of $(3.9)_4$, we are able to prove the theorem for $n \geq 2$.

Let us consider for system (3.8) the Cauchy problem on $\mathbb{R}^n$, $n \geq 1$. The following lemma holds:

LEMMA 3.4. – *Let* $\varphi_0(x) \in L^p(\mathbb{R})^n$, $p \geq 1$. *Then for the initial value problem* (3.8) *there exists a unique smooth solution* $\varphi(x, t)$ *such that* $\forall q \geq p \geq 1$

$$(3.10)\quad \begin{cases} |\varphi(t)|_q \leq M_1 |\varphi_0|_p t^{-\mu}, \quad \forall t > 0, \quad \mu = \dfrac{n}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right); \\ |\nabla\varphi(t)|_q \leq M_2 |\varphi_0|_p t^{-\mu'}, \quad \forall t > 0, \quad \mu' = \dfrac{1}{2} + \mu; \end{cases}$$

where

$$(3.11)\qquad \begin{cases} M_1 = (4\pi)^{-\mu}\beta^{-n/2\beta}; \\ M_2 = (4\pi)^{-\mu}\beta^{-1/2-n/2\beta}\,\dfrac{\Gamma^{1/\beta}(n/2+\beta/2)}{\Gamma^{1/\beta}(n/2)}, \qquad \dfrac{1}{\beta} = 1 + \dfrac{1}{q} - \dfrac{1}{p}. \end{cases}$$

PROOF. – The proof of (3.10) with some constants M_1, M_2 is well konwn. On the other hand we have the analogous of Theorem 3.2 already quoted for the initial boundary value problem (3.8). However here we want recall the numerical value of the constant M_1, M_2, not only their existence. The key tool to prove (3.10)-(3.11) is the Young theorem on convolution product. From the rapresentation of the solution by mean heat kernel,

$$\varphi(x, t) = (4\pi t)^{-n/2}\int_{\mathbb{R}^n} e^{-|x-y|^2/4t}\varphi_0(y)\,dy\,,$$

we have in virtue of the Young theorem (we set $\mathcal{H}(z, t) = (4\pi t)^{-n/2}e^{-|z|^2/4t}$)

$$|\varphi(t)|_q \leq |\mathcal{H}(t)|_{\beta(p,q)}|\varphi_0|_p\,,$$

$$|\nabla\varphi(t)|_a \leq |\nabla\mathcal{H}(t)|_{\beta(p,q)}|\varphi_0|_p\,,$$

with $1/\beta(p, q) = 1 + 1/q - 1/p$. Taking into account the properties of Gamma function, a simple computation gives

$$|\mathcal{H}(t)|_{\beta(p,q)} = (4\pi t)^{n/2\beta - n/2}\beta^{-n/2\beta} = (4\pi)^{-\mu}\beta^{-n/2\beta}t^{-\mu} = M_1 t^{-\mu}\,,$$

$$|\nabla\mathcal{H}(t)|_{\beta(p,q)} = (4\pi t)^{n/2\beta - n/2}t^{-1/2}\beta^{-1/2+n/2\beta}\,\frac{\Gamma^{1/\beta}(\beta/2)}{\Gamma^{1/\beta}(n/2)} = M_2 t^{-1/2-\mu}\,.$$

The lemma is proved completely.

LEMMA 3.5. – *Let $\Omega \subseteq \mathbb{R}^n$ be, $n \geq 2$, as in Theorem 2.1. Let $w(x) \in \mathring{H}^{PA}_{q,p}(\Omega)$ and assume $PAw(x) = 0$ a.e. in Ω. Then $w(x)$ is equal zero a.e. in Ω.*

PROOF. – In our hypotheses we have that $w(x)$ is a solution to system (3.1) with $f(x) = 0$. We consider in system (3.5) $\varphi_0(x) \in \mathcal{C}_0(\Omega)$ and multiply $(3.5)_1$ by $w(x)$. In virtue of the summability properties of $\varphi(x, t)$ given in Theorem 3.1, integrating by parts on $\Omega \times (0, t)$, we have

$$(3.12)\qquad (w, \varphi_0) = (w, \varphi(t))\,, \qquad \forall t > 0\,.$$

Applying Hölder inequality to (3.12), from $(3.7)_1$ we deduce

$$|(w, \varphi_0)| \leq |w|_q|\varphi(t)|_{q'} \leq C|w|_q|\varphi_0|_{1+\varepsilon}t^{-(n/2)(1/q+\varepsilon/(1+\varepsilon))}, \quad \forall t > 0\,, \quad \varepsilon \in \left(0, \frac{1}{q-1}\right].$$

Making $t \to \infty$, we obtain $(\boldsymbol{w}, \varphi_0) = 0$. For the arbitrarity of $\varphi_0(x) \in \mathcal{C}_0(\Omega)$ we conclude that $\boldsymbol{w}(x) = 0$ *a.e.* in Ω.

LEMMA 3.6. – *Let $\Omega \subseteq \mathbb{R}^n$ be, $n \geq 2$, $\partial\Omega$ C^2-class. Let $\boldsymbol{u}(x) \in L^q_{\mathrm{loc}}(\Omega)$, $q > 1$, with $\boldsymbol{u}(x) \cdot \vec{n}_{|\partial\Omega} = 0$ and*

$$(3.13) \qquad |(\boldsymbol{u}, \varphi)| \leq M |\varphi|_{q'}, \qquad \forall \varphi(x) \in \mathcal{C}_0(\Omega).$$

Then there exists a function $\pi(x)$ such that $\overline{\boldsymbol{u}}(x) = \boldsymbol{u}(x) - \nabla\pi(x) \in J^q(\Omega)$. Moreover, if $\boldsymbol{u}(x) \in J^p(\Omega)$, for some $p > 1$, then $\boldsymbol{u}(x) \in J^q(\Omega)$.

PROOF. – For any $R > \operatorname{diam}(\Omega^c)$ we can consider on $\Omega_R = \Omega \cap S_R$ ($S_R = \{x \in \mathbb{R}^n,\ |x| \leq R\}$) $L^q(\Omega_R) = J^q(\Omega_R) \oplus G^q(\Omega_R)$. We set $\boldsymbol{u}_R(x) = \boldsymbol{u}(x) - \nabla\pi_R(x)$. From (3.13) we deduce that $|\boldsymbol{u}_R|_q \leq M$, $\forall R > \operatorname{diam}(\Omega^c)$. We define $\overline{\boldsymbol{u}}_R(x) = \boldsymbol{u}_R(x)$ if $x \in \Omega_R$, otherwise 0. Since $\{\overline{\boldsymbol{u}}_R(x)\}_R$ is uniformely bounded from M in $J^q(\Omega)$, we can select a sub-sequence, again labelled by $\overline{R}$, such that $\overline{\boldsymbol{u}}_R(x)$ weakly converges to some $\overline{\boldsymbol{u}}(x)$ in $J^q(\Omega)$ and $|\overline{\boldsymbol{u}}|_q \leq M$. Now we have

$$(\boldsymbol{u} - \overline{\boldsymbol{u}}, \varphi) = \lim_R (\boldsymbol{u} - \boldsymbol{u}_R, \varphi) = \lim_R (\nabla\pi_R, \varphi) = 0, \qquad \forall\varphi(x) \in \mathcal{C}_0(\Omega).$$

Then [40] $\boldsymbol{u}(x) - \overline{\boldsymbol{u}}(x) = \nabla\pi(x)$ with $(d/d\vec{n})\,\pi_{|\partial\Omega} = 0$, which proves the first part of the lemma. As far as the latter is concerned, we observe that $\Delta\pi(x) = 0$ and, in virtue of summability, $\nabla\pi(x)$ tends to zero at infinity, then $\nabla\pi(x) = 0$.

4. – Proof of the theorems.

PROOF OF THEOREM 2.1. – The former part of the theorem is a easy consequence of Lemma 3.1, Lemma 3.2 and convexity theorem for L^r-spaces. In fact if $q \leq np/(n-2p)$ we have

$$|\boldsymbol{w}|_r \leq |\boldsymbol{w}|^a_{np/(n-2p)} |\boldsymbol{w}|_q^{1-a} \leq C |\nabla \boldsymbol{w}|^a_{np/(n-p)} |\boldsymbol{w}|_q^{1-a} \leq$$

$$\leq C|D^2\boldsymbol{w}|_p^a |\boldsymbol{w}|_q^{1-a} \leq C|P\Delta\boldsymbol{w}|_p^a |\boldsymbol{w}|_q^{1-a}, \qquad \forall r \in \left[q, \frac{np}{n-2p}\right];$$

the conversely, $q \geq np/(n-2p)$, is the same.

To proving the latter part of the theorem, we start for some $r \in [\overline{r}, \infty)$, where $\overline{r} = \max\{p, q\}$. Since $\boldsymbol{w}(x) \in \mathring{H}^{P\Delta}_{q,p}(\Omega)$ we multiply $P\Delta\boldsymbol{w}(x)$ by $\varphi(x, t)$ solution to system (3.5) with $\varphi_0(x) \in \mathcal{C}_0(\Omega)$. Integrating by parts on $\Omega \times (0, T)$ we obtain

$$\int_0^t (P_p\Delta\boldsymbol{w}, \varphi(\tau))\,d\tau = \int_0^t (\boldsymbol{w}, P_q\Delta\varphi(\tau))\,d\tau = \int_0^t (\boldsymbol{w}, \varphi_\tau(\tau))\,d\tau = (\boldsymbol{w}, \varphi(t)) - (\boldsymbol{w}, \varphi_0).$$

Thus we have

$$(4.1)\qquad |(\boldsymbol{w},\varphi_0)|\leq|(\boldsymbol{w},\varphi(t))|+\left|\int_0^t(P\Delta\boldsymbol{w},\varphi(\tau))\,d\tau\right|=J_1(t)+J_2(t).$$

Now we estimate $J_j(t)$ for $j=1,2$. Applying Hölder inequality and $(3.7)_1$ we have

$$(4.2)\qquad \begin{cases} J_1(t)\leq|\boldsymbol{w}|_q\,|\varphi(t)|_{q'}\leq M|\boldsymbol{w}|_q\,|\varphi_0|_{r'}\,t^{-\mu},\\ \mu=\dfrac{n}{2}\left(\dfrac{1}{q}-\dfrac{1}{r}\right),\qquad r'=r/(r-1),\qquad q'=q/(q-1).\end{cases}$$

For $J_2(t)$ we again apply Hölder inequality and after inequality $(3.8)_1$, then

$$(4.3)\quad J_2(t)\leq|P\Delta\boldsymbol{w}|_p\int_0^t|\varphi(\tau)|_{p'}\,d\tau\leq M|P\Delta\boldsymbol{w}|_p\,|\varphi_0|_{r'}\int_0^t\tau^{-\mu_1}d\tau,\qquad \mu_1=\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right);$$

we observe that, for any $a\in[0,1)$, (2.2) implies $1/p<1/r+2/n$, thus the right hand side of (4.3) is finite for any $r\geq\bar r$. Increasing the right hand side of (4.1) by mean (4.2)-(4.3), we deduce by simple computation:

$$|(\boldsymbol{w},\varphi_0)|\leq M|\varphi_0|_{r'}(|P\Delta\boldsymbol{w}|_p t^{1-\mu_1}+|\boldsymbol{w}|_q t^{-\mu}),$$

$$\mu_1=\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right),\qquad \mu=\frac{n}{2}\left(\frac{1}{q}-\frac{1}{r}\right).$$

This last inequality holds for any $\varphi_0(x)\in\mathcal{C}_0(\Omega)$. From Lemma 3.6 it follows that

$$(4.4)\qquad |\boldsymbol{w}|_r\leq M(|P\Delta\boldsymbol{w}|_p t^{1-\mu_1}+|\boldsymbol{w}|_q t^{-\mu}).$$

In virtue of Lemma 3.5 we can assume $|P\Delta\boldsymbol{w}|_p\neq 0$. Therefore, setting $t=(|\boldsymbol{w}|_q/|P\Delta\boldsymbol{w}|_p)^{\alpha}$, $\alpha=2pq/(2pq+n(p-q))$ (3) we have (2.1) with $1-a+\mu_1\alpha=\alpha\mu=a$ or conversely $(1-\mu_1)\,\alpha=1-\alpha\mu=1-a$, which are equivalent to (2.2). To extend the above result to the case of $r=\infty$ $(n\geq 3)$, it is sufficient to make the following observations. Lemma 3.1, inequality (4.4) and Sobolev imbbending theorem ensure that $\boldsymbol{w}(x)\in L^{\infty}(\Omega)$, moreover Theorem 3.1 ensures that inequality (4.4) holds for any $r\geq q$ ($\boldsymbol{w}(x)\in L^r(\Omega)$) with the right hand side depending on r as continuous and bounded function, thus making $r\to\infty$ we dedudce (2.1) for $r=\infty$.

Now, we remove the hypothesis $r\geq\bar r$. If it is the case of $p<q$, then the proof of theorem is complete. Otherwise for $q<p$ we consider the convexity theorem for L^p-space:

$$(4.5)\qquad |\boldsymbol{w}|_r\leq|\boldsymbol{w}|_{r_1}^{b}\,|\boldsymbol{w}|_q^{1-b},\qquad \frac{1}{r}=\frac{b}{r_1}+\frac{1-b}{q},\qquad \text{for } r_1>p.$$

---

(3) Since $q\leq r$ and $1/p<1/r+2/n$, we have $\alpha>0$.

Since $r_1 > p$, we increase the right hand side of (4.5) by (2.1), therefore

$$|\boldsymbol{w}|_r \leq M |P\Delta \boldsymbol{w}|_p^{bc} |\boldsymbol{w}|_q^{1-bc}, \qquad \forall r > q .$$

A simple computation gives $bc = a$, with a satisfying (2.2).

PROOF OF THEOREM 2.2. – In hypothesis $s \in [1, n)$ inequality (2.5) is a trivial implication of Sobolev inequality and convexity theorem for L^p-space. In fact if $q \leq ns/n - s$ we have

$$|w|_r \leq |w|_{ns/(n-s)}^{a} |w|_q^{1-a} \leq (C|\nabla w|_s)^a |w|_q^{1-a}, \qquad \forall r \in \left[q, \frac{ns}{n-s}\right];$$

the conversely, $q \geq ns/(n-s)$, is the same. As far as the value of constant C we refer to [1, 40].

Now, we consider the case of $s \geq n$. We start considering $r \in [\bar{r}, \infty]$, $\bar{r} = \max\{s, q\}$. The following equation holds

$$(\nabla w, \nabla \varphi(t)) = -(w, \Delta\varphi(t)) = -(w, \varphi_t(t)), \qquad \forall t > 0,$$

where $\varphi(x, t)$ is the solution to the initial value problem of heat equation (3.9) corresponding to $\varphi(x) \in L^{r'}(\mathbb{R}^n)$. Integrating on $(0, t)$ we have

$$(w, \varphi_0) = (w, \varphi(t)) + \int_0^t (\nabla w, \nabla\varphi(t))\, d\tau .$$

Applying the Hölder inequality and (3.10), we obtain

$$|(w, \varphi_0)| \leq |w|_q |\varphi(t)|_{q'} + |\nabla w|_s \int_0^t |\nabla\varphi(\tau)|_{s'}\, d\tau \leq$$

$$\leq M_1 |w|_q |\varphi_0|_{r'} t^{-\mu} + (1-\mu')^{-1} M_2 |\nabla w|_s |\varphi_0|_{r'} t^{1-\mu'},$$

where we have taken into account that (2.5) and $a \in [0, 1)$ imply $\mu' < 1$ $(1/s < 1/n + 1/r)$. The last integral inequality implies

$$|w|_r \leq M_1 |w|_q t^{-\mu} + \gamma(r, s) M_2 |\nabla w|_s t^{1-\mu'}, \qquad \forall t > 0 . \tag{4.6}$$

Since we can assume $|\nabla w|_s \neq 0$, we get $t = (|w|_q/(|\nabla w|_s \gamma(r, s)))^{\alpha} \xi$ with $\alpha = 2qs/(2sq + n(s-q))$ and $\xi > 0$. Therefore, after substuting this t in (4.6), we obtain

$$|w|_r \leq (M_1 \xi^{-\mu} + (1-\mu')^{-1} M_2 \xi^{1-\mu'}) |\nabla w|_s^a |w|_q^{1-a}, \tag{4.7}$$

with a deduced from (2.5). Making on the right hand side of (4.7) the minimum value with rispect to ξ, we deduce (2.4). If $s \leq q$ the proof is complete. Thus suppose $s > q$. For any $\tilde{r} \in [q, s]$, we have

$$(4.8) \qquad |w|_{\tilde{r}} \leq |w|_s^b \, |w|_q^{1-b}, \qquad \frac{1}{\tilde{r}} = \frac{b}{s} + \frac{1-b}{q}.$$

Increasing by (4.7) written for $r = s$ we deduce (2.4). The numerical value of the constant C in (2.4) is consequence of a simple computation which takes into account the exponent a and b in (4.7)-(4.8).

PROOF OF THEOREM 2.3. – To prove (2.6) it is sufficient to employ the convexity theorem for L^p-space and Lemma 3.2, as it has been already made in the proof of Theorem 2.2.

Let us consider the case $s \geq n$, $q \geq s$ and $r < \infty$. We denote by $d = \operatorname{diam}(\Omega^c)$. Moreover we define two cut-off functions as it follows $(R > d)$ $h_1(x) = 1$ for $|x| \leq R$, $h_1(x) = 0$ for $|x| \geq 2R$, $h_1(x) \in [0, 1]$ for $|x| \in [R, 2R]$; $h_2(x) = 1$ for $|x| \leq 2R$, $h_2(x) = 0$ for $|x| \geq 3R$, $h_2(x) \in [0, 1]$ for $|x| \in [2R, 3R]$. Moreover they satisfy the condition $|\nabla h_1(x)| \leq A_1/R$ and $|\nabla h_2(x)| \leq A_2/R$. Consider

$$w_1(x) = (1 - h_1(x))\, w(x), \qquad w_2(x) = h_2(x)\, w(x).$$

Since $w_1(x)$ is defined in the whole $\mathbb{R}^n$ we can apply Theorem 2.2, then

$$|w_1|_r \leq C_1 |\nabla w_1|_s^a \, |w_1|_q^{1-a} \leq C_1 \left(|\nabla w|_s + \frac{1}{R} |w|_{L^s(R \leq |x| \leq 2R)} \right)^a |w|_q^{1-a} \leq$$

$$\leq C_1 \left(|\nabla w|_s + \frac{1}{R^{1 - n(q-s)/qs}} |w|_q \right)^a |w|_q^{1-a}.$$

We employ Lemma 3.3, then

$$|w_2|_r \leq C_2 |\nabla w_2|_s^a \, |w_2|_q^{1-a} + C_3 |w_2|_q.$$

Now two cases are possible

$$\frac{|w|_q}{|\nabla w|_s} \leq d^{1 - n(q-s)/qs} \qquad \text{or} \qquad \frac{|w|_q}{|\nabla w|_s} > d^{1 - n(q-s)/qs}.$$

In the former case we fixed $R > d$ and increase $|w|_q$ in estimates of $|w_1|_r$ and $|w_2|_r$ with $d^{1 - n(q-s)/qs} |\nabla w|_s$, therefore

$$|w|_r \leq |w_1|_r + |w_2|_r \leq$$

$$\leq [C_1 (1 + (1/R)^{1 - n(q-s)/qs})^a + C_2 + C_3 d^{(1 - n(q-s)/qs)a}]\, |\nabla w|_s^a \, |w|_q^{1-a}.$$

In the latter case we modify the estimate for $w_2(x)$ and choose R subsequently in a suitable way. For any $r \geq q$ there exists an $\bar{s} \in [1, n)$ such that $n\bar{s}/(n - \bar{s} > r$. In virtue of

(2.6) we can get with C independent of R and $w_2(x)$

$$|w_2|_r \leq C|\nabla w_2|_{\bar{s}}^{b} |w_2|_q^{1-b}.$$

Since $s > \bar{s}$ and $w_2(x)$ is zero for $|x| \geq 3R$, applying Hölder inequality we deduce

$$|w_2|_r \leq CR^{bn(s-\bar{s})/s\bar{s}} |\nabla w_2|_s^b |w_2|_q^{1-b} \leq$$

$$\leq CR^{bn(s-\bar{s})/s\bar{s}} \left(|\nabla w|_s + \frac{1}{R} |w|_{L^s(2R \leq |x| \leq 3R)} \right)^b |w|_q^{1-b} \leq$$

$$\leq CR^{bn(s-\bar{s})/s\bar{s}} \left(|\nabla w|_s + \frac{1}{R^{1-n(q-s)/qs}} |w|_q \right)^b |w|_q^{1-b}.$$

We observe that for $s \geq n$ we have $1 > n(q-s)/qs$, for any $q \geq 1$. Now we are free to choose $R^{1-n(q-s)/qs} = |w|_q / |\nabla w|_s > d^{1-n(q-s)/qs}$. Therefore the estimate for $|w|_r$ becomes

$$|w|_r \leq |w_1|_r + |w_2|_r \leq 2^a C_1 |\nabla w|_s^a |w|_q^{1-a} + 2^b C |\nabla w|_s^{b(1-\beta)} |w|_q^{1-b(1-\beta)},$$

$$\text{with } \beta = \frac{1}{\bar{s}} \frac{qn(s-\bar{s})}{qs - n(q-s)}.$$

Now, taking into account the value of b in (2.7) and a in (2.8), a simple computation gives $a = b(1-\beta)$, then we have completed the proof in the case of $q \geq s \geq n$. Now, let us consider the case $q < s$. First of all we prove that $w(x) \in L^s(\Omega)$. Of course by Poincaré inequality $w(x) \in L^s_{\mathrm{loc}}(\Omega)$. Moreover, introducing a smooth cut-off function $k(x)$ such that $k(x) = 1$ for $|x| \leq \bar{R}$, $k(x) = 0$ for $|x| \geq 2\bar{R}$, $\bar{R} > \operatorname{diam}(\Omega^c)$, setting $\tilde{w}(x) = (1-k(x))\, w(x)$, from (2.4) we have

$$|\tilde{w}|_s \leq C|\nabla \tilde{w}|_s^{a_1} |\tilde{w}|_s^{1-a_1},$$

which implies $w(x) \in L^s(\Omega - S_{2\bar{R}})$. So we have proved $w(x) \in L^s(\Omega)$. Now, for $w(x) \in W^{1,s}(\Omega)$ we have already obtained estimate (2.6). Then for any $r \geq s$

$$|w|_r \leq C|\nabla w|_s^{a_2} |w|_s^{1-a_2}. \tag{4.9}$$

Applying to (4.9) the convexity theorem for L^p-space, we deduce

$$|w|_r \leq C|\nabla w|_s^{a_2} |w|_r^{b(1-a_2)} |w|_q^{(1-b)(1-a_2)}, \qquad \text{with } b = \frac{r(s-q)}{s(r-q)}.$$

It is immediate to deduce

$$|w|_r \leq C|\nabla w|_s^a |w|_q^{1-a},$$

with a given in (2.9). To obtain the cases of $r \in (q, s)$ it is sufficient to apply again the convexity theorem for L^p-spaces.

PROOF OF THEOREM 2.4. – The case of $r = ns/(n-s)$ with $s \in (1, n)$ is the ordinary Sobolev inequality. Thus we consider only the cases of inequalities (2.9), (2.11)-(2.14)

We introduce a smooth cut-off function $h(x)$ such that $h(x) = 1$ for $|x| \leqslant R$, $h(x) = 0$ for $|x| \geqslant 2R$ and $|\nabla h(x)| \leqslant C/|x|$. Let us consider the following relation

$$(4.10)\quad (\nabla w, \nabla\varphi(t)\, h) = -(w, \Delta\varphi(t)\, h) - (w, \nabla h\cdot\nabla\varphi(t)) = \\ = -(w, \varphi_t(t)\, h) - (w, \nabla h\cdot\nabla\varphi(t))\,, \qquad \forall t > 0\,,$$

Where $\varphi(x, t)$ is the solution of (3.8) corresponding to $\varphi_0(x) \in C_0^\infty(\Omega)$, with $\operatorname{supp}\{\varphi_0\} \subseteq S(O, \overline{R})$ for some $\overline{R} < R$. Integrating (4.10) on $(0, t)$ we deduce

$$(4.11)\quad (w, \varphi_0 h) = (w, \varphi(t)\, h) + \int_0^t (\nabla w, \nabla\varphi(\tau)\, h)\, d\tau + \int_0^t (\nabla w, \varphi(\tau)\, \nabla h)\, d\tau + \\ + \int_0^t (w, \nabla h\cdot\nabla\varphi(\tau))\, d\tau = I_1(t) + I_2(t) + I_3(t) + I_4(t)\,, \qquad \forall t > 0\,.$$

Now we increase $I_i(t)$, $i = 1, 2, 3, 4$. We have

$$(4.12)\quad |I_1(t)| \leqslant |w|_{-1,q}\, |\varphi(t)\, h|_{1,q'} \leqslant C|w|_{-1,q}\, |\varphi_0|_{r'} \left(t^{-\mu'} + \frac{1}{R} t^{-\mu} + t^{-\mu'}\right), \\ \forall t > 0\,, \quad \mu = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right), \quad \mu' = \frac{1}{2} + \mu$$

where increasing we have taken into account (3.9) and $r \geqslant q$. As far as $I_2(t)$ is concerned, applying the Hölder inequality and $(3.9)_2$ we obtain

$$(4.13)\quad |I_2(t)| \leqslant |\nabla w|_s \int_0^t |\nabla\varphi(\tau)|_{s'}\, d\tau \leqslant C\gamma\, |\nabla w|_s\, |\varphi_0|_{r'}\, t^{1-\mu_1'}\,, \\ \mu_1' = \frac{1}{2} + \frac{n}{2}\left(\frac{1}{s} - \frac{1}{r}\right), \qquad \gamma = (1 - \mu_1')^{-}1.$$

We stress that $(2.10)_1$ and $r \geqslant \bar{r}$ imply $\mu_1' < 1$. Finally, applying Hölder inequality, in virtue of Theorem 3.2, we get

$$(4.14)\quad |I_3(t) + I_4(t)| \leqslant |w|_{-1,q} \int_0^t |\nabla h\cdot\nabla\varphi(\tau)|_{1,q'}\, d\tau + |\nabla w|_s\, |\nabla h \varphi(t)|_{s'}\, d\tau \leqslant \\ \leqslant \frac{C}{R}(|w|_{-1,q} + |\nabla w|_s) \int_0^t |\varphi(\tau)|_{q'}\, d\tau\,.$$

Recalling that $\operatorname{supp}\{\varphi_0\} \subseteq S(0, \overline{R}) \cap \Omega$, from (4.11)-(4.14), we deduce

$$|(w, \varphi_0)| \leq C|w|_{-1,q}|\varphi_0|_{r'}\left(t^{-\mu} + \frac{1}{R}t^{-\mu} + t^{-\mu'}\right) + \tag{4.15}$$

$$+ C\gamma(r, s)|\nabla w|_s|\varphi_0|_{r'}t^{-\gamma(r,s)} + C(t)\frac{1}{R}(|w|_{-1,q}|\nabla w|_s).$$

Making $R \to \infty$ in (4.15) we arrive

$$|(w, \varphi_0)| \leq C|w|_{-1,q}|\varphi_0|_{r'}(t^{-\mu} + t^{-\mu'}) + \gamma(r, s)\, C|\nabla w|_s|\varphi_0|_{r'}t^{-\gamma(r,s)}.$$

Since $\operatorname{supp}\{\varphi_0(x)\} \subset S(O, \overline{R}) \cap \Omega$ for the arbitrarity of $\varphi_0(x)$ we get

$$|w|_{L^r S(O,\overline{R}) \cap \Omega)} \leq C|w|_{-1,q}(t^{-\mu} + t^{-\mu'}) + C\gamma(r, s)|\nabla w|_s t^{-\gamma(r,s)}. \tag{4.16}$$

On the other hand the right hand side of (4.16) is indipendent of $\overline{R}$, therefore

$$|w|_r \leq C|w|_{-1,q}(t^{-\mu} + t^{-\mu'}) + C\gamma(r, s)|\nabla w|_s t^{-\gamma(r,s)}. \tag{4.17}$$

We can assume $|\nabla w|_s \neq 0$. In fact if $|\nabla w|_s = 0$, then $w(x) = 0$. We set $t = (|w|_{-1,q}/|\nabla w|_s)^\alpha$ with $\alpha = 2qs/(2qs + n(s-q))$. We observe that from $(2.10)_1$ it follows that $\alpha > 0$. Substuting t in (4.17), after a simple computation, we conclude the proof of inequality (2.9). As far as (2.12) is concerned, then we modify estimate (4.12) as it follows

$$|I_1(t)| \leq \|w\|_{-1,q}|\nabla(h\varphi(t))|_{q'} \leq C\|w\|_{-1,q}|\varphi_0|_{r'}\left(t^{-\mu'} + \frac{1}{R}t^{-\mu}\right).$$

After which it is sufficient to repeat the above argument lines. Finally to prove (2.11) we again modify (4.12) and precisely we have

$$|I_1(t)| = |(\tilde{w}, D_j\varphi(t))| \leq |\tilde{w}|_q|\nabla\varphi(t)|_{q'} \leq C|\tilde{w}|_q|\varphi_0|_{r'}t^{-\mu'},$$

then we repeat the above arguments.

Taking into account Lemma 3.4 the case $\Omega \equiv \mathbb{R}^n$ is formally the same, thus the proof is omitted.

Since $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $W^{-1,q}(\mathbb{R}^n)$ and $D^{-1,q}(\mathbb{R}^n)$, we restrict the proof of (2.13)-(2.14) to $w(x) \in C_0(\mathbb{R}^n)$, after which by standard arguments of density one completes the proof. We consider the equation

$$(w, \varphi_t(t)) = (w, \Delta\varphi(t)),$$

where $\varphi(x, t)$ is the solution to (3.8) corresponding to $\varphi_0(x) \in W^{1,r'}(\mathbb{R}^n)$. Integrating the above equation on $(0, t)$ we have

$$|(w, \varphi_0)| \leq |(w, \varphi(t))| + \int_0^t |(w, \Delta\varphi(\tau))|d\tau = I_1(t) + I_2(t).$$

Now, with simple computation and taking into account (3.10), we obtain

$$I_1(t) \leqslant \begin{cases} |w|_{-1,q}\,|\varphi(t)|_{q'} \leqslant C|w|_{-1,q}\,t^{-(n/2)(1/q-1/r)}\,|\varphi|_{1,r'} \\ \|w\|_{-1,q}\,|\nabla\varphi(t)|_{q'} \leqslant C\|w\|_{-1,q}\,t^{-(n/2)(1/q-1/r)}\,|\nabla\varphi_0|_{r'}; \end{cases}$$

$$I_2(t) \leqslant \begin{cases} |w|_p \int_0^t |\Delta\varphi(\tau)|_{p'}\,d\tau \leqslant C|w|_p\,t^{1/2-(n/2)(1/p-1/r)}\,|\nabla\varphi_0|_{r'} \leqslant \\ \qquad \leqslant C|w|_p\,t^{1/2-(n/2)(1/p-1/r)}\,|\varphi_0|_{1,r'}; \\ |w|_p \int_0^t |\Delta\varphi(\tau)|_p\,d\tau \leqslant C|w|_p\,t^{1/2-(n/2)(1/p-1/r)}\,|\nabla\varphi_0|_{r'}. \end{cases}$$

In the above inequalities for $I_1(t)$ and $I_2(t)$ the former estimate on the right hand side is obtained to deduce (2.13). While the latter estimate is obtained to deduce (2.14). Therefore, following the arguments already employed to prove inequalities (2.9) and (2.12) one concludes the proof.

PROOF OF THEOREM 2.5. – In virtue of Lemma 3.1 we can assume $D^2\boldsymbol{w}(x) \in L^p(\Omega)$. We introduce a smooth cut-off function $h_1(x)$ such that $h_1(x) = 0$ for $|x| \geqslant 2R$, $h_1(x) \in [0, 1]$ for $|x| \in [R, 2R]$, $h_1(x) = 1$ for $|x| \leqslant R$, $R > \operatorname{diam}(\Omega^c)$. Define $\boldsymbol{w}_1(x) = (1 - h_1(x))\,\boldsymbol{w}(x)$. Estimate (2.11) implies

$$|\nabla \boldsymbol{w}|_r \leqslant C_2\,|D^2\boldsymbol{w}_1|_p^a\,|\boldsymbol{w}_1|_q^{1-a} \leqslant C(|D^2\boldsymbol{w}|_p^a + |\boldsymbol{w}|^a_{W^{1,p}(R\leqslant|x|\leqslant 2R)})|\boldsymbol{w}|_q^{1-a}, \tag{4.18}$$

with a as stated in $(2.10)_1$. Taking into account (3.4), estimate (4.18) becomes

$$|\nabla \boldsymbol{w}_1|_r \leqslant C(|D^2\boldsymbol{w}|_p^a + |\boldsymbol{w}|^a_{L^pR\leqslant(|x|)\leqslant 2R})|\boldsymbol{w}|_q^{1-a}. \tag{4.19}$$

If $p \in (1, n/2)$, applying Hölder inequality to the right hand side of (4.19), the Sobolev inequality (3.3) and inequality (3.1) we have

$$\begin{aligned} |\nabla \boldsymbol{w}_1|_r &\leqslant C(|D^2\boldsymbol{w}|_p^a + |\boldsymbol{w}|^a_{np/(n-2p)})|\boldsymbol{w}|_q^{1-a} \leqslant \\ &\leqslant C(|D^2\boldsymbol{w}|_p^a + |\nabla\boldsymbol{w}|^a_{np/(n-p)})|\boldsymbol{w}|_q^{1-a} \leqslant C|D^2\boldsymbol{w}|_p^a\,|\boldsymbol{w}|_q^{1-a} \leqslant C|P\Delta\boldsymbol{w}|_p^a\,|\boldsymbol{w}|_q^{1-a}. \end{aligned} \tag{4.20}$$

If $p \in [n/2, \infty)$, then we increase the right hand side of (4.19) by (3.2), after which we employ Hölder inequality. Then we obtain ($\Omega^* \subset \Omega$ is a bounded domain including Ω_1 and $R \leqslant |x| \leqslant 2R$).

$$\begin{aligned} |\nabla \boldsymbol{w}_1|_r &\leqslant C(|P\Delta\boldsymbol{w}|_p^a + |\boldsymbol{w}|^a_{L^p(\Omega)^*})|\boldsymbol{w}|_q^{1-a} \leqslant \\ &\leqslant C(|P\Delta\boldsymbol{w}|_p^a\,|\boldsymbol{w}|_q^{1-a} + |\boldsymbol{w}|^a_{L^{p_0}(\Omega^*)}\,|\boldsymbol{w}|_q^{1-a}), \qquad p_0 \geqslant 1. \end{aligned} \tag{4.21}$$

Now, if $n \geq 3$ and $p > n/2$ we can choose $p_0 = \infty$, otherwise for $p = n/2$ $(n \geq 3)$ and $n = 2$ exponent p_0 can be choosen sufficiently large but less then infinity. Therefore employing (2.1) we obtain from (4.21)

$$|\nabla w_1|_r \leq C(|PΔw|_p^a |w|_q^{1-a} + |PΔw|_p^{ab} |w|_q^{1-ab}), \tag{4.22}$$

with b as stated in (2.2). Let $h_2(x)$ be another smooth cut-off function with support in $S(O, 3R)$ with $h_2(x) = 1$ for $|x| \leq 2R$ and $h_2(x) \in [0, 1]$ for $2R \leq |x| \leq 3R$. Define $w_2(x) = h_2(x) w(x)$ and apply (3.4), thus we have ($\Omega_2 = \Omega \cap S(O, 3R)$)

$$|\nabla w_2|_r \leq C_2(|D^2 w_2|^a_{L^p(\Omega_2)} |w_2|^{1-a}_{L^q(\Omega_2)} + |w_2|_{L^q(\Omega_2)}) \leq$$

$$\leq C(|D^2 w|_p^a + |w|^a_{W^{1,p}(\Omega_2)}) |w|^{1-a}_{L^q(\Omega_2)} + C|w_2|_{L^q(\Omega_2)} .$$

Finally applying inequality (3.4), we have

$$|\nabla w_2|_r \leq C(|D^2 w|_p^a + |w|^a_{L^p(\Omega_2)}) |w|^{1-a}_{L^q(\Omega_2)} + C|w_2|_{L^q(\Omega_2)} . \tag{4.23}$$

Since Ω_2 is bounded, for $p \in (1, n/2)$ we apply Hölder inequality as it follows

$$|\nabla w_2|_r \leq C(|D^2 w|_p^a + |w|^a_{np/(n-2p)} |w|_q^{1-a} + C|w|_{L^s(\Omega_2)}, \qquad s \geq q . \tag{4.24}$$

From Sobolev inequality (3.3) $|w|_{np/(n-2p)} \leq C|\nabla w|_{np/(n-p)} \leq C|D^2 w|_p$; moreover for $1/s = 1/r - 1/n$ estimate (2.1) gives $|w|_s \leq C|PΔw|_p^a |w|_q^{1-a}$. Therefore, taking into account (3.2), for $p \in (1, n/2)$, inequality (4.24) becomes

$$|\nabla w_2|_r \leq C|PΔw|_p^a |w|_q^{1-a}, \qquad p \in (1, n/2). \tag{4.25}$$

If $p \geq n/2$ from (4.23) applying (3.3) and Hölder inequality we have, $\bar{r} = \max\{p, q\}$,

$$|\nabla w_2|_r \leq C(|PΔw|_p^a |w|_q^{1-a} + |w|^a_{L^p(\Omega_2)} |w|^{1-a}_{L^q(\Omega_2)} + |w|_{L^q(\Omega_2)}) \leq$$

$$\leq C(|PΔw|_p^a |w|_q^{1-a} + |w|_{L^{s_1}(\Omega_2)}), \qquad s_1 \geq \bar{r} .$$

Now, for $n \geq 3$ and $p > n/2$ we set $s_1 \leq \infty$, for $p = n/2$ $(n \geq 3)$ and $n = 2$ we choose $s_1 < \infty$ but arbitrary greater then $\bar{r}$. Thus from (2.1) we deduce for $p \geq n/2$

$$|\nabla w_2|_r \leq C(|PΔw|_p^a |w|_q^{1-a} + |PΔw|_p^b |w|_q^{1-b}), \tag{4.26}$$

with b as stated in (2.15). Coupling (4.20) and (4.25) we deduce (2.15), while coupling (4.25)-(4.26) we deduce (2.16). In the case of $r < n$, then we can modify (4.26). Indeed, we can choose $1/s_1 = 1/r - 1/n$, thus via (2.1) (4.26) becomes

$$|\nabla w_2|_r \leq C|PΔw|_p^a |w|_q^{1-a} .$$

This last estimate and (4.25) ensure (2.17) with $C_7 = 0$.

To complete the proof of the theorem we must prove that (2.15) is not true for $r > n$ with $p \geqslant q \geqslant n/2$ and $r = np/(n-p)$ with $p = \in [n/2, n)$. Assume that inequality (2.15) holds for some $r > n$ and $p \geqslant q$. Applying this inequality to solutions of system (3.5), we obtain

$$|\nabla\varphi(t)|_r \leqslant C|P\Delta\varphi(t)|_p^a\,|\varphi(t)|_q^{1-a} = C|\varphi_t(t)|_p^a\,|\varphi(t)|_q^{1-a} \leqslant Ct^{-a(1+(n/2)(1/q-1/p))}\,|\varphi_0|_q,$$

with $1/r = 1/n + a(1/p - 2/n) + (1-a)(1/q)$. A simple computation gives $a(1 + (n/2)(1/q - 1/p)) = 1/2 + (n/2)(1/q - 1/r) > n/2q$, which denies the optimality of $(3.8)_2$. Therefore (2.15) is not true. In the case of $p \in [n/2, n)$ and $r = np/(n-p)$ again we assume *ab absurdum* that (2.15) is true. As consequence from inequality (3.2), applying the Poincaré inequality and Hölder inequality, we deduce

$$|D^2 w|_p \leqslant C(|P\Delta w|_p + |w|_{L^p(\Omega^*)}) \leqslant C(|P\Delta w|_p + |\nabla w|_{L^p(\Omega^*)}) \leqslant$$

$$\leqslant C((|P\Delta w|_p + |\nabla w|_{L^{np/(n-p)}(\Omega^*)}) \leqslant C|P\Delta w|_p\,.$$

This last result contradicts the optimality of (3.1) stated in Lemma 3.1. Thus (2.15) for $r = np/(n-p)$ and $p \in [n/2, n)$ is not true.

PROOF OF THEOREM 2.6. – The proof of the former part of the theorem is very similar to one of Theorem 2.1. We assume $w(x) \in \overset{\circ}{H}{}^{P\Delta,1}_{q,p}(\Omega)$ and multiply $P\Delta w(x)$ by $(\partial/\partial x_j)\,\varphi(x,t) = D\varphi(x,t)$, where $\varphi(x,t)$ is the solution to system (3.5) corresponding to $\varphi_0(x) \in \mathcal{C}_0(\Omega)$. Integrating on $\Omega \times (0, T)$ we obtain

$$\int_0^t (P\Delta w, D\varphi(\tau))\,d\tau = \int_0^t [(\Delta w, D\varphi(\tau)) - (\nabla\pi, D\varphi(\tau))]\,d\tau =$$

$$= \int_0^t (\Delta w, D\varphi(\tau))\,d\tau - \int_0^t \lim_\varepsilon (\nabla\tilde\pi_\varepsilon, D\varphi(\tau))\,d\tau =$$

$$= \int_0^t (\nabla D w, \nabla\varphi(\tau))\,d\tau + \int_0^t \lim_\varepsilon (\nabla D\tilde\pi_\varepsilon, \varphi(\tau))\,d\tau =$$

$$= -\int_0^t (Dw, \Delta\varphi(\tau))\,\tau = -\int_0^t (Dw, \varphi_\tau(\tau) - \nabla\tilde p(\tau))\,d\tau =$$

$$= (Dw, \varphi_0) + (w, D\varphi(\tau)) - \int_0^t \lim_\varepsilon (w, \nabla D\tilde p_\varepsilon(\tau)) =$$

$$= (Dw, \varphi_0) + (w, D\varphi(\tau))\,, \quad \forall t > 0;$$

in above equation $\tilde\pi(x)$ is an extension of $\pi(x)$ inside of Ω^c, in such a way that we can consider the mollification $(\nabla\tilde\pi(x))_\varepsilon = \nabla\tilde\pi_\varepsilon(x)$, quite analogous is the meaning of $\tilde p(x, t)$.

Therefore we deduce

$$(4.27)\qquad |(Dw, \varphi_0)| \leq |(w, D\varphi(t))| + \Big|\int_0^t (P\Delta w, D\varphi(\tau))\, d\tau\Big| = I_1(t) + I_2(t), \qquad \forall t > 0 .$$

Now, we apply the Hölder inequality and $(3.7)_2$ to $I_i(t)$, $i = 1, 2$. Then we have

$$(4.28)\qquad I_1(t) \leq |w|_q |\nabla\varphi(t)|_{q'} \leq M|w|_q |\varphi_0|_{r'} t^{-\mu'},$$

$$r' \in (1, q'), \qquad \mu' = \frac{1}{2} + \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right).$$

Since $r \geq q$ and (2.18) imply $1/p < 1/r + 1/n$, it is possible to increase $I_2(t)$ as it follows:

$$(4.29)\qquad I_2(t) \leq |P\Delta w|_p \int_0^t |\nabla\varphi(\tau)|_{p'}\, d\tau \leq M|P\Delta w|_p |\varphi_0|_{r'} t^{1-\mu_1'},$$

$$r' \in (1, p'), \qquad \mu_1' = \frac{1}{2} + \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right).$$

Since Lemma 3.1 and (2.11) imply $Dw(x) \in J^r(\Omega)$, increasing the right hand side of (4.27) by mean (4.28)-(4.29), for the arbitrarity of $\varphi_0(x) \in \mathcal{C}_0(\Omega)$, in virtue of Lemma 3.6, we get

$$(4.30)\qquad |Dw|_r \leq M(|P\Delta w|_p t^{1-\mu_1'} + |w|_q t^{-\mu'}), \qquad \forall t > 0 .$$

We again set $t = (|w|_q / |P\Delta w|_p)^\alpha$ with $\alpha = 2pq/(2pq + n(p-q))$, therefore the above inequality implies (2.19). If $n \geq 3$, to obtain the case of $r = \infty$ it is sufficient to observe that for any $r \geq q$ $Dw(x) \in L^r(\Omega)$ and the right hand side of (4.30) depend on r as continuous and bounded function, thus making $r \to \infty$ we deduce (2.19) for $r = \infty$.

Now we prove the latter part of the theorem. We begin considering the case of $p \in [n/2, n)$. Let $k(x)$ be a smooth cut-off function with $k(x) = 0$ for $|x| \leq R$, $R > \operatorname{diam}(\Omega^c)$, $k(x) = 1$ for $|x| \geq 2R$ and $k(x) \in [0, 1]$ for $|x| \in [R, 2R]$. Define $w_1(x) = (1 - k(x))\, w(x)$. Estimate (2.11) implies

$$(4.31)\qquad |\nabla w_1|_r \leq C|D^2 w_1|_p^a |w_1|_q^{1-a} \leq$$

$$\leq C(|D^2 w|_p + |\nabla w|_{L^p(R \leq |x| \leq 2R)} + |w|_{L^p(R \leq |x| \leq 2R)})^a |w|_q^{1-a}.$$

Since it is $np/(n-p) \geq r \geq p$, after applying the Hölder inequality to the right hand side of (4.31), taking into account Lemma 3.2 and the Poincarè inequality, we have

$$(4.32)\qquad |\nabla w_1|_r \leq$$

$$\leq C(|D^2 w|_p + |\nabla w|_{L^{np/(n-p)}(R \leq |x| \leq 2R)} + |w|_{L^{np/(n-p)}(R \leq |x| \leq 2R)})^a |w|_q^{1-a} \leq$$

$$\leq C(|D^2 w|_p + |\nabla w|_{L^{np/(n-p)}(R \leq |x| \leq 2R)})^a |w|_q^{1-a} \leq$$

$$\leq C|D^2 w|_p^a |w|_q^{1-a} \leq C|P\Delta w|_p^a |w|_q^{1-a}.$$

Now, we consider $|\nabla w|_{L^r(\Omega\cap S_{2R})}$. From Lemma 3.3, applying the Hölder inequality and the Poincaré inequality, we deduce

$$|\nabla w|_{L^r(\Omega\cap S_{2R})}\leq C(|D^2 w|_p^a |w|_q^{1-a}+|w|_{L^q(\Omega\cap S_{2R})})\leq$$
$$\leq C(|D^2 w|_p+|\nabla w|_{L^{np/(n-p)}(\Omega\cap S_{2R})})^a |w|_q^{1-a}.$$

From Lemma 3.2 and inequality (3.1), we obtain

$$|\nabla w_1|_{L^r(\Omega\cap S_{2R})}\leq C|P\Delta w|_p^a |w|_q^{1-a}. \tag{4.33}$$

Estimates (4.32) and (4.33) imply (2.19).

Now we consider the case of $p\geq n$. We commence cosidering the case of $r=p=q$. Making an integration by parts and applying the Hölder inequality, we have

$$|\nabla w|_p^p=\int_\Omega |\nabla w(x)|^{p-2}\nabla w(x):\ \nabla w(x)\,dx\leq$$
$$\leq (p-1)\int_\Omega |\nabla w(x)|^{p-2}|D^2 w(x)|\ |w(x)|\,dx\leq (p-1)|\nabla w|_p^{p-2}|D^2 w|_p|w|_p.$$

Therefore, from (3.1)

$$|\nabla w|_p\leq C|P\Delta w|_p^{1/2}|w|_p^{1/2}. \tag{4.34}$$

Now, we consider the case $q\leq p$. From (2.1) it follows that $w(x)\in J^p(\Omega)$ and $|w|_p\leq \leq C|P\Delta w|_p^b |w|_q^{1-b}$ with b given in (2.2). Thus, from (4.34) we deduce also

$$|\nabla w|_p\leq C|P\Delta w|_p^{1/2+b/2}|w|_q^{1/2-b/2}. \tag{4.35}$$

Finally, if $r\geq p\geq q$, before we note that inequality (4.35) ensures that $\nabla w(x)\in L^p(\Omega)$, after which we can apply inequality (2.6) and obtain

$$|\nabla w|_r\leq C|D^2 w|_p^c|\nabla w|_p^{1-c}, \tag{4.36}$$

with c given in (2.7). We estimate the right hand side of (4.33) by inequality (3.1) and (4.32):

$$|\nabla w|_r\leq C|P\Delta w|_p^{1/2+c/2+(b/2)(1-c)}|w|_q^{(1/2)(1-b)(1-c)}.$$

Making a simple computation we have $a=1/2+c/2+(b/2)(1-c)$ and $1-a= =(1/2)(1-b)(1-c)$. The theorem is completely proved.

PROOF OF COROLLARY 2.1. – The proof of the corollary is very easy. Infact to prove (2.21) we observe that from (2.1) and (2.6) the following inequalities holds:

$$|w|_r^\xi\leq C|P\Delta w|_p^{a\xi}|w|_q^{\xi-a\xi},\qquad \xi\in[0,1], \tag{4.31}$$

$$|w|_r^\eta\leq C|\nabla w|_p^{b\eta}|w|_q^{\eta-b\eta},\qquad \eta\in[0,1], \tag{4.32}$$

Therefore a suitable coupling (4.31) and (4.32) implies (2.21) with $\chi = \xi/(\xi + \eta)$ and $\chi' = \eta/(\xi + \eta)$. To prove (2.23) we observe that $\forall \bar{r} \in [q, ns/(n-s)]$ we have from (2.1)

$$|w|_r \le C|P\Delta w|_p^a |w|_{\bar{r}}^{1-a}, \tag{4.33}$$

with a satysfying (2.2). On the other hand in virtue of Theorem 2.3

$$|w|_{\bar{r}} \le C|\nabla w|_s^\chi |w|_q^{1-\chi}.$$

Substuting this last inequality (4.33) we have (2.23). The arbitrarity choosen of $\bar{r} \in [q, ns/n-s]$ implies one of $\chi \in [0, 1]$. For $s \in [1, n)$, $q > ns/(n-s)$, $r \in [ns/(n-s), q]$, we have (2.6) since (2.1) does not hold. Finally for $s > n$ and $r \ge q$ to prove (2.21) it is sufficient to repeat the argument lines already employed for the above case of $s \in [1, n)$, $ns/(n-s) > q$, $r \in [q, ns/(n-s)]$.

Acknowledgment – This paper is performed under the auspices of Italian CNR and Murst Contracts 40%-60% at the University of Basilicata (Potenza).

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