The Annals of Statistics
1977, Vol. 5, No. 2, 316–329

SOME INVARIANCE PRINCIPLES RELATING TO JACKKNIFING AND THEIR ROLE IN SEQUENTIAL ANALYSIS[1]

BY PRANAB KUMAR SEN

University of North Carolina at Chapel Hill

For a broad class of jackknife statistics, it is shown that the Tukey estimator of the variance converges almost surely to its population counterpart. Moreover, the usual invariance principles (relating to the Wiener process approximations) usually filter through jackknifing under no extra regularity conditions. These results are then incorporated in providing a bounded-length (sequential) confidence interval and a preassigned-strength sequential test for a suitable parameter based on jackknife estimators.

1. Introduction. The jackknife estimator, originally introduced for bias reduction by Quenouille and extended by Tukey for robust interval estimation, has been studied thoroughly by a host of workers during the past twenty years; along with some extensive bibliography, detailed studies are made in the recent papers of Arvesen (1969), Schucany, Gray and Owen (1971), Gray, Watkins and Adams (1972) and Miller (1974). One of the major concerns is the asymptotic normality of the studentized form of the jackknife statistics. The purpose of the present investigation is to focus on some deeper asymptotic properties of jackknife estimators and to stress their role in the asymptotic theory of sequential procedures based on jackknifing. Specifically, the almost sure convergence of the Tukey estimator of the variance is established here for a broad class of jackknife statistics and their asymptotic normality results are strengthened to appropriate (weak as well as strong) invariance principles yielding Wiener process approximations for the tail-sequence of jackknife estimators. These results are then incorporated in providing (i) a bounded-length (sequential) confidence interval and (ii) a prescribed-strength sequential test for a suitable parameter based on jackknife estimators.

Section 2 deals with the preliminary notions along with some new interpretations of the jackknife estimator and the Tukey estimator of the variance. For convenience of presentation, in Section 3 we adopt the framework of Arvesen (1969) and present the invariance principles for jackknifing U-statistics. Section 4 displays parallel results for general estimators. The two sequential problems of estimation and testing are treated in the last two sections of the paper.

Received March 1976; revised August 1976.

[1] Work supported by the Air Force Office of Scientific Research, A.F.S.C., U.S.A.F., Grant No. AFOSR-74-2736.

AMS 1970 *subject classifications.* Primary 60B10, 62E20, 62G35, 62L10.

Key words and phrases. Almost sure convergence, Brownian motions, bounded-length confidence intervals, invariance principles, preassigned strength sequential tests, jackknife, stopping time, tightness, Tukey estimator of the variance, U-statistics, von Mises' functionals.

2. Preliminary notions. Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv) with a distribution function (df) F, and let

$$\hat{\theta}_n = T_n(X_1, \cdots, X_n), \qquad n \geqq 1 \tag{2.1}$$

be a sequence of estimators of a parameter θ, such that

$$E\hat{\theta}_n = \theta + n^{-1}\beta_1 + n^{-2}\beta_2 + \cdots (\Rightarrow E(\hat{\theta}_n - \theta) = O(n^{-1})) \tag{2.2}$$

where the β_j are unknown constants. Let us denote by

$$\hat{\theta}^i_{n-1} = T_{n-1}(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n), \qquad 1 \leqq i \leqq n, \tag{2.3}$$

$$\hat{\theta}_{n,i} = n\hat{\theta}_n - (n-1)\hat{\theta}^i_{n-1}, \qquad 1 \leqq i \leqq n, \tag{2.4}$$

$$\theta_n^* = n^{-1} \sum_{i=1}^n \hat{\theta}_{n,i} = n\hat{\theta}_n - (n-1)\{n^{-1} \sum_{i=1}^n \hat{\theta}^i_{n-1}\}. \tag{2.5}$$

Then, θ_n^* is termed the *jackknife estimator* of θ. Clearly, by (2.2), (2.3) and (2.5),

$$E\theta_n^* = \theta - \beta_2/n(n-1) + \cdots (\Rightarrow E(\theta_n^* - \theta) = O(n^{-2})). \tag{2.6}$$

Further, let

$$V_n^* = \frac{1}{n-1} \sum_{i=1}^n [\hat{\theta}_{n,i} - \theta_n^*]^2 = (n-1) \sum_{i=1}^n \left(\hat{\theta}^i_{n-1} - \frac{1}{n} \sum_{j=1}^n \hat{\theta}^j_{n-1}\right)^2 \tag{2.7}$$

Tukey has suggested that V_n^* may be used as an estimator of the variance of $n^{\frac{1}{2}}(\theta_n^* - \theta)$, and further,

$$n^{\frac{1}{2}}(\theta_n^* - \theta)/[V_n^*]^{\frac{1}{2}} \to_{\mathscr{D}} \mathscr{N}(0, 1) \quad \text{as} \quad n \to \infty. \tag{2.8}$$

Various authors have established (2.8) under suitable regularity conditions. Our intention is to obtain stronger results concerning (i) the almost sure (a.s.) convergence of V_n^* and (ii) Wiener process approximations for the tail-sequence $\{\theta_k^* - \theta; k \geqq n\}$.

For simplicity, we assume that $p = 1$, i.e., the X_i are real valued and $R = (-\infty, \infty)$. For every $n\,(\geqq 1)$, the order statistics corresponding to $X_1, \cdots, X_n$ are denoted by $X_{n,1} \leqq \cdots \leqq X_{n,n}$. Let $\mathscr{C}_n = \mathscr{C}(X_{n,1}, \cdots, X_{n,n}, X_{n+1}, \cdots)$ be the σ-field generated by $(X_{n,1}, \cdots, X_{n,n})$ and by X_{n+j}, $j \geqq 1$. Then $\mathscr{C}_n$ is nonincreasing in $n\,(\geqq 1)$. Note that given $\mathscr{C}_n$, X_{n+j}, $j \geqq 1$ are all held fixed while $(X_1, \cdots, X_n)$ are interchangeable and assume all possible permutations of $(X_{n,1}, \cdots, X_{n,n})$ with equal conditional probability $(n!)^{-1}$. Hence,

$$E(\hat{\theta}_{n-1} \mid \mathscr{C}_n) = n^{-1} \sum_{i=1}^n \hat{\theta}^i_{n-1} \quad \text{a.e.}, \tag{2.9}$$

and, therefore, by (2.5) and (2.9),

$$\begin{aligned} \theta_n^* &= n\hat{\theta}_n - (n-1)E(\hat{\theta}_{n-1} \mid \mathscr{C}_n) \\ &= \hat{\theta}_n + (n-1)E\{\hat{\theta}_n - \hat{\theta}_{n-1} \mid \mathscr{C}_n\} \quad \text{a.e.} \end{aligned} \tag{2.10}$$

Clearly, if $\{\hat{\theta}_n, \mathscr{C}_n\}$ is a reverse-martingale, $\theta_n^* = \hat{\theta}_n$; otherwise, the jackknifing consists in adding on the correction factor

$$\theta_n^* - \hat{\theta}_n = (n-1)E\{(\hat{\theta}_n - \hat{\theta}_{n-1}) \mid \mathscr{C}_n\}. \tag{2.11}$$

It follows by similar arguments that

$$(2.12)\qquad V_n^* = n(n-1)\,\mathrm{Var}\,\{(\hat\theta_n - \hat\theta_{n-1}) \mid \mathscr{C}_n\} = n(n-1)\{E[(\hat\theta_n - \hat\theta_{n-1})^2 \mid \mathscr{C}_n] - (E[(\hat\theta_n - \hat\theta_{n-1}) \mid \mathscr{C}_n])^2\}\,.$$

These interpretations and representations for jackknifing are quite useful for our subsequent results.

For further reduction of bias, higher order jackknife estimators have been proposed by various workers (see [6, 12]). The second order jackknife estimator (see (4.20) of [12]) can be written in our notations as

$$(2.13)\qquad \theta_n^{**} = \tfrac{1}{2}\{n^2\hat\theta_n - 2(n-1)^2 E(\hat\theta_{n-1} \mid \mathscr{C}_n) + (n-2)^2 E(\hat\theta_{n-2} \mid \mathscr{C}_n)\}$$

and a similar expression holds for the higher order jackknifing. In fact, we have also a second interpretation for θ_n^*, θ_n^{**} etc. from the classical least squares point of view. We denote by θ_n^{k*} the kth order jackknife estimator of θ based on $X_1, \cdots, X_n$, for $k = 0, 1, \cdots, n-1$. Then, we have the following.

THEOREM 2.1. *Consider the following $k+1$ simultaneous equations in the $k+1$ unknown parameters θ and $\beta_1, \cdots, \beta_k$:*

$$(2.14)\qquad \hat\theta_{n-i} = \theta + (n-i)^{-1}\beta_1 + \cdots + (n-i)^{-k}\beta_k \qquad \text{for} \quad i = 0, 1, \cdots, k$$

where the $\hat\theta_{n-i}$ are defined by (2.1), *and let $\tilde\theta_n^{\,k}$ be the solution for θ. Then*

$$(2.15)\qquad \theta_n^{k*} = E(\tilde\theta_n^{\,k} \mid \mathscr{C}_n) \qquad \textit{for every} \quad 0 \leqq k < n\,.$$

PROOF. Note that by (2.2), neglecting terms of the order n^{-k-1},

$$(2.16)\qquad E(\hat\theta_{n-i}) = \theta + (n-i)^{-1}\beta_1 + \cdots + (n-i)^{-k}\beta_k \qquad \text{for} \quad i = 0, 1, \cdots, k$$

where

$$(2.17)\qquad \mathbf{A}_{n,k} = ((n-i)^{-i'})_{i,i'=0,\cdots,k} \quad \text{is nonsingular.}$$

Let $a_{n,k}^{ii'}$ be the element in the i, i' position of the matrix $\mathbf{A}_{n,k}^{-1}$ for $i, i' = 0, \cdots, k$. Then, under (2.16), the classical least squares estimator of θ based on the $k+1$ estimators $\hat\theta_{n-i}$, $i = 0, \cdots, k$ $(n > k)$ is given by

$$(2.18)\qquad \hat\theta_n^{\,k} = \textstyle\sum_{i=0}^{k} a_{n,k}^{0i}\hat\theta_{n-i}\,.$$

On the other hand, by the same conditional arguments as leading to (2.9), for every $i: 0 \leqq i \leqq k$, we have

$$(2.19)\qquad E(\hat\theta_{n-1} \mid \mathscr{C}_n) = \binom{n}{n-i}^{-1} \textstyle\sum_{n,i} T_{n-i}(X_{j_1}, \cdots, X_{j_{n-i}}) = T_{n,i}^*\,, \quad \text{say,}$$

where the summation $\sum_{n,i}$ extends over all $1 \leqq j_1 < \cdots < j_{n-i} \leqq n$ for $i = 0, \cdots, k$. Thus, by (2.18) and (2.19),

$$(2.20)\qquad E(\hat\theta_{n-i} \mid \mathscr{C}_n) = \textstyle\sum_{i=0}^{k} a_{n,k}^{0i}\, T_{n,i}^*\,.$$

For the model (2.16), the kth order jackknife estimator defined by Schucany, Gray and Owen (1971) (cf. their Definition 4.1) agrees with our (2.20), and hence the proof of the theorem is complete. In fact, for the comparatively more general model (4.16) of Schucany et al. (1971), the same equivalence follows on parallel lines. □

Since in Section 3 we shall be concerned with jackknifing functions of U-statistics, we find it convenient to introduce the following notations at this stage. Let $\phi(X_1, \cdots, X_m)$, symmetric in its m arguments, be a Borel measurable *kernel* of *degree* m ($\geqq 1$) and consider the *regular functional* (estimable parameter)

$$(2.21) \qquad \xi = \xi(F) = \int_{R^{mp}} \cdots \int \phi(x_1, \cdots, x_m)\, dF(x_1) \cdots dF(x_m)\,, \qquad F \in \mathscr{F}$$

where $\mathscr{F} = \{F : |\xi(F)| < \infty\}$. Then, for $n \geqq m$, the U-statistic corresponding to ξ is defined by

$$(2.22) \qquad U_n = \binom{n}{m}^{-1} \sum_{C_{n,m}} \phi(X_{i_1}, \cdots, X_{i_m})\,; \quad C_{n,m} = \{1 \leqq i_1 < \cdots < i_m \leqq n\}\,.$$

Note that $EU_n = \xi(F)$ and U_n is a symmetric function of $X_1, \cdots, X_n$. Further, let

$$(2.23) \qquad \zeta_h = \mathrm{Var}\,\{\phi_h(X_1, \cdots, X_h)\}\,;$$

$$(2.24) \qquad \phi_h(x_1, \cdots, x_h) = E\phi(x_1, \cdots, x_h, X_{h+1}, \cdots, X_m)$$

for $h = 0, \cdots, m$, where $\zeta_0 = 0$ and $\phi_0 = \xi$. We assume that

$$(2.25) \qquad 0 < \zeta_1\,, \qquad \zeta_m < \infty \quad (\text{where } \zeta_1 \leqq m^{-1}\zeta_m)\,.$$

3. Invariance principles relating to jackknifing U-statistics. We shall be concerned here mainly with the following two types of estimators:

(i) Let g, defined on R, have a bounded second derivative in some neighborhood of ξ, and

$$(3.1) \qquad \hat{\theta}_n = g(U_n)\,, \quad \forall\, n \geqq m\,.$$

(ii) For some positive integer q, we have

$$(3.2) \qquad \hat{\theta}_n = \sum_{s=0}^{q} \alpha_{n,s} U_n^{(s)}\,, \qquad n \geqq m\,,$$

where $U_n^{(0)} = U_n$ is an unbiased estimator of $\theta = \xi(F)$,

$$(3.3) \qquad \alpha_{n,0} = 1 + n^{-1}c_{0,1} + n^{-2}c_{0,2} + O(n^{-3})\,,$$

$U_n^{(1)}, \cdots, U_n^{(q)}$ are appropriate U-statistics with expectations $\theta_1, \cdots, \theta_q$ (unknown but finite) and

$$(3.4) \qquad \alpha_{n,h} = n^{-h}c_{h,0} + O(n^{-h-1})\,, \qquad h \geqq 1\,;$$

the $c_{s,j}$ are real constants; possibly, some being equal to 0. The classical von Mises' (1947) differentiable statistical function (corresponding to $\xi(F)$) is a special case of (3.2) with $q = m$ and $c_{0,1} = -\binom{m}{2}$.

First, we consider the following.

THEOREM 3.1. *For* $\{\hat{\theta}_n\}$ *defined by* (3.1) *or* (3.2)—(3.4),

$$(3.5) \qquad V_n^* \to \gamma^2 \quad \text{a.s.,} \quad as \quad n \to \infty\,,$$

where

$$\gamma^2 = [g'(\xi)]^2 m^2\zeta_1\,, \quad \textit{for } (3.1) \qquad (3.6)$$
$$\qquad = m^2\zeta_1\,, \quad \textit{for } (3.2).$$

PROOF. In the context of weak convergence of Rao–Blackwell estimator of distribution functions, Bhattacharyya and Sen (1977) have shown that under (2.25),

$$n(n-1)E[(U_{n-1}-U_n)^2 \mid \mathscr{C}_n] \to m^2\zeta_1 \quad \text{a.s.,} \quad \text{as} \quad n\to\infty\,. \qquad (3.7)$$

On the other hand, as in Section 2,

$$n(n-1)E[(U_{n-1}-U_n)^2 \mid \mathscr{C}_n] = (n-1)\textstyle\sum_{i=1}^n [U_{n-1}^i - U_n]^2 \qquad (3.8)$$

where the U_{n-1}^i are defined as in (2.3) with T_{n-1} being replaced by U_{n-1}. Hence, from (3.7) and (3.8), we obtain that

$$\max_{1\le i\le n}(U_{n-1}^i - U_n)^2 = O(n^{-1}) \quad \text{a.s.,} \quad \text{as} \quad n\to\infty\,. \qquad (3.9)$$

Further, $\{U_n, \mathscr{C}_n, n \geqq m\}$ is a reverse martingale, so that $U_n \to \xi(F)$ a.s., as $n\to\infty$, and hence, by (3.9),

$$\max_{1\le i\le n}|U_{n-1}^i - \xi(F)| \to 0 \quad \text{a.s.,} \quad \text{as} \quad n\to\infty\,. \qquad (3.10)$$

First, consider the case of (3.1). Then, we have

$$\begin{aligned}\hat\theta_{n-1} - \hat\theta_n &= g(U_{n-1}) - g(U_n)\\ &= g'(U_n)[U_{n-1}-U_n] + \tfrac12 g''(hU_n + (1-h)U_{n-1})[U_{n-1}-U_n]^2\,,\\ &\qquad 0<h<1\,.\end{aligned} \qquad (3.11)$$

Note that $E[U_{n-1}\mid\mathscr{C}_n] = U_n$ a.e. and further by (3.7), (3.8), (3.10) and the boundedness of g'' (in a neighborhood of ξ), we have

$$\begin{aligned}&|E\{g''(hU_n + (1-h)U_{n-1})[U_{n-1}-U_n]^2 \mid \mathscr{C}_n\}|\\ &\leqq \max_{1\le i\le n}|g''(h_iU_n + (1-h_i)U_{n-1}^i)|\{n^{-1}\textstyle\sum_{i=1}^n (U_{n-1}^i - U_n)^2\}\\ &= O(n^{-2}) \quad \text{a.s.,} \quad \text{as} \quad n\to\infty\,, \quad \text{where} \quad 0<h_i<1\\ &\qquad \text{for every} \quad i=1,\cdots,n\,.\end{aligned} \qquad (3.12)$$

Hence, we obtain from (3.11) and (3.12) that

$$E(\hat\theta_{n-1} - \hat\theta_n \mid \mathscr{C}_n) = O(n^{-2}) \quad \text{a.s.,} \quad \text{as} \quad n\to\infty\,. \qquad (3.13)$$

Similarly,

$$\begin{aligned}&\operatorname{Var}\{(\hat\theta_{n-1} - \hat\theta_n)\mid\mathscr{C}_n\}\\ &= E\{(\hat\theta_{n-1} - \hat\theta_n)^2\mid\mathscr{C}_n\} + O(n^{-4}) \quad \text{a.s.}\\ &= n^{-1}\textstyle\sum_{i=1}^n [g(U_{n-1}^i) - g(U_n)]^2 + O(n^{-4}) \quad \text{a.s.,} \quad \text{as} \quad n\to\infty\,.\end{aligned} \qquad (3.14)$$

Further, as in (3.12),

$$\begin{aligned}&|n^{-1}\textstyle\sum_{i=1}^n [g(U_{n-1}^i) - g(U_n)]^2 - [g'(U_n)]^2 n^{-1}\sum_{i=1}^n (U_{n-1}^i - U_n)^2|\\ &\leqq \{\max_{1\le i\le n}|(g'(h_iU_n + (1-h_i)U_{n-1}^i))^2 - (g'(U_n))^2|\}\\ &\qquad \times\left\{\frac1n \textstyle\sum_{i=1}^n (U_{n-1}^i - U_n)^2\right\} \quad (0<h_i<1)\\ &= \{o(1) \text{ a.s.}\}\{O(n^{-2}) \text{ a.s.}\} = o(n^{-2}) \quad \text{a.s.,} \quad \text{as} \quad n\to\infty\,,\end{aligned} \qquad (3.15)$$

where by (3.9), (3.8) and the a.s. convergence of U_n to $\xi(F)$,

$$[g'(U_n)]^2(n-1)\sum_{i=1}^{n}[U_{n-1}^i - U_n]^2 \to m^2\zeta_1[g'(\xi)]^2 \quad \text{a.s.,} \quad \text{as} \quad n\to\infty . \tag{3.16}$$

Hence, from (3.14)–(3.16), we obtain that

$$\begin{aligned} V_n^* &= (n-1)\sum_{i=1}^{n}[g(U_{n-1}^i) - g(U_n)]^2 \\ &= n(n-1)\,\mathrm{Var}\,\{(\hat\theta_{n-1} - \hat\theta_n)\,|\,\mathscr{C}_n\} \\ &\to m^2\zeta_1[g'(\xi)]^2 = \gamma^2 \quad \text{a.s.,} \quad \text{as} \quad n\to\infty . \end{aligned} \tag{3.17}$$

For the case of (3.2), we note that

$$\begin{aligned} &\alpha_{n-1,0}U_{n-1}^{(0)} - \alpha_{n,0}U_n^{(0)} \\ &\qquad = (U_{n-1} - U_n) + c_{0,1}\{(n-1)^{-1}U_{n-1} - n^{-1}U_n\} + O(n^{-2}) \quad \text{a.s.,} \end{aligned} \tag{3.18}$$

$$\alpha_{n-1,h}U_{n-1}^{(h)} - \alpha_{n,h}U_n^{(h)} = \alpha_{n-1,h}(U_{n-1}^{(h)} - U_n^{(h)}) + O(n^{-h-1})U_n^{(h)} , \quad h \geqq 1 , \tag{3.19}$$

and hence, the proof of (3.5) follows on parallel lines. □

REMARK 1. From (2.11) and (3.13), we obtain that for every $\varepsilon > 0$,

$$n^{1-\varepsilon}|\theta_n^* - \hat\theta_n| \to 0 \quad \text{a.s.,} \quad \text{as} \quad n\to\infty . \tag{3.20}$$

The last result is of fundamental importance to the main results of this section.

REMARK 2. By virtue of (3.14)–(3.16), V_n^* is asymptotically equivalent to

$$[g'(U_n)]^2 s_n^2 \qquad \text{where} \quad s_n^2 = (n-1)\sum_{i=1}^{n}[U_{n-1}^i - U_n]^2 ; \tag{3.21}$$

in case (3.2), (3.21) holds with $g'(U_n) \equiv 1$. Let us also denote by

$$V_{ni} = \binom{n-1}{m-1}^{-1}\sum\nolimits_{n,i}\phi(X_i, X_{i_2}, \cdots X_{i_m}) , \qquad 1 \leqq i \leqq n , \tag{3.22}$$

where the summation $\sum_{n,i}$ extends over all $1 \leqq i_2 < \cdots < i_m \leqq n$ with $i_j \neq i$ for $2 \leqq j \leqq m$. Then, $U_n = n^{-1}\sum_{i=1}^{n} V_{ni}$. Further, let

$$V_n = (n-1)^{-1}\sum_{i=1}^{n}[V_{ni} - U_n]^2 . \tag{3.23}$$

Sen (1960) has shown that V_n is a distribution-free estimator of ζ_1. It is interesting to note that by definition

$$\binom{n-1}{m}U_{n-1}^i + \binom{n-1}{m-1}V_{ni} = \binom{n}{m}U_n , \quad \forall\, 1 \leqq i \leqq n , \tag{3.24}$$

and, as a result, it follows by routine steps that

$$s_n^2 = m^2(n-1)^2(n-m)^{-2}V_n , \quad \forall\, n > m . \tag{3.25}$$

Hence, the a.s. convergence of s_n^2 (to $m^2\zeta_1$) insures the same for V_n (to ζ_1). However, from the computational point of view, the labor involved in the computation of V_n is $O(n^m)$ whereas for s_n^2, it is $O(n^{m+1})$. Hence, V_n should be preferable to s_n^2. (3.25) will be of use in Section 5.

By virtue of (3.5) and (3.20) and the invariance principles for U-statistics, studied by Loynes (1970), Miller and Sen (1972) and Sen (1974b), we are in a position to present the following results (without derivation):

(i) Consider a sequence $\{W_n^*\}$ of stochastic processes, where

$$W_n^* = \{W_n^*(t) = n^{\frac{1}{2}}[\theta^*_{k_n(t)} - \theta]/\gamma, 0 \leqq t \leqq 1\}, \qquad n > m, \tag{3.26}$$

and $k_n(t) = \min\{k : n/k \leqq t\}$, $0 \leqq t \leqq 1$. Note that $W_n^*(0)$ is equal to 0 with probability 1 and for every $n\,(\geqq m)$, W_n^* belongs to the $D[0, 1]$ space with which we associate the J_1-topology. Further, let $W^* = \{W^*(t), 0 \leqq t \leqq 1\}$ be a standard Brownian motion on $[0, 1]$. Then, as $n \to \infty$,

$$W_n^* \to_{\mathscr{D}} W^*, \quad \text{in the } J_1\text{-topology on} \quad D[0, 1]. \tag{3.27}$$

(ii) Let $S = \{S(t), t \in [0, \infty)\}$ be a random process defined by

$$\begin{aligned} S(t) &= 0, & 0 \leqq t < m+1, \\ &= k(\theta_k^* - \theta)/\gamma, & k \leqq t < k+1, \quad k \geqq m+1, \end{aligned} \tag{3.28}$$

and, we assume that for some $r > 2$,

$$E|\phi(X_1, \cdots, X_m)|^r < \infty. \tag{3.29}$$

Then, there exists a standard Wiener process $W = \{W(t), t \in [0, \infty)\}$ on $[0, \infty)$, such that

$$S(t) = W(t) + o(t^{\frac{1}{2}}) \quad \text{a.s.,} \quad \text{as} \quad t \to \infty. \tag{3.30}$$

(iii) In (3.1), we have considered $\hat{\theta}_n = g(U_n)$. It is possible to take $\hat{\theta}_n = g(U_n^{(1)}, \cdots, U_n^{(k)})$, for some $k \geqq 1$, where g has bounded second order partial derivatives in a neighborhood of the point $(EU_n^{(1)}, \cdots, EU_n^{(k)})(\in R^k)$. The proof follows as a straightforward extension of what has been done before, and hence, for intended brevity, the details are omitted.

(iv) Jackknifing functions of generalized U-statistics have been considered by Arvesen (1969). Here also, as in Sen (1974c), we may consider the product sigma-field formed by the individual sample sequence $\{\mathscr{C}_n\}$ and express the usual jackknife estimator as the conditional expectation of a linear combination of original estimators for adjacent sample sizes. Further, a result parallel to (3.20) holds in this case. Hence, by virtue of Theorem 2.2 of Sen (1974a), we are in a position to derive a similar invariance principle for the jackknife estimators. Further, by virtue of (3.19)—(3.23) of Sen (1974a), it can be shown that (3.30) extends to a multiparameter Gaussian process. For intended brevity, the details are omitted again.

4. Invariance principles for general $\{\theta_n^*\}$. Structural properties of U-statistics have enabled us to study the invariance principles in Section 3 without having any extra regularity conditions. If $\hat{\theta}_n$ is not a function of U-statistics, we need, however, a few extra regularity conditions to derive similar results. These will be studied here.

Concerning the original sequence of estimators $\{\hat{\theta}_n\}$, we assume that

$$\hat{\theta}_n \to \theta \quad \text{a.s.,} \quad \text{as} \quad n \to \infty, \tag{4.1}$$

$$\delta_n^2 = \operatorname{Var}(\hat{\theta}_n) \downarrow 0 \quad \text{as} \quad n \to \infty; \qquad \lim_{n\to\infty} n\delta_n^2 = \delta^2, \quad 0 < \delta < \infty. \tag{4.2}$$

Let us also define

$$Y_n = n(n-1)(\hat{\theta}_{n-1} - \hat{\theta}_n)^2, \qquad n \geqq 2, \tag{4.3}$$

and assume that

$$E[Y_n \mid \mathscr{C}_n] \to \delta^2 \quad \text{a.s.,} \quad \text{as} \quad n \to \infty, \tag{4.4}$$

$$Y_n \quad \text{is uniformly (in } n\text{) integrable}, \tag{4.5}$$

$$|E(\hat{\theta}_n - \hat{\theta}_{n+1} \mid \mathscr{C}_{n+1})| = o(n^{-\frac{3}{2}}) \quad \text{a.s.,} \quad \text{as} \quad n \to \infty. \tag{4.6}$$

Consider now a sequence of stochastic processes $\{W_n\}$, where

$$W_n = \{W_n(t) = \delta_n^{-1}(\hat{\theta}_{k_n(t)} - \theta), 0 \leqq t \leqq 1\}; \tag{4.7}$$

$$k_n(t) = \min\{k : \delta_k^2/\delta_n^2 \leqq t\}, \qquad 0 \leqq t \leqq 1. \tag{4.8}$$

Then we have the following.

THEOREM 4.1. *Under the assumptions made above, as $n \to \infty$*

$$W_n \to_{\mathscr{D}} W^*, \quad \textit{in the } J_1\textit{-topology on} \quad D[0, 1], \tag{4.9}$$

where W^* is a standard Brownian motion on $[0, 1]$.

OUTLINE OF THE PROOF. Let us write $Q_k = \hat{\theta}_k - \hat{\theta}_{k+1}$, $k \geqq m$. Then, by (4.1),

$$\hat{\theta}_N - \theta = \sum_{k \geq N} Q_k \quad \text{a.s.,} \quad \forall\, N \geqq m. \tag{4.10}$$

Also, let $\tilde{Q}_k = Q_k - E(Q_k \mid \mathscr{C}_{k+1})$, $Z_k = \sum_{s \geq k} \tilde{Q}_s$, $k \geqq m$, and

$$\tilde{W}_n = \{\tilde{W}_n(t) = \delta_n^{-1} Z_{k_n(t)}, 0 \leqq t \leqq 1\}, \tag{4.11}$$

where $\{k_n(t), 0 \leqq t \leqq 1\}$ is defined by (4.8). Then, by (4.2) and (4.6),

$$\begin{aligned} &\sup_{0 \leq t \leq 1} |W_n(t) - \tilde{W}_n(t)| \\ &\quad = \delta_n^{-1}\{\sup_{N \geq n} |\textstyle\sum_{k \geq N} E(Q_k | \mathscr{C}_{k+1})|\} \to_p 0, \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{4.12}$$

On the other hand, $\{Z_n, \mathscr{C}_n; n \geqq m\}$ is a reverse martingale, so that the backward invariance principle of Loynes (1970) holds provided the following conditions hold: as $n \to \infty$,

$$A_n = \delta_n^{-2}\{\textstyle\sum_{k \geq n} E(\tilde{Q}_k^2 \mid \mathscr{C}_{k+1})\} \to_p 1, \tag{4.13}$$

$$B_n = \delta_n^{-2}\{\textstyle\sum_{k \geq n} E(\tilde{Q}_k^2 I(\tilde{Q}_k^2 > \varepsilon\delta_n^2) \mid \mathscr{C}_{k+1})\} \to_p 0, \quad \forall\, \varepsilon > 0. \tag{4.14}$$

(Though Loynes (1970) has assumed a.s. convergence in this respect, his result holds even under the convergence in probability.) Note that by (4.2), (4.3) and (4.6),

$$\begin{aligned} A_n &= \delta_n^{-2}(\textstyle\sum_{k \geq n} \{E(Q_k^2 \mid \mathscr{C}_{k+1}) - [E(Q_k \mid \mathscr{C}_{k+1})]^2\}) \\ &= \delta_n^{-2}(\textstyle\sum_{k \geq n} \{E(Y_{k+1} \mid \mathscr{C}_{k+1})/k(k+1) - o(k^{-3})\}) \\ &\to 1 \quad \text{a.s.,} \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.15}$$

Again, by (4.2) through (4.6), for every $\varepsilon > 0$, as $n \to \infty$,

$$\begin{aligned} &\delta_n^{-2} \textstyle\sum_{k\geq n} E\{Q_k^2 I(Q_k^2 > \varepsilon\delta_n^2)\} \\ (4.16) \qquad &= \delta_n^{-2} \textstyle\sum_{k\geq n} \frac{1}{k(k+1)} E\{Y_k I(Y_k > \varepsilon k(k+1)\delta_n^2)\} \\ &= (\delta_n^{-2})(o(1)) \left(\textstyle\sum_{k\geq n} \frac{1}{k(k+1)}\right) = o(1)\,, \end{aligned}$$

and hence,

$$(4.17) \qquad \delta_n^{-2} \textstyle\sum_{k\geq n} E\{Q_k^2 I(Q_k^2 > \varepsilon\delta_n^2) \mid \mathscr{C}_{k+1}\} \to_p 0 \quad \text{as} \quad n \to \infty\,.$$

Since, by (4.2) and (4.3), $\delta_n^{-2} \sum_{k\geq n} \{E(Q_k \mid \mathscr{C}_{k+1})\}^2 \to 0$ a.s., as $n \to \infty$, (4.14) follows from (4.17), the definition of the $\tilde{Q}_k$ and the inequality that for every $\varepsilon' > 0$, $(\sum_{i=1}^2 u_i^2) I(\sum_{i=1}^2 u_i^2 > 2\varepsilon') \leqq 4\{\sum_{i=1}^2 u_i^2 I(u_i^2 > \varepsilon')\}$. Hence, the proof of the theorem is complete.

REMARK. For dependent random variables, McLeish (1974) has considered some (forward) invariance principles. In the spirit of Loynes (1970), our Theorem 4.1 provides an analogous backward invariance principle.

Since (4.4) corresponds to (3.7), virtually repeating the proof of Theorem 3.1, it follows that under the same conditions on g, as in Section 3,

$$(4.18) \qquad |\theta_n^* - \hat{\theta}_n| = (n-1)|E\{(\hat{\theta}_{n-1} - \hat{\theta}_n) \mid \mathscr{C}_n\}| = o(n^{-\frac{1}{2}}) \quad \text{a.s.},$$

by (4.6), and

$$(4.19) \qquad V_n^* \to \delta^2 \quad \text{a.s.,} \quad \text{as} \quad n \to \infty\,.$$

Hence, if in (4.7), we replace $\{\hat{\theta}_k\}$ by $\{\theta_k^*\}$ and denote the corresponding process by W_n^*, then (4.9) holds for $\{W_n^*\}$ as well. Further, δ_n^{-1} may also be replaced by $n^{\frac{1}{2}}(V_n^*)^{-\frac{1}{2}}$.

The conditions (4.1), (4.2), (4.4), (4.5) and (4.6) are most conveniently verifiable if $\hat{\theta}_n$ can be expressed as

$$(4.20) \qquad \hat{\theta}_n = m_n + r_n\,,$$

where $\{m_n, \mathscr{C}_n\}$ is a reverse martingale and $|r_{n-1} - r_n| = o(n^{-\frac{3}{2}})$ a.s., as $n \to \infty$.

5. Asymptotic sequential confidence interval based on jackknifing. Tukey proposed the use of (2.8) for a robust confidence interval for θ. By virtue of our invariance principles, we are in a position to consider the following robust sequential interval estimation problem.

Suppose θ, γ^2, $\hat{\theta}_n$, θ_n^* and V_n^* are defined as before. The underlying *df* F, and hence, θ and γ^2 being unknown, it is desired to determine (sequentially) a confidence interval for θ having a maximum-width $2d$, $d > 0$ being predetermined, and a preassigned confidence coefficient $1 - \alpha$, $0 < \alpha < 1$. For every $n \geqq 1$ and $d > 0$, let

$$(5.1) \qquad I_n(d) = \{\theta : \theta_n^* - d \leqq \theta \leqq \theta_n^* + d\}\,,$$

and let τ_α be the upper $100\alpha\%$ point of the standard normal df. Finally, let $n_0\,(=n_0(d))$ be the initial sample size. Then, we consider a *stopping variable* $N\,(=N(d))$, defined by

$$N = \text{smallest integer} \quad n\,(\geqq n_0) \quad \text{such that} \quad V_n^* \leqq nd^2/\tau_{\alpha/2}^2\,; \tag{5.2}$$

if no such n exist, we let $N=\infty$. Whenever $N<\infty$, the proposed confidence interval for θ is $I_N(d)$, defined by (5.1) for $n=N=N(d)$. The above procedure is a direct adaptation of the Chow-Robbins (1965) procedure under our jackknifing setup.

THEOREM 5.1. *Under the hypothesis of Theorem* 3.1 (*or* 4.1),

$$\lim_{d\to 0} P\{\theta \in I_{N(d)}(d)\} = 1-\alpha\,, \tag{5.3}$$

$$\lim_{d\to 0} \{(N(d)d^2)/(\tau_{\alpha/2}^2\gamma^2)\} = 1 \quad \text{a.s.} \tag{5.4}$$

If, in addition $E\{\sup_n V_n^*\}<\infty$, *then*

$$\lim_{d\to 0} \{(d^2EN(d)/(\tau_{\alpha/2}^2\gamma^2)\} = 1\,. \tag{5.5}$$

OUTLINE OF THE PROOF. We follow the line of attack of Chow and Robbins (1965). We need to show that (a) $V_n^* \to \gamma^2$ a.s., as $n\to\infty$, (b) (2.8) holds and (c) for every $\varepsilon>0$ and $\eta>0$, there exist a $\delta(0<\delta<1)$ and an n^*, such that for $n\geqq n^*$,

$$P\{\max_{n':|n-n'|\leqq \delta n} n^{\frac{1}{2}}|\theta_{n'}^* - \theta_n^*| > \varepsilon\} < \eta\,. \tag{5.6}$$

Now (a) has already been proved, (b) is a direct consequence of (3.27) and finally (c) follows from the *tightness* property of $\{W_n^*\}$ which, in turn, is insured by (3.27). □

The condition that $E(\sup_n V_n^*)<\infty$, needed for (5.5), however, does not follow from the hypothesis of Theorem 3.1 (or 4.1); nor is it a very readily verifiable one. It is possible to obtain (5.5) under a somewhat different condition which is more easily verifiable.

THEOREM 5.2. *If* $\{\hat\theta_n\}$ *is defined by* (3.1) *or* (3.2)—(3.4) *and the hypothesis of Theorem* 3.1 *holds, then* $E\{|\phi(X_1,\cdots,X_m)|^r) < \infty$ *for some* $r>4$ *insures* (5.5).

PROOF. Consider the estimator V_n, defined by (3.23). It follows from Sproule (1969) that V_n can be expressed as a linear combination of several U-statistics whose moments of the order $q\,(>0)$ exist whenever $E|\phi|^{2q}<\infty$. As such, using Theorem 1 of Sen (1974c), it follows that for every $\varepsilon>0$, there exist a positive $K_\varepsilon\,(<\infty)$ and an $n_0(\varepsilon)$ such for $n\geqq n_0(\varepsilon)$,

$$P\{|V_n - \zeta_1| > \varepsilon/2\} \leqq K_\varepsilon n^{-s}\,, \quad s = r/4 > 1\,. \tag{5.7}$$

Further, $g'(t)$ has a bounded derivative in a neighborhood of $t=\xi$, and hence, by Theorem 1 of Sen (1974c), again, for $\hat\theta_n$ defined as in (3.1),

$$P\{|g'(U_n) - g'(\xi)| > \varepsilon/2\} \leqq K_\varepsilon n^{-\frac{1}{2}r}\,, \quad \forall\, n \geqq n_0(\varepsilon)\,. \tag{5.8}$$

From (3.15), (3.21), (3.25), (5,7) and (5.8), it follows by some standard steps that for every $\eta > 0$, there exist a constant K_η ($< \infty$) and an $n_0(\eta)$, such that for $n \geqq n_0(\eta)$,

$$P\{|V_n^* - \gamma^2| > \eta\} \leqq K_\eta n^{-s}; \quad s = r/4 > 1 . \tag{5.9}$$

Having established this, we may proceed as in the proof of Theorem 3.1 of Sen and Ghosh (1971) [namely, as in their (5.16)—(5.19)], and complete the proof of (5.5) by using (5.3) and (5.9). For brevity, the details are omitted. □

6. Sequential tests based on jackknife estimators. The embedding of Wiener processes in (3.30) and the strong convergence of V_n^* (to γ^2) in (3.5) enable us to construct the following type of asymptotic sequential tests; for further motivation of this type of procedure, we may refer to Sen (1973) and Sen and Ghosh (1974).

Consider a suitable parameter θ (for which the sequences $\{\hat{\theta}_n\}$ and $\{\theta_n^*\}$ of estimators are available sequentially), and suppose that we desire to test

$$H_0: \theta = \theta_0 \quad \text{vs.} \quad H_1: \theta = \theta_1 = \theta_0 + \Delta , \qquad \Delta > 0 , \tag{6.1}$$

where θ_0 and Δ are specified and we like the test to have the prescribed strength (α, β). Since the df F is not known, no fixed sample size test sounds feasible and we take recourse to the following sequential procedure:

Suppose that $0 < \alpha, \beta < \frac{1}{2}$ and consider two positive numbers $(A, B): 0 < B < 1 < A < \infty$, where $\beta/(1-\alpha) \leqq B$ and $(1-\beta)/\alpha \geqq A$. Starting with an initial sample of size n_0 $(= n_0(\Delta))$, continue drawing observations one by one as long as

$$bV_m^* < m\Delta[\theta_m^* - (\theta_0 + \theta_1)/2] < aV_m^* , \qquad m \geqq n_0(\Delta) , \tag{6.2}$$

where $a = \log A$, $b = \log B$ $(\Rightarrow -\infty < b < 0 < a < \infty)$, θ_m^* is the jackknife estimator of θ based on $X_1, \cdots, X_m$ [viz., (2.5)] and V_m^* is defined as in (2.7). If, for the first time, (6.2) is vitiated for $m = N$ and $\Delta[\theta_N^* - (\theta_0 + \theta_1)/2]$ is $\leqq bV_N^*$ (or $\geqq aV_N^*$), accept H_0 (or H_1); the *stopping variable* N is denoted by $N(\Delta)$.

Since $m^{-\frac{1}{2}}V_m^* \to 0$ a.s., as $m \to \infty$ (by (3.5)) and $m^{\frac{1}{2}}(\theta_m^* - \theta)$ is asymptotically normal with mean 0 and variance γ^2, it is easy to see that for every fixed θ and Δ,

$$\begin{aligned} P_\theta\{N(\Delta) > n\} &\leqq P\{n^{-\frac{1}{2}}bV_n^* < \Delta n^{\frac{1}{2}}[\theta_n^* - \tfrac{1}{2}(\theta_0 + \theta_1)] < n^{-\frac{1}{2}}aV_n^*\} \\ &\to 0 \quad \text{as} \quad n \to \infty , \end{aligned} \tag{6.3}$$

and hence, the proposed test terminates with probability one. For the OC and ASN function, as in Sen (1973) and Sen and Ghosh (1974), we consider the asymptotic situation where we let $\Delta \to 0$ (comparable to $d \to 0$ in Section 5) and set

$$\theta = \theta_0 + \phi\Delta \qquad \text{where} \quad \phi \in \Phi = \{\phi: |\phi| < K < \infty\} , \tag{6.4}$$

(6.5) $\lim_{\Delta\to 0} n_0(\Delta) = \infty$ but $\lim_{\Delta\to 0} \Delta^2 n_0(\Delta) = 0$,

(6.6) $e^a = A = (1-\beta)/\alpha$, $e^b = B = \beta/(1-\alpha)$, $0 < \alpha, \beta < \frac{1}{2}$.

Finally, let us denote by $L_F(\phi, \Delta)$ the OC (i.e., probability of accepting H_0 when actually $\theta = \theta_0 + \phi\Delta$) of the test based on (6.2). Then, we have the following:

THEOREM 6.1. *Under* (6.4)—(6.6) *and the hypothesis of Theorem* 3.1 (*or* 4.1)

(6.7) $$\lim_{\Delta\to 0} L_F(\phi, \Delta) = P(\phi) = (A^{1-2\phi} - 1)/(A^{1-2\phi} - B^{1-2\phi}) , \quad \phi \neq \tfrac{1}{2} ,$$
$$= a/(a-b) , \quad \phi = \tfrac{1}{2} ,$$

and hence, asymptotically the OC *does not depend on F. Further*

(6.8) $P(0) = 1 - \alpha$ *and* $P(1) = \beta$,

so that the test has asymptotic strength (α, β).

PROOF. Let us choose a sequence $\{n^* = n^*(\Delta)\}$ such that

(6.9) $n^*(\Delta) \sim K\Delta^{-2}$ as $\Delta \to 0$, where $K(<\infty)$ is arbitrarily large.

Then, by (3.27), (6.4), (6.5) and (6.9), for $\theta = \theta_0 + \phi\Delta$, defining $U_\Delta^* = \{U_\Delta(t) = \Delta n[\theta_n^* - \frac{1}{2}(\theta_0 + \theta_1)]/\gamma, \Delta^2 n \leqq t < \Delta^2(n+1), n_0(\Delta) \leqq n < n^*(\Delta)\}$, it follows that as $\Delta \to 0$,

(6.10) $$U_\Delta^* \to_{\mathscr{D}} \{W(t) + (\phi - \tfrac{1}{2})t/\gamma, 0 < t \leqq K\}$$

where $\{W(t), t > 0\}$ is a standard Wiener process on $[0, \infty)$. Also, by (3.5), $\sup\{|V_n^*/\gamma^2 - 1| : n_0(\Delta) \leqq n \leqq n^*(\Delta)\} \to_p 0$ as $\Delta \to 0$. Finally, by (6.3), for every $\eta > 0$, there exists a $K = K_\eta (< \infty)$, such that defining $n^*(\Delta)$ by (6.9) with $K = K_\eta$, we have $P\{N(\Delta) > n^*(\Delta)\} < \eta$. Hence, using (6.2) and (6.10), it follows that

(6.11) $$\lim_{\Delta\to 0} L_F(\phi, \Delta) = P\{W(t) + (\phi - \tfrac{1}{2})t/\gamma \text{ is } \leqq b\gamma \text{ for a smaller } t (\geqq 0) \text{ than any other } t (> 0) \text{ for which } W(t) + (\phi - \tfrac{1}{2})t/\gamma \text{ is } \geqq a\gamma\} .$$

By the classical result of Dvoretzky, Kiefer and Wolfowitz (1953), the right-hand side of (6.11) is equal to $P(\phi)$, defined by (6.7), and hence, the proof of (6.7) is complete. (6.8) follows from (6.7) by sustituting $\phi = 0$ and 1, respectively. □

As in Theorem 5.2, for the study of the ASN (i.e., $E\{N(\Delta) \mid \theta = \theta_0 + \phi\Delta\}$ for $\Delta \to 0$) function, the weak (or a.s.) convergence results of Section 3 (or 4) are not enough and we need some analogous moment convergence results which, in turn, may demand more restrictive conditions on the df F. Suppose that as in Theorem 5.2, we assume $\hat{\theta}_n = g(U_n)$ and that

(6.12) $E|\phi(X_1, \cdots, X_m)|^r < \infty$ for some $r > 4$.

Then, we not only have (5.9), but also, it can be shown by steps similar to those in Section 3 that

(6.13) $$P\{|\theta_n^* - \theta| > \varepsilon\} \leqq C_\varepsilon n^{-s} , \quad s > 1 ,$$

for n sufficiently large. Further, for $n_0(\Delta) \leqq n \leqq n^*(\Delta)$, we may write

$$n\Delta[\theta_n^* - \tfrac{1}{2}(\theta_0 + \theta_1)] = \Delta Z_n + n\Delta^2(\phi - \tfrac{1}{2}) + R_n^{\Delta}, \tag{6.14}$$

where for every $\varepsilon > 0$,

$$P\{\max_{n_0(\Delta) \leqq n \leqq n^*(\Delta)} |R_n^{\Delta}| > \varepsilon\} \leqq C_\varepsilon n^{-s}, \quad \text{for} \quad \forall \Delta : 0 < \Delta \leqq \Delta_0, \tag{6.15}$$

and where $\{Z_n, \mathscr{B}_n; n \geqq 1\}$ is a martingale; $\mathscr{B}_n$ being the σ-field generated by $X_1, \cdots, X_n$, $n \geqq 1$ ($\Rightarrow \mathscr{B}_n$ is $\nearrow$ in n). As such the method of attack of Sen (1973) and Ghosh and Sen (1976) can directly be adapted to arrive at the following.

THEOREM 6.2. *Under the assumptions made earlier, for every* $\phi \in \Phi$,

$$\lim_{\Delta \to 0} \{\Delta^2 E[N(\Delta) \mid \theta = \theta_0 + \phi\Delta]\} = \psi(\phi, \gamma) \tag{6.16}$$

where

$$\begin{aligned} \psi(\phi, \gamma) &= \{bP(\phi) + a[1 - P(\phi)]\}\{\gamma^2/(\phi - \tfrac{1}{2})\}, & \phi \neq \tfrac{1}{2} \\ &= -\gamma^2 ab, & \phi = \tfrac{1}{2} \end{aligned} \tag{6.17}$$

and γ^2 *is defined by* (3.6).

Acknowledgment. Thanks are due to the referee for his critical reading of the manuscript and useful comments of it.

REFERENCES

[1] ARVESEN, J. M. (1969). Jackknifing U-statistics. *Ann. Math. Statist.* **40** 2076–2100.

[2] BHATTACHARYYA, B. B. and SEN, P. K. (1977). Weak convergence of the Rao–Blackwell estimator of a distribution function. *Ann. Probability* **5**. To appear.

[3] CHOW, Y. S. and ROBBINS, H. (1965). On the asymptotic theory of fixed-width sequential intervals for the mean. *Ann. Math. Statist.* **36** 457–462.

[4] DVORETZKY, A., KIEFER, J. and WOLFOWITZ, J. (1953). Sequential decision problems for processes with continuous time parameter: testing hypotheses. *Ann. Math. Statist.* **24** 254–264.

[5] GHOSH, M. and SEN, P. K. (1976). Sequential rank tests for regression. *Sankhyā Ser. A.* To appear.

[6] GRAY, H. L., WATKINS, T. A. and ADAMS, J. E. (1972). On the jackknife statistic, its extensions and its relation to e_n-transformations. *Ann. Math. Statist.* **43** 1–30.

[7] LOYNES, R. M. (1970). An invariance principle for reverse martingales. *Proc. Amer. Math. Soc.* **25** 56–64.

[8] MCLEISH, D. L. (1974). Dependent central limit theorems and invariance principles. *Ann. Probability.* **2** 620–628.

[9] MILLER, R. G., JR. (1974). An unbalanced jackknife. *Ann. Statist.* **2** 880–891.

[10] MILLER, R. G., JR. and SEN, P. K. (1972). Weak convergence of U-statistics and von Mises' differentiable statistical functions. *Ann. Math. Statist.* **43** 31–41.

[11] MISES, R. V. (1947). On the asymptotic distribution of differentiable statistical functions. *Ann. Math. Statist.* **18** 309–348.

[12] SCHUCANY, W. R., GRAY, H. L. and OWEN, D. B. (1971). Bias reduction in estimation. *J. Amer. Statist. Assoc.* **66** 524–533.

[13] SEN, P. K. (1960). On some convergence properties of U-statistics. *Calcutta Statist. Assoc. Bull.* **10** 1–18.

[14] SEN, P. K. (1973). Asymptotic sequential tests for regular functionals of distribution functions. *Theor. Probability Appl.* **18** 226–240.

[15] SEN, P. K. (1974a). Weak convergence of generalized U-statistics. *Ann. Probability* **2** 90–102.

[16] SEN, P. K. (1974b). Almost sure behavior of U-statistics and von Mises' differentiable statistical functions. *Ann. Statist.* **2** 387–395.

[17] SEN, P. K. (1974c). On L^p-convergence of U-statistics. *Ann. Inst. Statist. Math.* **26** 55–60.

[18] SEN, P. K. and Ghosh, M. (1971). On bounded length sequential confidence intervals based on one-sample rank order statistics. *Ann. Math. Statist.* **42** 189–203.

[19] SEN, P. K. and GHOSH, M. (1974). Sequential rank tests for location. *Ann. Statist.* **2** 540–552.

[20] SPROULE, R. N. (1969). A sequential fixed-width confidence interval for the mean of a U-statistic. Ph. D. dissertation, Univ. North Carolina, Chapel Hill.

DEPARTMENT OF BIOSTATISTICS
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA 27514