The Annals of Probability
1974, Vol. 2, No. 6, 1139–1151

SOME ITERATED LOGARITHM RESULTS FOR SUMS OF INDEPENDENT TWO-DIMENSIONAL RANDOM VARIABLES[1]

BY SHEY SHIUNG SHEU

University of California, Berkeley

Let $Z_i = (X_i, Y_i)$, $i \geqq 1$, be independent two-dimensional random variables, defined on a probability triple $(\Omega, \mathscr{A}, P)$, such that $E(X_i) = E(Y_i) = E(X_i Y_i) = 0$, $E(X_i^2) < \infty$, $E(Y_i^2) < \infty$ for all i. The purpose of this paper is to investigate the limit points of $\{(S_n(\omega)/L(n), T_n(\omega)/M(n)), n = 1, 2, \cdots\}$, where $\omega \in \Omega$, $S_n = \sum_{i=1}^n X_i$, $T_n = \sum_{i=1}^n Y_i$, $L(n) = [2E(S_n^2) \log\log E(S_n^2)]^{\frac{1}{2}}$, $M(n) = [2E(T_n^2) \log\log E(T_n^2)]^{\frac{1}{2}}$. The author will show the limit sets are the closed unit disk almost surely under some general conditions. An example with all limit points lying on the two axes with probability one will be constructed.

1. Introduction. Let R^k be the k-dimensional Euclidean space. Denote the *one-point compactification* of R^k by $\bar{R}^k = R^k \cup \{\infty\}$ with the usual topology. Let $\{a_n, n \geqq 1\}$ be a sequence of points in R^k. A point x_0 in $\bar{R}^k$ is a *limit point* (*accumulation point*) of $\{a_n\}$ if either $x_0 \in R^k$ and $\forall \varepsilon > 0$, $\forall n$, $\exists m \ni m \geqq n$ and $|a_m - x_0| < \varepsilon$, where $|\ |$ is the Euclidean norm, i.e. $|(y_1, \cdots, y_k)| = (y_1^2 + \cdots + y_k^2)^{\frac{1}{2}}$, or $x_0 = \infty$ and $\{a_n\}$ is unbounded. The collection of all such x_0 is called the *limit set* of $\{\alpha_n\}$. Now Hartman–Wintner's law of the iterated logarithm [6] can be stated in the following manner: suppose $X_1, X_2, \cdots$ are i.i.d. real-valued random variables, defined on a probability triple $(\Omega, \mathscr{A}, P)$, with means zero and variance one. Then, with probability 1, the limit set of $\{[X_1(\omega) + \cdots + X_n(\omega)]/(2n \log\log n)^{\frac{1}{2}}, n \geqq 3\}$ is $[-1, 1]$. The author is interested in the analogue of this result for the multidimensional variables. Namely, suppose we have, say, 2-dimensional random variables $Z_n = (X_n, Y_n)$, $n \geqq 1$; let us assume $E(X_n) = E(Y_n) = E(X_n Y_n) = 0$ for all n; let $S_n = \sum_{i=1}^n X_i$, $T_n = \sum_{i=1}^n Y_i$, $L(n) = [2E(S_n^2) \log\log E(S_n^2)]^{\frac{1}{2}}$, $M(n) = [2E(T_n^2) \log\log E(T_n^2)]^{\frac{1}{2}}$. What can be said about the limit sets of $\{(Sn/L(n), T_n/M(n))\}$ under reasonable conditions? The main results of this paper are Theorem (2.10) and Theorem (2.14).

The following notations and conventions will be used throughout the article.

(1.1 a) $(\Omega, \mathscr{A}, P)$ is the probability triple on which the random variables considered in each statement are defined.

Received October 1, 1973; revised January 15, 1974.

[1] This paper was prepared with the partial support of the U.S. Army Research Office (Durham), Grants DA-ARO-D-31-124-G816 and DA-ARO-D31-124-G1135.

AMS 1970 *subject classifications*. 60F15, 60G50.

Key words and phrases. Law of the iterated logarithm, limit set, sums of independent random variables.

(1.1 b) $Z_n = (X_n, Y_n)$, $n \geqq 1$, is a sequence of *independent* random variables in R^2.

(1.1 c) $$\log_2 = \log\log .$$

2. Statement of results. The limits sets related to the law of the iterated logarithm are not random. In fact,

(2.1) PROPOSITION. *Recall* (1.1 b). *Let* $\{\phi(n)\}$ *and* $\{\psi(n)\}$ *be two sequences of numbers such that*

(2.2) $$\lim_{n\to\infty} \phi(n) = \infty , \qquad \lim_{n\to\infty} \psi(n) = \infty .$$

Let

$$S_n = \sum_{i=1}^n X_i , \qquad T_n = \sum_{i=1}^n Y_i .$$

Then there exists a unique nonempty closed subset H *of* $\bar{R}^2$ *such that*

(2.3) $$P\{\omega \mid \textit{the limit set of } \{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\} \textit{ is } H\} = 1 .$$

In fact,

(2.4) $$H = \{x_0 \in \bar{R}^2 \mid x_0 \textit{ is a limit point of } \{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\} \textit{ for almost all } \omega \in \Omega\} .$$

The proof is based on the Kolmogorov 0–1 law and the fact that $\bar{R}^2$ is a separable metric space. Following (2.1), we make the

(2.5) DEFINITION. In Proposition (2.1), H is called the a.s. limit set of $\{(S_n/\phi(n), T_n/\psi(n))\}$.

Thus, the goal is to determine H with given ϕ and ψ.

For the case when Z_n's are i.i.d., it has been known in the literature that the a.s. limit set of $\{\sum_{i=1}^n Z_i/(2n \log_2 n)^{\frac{1}{2}}\}$ is the closed unit disk if Z_1 has mean 0 and covariance matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. We use a result due to Strassen [13] to prove that the converse is also true. Hence,

(2.6) PROPOSITION. *Suppose* $Z_n = (X_n, Y_n)$, $n \geqq 1$, *are* i.i.d. *Let*

$$W_n = \sum_{i=1}^n Z_i .$$

Then the a.s. *limit set* H *of* $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ *is the closed unit disk if and only if*

$$E(Z_1) = (0, 0) , \qquad \operatorname{Cov}(Z_1) = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) .$$

Hence, whenever $E(Z_1) = (0, 0)$ and $\operatorname{Cov}(Z_1)$ exists, then a.s. limit set of $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ is an ellipsoid.

When the variables do not have the same distribution but the two coordinates for each variable have equal variances, Kolmogorov's result [8] can be easily extended. We remind you of (1.1 b).

(2.7) PROPOSITION. *Suppose* Z_n *satisfies*

(2.8 a) $$E(X_i) = E(Y_i) = E(X_i Y_i) = 0 ,$$

(2.8 b) $$E(X_i^2) = E(Y_i^2) = \sigma_i^2 < \infty , \qquad \forall i .$$

Let

(2.8 c) $$s_n^2 = \sum_{i=1}^n \sigma_i^2\,, \qquad W_n = \sum_{i=1}^n Z_i\,.$$

Let

$$|X_k| = \text{ess sup}_{\omega\in\Omega} |X_k(\omega)|\,,$$
$$|Y_k| = \text{ess sup}_{\omega\in\Omega} |Y_k(\omega)|\,.$$

Then the a.s. *limit set* H *of* $\{W_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ *is theclosed unit disk if*

(2.9) $$s_n^2 \to \infty\,, \qquad \sup_{k\leq n} (|X_k| + |Y_k|) = o((s_n^2/\log_2 s_n^2)^{\frac{1}{2}})\,.$$

Proposition (2.7) is proved by using linear functionals on R^2 then applying the one-dimensional result.

For the case when two coordinates have unequal variances, we construct an example in Section 4 which amounts to saying the following:

(2.10) THEOREM. *There exists a sequence of independent random variables,* $Z_n = (X_n, Y_n)$, $n \geq 1$ *such that* X_n *and* Y_n *are independent for each n and*

$$E(X_i) = E(Y_i) = 0\,, \qquad E(X_i^2) < \infty\,, \qquad E(Y_i^2) < \infty\,, \qquad \forall i\,,$$

but the a.s. *limit set of*

$$\{(S_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}, T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}\} \quad \textit{is} \quad \{(a, b) \mid ab = 0, |a| \leq 1, |b| \leq 1\}\,,$$

where

$$s_n^2 = \sum_{i=1}^n E(X_i^2)\,, \qquad t_n^2 = \sum_{i=1}^n E(Y_i^2)\,, \qquad S_n = \sum_{i=1}^n X_i\,, \qquad T_n = \sum_{i=1}^n Y_i\,.$$

It is well known that the law of the iterated logarithm is strongly related to the Central Limit Theorem, and conditions stronger than Lindeberg's are needed. We present in Theorem (2.14) a condition on the rate of convergence to the normal law.

To state (2.14), assume (1.1 b) and

$$E(X_i) = 0\,, \qquad E(Y_i) = 0\,, \qquad E(X_i Y_i) = 0\,,$$
$$E(X_i^2) = \sigma_i^2 < \infty\,, \qquad E(Y_i^2) = \tau_i^2 < \infty\,.$$

Define

$$s^2(n) = s_n^2 = \sum_{i=1}^n \sigma_i^2\,, \qquad t^2(n) = t_n^2 = \sum_{i=1}^n \tau_i^2\,,$$
$$S(n) = S_n = \sum_{i=1}^n X_i\,, \qquad T(n) = T_n = \sum_{i=1}^n Y_i\,.$$

To avoid triviality, assume that $s_n^2 \to \infty$ and $t_n^2 \to \infty$. Let

(2.11) $$\rho(n) = \rho(F_n, \Phi) = \sup_{-\infty<x,y<\infty} |F_n(x, y) - \Phi(x, y)|\,,$$

where Φ is the standard 2-dim normal distribution function and F_n the distribution function of $(S_n/s_n, T_n/t_n)$.

Define two subsequences $\{n_k\}$ and $\{m_k\}$ by induction. Take any γ_0, $\gamma > 1$, γ_0 fixed. Let

(2.12) n_0 be the first n such that $s^2(n) \geq 3$ and $t^2(n) \geq 3$, when $n_0, \cdots, n_{k-1}$

are defined, n_k is the first n such that

$$s^2(n) \geqq \gamma_0 s^2(n_{k-1}) \quad or \quad t^2(n) \geqq \gamma_0 t^2(n_{k-1}) .$$

(2.13) Let $m_0 = n_0$; when $m_0, \cdots, m_{k-1}$ are defined, m_k is the first n such that $s^2(n) \geqq \gamma_0 s^2(m_{k-1})$ and $t^2(n) \geqq \gamma_0 t^2(m_{k-1})$.

(2.14) THEOREM. *Suppose* $\exists \beta > 0 \ni$

(2.15) $$\rho(n) = O(\min\{1/(\log s_n^2)^{1+\beta}, 1/(\log t_n^2)^{1+\beta}\}) .$$

Let H be the a.s. *limit set of*

$$\{(S_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}, T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}\} \quad and \quad (a, b) \in R^2 .$$

Then $(a, b) \notin H$ *if* $\exists \delta > 0 \ni$

(2.16) $$\sum_{k=0}^{\infty} 1/(\log s^2(n_k))^{a^2-\delta}(\log t^2(n_k))^{b^2-\delta} < \infty ,$$

and $(a, b) \in H$ *if* $\exists \delta > 0 \ni$

(2.17) $$\sum_{k=0}^{\infty} 1/(\log s^2(m_k))^{a^2+\delta}(\log t^2(m_k))^{b^2+\delta} = \infty .$$

(2.18) REMARK. In (2.16), when $a = 0$ or $b = 0$, $(\log s^2(n_k))^{a^2-\delta}$ or $(\log t^2(n_k))^{b^2-\delta}$ should be considered equal to 1 respectively.

As applications of (2.14), we show

(2.19) COROLLARY. *Assume* (1.1 b), (2.8 a), (2.8 b), (2.8 c), *and* $s_n^2 \to \infty$. *Suppose, furthermore, all* Z_n*'s are normal. Then the* a.s. *limit set H of* $\{W_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ *is the closed disk with center origin and radius d if and only if*

(2.20a) $$\sum_{k=0}^{\infty} 1/(\log s^2(n_k))^{d^2-\delta} = \infty , \qquad \forall \delta > 0 ,$$

(2.20b) $$\sum_{k=0}^{\infty} 1/(\log s^2(n_k))^{d^2+\delta} < \infty , \qquad \forall \delta > 0 .$$

(2.21) COROLLARY. *Under* (1.1 b), (2.8 a), (2.8 b), (2.8 c), *and* $s_n^2 \to \infty$, *the* a.s. *limit set of* $\{W_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ *is the closed unit disk if* $\exists \beta > 0$, $\lambda < \infty \ni$

(2.22) $$E[(X_i^2 + Y_i^2)(\log (X_i^2 + Y_i^2))^{1+\beta}] \leqq \lambda \sigma_i^2 , \qquad \forall i .$$

3. The proofs.

(3.1) LEMMA. *Let* $\{U_n, n \geqq 1\}$ *be a sequence of random variables in* R^2. *Then for any* $x_0 \in \bar{R}^2$, *the set*

(3.2) $$A(x_0) = \{\omega \in \Omega \mid x_0 \text{ is a limit point of } \{U_n(\omega)\}\} \text{ is measurable.}$$

PROOF. Since $\bar{R}^2$ satisfies the first axiom of countability,

$$A(x_0) = \bigcap_{V \in \mathscr{V}} \{U_n \in V \text{ i.o.}\} ,$$

where $\mathscr{V}$ is a countable family of open neighborhoods at x_0. □

(3.3) THE PROOF OF (2.1). Define $A(x_0)$ by (3.2) for each $x_0 \in \bar{R}^2$ by taking $U_n(\omega) = (S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))$. By (2.2), $A(x_0)$ is a tail event. Applying the Kolmogorov 0-1 law, $P(A(x_0)) = 0$ or 1. Let $H = \{x_0 \in \bar{R}^2 \mid P(A(x_0)) = 1\}$.

(a) Claim: H is closed.
Indeed, take any $\{x_n\}$ in H such that

$$x_n \to \bar{x}\,, \qquad \text{some } \bar{x} \in \bar{R}^2 .$$

Because any limit set is closed,

$$\bigcap_n A(x_n) \subset A(\bar{x}) .$$

Hence,

$$P(\bigcap_n A(x_n)) = 1 \Rightarrow P(A(\bar{x})) = 1 \Rightarrow \bar{x} \in H .$$

(b) Claim: $P\{\omega \mid H$ is the limit set of $\{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\}\} = 1$.
Note that $y \in H$ if for all open nbd V of y we have $P((S_n/\phi(n), T_n/\psi(n)) \in V$, i.o.$) = 1$. Since H^c is open, $\forall y \in H^c$ $\exists$ open nbd $D(y)$ of $y \ni D(y) \subset H^c$ and

$$P((S_n/\phi(n), T_n/\psi(n)) \in D(y),\ \text{i.o.}) < 1.$$

Again, by (2.2), the event $\{(S_n/\phi(n), T_n/\psi(n)) \in D(y)$, i.o.$\}$ is a tail event. Hence, by the Kolmogorov 0–1 law,

$$P((S_n/\phi(n), T_n/\psi(n)) \in D(y),\ \text{i.o.}) = 0 . \tag{3.4}$$

Since $\bigcup_{y \in H^c} D(y) = H^c$ and H^c is Lindelöf, there exists a countable subcovering $D(y_1), D(y_2), \cdots$, i.e.

$$\bigcup_n D(y_n) = H^c .$$

Let $B_m = \{\omega \mid S_n(\omega)/\phi(n), T_n(\omega)/\psi(n)) \in \bigcup_{k=1}^m D(y_k)$, i.o.$\}^c$ for each m. Then (3.4) implies $P(B_m) = 1$, $\forall m$. Let G be a countable dense subset of H. Let $C_m = B_m \cap (\bigcap_{x \in G} A(x))$. Then $P(C_m) = 1$, $\forall m$. Observe that $\forall \omega \in C_m$, the limit set of $\{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\}$ contains H but is contained in $(\bigcup_{k=1}^m D(y_k))^c = \bigcap_{k=1}^m D(y_k)^c$. Let $C = \bigcap_m C_m$. Note $\bigcap_{k=1}^\infty (D(y_k))^c = H$. Hence, $P(C) = 1$ and $\forall \omega \in C$, the limit set of $\{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\}$ is exactly H. Moreover, if we choose $D(y)$ in such a way that $\overline{D(y)} \subset H^c$, then by the compactness of $\overline{D(y)}$ (in $\bar{R}^2$) we see

$$C = \{\omega \mid \text{the limit set of } \{(S_n(\omega)/\phi(n), T_n(\omega)/\psi(n))\} \text{ is exactly } H\} .$$

Since for each ω, the limit set of $\{U_n(\omega)\}$ is unique and nonempty, H is unique and nonempty. □

(3.5) REMARK. From the above proof, one sees that if $E \subset \bar{R}^2$, E is closed, $E \cap H = \phi$, then $P((S_n/\phi(n), T_n/\psi(n)) \in E$, i.o.$) = 0$ by the compactness of E. To prove (2.6), observe a simple fact:

(3.6) LEMMA. *Let $\{a_n\}$ be an indexed set in R^2. Suppose T: $R^2 \to R^2$ is a continuous map and x_0, $x_0 \in R^2$, is a limit point of $\{a_n\}$. Then $T(x_0)$ is a limit point of $\{T(a_n)\}$.*

(3.7) THE PROOF OF (2.6). The proof of the "if" part can be found in a paper by Finkelstein [4]. For the "only if" part, suppose the a.s. limit set H of $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ is the closed unit disk. Let $S_n = \sum_{i=1}^n X_i$. Since $\infty \notin H$, $\{W_n(\omega)/(2n \log_2 n)^{\frac{1}{2}}\}$ is bounded almost surely. By (3.6) and Remark (3.5), the a.s. limit set of $\{S_n/(2n \log_2 n)^{\frac{1}{2}}\}$ is $[-1, 1]$. Applying a result by Strassen [13], $E(X_1) = 0$, $E(X_1^2) = 1$. Similarly, $E(Y_1) = 0$, $E(Y_1^2) = 1$. We know $\exists\, \lambda_1, \lambda_2 > 0$ and an orthogonal transformation A such that the cavariance matrix of $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} AZ_1$ is $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Let $Z_i' = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} AZ_i$, $W_n' = \sum_{i=1}^n Z_i'$. By "if" part, we see the a.s. limit set of $\{W_n'/(2n \log_2 n)^{\frac{1}{2}}\}$ is the closed unit disk. But $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} A$ is a 1–1 onto linear map, the a.s. limit set of $\{W_n'/(2n \log_2 n)^{\frac{1}{2}}\}$ is the image of the a.s. limit set of $\{W_n/(2n \log_2 n)^{\frac{1}{2}}\}$ under the map $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} A$. Hence, thh unit disk is invariant under the map $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} A$, which cannot hold unless $\lambda_1 = \lambda_2 = 1$. Therefore,

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \operatorname{Cov}(Z_1') = A \operatorname{Cov}(Z_1) A^t = \operatorname{Cov}(Z_1)\,. \qquad \square$$

(3.8) THE PROOF OF (2.7). Take a point (a, b) on the unit circle. Consider $V_n = aX_n + bY_n$, $n \geqq 1$. Let

$$U_n = \sum_{i=1}^n V_i\,.$$

By (2.8a, b, c), and (2.9), one sees $E(V_i) = 0$, $E(V_i^2) = \sigma_i^2$, and $\sup_{k \leqq n} |V_k| = o((s_n^2/\log_2 s_n^2)^{\frac{1}{2}})$, $s_n^2 \to \infty$. By Kolmogorov,

(3.9) $$\limsup U_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

This is true as long as $a^2 + b^2 = 1$. Using (3.6), we deduce

(3.10) $$H \subset \{(x, y) \mid x^2 + y^2 \leqq 1\}\,.$$

(3.11) Claim: $H \supset \{(x, y) \mid x^2 + y^2 = 1\}$.

Suppose the above claim is not true. Then $\exists\, (a_0, b_0)$ and $B_\delta(a_0, b_0)$, $\delta > 0$, such that

(3.12) $$P(W_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}} \in B_\delta(a_0, b_0),\ \text{i.o.}) = 0$$

by Proposition (2.1), where $a_0^2 + b_0^2 = 1$ and $B_\delta(a_0, b_0) = \{(a, b) \mid |(a, b) - (a_0, b_0)| < \delta\}$. Let $V_n^0 = a_0 X_n + b_0 Y_n$, $n \geqq 1$, and $U_n^0 = \sum_{i=1}^n V_i^0$. Using elementary geometry and (3.10), (3.12), we have

$$\limsup U_n^0/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}} \leqq 1 - \delta^2/2 < 1 \quad \text{a.s.}$$

which contradicts (3.9).

To probe $H \supset \{(x, y) \mid x^2 + y^2 \leqq 1\}$, take another independent copy $\{Z_n' = (X_n', Y_n')\}$ or $\{Z_n\}$. Consider $\Lambda_n = \sum_{i=1}^n (X_i, Y_i, X_i', Y_i')$. Corrying out the same arguments as before, we see the a.s. limit of $\{\Lambda_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ contains $\{(a, b, c, d) \mid a^2 + b^2 + c^2 + d^2 = 1\}$. Then taking the projection, we have

(3.13) $$H \supset \{(x, y) \mid x^2 + y^2 \leqq 1\}\,.$$

The theorem follows from (3.10) and (3.13). $\square$

We shall work on Theorem (2.14). The following lemma (3.14) is known.

(3.14) LEMMA. *Let Y be $N(0, 1)$.*

(a) $P(Y \geqq t) \leqq (2\pi)^{-\frac{1}{2}} t^{-1} e^{-t^2/2}$, $t \geqq 0$. *In particular, when* $t \geqq 1$, $P(Y \geqq t) \leqq e^{-t^2/2}$.

(b) *Let* $-\infty < a < b > \infty$, $m = \max\{a^2, b^2\}$.

Then $P(a \leqq Y \leqq b) \geqq (2\pi)^{-\frac{1}{2}}(b-a)e^{-m/2}$. *In particular, when* $b - a \geqq (2\pi)^{\frac{1}{2}}$, $P(a \leqq Y \leqq b) \geqq e^{-m/2}$.

(3.15) LEMMA. *Let* $Z_i = (X_i, Y_i)$, $1 \leqq i \leqq n$, *be independent such that* $E(X_i) = E(Y_i) = 0$, $E(X_i^2) = \sigma_i^2 < \infty$, $E(Y_i^2) = \tau_i^2 < \infty$, $1 \leqq i \leqq n$. *Let* $s_n^2 = \sum_{i=1}^n \sigma_i^2$, $t_n^2 = \sum_{i=1}^n \tau_i^2$, $S_k = \sum_{i=1}^k X_i$, $T_k = \sum_{i=1}^k Y_i$, $1 \leqq k \leqq n$. *Then for any* x, y, $-\infty < x < \infty$, $-\infty < y < \infty$,

$$P(\bigcup_{k=1}^n [S_k \geqq xs_n, T_k \geqq yt_n]) \leqq 3P(S_n \geqq (x - 3^{\frac{1}{2}})s_n, T_n \geqq (y - 3^{\frac{1}{2}})t_n)\,.$$

PROOF. Let

$$\begin{aligned} A_k &= [S_k \geqq xs_n, T_k \geqq yt_n]\,, && 1 \leqq k \leqq n\,,\\ B_k &= A_1^c \cap \cdots \cap A_{k-1}^c \cap A_k\,, && 2 \leqq k \leqq n\,,\\ &= A_1\,, && k = 1\,,\\ C_k &= [\textstyle\sum_{i=k+1}^n X_i \geqq -3^{\frac{1}{2}}s_n, \sum_{i=k+1}^n Y_i \geqq -3^{\frac{1}{2}}t_n]\,, && 1 \leqq k \leqq n-1\,,\\ &= \Omega\,, && k = n\,, \end{aligned}$$

and also, let $D = [S_n \geqq (x - 3^{\frac{1}{2}})s_n, T_n \geqq (y - 3^{\frac{1}{2}})t_n]$.
Then

$$\begin{aligned} P(D) &\geqq P(D \cap (\textstyle\bigcup_{k=1}^n A_k)) = P(D \cap (\bigcup_{k=1}^n B_k))\\ &= \textstyle\sum_{k=1}^n P(D \cap B_k) \geqq \sum_{k=1}^n P(B_k \cap C_k)\\ &= \textstyle\sum_{k=1}^n P(B_k)P(C_k) \quad \text{by independence.} \end{aligned}$$

But

$$\begin{aligned} P(C_k) &= P(\textstyle\sum_{i=k+1}^n X_i \geqq -3^{\frac{1}{2}}s_n, \sum_{i=k+1}^n Y_i \geqq -3^{\frac{1}{2}}t_n)\\ &\geqq 1 - P(\textstyle\sum_{i=k+1}^n X_i < -3^{\frac{1}{2}}s_n) - P(\sum_{i=k+1}^n Y_i < -3^{\frac{1}{2}}t_n)\\ &\geqq 1 - \frac{\sum_{i=k+1}^n \sigma_i^2}{3s_n^2} - \frac{\sum_{i=k+1}^n \tau_i^2}{3t_n^2} && \text{by Chebyshev}\\ &\geqq 1 - \tfrac{1}{3} - \tfrac{1}{3} = \tfrac{1}{3}\,, && 1 \leqq k \leqq n-1\,. \end{aligned}$$

Of course,

$$P(C_n) = P(\Omega) > \tfrac{1}{3}\,.$$

Hence,

$$P(D) \geqq \tfrac{1}{3}\textstyle\sum_{k=1}^n P(B_k) = \tfrac{1}{3}P(\bigcup_{k=1}^n B_k) = \tfrac{1}{3}P(\bigcup_{k=1}^n A_k)\,.$$

(3.16) THE PROOF OF (2.14). Remember (2.12) and (2.13). First note that the convergence or divergence of the series in (2.16) does not really depend on γ_0 in the sense that given any other γ, $\gamma > 1$, define a subsequence $\{\nu_k\}$ by the procedure described in (2.12) but using γ instead of γ_0, then

(3.17) $\sum_k 1/(\log s^2(\nu_k))^{a^2-\delta}(\log t^2(\nu_k))^{b^2-\delta} < \infty$ if and only if (2.16) holds. This is true because if $\gamma > \gamma_0$, thed $\nu_k \geqq n_k$, $\forall k$; therefore, (2.16) $\Rightarrow$ (3.17); if $\gamma < \gamma_0$, then $\exists$ integer $l \ni \gamma^l \geqq \gamma_0$, hence, $\nu_{2lk} \geqq n_k$, $\forall k$; therefore, (2.16) $\Rightarrow$ $\sum_k 1/(\log s^2(\nu_{2lk}))^{a^2-\delta}(\log t^2(\nu_{2lk}))^{b^2-\delta} < \infty \Rightarrow$ (3.17) by the fact that both $s^2(n)$ and $t^2(n)$ are non-decreasing. Reversing the roles of γ_0 and γ, one sees (2.16 $\Leftrightarrow$ (3.17). The same argument can be applied to (2.17), too. Hence, we may take γ_0 to be any value we like as long as $\gamma_0 > 1$. Without loss of generality, assume that $a > 0$, $b > 0$. Given $\varepsilon > 0$, ε is small. Let $L(j) = (2s_j^2 \log_2 s_j^2)^{\frac{1}{2}}$, $M(j) = (2t_j^2 \log_2 t_j^2)^{\frac{1}{2}}, j \geqq 1$. Define the event $A_j = [S_j \geqq (a-\varepsilon)L(j), T_j \geqq (b-\varepsilon)M(j)]$. Let $C_k = \bigcup_{n_{k-1} \leqq j \leqq n_k - 1} A_j$. Then

$$P(C_k) \leqq P(\textstyle\bigcup_{n_{k-1} \leqq j \leqq n_k - 1} [S_j \geqq (a-\varepsilon)L(n_{k-1}), T_j \geqq (b-\varepsilon)M(n_{k-1})]) .$$

Since

$$s^2(n_{k-1}) \leqq s^2(n_k - 1) \leqq \gamma_0 s^2(n_{k-1}) , \qquad t^2(n_{k-1}) \leqq t^2(n_k - 1) \leqq \gamma_0 t^2(n_{k-1}) ,$$

we can choose γ_0 so close to 1 that $(a-\varepsilon)L(n_{k-1}) \geqq (a-2\varepsilon)L(n_k - 1)$ and $(b-\varepsilon)M(n_{k-1}) \geqq (b-2\varepsilon)M(n_k - 1)$.
Thus,

$$\begin{aligned} P(C_k) &\leqq P(\textstyle\bigcup_{n_{k-1} \leqq j \leqq n_k - 1} [S_j \geqq (a-2\varepsilon)L(n_k - 1), T_j \geqq (b-2\varepsilon)M(n_k - 1)]) \\ &\leqq 3P(S(n_{k-1}) \geqq (a-2\varepsilon)L(n_k - 1) - 3^{\frac{1}{2}}s(n_k - 1), \\ &\qquad T(n_k - 1) \geqq (a-2\varepsilon)M(n_k - 1) - 3^{\frac{1}{2}}t(n_k - 1)) \end{aligned}$$

by (3.15).
If k is sufficiently large, we have

$$\begin{aligned} (a-2\varepsilon)L(n_k - 1) - 3^{\frac{1}{2}}s(n_k - 1) &\geqq (a-3\varepsilon)L(n_k - 1) , \\ (b-2\varepsilon)M(n_k - 1) - 3^{\frac{1}{2}}t(n_k - 1) &\geqq (b-3\varepsilon)M(n_k - 1) . \end{aligned}$$

Hence,

$$\begin{aligned} P(C_k) &\leqq 3P(S(n_{k-1}) \geqq (a-3\varepsilon)L(n_k - 1), \\ &\qquad T(n_k - 1) \geqq (b-3\varepsilon)M(n_k - 1)) \\ &\leqq 3[3\rho(n_k - 1) + P(X \geqq (a-3\varepsilon)(2\log_2 s^2(n_k - 1))^{\frac{1}{2}}, \\ &\qquad Y \geqq (b-3\varepsilon)(2\log_2 t^2(n_k - 1))^{\frac{1}{2}})] \end{aligned} \tag{3.18}$$

by (2.11), where X, Y are two independent $N(0, 1)$. Now by (2.15),

$$\begin{aligned} \rho(n_k - 1) &\leqq K_0 \min \{1/(\log s^2(n_k - 1))^{1+\beta}, 1/(\log t^2(n_k - 1))^{1+\beta}\} \\ &\leqq K_0 \min \{1/(\log s^2(n_{k-1}))^{1+\beta}, 1/(\log t^2(n_{k-1}))^{1+\beta}\} , \end{aligned}$$

where K_0 is an absolute constant. Since for each i, $s^2(n_i) \geqq \gamma_0 s^2(n_{i-1})$ and $t^2(n_i) \geqq \gamma_0 t^2(n_{i-1})$, we have $s^2(n_{k-1}) \geqq \gamma_0^{\frac{1}{2}(k-1)} s^2(n_0)$ and $t^2(n_{k-1}) \geqq \gamma_0^{\frac{1}{2}(k-1)} t^2(n_0)$. Hence,

$$\rho(n_{k-1}) \leqq K_1/[\tfrac{1}{2}(k-1)]^{1+\beta} , \qquad \text{some } K_1 > 0 . \tag{3.19}$$

Next,

$$\begin{aligned} &P(X \geqq (a-3\varepsilon)(2\log_2 s^2(n_k - 1))^{\frac{1}{2}}, Y \geqq (b-3\varepsilon)(2\log_2 t^2(n_k - 1))^{\frac{1}{2}}) \\ &\quad \leqq 1/(\log s^2(n_k - 1))^{(a-3\varepsilon)^2}(\log t^2(n_k - 1))^{(b-3\varepsilon)^2} \qquad \text{by (3.14a)} \\ &\quad \leqq 1/(\log s^2(n_{k-1}))^{(a-3\varepsilon)^2}(\log t^2(n_{k-1}))^{(b-3\varepsilon)^2} \\ &\quad \leqq 1/(\log s^2(n_{k-1}))^{a^2-\delta}(\log t^2(n_{k-1}))^{b^2-\delta} \end{aligned} \tag{3.20}$$

when ε is sufficiently small. With (3.18), (3.19), (3.20) and (2.16), we obtain $\sum_k P(C_k) < \infty$.

By Borel–Cantelli,

$$P(C_k, \text{ i.o.}) = 0 \Rightarrow P(A_j, \text{ i.o.}) = 0 \Rightarrow (a, b) \notin H .$$

To prove the second part of the theorem, let $\| \;\|$ be the max norm; i.e. $\|(a, b)\| = \max(|a|, |b|)$. Given $\varepsilon > 0$. Clearly,

$$\left[\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| > 2\varepsilon\right]$$
$$\subset \left[\left\|\left(\frac{S(m_k)}{L(m_k)}, \frac{T(m_k)}{M(m_k)}\right) - (a, b)\right\| > \varepsilon\right] \cup \left[\left\|\left(\frac{S(m_{k-1})}{L(m_k)}, \frac{T(m_{k-1})}{M(m_k)}\right)\right\| > \varepsilon\right].$$

Therefore,

$$P\left(\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| > 2\varepsilon\right)$$
$$\leqq P\left(\left\|\left(\frac{S(m_k)}{L(m_k)}, \frac{T(m_k)}{M(m_k)}\right) - (a, b)\right\| > \varepsilon\right) + P\left(\left\|\left(\frac{S(m_{k-1})}{L(m_k)}, \frac{T(m_{k-1})}{M(m_k)}\right)\right\| > \varepsilon\right).$$

Subtracting both sides from 1,

$$P\left(\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| \leqq 2\varepsilon\right)$$
$$\geqq P\left(\left\|\left(\frac{S(m_k)}{L(m_k)}, \frac{T(m_k)}{M(m_k)}\right) - (a, b)\right\| \leqq \varepsilon\right) - P\left(\left\|\left(\frac{S(m_{k-1})}{L(m_k)}, \frac{T(m_{k-1})}{M(m_k)}\right)\right\| \geqq \varepsilon\right). \tag{3.21}$$

But

$$P(\|(S(m_{k-1})/L(m_k), T(m_{k-1})/M(m_k))\| \geqq \varepsilon) \leqq P(|S(m_{k-1})/L(m_k)| \geqq \varepsilon) + P(|T(m_{k-1})/M(m_k)| \geqq \varepsilon) . \tag{3.22}$$

By (2.11) and (3.14a),

$$P(|S(m_{k-1})/L(m_k)| \geqq \varepsilon) \leqq 2\rho(m_{k-1}) + \exp(-[\varepsilon^2 s^2(m_k) \log_2 s^2(m_k)]/s^2(m_{k-1})) , \tag{3.23}$$

where

$$\rho(m_{k-1}) \leqq K_0 \min\{1/(\log s^2(m_{k-1}))^{1+\beta}, 1/(\log t^2(m_{k-1}))^{1+\beta}\} \leqq K_2/(k-1)^{1+\beta} , \quad \text{some } K_2 > 0 , \tag{3.24}$$

by the definition of $\{m_k\}$,
and

$$\exp(-[\varepsilon^2 s^2(m_k) \log_2 s^2(m_k)]/s^2(m_{k-1})) \leqq 1/(\log s^2(m_k))^{\gamma_0 \varepsilon^2} \leqq K_3/k^{\gamma_0 \varepsilon^2} , \quad \text{some } K_3 > 0 . \tag{3.25}$$

If we choose γ_0 so large that $\gamma_0 \varepsilon^2 \geqq 2$, then by (3.23), (3.24) and (3.25),

$$\sum_k P(|S(m_{k-1})/L(m_k)| \geqq \varepsilon) < \infty . \tag{3.26}$$

Similarly,

$$\sum_k P(|T(m_{k-1})/M(m_k)| \geqq \varepsilon) < \infty . \tag{3.27}$$

With (3.22), (3.26) and (3.27), one sees that

$$\sum_k P(\|(S(m_{k-1})/L(m_k), T(m_{k-1})/M(m_k))\| \geqq \varepsilon) < \infty . \tag{3.28}$$

Again, applying (2.11), (2.15) and (3.14b),

$$\begin{aligned} P(\|(S(m_k)/L(m_k), T(m_k)/M(m_k)) - (a, b)\| \leqq \varepsilon) & \\ \geqq -4\rho(m_k) + 1/(\log s^2(m_k))^{a^2+2\varepsilon|a|+\varepsilon^2}(\log t^2(m_k))^{b^2+2\varepsilon|b|+\varepsilon^2} & \\ \geqq -K_4/k^{1+\beta} + 1/(\log s^2(m_k))^{a^2+\delta}(\log t^2(m_k))^{b^2+\delta} & \end{aligned} \tag{3.29}$$

if ε is sufficiently small. By (2.17), (3.21), (3.28) and (3.29), we conclude that

$$\sum_k P\left(\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| \leqq 2\varepsilon\right) = \infty .$$

By independence and Borel–Cantelli,

$$P\left(\left\|\left(\frac{S(m_k) - S(m_{k-1})}{L(m_k)}, \frac{T(m_k) - T(m_{k-1})}{M(m_k)}\right) - (a, b)\right\| \leqq 2\varepsilon, \text{ i.o.}\right) = 1 . \tag{3.30}$$

Using (2.15) and a result of Petrov [11] in the one-dimensional case, we see that for almost all ω, $\exists k_0(\omega) \ni \forall k \geqq k_0(\omega)$,

$$S(m_{k-1}, \omega) \leqq 2L(m_{k-1}) \leqq 2/\gamma_0^{\frac{1}{2}} L(m_k)$$
$$T(m_{k-1}, \omega) \leqq 2M(m_{k-1}) \leqq 2/\gamma_0^{\frac{1}{2}} M(m_k) .$$

Thus, if γ_0 is sufficiently large, then (3.30) implies

$$P(\|(S(m_k)/L(M_k), T(m_k)/M(m_k)) - (a, b)\| \leqq 3\varepsilon, \text{ i.o.}) = 1 .$$

Since ε is arbitrary, $(a, b) \in H$. □

(3.31) THE PROOF OF (2.19). In this case, Z_n's are normal; therefore, (2.15) is trivially satisfied. Since $s_n^2 = t_n^2$, $n_k = m_k$. Suppose (2.20a, b). With (2.20a) and (2.17), we see that $(a, b) \in H$ if $a^2 + b^2 < d^2$. With (2.20b) and (2.16), we see that $(a, b) \notin H$ if $a^2 + b^2 > d^2$.
Hence,

$$H = \{(a, b) \mid a^2 + b^2 \leqq d^2\} .$$

Conversely, suppose $H = \{(a, b) \mid a^2 + b^2 \leqq d^2\}$. Then (2.16) $\Rightarrow$ (2.20a) and (2.17) $\Rightarrow$ (2.20b). □

(3.32) THE PROOF OF (2.21). By a multidimensional CLT, viz, Bhattacharya [1], we know

$$\rho(F_n, \Phi) \leqq K(\Lambda + \Delta + \Gamma) , \qquad \text{for some } K > 0 . \tag{3.33}$$

where

$$\Gamma = \max_{i \leq n} \sigma_i / s_n ,$$

$$\Delta = \sum_{i=1}^n \sup_{(a,b):a^2+b^2=1} \int_{[|aX_i+bY_i| \leq s_n]} \left| \frac{aX_i + bY_i}{s_n} \right|^3 dP ,$$

$$\Lambda = \sum_{i=1}^n \int_{[X_i^2+Y_i^2 \geq s_n^2]} \frac{X_i^2 + Y_i^2}{s_n^2} dP .$$

By (2.22) and Jensen's inequality,

$$E(X_i^2)(\log E(X_i^2))^{1+\beta} \leq E[X_i^2(\log X_i^2)^{1+\beta}] \leq \lambda \sigma_i^2 , \qquad \forall i .$$

Hence,

$$\sigma_i^2 \leq \exp(\lambda^{1/(1+\beta)}) , \qquad \forall i ,$$

$$\Gamma \leq \frac{1}{s_n} \exp(\tfrac{1}{2}\lambda^{1/(1+\beta)}) . \tag{3.34}$$

Next,

$$\Lambda = \sum_{i=1}^n \int_{[X_i^2+Y_i^2 \geq s_n^2]} \frac{X_i^2 + Y_i^2}{s_n^2} dP$$

$$\leq \frac{1}{s_n^2(\log s_n^2)^{1+\beta}} \sum_{i=1}^n \int (X_i^2 + Y_i^2)(\log (X_i^2 + Y_i^2))^{1+\beta} dP \tag{3.35}$$

$$\leq \lambda / (\log s_n^2)^{1+\beta}$$

by (2.22).

And,

$$\Delta = \sum_{i=1}^n \sup_{(a,b)} \int_{[aX_i+bY_i| \leq s_n]} \left| \frac{aX_i + bY_i}{s_n} \right|^3 dP$$

$$\leq \frac{1}{s_n^2(\log s_n^2)^{1+\beta}} \sum_{i=1}^n \int (X_i^2 + Y_i^2)(\log (X_i^2 + Y_i^2))^{1+\beta} dP \tag{3.36}$$

$$\leq \lambda / (\log s_n^2)^{1+\beta} .$$

Put (3.34), (3.35), (3.36) in (3.33), to see that

$$\rho(F_n, \Phi) = O(1/(\log s_n^2)^{1+\beta}) .$$

Note that $s_n^2 = t_n^2$, $n_k = m_k$, and $\log s^2(n_k)$ grows linearly.
Hence,

$$\sum_k 1/(\log s^2(n_k))^{a^2+b^2} = \infty \qquad \text{if} \quad a^2 + b^2 < 1 ,$$
$$< \infty \qquad \text{if} \quad a^2 + b^2 > 1 .$$

By Theorem (2.14), we conclude that the a.s. limit set is

$$\{(a, b) \mid a^2 + b^2 \leq 1\} . \qquad \square$$

4. An example. We present here an example for Theorem (2.10). The idea is to construct Z_n, $n \geq 1$, so that when $\{S_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}\}$ clusters around any point, $\{T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}\}$ will stay around 0; and vice versa. To achieve this, let us take two independent copies of fair-coin-tossing variables; i.e. $V_1, V_2, \cdots, U_1, U_2, \cdots$ i.i.d. ± 1 with probability $\frac{1}{2}$ each. We know

$$\sum_{i=1}^{n} V_i/(2n \log_2 n)^{\frac{1}{2}} \to 0 \quad \text{in probability.}$$

Therefore, $\exists \{n_k\} \ni$

$$\sum_{i=1}^{n_k} V_i(2n_k \log_2 n_k)^{\frac{1}{2}} \to 0 \quad \text{a.s.} \tag{4.1}$$

Define $Z_1, Z_2, \cdots$:

$$\binom{V_1}{0}, \binom{V_2}{0}, \cdots, \binom{V_{n_1}}{0}, \quad \binom{0}{U_1}, \binom{0}{U_2}, \cdots, \binom{0}{U_{n_1}},$$
$$\binom{V_{n_1+1}}{0}, \cdots, \binom{V_{n_2}}{0}, \cdots .$$

Write $Z_n = (X_n, Y_n)$, $S_n = \sum_{i=1}^n X_i$, $T_n = \sum_{i=1}^n Y_i$,

$$s_n^3 = E(S_n^2), \qquad t_n^2 = E(T_n^2), \qquad n \geqq 1 .$$

When $S_n/(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}$ visits around a, $-\infty < a < \infty$, $T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}$ is equal to $\sum_{i=1}^{n_k} U_i/(2n_k \log_2 n_k)^{\frac{1}{2}}$, for some n_k, which tends to zero almost surely. Similarly, if we look at those n's when $T_n/(2t_n^2 \log_2 t_n^2)^{\frac{1}{2}}$ visits around b, $-\infty < b < \infty$, $S_n(2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}$ is equal to $\sum_{i=1}^{n_k} V_i/(2n_k \log_2 n_k)^{\frac{1}{2}}$, for some n_k, which also tends to zero almost surely. Therefore, the a.s. limit set sits in the two axes. Applying the one-dimensional result, one argues that the a.s. limit set is exactly $\{(a, b) \mid ab = 0, |a| \leqq 1, |b| \leqq 1\}$. Formally,

(4.2) Claim: the a.s. limit set H of $\{(S_n/L(n), T_n/M(n))\}$ is $\{(a, b) \mid ab = 0, |a| \leqq 1, |b| \leqq 1\}$. To show this claim, let $\varepsilon > 0$ and let

$$A_k = [|S_k/L(k)| \geqq \varepsilon, |T_k/M(k)| \geqq \varepsilon],$$
$$B_k = \bigcup_{2n_{k-1} < i \leqq 2n_{k-1}+m_k} A_i ,$$
$$C_k = \bigcup_{2n_{k-1}+m_k < i \leqq 2n_k} A_i .$$

where $m_k = n_k - n_{k-1}$ and $n_0 = 0$. Since $\{A_k, \text{ i.o.}\} \subset \{B_k, \text{ i.o.}\} \cup \{C_k, \text{ i.o.}\}$, we have

$$P(A_k, \text{ i.o.}) \leqq P(B_k, \text{ i.o.}) + P(C_k, \text{ i.o.}) .$$

But

$$P(B_k, \text{ i.o.}) \leqq P(|T_{n_{k-1}}/M(n_{k-1})| \geqq \varepsilon, \text{ i.o.}) = 0 ,$$
$$P(C_k, \text{ i.o.}) \leqq P(|S_{n_k}/L(n_k)| \geqq \varepsilon, \text{ i.o.}) = 0$$

by (4.1). Hence, $P(A_k, \text{ i.o.}) = 0$. Since ε is arbitrary, $H \subset \{(a, b) \mid ab = 0\}$. Furthermore, all X_n and Y_n are bounded by 1 and $s_n^2 \to \infty$, $t_n^2 \to \infty$. Using Kolmogorov's result and Lemma (3.6), one sees

$$H \subset \{(a, b) \mid |a| \leqq 1, |b| \leqq 1\} .$$

Hence,

$$H \subset \{(a, b) \mid ab = 0, |a| \leqq 1, |b| \leqq 1\} . \tag{4.3}$$

To show the other direction of inclusion, suppose

(4.4) $\quad (a_0, b_0) \notin H$ for some (a_0, b_0) such that $a_0 b_0 = 0$, $|a_0| \leqq 1$, $|b_0| \leqq 1$.

Without loss of generality, assume $b_0 = 0$. Remember H is closed. By (4.3) and (4.4), $\exists \varepsilon > 0 \ni$ if $E = \{(x, y) \mid |x - a_0| \leqq \varepsilon\}$, then $E \cap H = \emptyset$. Applying Remark (3.5),

$$P((S_n/L(n), T_n/M(n)) \in E, \text{ i.o.}) = 0 .$$

Hence,

$$P(|[S_n/L(n)] - a_0| \leqq \varepsilon, \text{ i.o.}) = 0 ,$$

which contradicts the one-dimensional result. □

(4.5) REMARK. It is obvious that we can make $E(X_n^2) > 0$, $E(Y_n^2) > 0$ for all n by adding "small" variables to each Z_n and the result remains valid.

Acknowledgment. I am deeply indebted to Professor David A. Freedman for suggesting the investigation and for guidance and encouragement.

This paper was prepared during the author's visit to the Department of Mathematics, Indiana University, in 1972–73.

I would also like to thank the referee for useful comments.

REFERENCES

[1] BHATTACHARYA, R. N. (1970). Rates of weak convergence for the multidimensional central limit theorem. *Theor. Probability Appl.* **15** 68–88.

[2] FELLER, W. (1943). The general form of the so-called law of the iterated logarithm. *Trans. Amer. Math. Soc.* **54** 373–402.

[3] FELLER, W. (1969). Limit theorems for probabilities of large deviations. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **14** 1–20.

[4] FINKELSTEIN, H. (1971). The law of the iterated logarithm for empirical distributions. *Ann. Math. Statist.* **42** 607–615.

[5] FREEDMAN, D. A. (1972). On tail probabilities for martingales. To appear.

[6] HARTMAN, P. and WINTNER, A. (1941). On the law of the iterated logarithm. *Amer. J. Math.* **63** 169–176.

[7] KESTEN, H. (1970). The limit points of a normalized random walk. *Ann. Math. Statist.* **41** 1173–1205.

[8] KOLMOGOROV, A. N. (1929). Über das Gesetz des iterierten logarithmus. *Math. Ann.* **101** 126–135.

[9] LÉVY, P. (1937). *Théorie de l'addition des variables aléatoires.* Gauthier-Villars, Paris.

[10] MARCINKIEWICZ, J. and ZYGMUND A. (1937). Remàrque sur la loi du logarithme itere. *Fund. Math.* **29** 215–222.

[11] PETROV, V. V. (1971). A theorem on the law of the iterated logarithm. *Tero. Verojatnost. i. Primenen* **16** 715–718.

[12] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 211–226.

[13] STRASSEN, V. (1966). A converse to the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** 265–268.

DEPARTMENT OF MATHEMATICS
SWAIN HALL-EAST
INDIANA UNIVERSITY
BLOOMINGTON, INDIANA 47401