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Some linear relations between values of trigonometric functions at $k\pi/n$

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1. Introduction. We explain the purpose of this note with the example of the cotangent. The meromorphic function cot is periodic modulo π. If n is a natural number, one says that the numbers

$$k\pi/n, \quad k = 0, 1, \ldots, n-1,$$

are the *n-division points* of the period π. Accordingly,

$$\cot\frac{k\pi}{n}, \quad k = 1, \ldots, n-1, \tag{1}$$

are called the *n-division values* of the cotangent (of course, $k = 0$ does not occur here). An n-division value (1) is said to be *primitive* if $(k, n) = 1$. In 1949 Siegel showed that there are no rational linear relations between primitve n-division values of the cotangent except the trivial ones, namely,

$$\cot\frac{(n-k)\pi}{n} = -\cot\frac{k\pi}{n}.$$

In other words, for each $n \geq 3$, the numbers

$$\cot\frac{k\pi}{n}, \quad 1 \leq k < n/2,\ (k, n) = 1, \tag{2}$$

are linearly independent over $\mathbb{Q}$ (cf. [1] for the history of this and related results).

If this is known, it is not very hard to see that *each* n-division value of cot must be a $\mathbb{Q}$-linear combination of the primitive ones (cf. Section 2). But what do these combinations look like? This question comes down to the

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following task: Let $d \geq 2$ be a divisor of $n \geq 3$. Then

$$\cot \frac{\pi}{d} = \sum_{\substack{1 \leq k < n/2 \\ (k,n)=1}} a_k \cot \frac{k\pi}{n} \tag{3}$$

with uniquely determined rational coefficients a_k. Describe these coefficients!

Let us consider a numerical example. Take $n = 1001 = 7 \cdot 11 \cdot 13$ and $d = 13$. Define

$$N = 5^2 \cdot 13^2 \cdot 61 \cdot 181 \cdot 1117.$$

Then each of the 360 coefficients a_k in (3) equals—up to the $\pm$ sign—one of the following six numbers:

$$a(1) = \frac{3 \cdot 19 \cdot 1951 \cdot 6007}{N}, \quad a(2) = \frac{190871389}{2 \cdot N}, \quad a(3) = -\frac{19 \cdot 275929}{N},$$
$$a(4) = \frac{19 \cdot 53 \cdot 103 \cdot 131}{N}, \quad a(5) = -\frac{3^4 \cdot 31 \cdot 1951}{2 \cdot N}, \quad a(6) = \frac{4373 \cdot 27793}{2 \cdot N}.$$

More precisely: Let the index k, $1 \leq k \leq 500$, $(k, 1001) = 1$, be given. If $j \in \{1, \ldots, 6\}$ is such that $k \equiv j \bmod 13$, then $a_k = a(j)$; however, if $k \equiv -j \bmod 13$, then $a_k = -a(j)$. The above formulas render the complete factorization of the numerators and denominators of $a(1), \ldots, a(6)$ (e.g., the numerator of $a(2)$ is a prime). But from these formulas one can hardly see how to find the coefficients a_k in general. Nevertheless, there exists a sort of closed formula for them.

In Section 2 we display such formulas for all derivatives of the cotangent (Theorem 1). These formulas involve group theoretic quantities such as the order modulo d of a prime divisor p of n, $p \nmid d$. For this reason it is not surprising that the proof strongly relies on character theory.

We think, for the same reason, that it is hardly possible to obtain theorems of this kind by elementary manipulations with known trigonometric formulas except in very special cases, e.g., if $n = d \cdot p$ and the prime p also divides d. Here the coefficients a_k can be read from

$$\sum_{k=0}^{q-1} \cot(x + k\pi/q) = q \cot qx,$$

which holds for any natural number q (cf. [4], p. 646, formulas no. 2, 4).

Section 3 contains the respective results for the derivatives of the tangent, cosecant, and the square of the cotangent. The proof of the last-mentioned formula serves as a paradigm for higher powers of cot, tan, etc. Finally, we briefly discuss the sine and the cosine. Here formulas of type (3) only hold under strong restrictions on n and d. They look very simple and are known, in principle. All formulas included in this paper have been tested with numerical examples—so they ought to be correct in detail.

2. The cotangent theorem. We need some notations. Throughout this paper p denotes a prime number. If n is an integer, we write $v_p(n)$ for the maximal integer $k \geq 0$ such that $p^k \mid n$. Moreover,

$$n^* = \prod_{p\mid n} p \quad \text{and} \quad \psi(n) = \prod_{p\mid n} (p^{v_p(n)} - 1) \tag{4}$$

(so ψ is formally similar to Euler's φ). Let d be a divisor of n. We put

$$\lambda(n,d) = \prod_{\substack{p\mid n \\ p\nmid d}} p^{l(p,d)}, \tag{5}$$

where $l(p,d)$ means the *order* of p mod d, i.e., the minimal exponent $k \geq 1$ with $p^k \equiv 1 \bmod d$.

For an integer $r \geq 1$ let $\cot(r,x)$ denote the $(r-1)$th derivative of the function $\cot x$; in particular, $\cot(1,x) = \cot x$. If the condition "$k \equiv j \bmod d$" occurs in a summation index, we write "$k \equiv j\,(d)$" instead (for typographical reasons).

Theorem 1. *Let $n \geq 3$ be a natural number and $d \geq 2$ a divisor of n. Put $m = \lambda(n,d)$. For each integer $r \geq 1$,*

$$\cot\left(r, \frac{\pi}{d}\right) = \sum_{\substack{1\leq k<n/2 \\ (k,n)=1}} a_{r,k} \cot\left(r, \frac{k\pi}{n}\right),$$

with

$$a_{r,k} = \left(\frac{d}{n}\right)^r \frac{1}{\psi(m^r)} \Big(\sum_{q\equiv k\,(d)} q^r + (-1)^r \sum_{q\equiv -k\,(d)} q^r\Big).$$

In these sums q runs through all natural numbers satisfying both $m^* \mid q \mid m$ and the respective congruence condition.

Remarks. 1. The coefficients $a_{r,k}$ obviously depend on the residue class of k mod d only; therefore we may write

$$a_{r,k} = a(r,j),$$

where $j \in \{1, \ldots, d-1\}$, $(j,d) = 1$, is uniquely determined by the congruence $k \equiv j \bmod d$. Since

$$a(r, d-j) = (-1)^r a(r,j),$$

only the numbers $a(r,j)$, $1 \leq j < d/2$, are of actual interest. This is what we observed in the numerical example of the Introduction.

2. Fix, for a moment, the number d and a finite set P of primes not dividing d. If n runs through those multiples of d for which $\{p : p \mid n,\, p \nmid d\} = P$, the above coefficients $a(r,j), 1 \leq j < d$, $(j,d) = 1$, remain *unchanged* up to the trivial factor $(d/n)^r$.

3. The example of the Introduction shows that the coefficients $a_{r,k}$ may look quite complicated if there are at least two prime divisors of n not dividing d. However, these coefficients have a very simple form if all primes of this kind are $\equiv 1 \bmod d$. For example, let n/d be a product of distinct primes $p \equiv 1 \bmod d$. By Theorem 1,

$$a_{1,k} = \begin{cases} \pm 1/\prod_{\substack{p\mid n\\ p\nmid d}}(p-1) & \text{if } k \equiv \pm 1 \bmod d,\\ 0 & \text{otherwise.}\end{cases}$$

4. The coefficients $a_{r,k}$ of Theorem 1 being known, it is easy to express an *arbitrary* n-division value $\cot(r, s\pi/d)$, $(s,d)=1$, in terms of the primitive ones: Let $s_1 \in \mathbb{Z}$ be such that $s_1 \equiv s \bmod d$ and $(s_1, n) = 1$. If k is an integer with $(k,n)=1$, define $k' \in \{1,\ldots,n-1\}$, $(k',n)=1$, by the congruence $s_1 k' \equiv k \bmod n$. Then

$$\cot\left(r, \frac{s\pi}{d}\right) = \sum_{\substack{1\le k<n/2\\ (k,n)=1}} b_{r,k} \cot\left(r, \frac{k\pi}{n}\right),$$

with

$$b_{r,k} = \begin{cases} a_{r,k'} & \text{if } k' < n/2,\\ (-1)^r a_{r,n-k'} & \text{otherwise.}\end{cases} \tag{6}$$

We postpone the proof of Theorem 1 (and (6)) for a moment and return to a remark of the Introduction: namely, that the $\mathbb{Q}$-linear independence of the primitive n-division values (2) implies that *each* n-division value of the cotangent is a rational linear combination of them. Although this follows from Theorem 1, we give a (short) independent proof here. Consider the nth cyclotomic field

$$\mathbb{Q}^{(n)} = \mathbb{Q}(\zeta_n) \quad \text{with } \zeta_n = e^{2\pi i/n}.$$

One readily checks that

$$i \cot \frac{k\pi}{n} = (1+\zeta_n^k)/(1-\zeta_n^k), \qquad (k,n)=1. \tag{7}$$

In particular, these numbers lie in $\mathbb{Q}^{(n)}$, more precisely, in the $\mathbb{Q}$-subspace $V = \mathbb{Q}^{(n)} \cap \mathbb{R}i$. It is easy to see that $\mathbb{Q}^{(n)}$ is the direct sum of the $\mathbb{Q}$-subspaces $\mathbb{Q}^{(n)} \cap \mathbb{R}$ and V. The first of these subspaces is a subfield of $\mathbb{Q}^{(n)}$ of well-known $\mathbb{Q}$-degree; indeed, for $n \ge 3$, $[\mathbb{Q}^{(n)} \cap \mathbb{R} : \mathbb{Q}] = [\mathbb{Q}^{(n)} : \mathbb{Q}]/2 = \varphi(n)/2$. Hence V must also have the $\mathbb{Q}$-dimension $\varphi(n)/2$. But then the $\mathbb{Q}$-linear independent numbers (7) with $1 \le k < n/2$ form a basis of V. Now let $d \ge 2$ be a divisor of n. Since $\mathbb{Q}^{(d)} \subseteq \mathbb{Q}^{(n)}$, each number $i\cot(s\pi/d)$ lies in V, so it is a $\mathbb{Q}$-linear combination of the elements of the above basis.

The proof of Theorem 1 requires some preparations, in particular, some additional notation. By $G^{(n)}$ we denote the Galois group of $\mathbb{Q}^{(n)}$ over $\mathbb{Q}$. Its elements can be written as $\sigma_k^{(n)}$, $k \in \mathbb{Z}$, $(k,n)=1$, where $\sigma_k^{(n)}$ acts on ζ_n by $\sigma_k^{(n)}(\zeta_n) = \zeta_n^k$. Hence, by (7),

$$\sigma_k^{(n)}\left(i\cot\frac{\pi}{n}\right) = i\cot\frac{k\pi}{n}. \tag{8}$$

Let $X^{(n)}$ denote the character group of $G^{(n)}$. As usual, we consider each $\chi \in X^{(n)}$ as a Dirichlet character modulo n, on putting $\chi(k) = \chi(\sigma^{(n)}(k))$ if $(k,n)=1$, and $\chi(k)=0$ otherwise. For each divisor d of n there is a canonical epimorphism

$$G^{(n)} \to G^{(d)} : \sigma_k^{(n)} \mapsto \sigma_k^{(d)} = \sigma_k^{(n)}|_{\mathbb{Q}^{(d)}}.$$

By virtue of this map, the group $X^{(d)}$ can be considered as a subgroup of $X^{(n)}$. Hence certain characters $\chi \in X^{(n)}$ can also be regarded as elements of $X^{(d)}$. If we do so, we sometimes write $\chi = \chi_d$ (in order to avoid ambiguities). The smallest divisor d for which $\chi \in X^{(d)}$ makes sense is the *conductor* f_χ of $\chi \in X^{(n)}$. Instead of χ_{f_χ} we simply write χ_f.

We also need the rational group ring $\mathbb{Q}[G^{(n)}]$ of $G^{(n)}$, whose elements have the shape

$$\gamma = \sum_{\sigma \in G^{(n)}} c_\sigma \sigma, \qquad c_\sigma \in \mathbb{Q}. \tag{9}$$

The field $\mathbb{Q}^{(n)}$ is a $\mathbb{Q}[G^{(n)}]$-module in the usual way. The *trace*

$$T_d^{(n)} = \sum_{\substack{k \bmod^* n \\ k \equiv 1\ (d)}} \sigma_k^{(n)},$$

is an element of $\mathbb{Q}[G^{(n)}]$ of particular importance; here d means a divisor of n and "$k \bmod^* n$" stands for "$1 \le k \le n$, $(k,n)=1$". For any $u \in \mathbb{Q}^{(n)}$, $T_d^{(n)}u$ is in $\mathbb{Q}^{(d)}$. Note the rule

$$\sigma_q^{(d)}(T_d^{(n)}u) = \sum_{\substack{k \bmod^* n \\ k \equiv q\ (d)}} \sigma_k^{(n)}(u), \tag{10}$$

which holds for any q with $(q,d)=1$.

Take a character $\chi \in X^{(n)}$. If $\gamma \in \mathbb{Q}[G^{(n)}]$ is as in (9), then $\chi(\gamma)$ is defined by

$$\chi(\gamma) = \sum_{\sigma \in G^{(n)}} c_\sigma \chi(\sigma).$$

Thus $\chi(\gamma)$ lies in the field $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(\sigma) : \sigma \in G^{(n)})$. This field is a (simple) $\mathbb{Q}[G^{(n)}]$-module, whose scalar multiplication is given by

$$\gamma u = \chi(\gamma) u.$$

Our main technical device in the proof of Theorem 1 is the character coordinates introduced by Leopoldt. We do not repeat their definition (cf. [1], [3]) but only recall those of their properties that enter the proof:

1. For each $\chi \in X^{(n)}$, the χ-coordinate is a $\mathbb{Q}[G^{(n)}]$-linear map

$$y_\chi^{(n)} : \mathbb{Q}^{(n)} \to \mathbb{Q}(\chi).$$

Accordingly, $y_\chi^{(n)}(\gamma u) = \chi(\gamma) y_\chi^{(n)}(u)$ for all $\gamma \in \mathbb{Q}[G^{(n)}]$ and $u \in \mathbb{Q}^{(n)}$. In particular, $y_\chi^{(n)}(\sigma_k^{(n)} u) = \chi(k) y_\chi^{(n)}(u)$ for each $k \in \mathbb{Z}$ with $(k, n) = 1$.

2. The totality of all χ-coordinates characterizes an element u of $\mathbb{Q}^{(n)}$ completely: If u and v are in $\mathbb{Q}^{(n)}$, then u equals v if, and only if, $y_\chi^{(n)}(u)$ equals $y_\chi^{(n)}(v)$ for all $\chi \in X^{(n)}$.

3. The *reduction property* reads as follows: Let d divide n and χ be a character in $X^{(d)} \subseteq X^{(n)}$. For each $u \in \mathbb{Q}^{(n)}$, $y_\chi^{(n)}(u) = y_\chi^{(d)}(T_d^{(n)} u)$.

Let n be ≥ 2. We observe that the numbers $i^r \cot(r, k\pi/n)$, $k \geq 1$, all lie in $\mathbb{Q}^{(n)}$. For $r = 1$ this was shown above (cf. (7)); for $r \geq 2$ this follows from the fact that the functions $i^r \cot(r, x)$ can be written as polynomials in $i \cot x$ with rational coefficients. Combined with (8), this argument also shows

$$\sigma_k^{(n)}\left(i^r \cot\left(r, \frac{\pi}{n}\right)\right) = i^r \cot\left(r, \frac{k\pi}{n}\right). \tag{11}$$

Furthermore, the coordinate $y_\chi^{(n)}(i^r \cot(r, \pi/n))$ is well-defined for each $\chi \in X^{(n)}$. In [1] we computed its value: If $\chi(-1) = (-1)^r$, then

$$y_\chi^{(n)}\left(i^r \cot\left(r, \frac{\pi}{n}\right)\right) = \left(\frac{2n}{f_\chi}\right)^r \prod_{p|n}\left(1 - \frac{\overline{\chi}_f(p)}{p^r}\right) B_{r,\overline{\chi}_f}/r, \tag{12}$$

where $\overline{\chi}$ is the complex-conjugate character of χ and B_{r,χ_f} is the rth Bernoulli number belonging to $\chi_f \in X^{(f_\chi)}$; if $\chi(-1) \neq (-1)^r$, then $y_\chi^{(n)}(i^r \cot(r, \pi/n)) = 0$. Formula (12) is the cornerstone of the

P r o o f o f T h e o r e m 1. Suppose that $d \geq 2$ divides n. Take a character $\chi \in X^{(d)}$. On comparing the χ-coordinates of $i^r \cot(r, \pi/d)$ and $i^r \cot(r, \pi/n)$, one obtains from (12)

$$y_\chi^{(d)}\left(i^r \cot\left(r, \frac{\pi}{d}\right)\right) = \left(\frac{d}{n}\right)^r \prod_{\substack{p|n \\ p\nmid d}}\left(1 - \frac{\overline{\chi}_f(p)}{p^r}\right)^{-1} y_\chi^{(n)}\left(i^r \cot\left(r, \frac{\pi}{n}\right)\right). \tag{13}$$

As regards the right side of (13), we may replace χ_f by χ since f_χ divides d and all p's in question do not divide d. In addition, we apply property 3 of character coordinates. Thereby we get

$$(14)\qquad y_\chi^{(d)}\left(i^r\cot\left(r,\frac{\pi}{d}\right)\right) = \left(\frac{d}{n}\right)^r \prod_{\substack{p\mid n\\ p\nmid d}}\left(1-\frac{\overline{\chi}(p)}{p^r}\right)^{-1} y_\chi^{(d)}\left(T_d^{(n)} i^r\cot\left(r,\frac{\pi}{n}\right)\right).$$

Suppose now that $\eta\in\mathbb{C}$ is such that $\eta^l=1$, $l\geq 1$ being a natural number. One readily verifies the formula

$$(15)\qquad \left(1-\frac{1}{\eta z}\right)^{-1} = \frac{1}{z^l-1}\sum_{j=1}^{l}\eta^j z^j,$$

which holds for all $z\in\mathbb{C}$ fulfilling $|z|\neq 1$, say. Take $\eta=\chi(p)$ and $z=p^r$ as in (14). Then formula (15) can be applied with $l=l(p,d)$. This yields

$$\prod_{\substack{p\mid n\\ p\nmid d}}\left(1-\frac{\overline{\chi}(p)}{p^r}\right)^{-1} = \frac{1}{\psi(m^r)}\sum_{m^*\mid q\mid m}\chi(q)q^r,$$

with $m=\lambda(n,d)$ as in (5) and ψ as in (4). We insert this in formula (14) and use property 1 of character coordinates. Then

$$y_\chi^{(d)}\left(i^r\cot\left(r,\frac{\pi}{d}\right)\right) = y_\chi^{(d)}\left(\left(\frac{d}{n}\right)^r\frac{1}{\psi(m^r)}\sum_{m^*\mid q\mid m} q^r\sigma_q^{(d)}\left(T_d^{(n)} i^r\cot\left(r,\frac{\pi}{n}\right)\right)\right).$$

Next one applies the identity (10) to the right side of this equation and recalls that $\chi\in X^{(d)}$ was arbitrary. Therefore, property 2 of character coordinates gives

$$i^r\cot\left(r,\frac{\pi}{d}\right) = \left(\frac{d}{n}\right)^r\frac{1}{\psi(m^r)}\sum_{m^*\mid q\mid m} q^r \sum_{\substack{k \bmod^* n\\ k\equiv q\ (d)}}\sigma_k^{(n)}\left(i^r\cot\left(r,\frac{\pi}{n}\right)\right).$$

By (11), one obtains

$$(16)\qquad i^r\cot\left(r,\frac{\pi}{d}\right) = \left(\frac{d}{n}\right)^r\frac{1}{\psi(m^r)}\sum_{m^*\mid q\mid m} q^r \sum_{\substack{k \bmod^* n\\ k\equiv q\ (d)}} i^r\cot\left(r,\frac{k\pi}{n}\right),$$

which readily implies Theorem 1. ■

The proof of formula (6) is still missing. However, this is also an easy consequence of (16): just apply $\sigma_{s_1}^{(n)}$ to both sides of this equation and change the summation index.

3. Other trigonometric functions. The method of the foregoing section also works for other trigonometric functions, provided that the χ-coordinates of the (suitably modified) division values are known and interrelated by formulas like (13). This is the case for the derivatives of the tangent. Let $\tan(r,x)$ denote the $(r-1)$th derivative of $\tan x$, so $\tan(1,x)=\tan x$. In [1] we showed that, for $n\ge 3$, $y_\chi^{(n)}(i^r\tan(r,\pi/n))$ equals $y_\chi^{(n)}(i^r\cot(r,\pi/n))$ times the following factor:

$$c_\chi(r,n)=\begin{cases}1-2^r\chi(2) & \text{if } 2\nmid n,\\ (1-2^r\chi_f(2))^{-1} & \text{if } n\equiv 2 \bmod 4,\\ -\chi(n/2+1) & \text{otherwise.}\end{cases}\tag{17}$$

Formula (17) implies that the numbers $\tan(r,k\pi/n)$, $1\le k<n/2$, $(k,n)=1$, are also linearly independent over $\mathbb{Q}$. But it requires a number of cases to be distinguished in the $\tan(r,x)$-formula of type (13). Finally, this phenomenon occurs in the consequent analogue of Theorem 1. For a divisor d of the natural number n put

$$\widetilde{d}=\begin{cases}d & \text{if } v_2(d)=v_2(n) \text{ or } v_2(d)\ge 2,\\ 2d & \text{if } v_2(d)=0<v_2(n),\\ d/2 & \text{if } v_2(d)=1<v_2(n).\end{cases}$$

Moreover, put $\widetilde{m}=\lambda(n,\widetilde{d})$.

THEOREM 2. *Let $d\ge 3$ be a divisor of n and $\widetilde{d}$, $\widetilde{m}$ as above. Then*

$$\tan\left(r,\frac{\pi}{d}\right)=\left(\frac{\widetilde{d}}{n}\right)^r\frac{1}{\psi(\widetilde{m}^r)}\sum_{\substack{1\le k<n/2\\ (k,n)=1}} b_{r,k}\tan\left(r,\frac{k\pi}{n}\right),$$

where $b_{r,k}$ takes the following values:

(a) *If $v_2(d)=v_2(n)$, then*

$$b_{r,k}=\sum_{q\equiv k\,(d)}q^r+\sum_{q\equiv -k\,(d)}(-q)^r.$$

(b) *If $2\le v_2(d)<v_2(n)$, then*

$$b_{r,k}=\sum_{q\equiv k+d/2\,(d)}q^r+\sum_{q\equiv -k+d/2\,(d)}(-q)^r.$$

(c) *If $v_2(d)=0$ and $v_2(n)=1$, then*

$$b_{r,k}=\sum_{4q\equiv k\,(d)}(2q)^r+\sum_{4q\equiv -k\,(d)}(-2q)^r-\sum_{2q\equiv k\,(d)}q^r-\sum_{2q\equiv -k\,(d)}(-q)^r.$$

(d) *If $v_2(d)=0$ and $v_2(n)\ge 2$, then*

$$b_{r,k}=\sum_{2q\equiv k\,(d)}q^r+\sum_{2q\equiv -k\,(d)}(-q)^r.$$

(e) *If* $v_2(d) = 1 < v_2(n)$, *then*

$$b_{r,k} = \sum_{q/2 \equiv k\,(d/2)} q^r + \sum_{q/2 \equiv -k\,(d/2)} (-q)^r.$$

In all cases q *runs through all natural numbers satisfying both* $\widetilde{m}^* \mid q \mid \widetilde{m}$ *and the respective congruence condition.*

Remark. An inspection of Theorem 2 shows the following: if $v_2(d) = v_2(n)$, then Theorem 1 applies word for word to the function $\tan(r,x)$; in the other cases the result may be quite different. For example, take $n_1 = 4004 = 4n$ and $d_1 = 26 = 2d$, where n and d are as in the Introduction. Let the coefficients $a_{1,k}$, $1 \le k < n_1/2$, $(k, n_1) = 1$, be defined as in Theorem 1; so they relate $\cot(\pi/26)$ with the primitive 4004-division values of cot. Up to sign, each coefficient $a_{1,k}$ equals one of the following six numbers:

$$\begin{aligned} &a(1,1) = a(1)/2, && a(1,3) = a(3)/2, && a(1,5) = a(5)/2,\\ &a(1,7) = -a(6)/2, && a(1,9) = -a(4)/2, && a(1,11) = -a(2)/2, \end{aligned}$$

where $a(j)$ is defined as in the Introduction; in fact, $a_{1,k} = \pm a(1,j)$ whenever $k \equiv \pm j \bmod d_1$. In the case of the tangent, let $c_{1,k}$ be the coefficient that plays the role of $a_{1,k}$ and let $c(1,j)$ correspond to $a(1,j)$ in the same way. Take $N = 5^2 \cdot 13^2 \cdot 61 \cdot 181 \cdot 1117$ as in the Introduction. Then the values of the numbers $c(1,j)$ are as follows:

$$\begin{aligned} c(1,1) &= \frac{2 \cdot 3 \cdot 347 \cdot 130811}{13 \cdot N}, & c(1,3) &= -\frac{709 \cdot 1348547}{13 \cdot N},\\ c(1,5) &= \frac{1980394109}{2^2 \cdot 13 \cdot N}, & c(1,7) &= \frac{3 \cdot 7 \cdot 59 \cdot 7448719}{2^2 \cdot 13 \cdot N},\\ c(1,9) &= \frac{2^2 \cdot 2777 \cdot 21569}{13 \cdot N}, & c(1,11) &= \frac{17 \cdot 67 \cdot 265619}{2^2 \cdot 13 \cdot N}. \end{aligned}$$

The above factorization of each $c(1,j)$ is complete. Obviously, the numerators of $c(1,j)$ have almost nothing in common with those of $a(1,j)$.

We do not include the whole proof of Theorem 2 here but pick out one case that may serve as a model: Suppose that $v_2(d) = 1 < v_2(n)$ (corresponding to case (e)). By (13) and (17), one obtains $y_\chi^{(d)}(i^r \tan(r, \pi/d))$, $\chi \in X^{(d)}$, if one multiplies $y_\chi^{(n)}(i^r \tan(r, \pi/n))$ by the factor

$$F = \left(\frac{d}{n}\right)^r \prod_{\substack{p \mid n\\ p \nmid d}} \left(1 - \frac{\overline{\chi}_f(p)}{p^r}\right)^{-1} (1 - 2^r \chi_f(2))^{-1} (-\chi(n/2+1))^{-1}.$$

Since $v_2(n) > v_2(d)$, d divides $n/2$, so $\chi(n/2+1) = 1$. Therefore

$$F = \left(\frac{\widetilde{d}}{n}\right)^r \prod_{\substack{p\mid n\\ p\nmid \widetilde{d}}} \left(1 - \frac{\overline{\chi}_f(p)}{p^r}\right)^{-1} \overline{\chi}_f(2), \tag{18}$$

with $\widetilde{d} = d/2$ (note that $2 \mid n$ but $2 \nmid \widetilde{d}$). Since $d \equiv 2 \bmod 4$, $X^{(d)}$ equals $X^{(\widetilde{d})}$ (in the usual sense). In particular, $\chi_{\widetilde{d}}$ is well-defined. We may further replace χ_f by $\chi_{\widetilde{d}}$ everywhere in (18) and get

$$F = \left(\frac{\widetilde{d}}{n}\right)^r \frac{1}{\psi(\widetilde{m}^r)} \sum_{\widetilde{m}^* \mid q \mid \widetilde{m}} \chi_{\widetilde{d}}(q) q^r \cdot \overline{\chi}_{\widetilde{d}}(2),$$

with $\widetilde{m} = \lambda(n, \widetilde{d})$. The q's occurring in this formula are all even; hence $\chi_{\widetilde{d}}(q) = \chi_{\widetilde{d}}(q/2)\chi_{\widetilde{d}}(2)$, and so

$$F = \left(\frac{\widetilde{d}}{n}\right)^r \frac{1}{\psi(\widetilde{m}^r)} \sum_{\widetilde{m}^* \mid q \mid \widetilde{m}} \chi_{\widetilde{d}}(q/2) q^r.$$

Since $\mathbb{Q}^{(d)} = \mathbb{Q}^{(\widetilde{d})}$, we have $y_\chi^{(d)}(i^r \tan(r, \pi/d)) = y_\chi^{(\widetilde{d})}(i^r \tan(r, \pi/d))$ (cf. property 3). With this in mind we obtain, in the same way as in the proof of Theorem 1,

$$i^r \tan\left(r, \frac{\pi}{d}\right) = \left(\frac{\widetilde{d}}{n}\right)^r \frac{1}{\psi(\widetilde{m}^r)} \sum_{\widetilde{m}^* \mid q \mid \widetilde{m}} q^r \sigma_{q/2}^{(\widetilde{d})}\left(T_{\widetilde{d}}^{(n)} i^r \tan\left(r, \frac{\pi}{n}\right)\right).$$

Now (10) yields assertion (e) of Theorem 2.

The above method also works for the derivatives of the cosecant. Let $\csc(r, x)$ denote the $(r-1)$th derivative of $\csc x = 1/\sin x$. In [1] the character coordinates of $i^r \csc(r, 2\pi/n)$, $n \geq 3$, were computed. It is not always true, however, that $\csc(r, 2\pi/d)$ is a $\mathbb{Q}$-linear combination of the primitive n-division values of $\csc(r, 2x)$ ($d \geq 3$, $d \mid n$), in contrast with the cases considered so far. Indeed, the methods of [1] show that

$$\csc\left(r, \frac{2\pi}{d}\right) \in \sum_{k \bmod^* n} \mathbb{Q} \csc\left(r, \frac{2k\pi}{n}\right)$$

if, and only if, $v_2(d) = v_2(n)$ or $4 \nmid n$. In the case $v_2(d) = v_2(n)$ Theorem 1 holds for $\csc(r, 2x)$ in place of $\tan(r, x)$ without any changes. If d is odd and $n \equiv 2 \bmod 4$, one obtains the same formula as in case (d) of Theorem 2, up to sign: the coefficients $b_{r,k}$ must be multiplied by -1.

From Theorems 1, 2 one can derive analogous theorems for the powers $\cot^r x$, $\tan^r x$, etc., since these functions are, essentially, rational linear combinations of the derivatives of $\cot x$, $\tan x$, etc. If r is even, an additional consideration is required, as the case $r = 2$ will show. We confine ourselves to this case here. For a natural number n define $\varrho(n)$ by

$$\varrho(n) = n \prod_{p|n} (1 + 1/p),$$

so this is another analogue of Euler's φ.

THEOREM 3. *Let $d \geq 2$ be a divisor of $n \geq 3$. Then*

$$\cot^2 \frac{\pi}{d} = \sum_{\substack{1 \leq k < n/2 \\ (k,n)=1}} c_k \cot^2 \frac{k\pi}{n},$$

with $c_k = 2u/v + a_{2,k}$. Here $a_{2,k}$ is the coefficient given in Theorem 1 (case $r = 2$) and

$$u = -1 + \varrho(d)/\varrho(n), \qquad v = \varphi(n)(-1 + \varrho(n)/3).$$

Proof. Since $i^2 \cot(2, x) = 1 + \cot^2 x$,

$$\cot^2 \frac{\pi}{d} = -1 + \sum_{\substack{1 \leq k < n/2 \\ (k,n)=1}} a_{2,k} \left(1 + \cot^2 \frac{k\pi}{n}\right). \tag{19}$$

It is not hard to verify that

$$-1 + \sum_{\substack{1 \leq k < n/2 \\ (k,n)=1}} a_{2,k} = u.$$

Let $\chi_0 \in X^{(1)}$ be the principal character. Then

$$T_1^{(n)} i^2 \cot\left(2, \frac{\pi}{n}\right) = y_{\chi_0}^{(1)}\left(T_1^{(n)} i^2 \cot\left(2, \frac{\pi}{n}\right)\right),$$

so property 3 of character coordinates gives

$$T_1^{(n)} i^2 \cot\left(2, \frac{\pi}{n}\right) = y_{\chi_0}^{(n)}\left(i^2 \cot\left(2, \frac{\pi}{n}\right)\right).$$

Now (12) implies

$$T_1^{(n)} i^2 \cot\left(2, \frac{\pi}{n}\right) = 4n^2 \prod_{p|n} (1 - 1/p^2) B_2/2,$$

where $B_2 = 1/6$ is the (ordinary) second Bernoulli number. Thus

$$\sum_{\substack{1 \leq k < n/2 \\ (k,n)=1}} \cot^2 \frac{k\pi}{n} = \frac{1}{2} T_1^{(n)} \cot^2 \frac{\pi}{n} = \frac{v}{2}.$$

The number v being > 0, (19) may be written

$$\cot^2\frac{\pi}{d} = \frac{2u}{v}\sum_{\substack{1\le k<n/2\\(k,n)=1}}\cot^2\frac{k\pi}{n} + \sum_{\substack{1\le k<n/2\\(k,n)=1}} a_{2,k}\cot^2\frac{k\pi}{n}.$$

This concludes the proof. ∎

The reader may ask what the respective formulas for the sine and cosine look like. They can be found—in an implicit form at least—in the literature. Let $\zeta_n = \cos(2\pi/n) + i\sin(2\pi/n)$ be as usual. Then ζ_d is in $\mathbb{Q}[G^{(n)}]\zeta_n$ if, and only if, n/d is square-free and $(d, n/d) = 1$. In this case

$$T_d^{(n)}\zeta_n = \mu(n/d)\sigma_{n/d}^{(d)\,-1}(\zeta_d),$$

where μ is the Möbius function (cf. formula (34) in [2], cf. also [3]). Combined with (10), this gives

$$\sin\frac{2\pi}{d} = \mu(n/d)\biggl(\sum_{k\equiv n/d\,(d)}\sin\frac{2\pi k}{n} - \sum_{k\equiv -n/d\,(d)}\sin\frac{2\pi k}{n}\biggr),$$

with $1 \le k < n/2$, $(k,n) = 1$. This formula remains valid for the cosine if the minus sign between both sums is replaced by "+". Whenever n and d do not satisfy the above condition, no formulas of this kind exist.

References

[1] K. Girstmair, *Character coordinates and annihilators of cyclotomic numbers*, Manuscripta Math. 59 (1987), 375–389.
[2] —, *Dirichlet convolution of cotangent numbers and relative class number formulas*, Monatsh. Math. 110 (1990), 231–256.
[3] G. Lettl, *The ring of integers of an abelian number field*, J. Reine Angew. Math. 404 (1990), 162–170.
[4] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series*, Vol. I, Gordon and Breach, New York, 1986.

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