SOME MANIFOLDS OF PERIODIC ORBITS IN THE RESTRICTED THREE BODY PROBLEM

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ABSTRACT

In the present paper we give some numerical results about natural families of periodic orbits, which emanate from limiting orbits around the equilateral equilibrium points of the Restricted Three Body Problem, when the mass ratio is greater than Routh's critical one.

RESUM

En aquest article es presenten els resultats numèrics obtinguts sobre les famílies naturals d'òrbites periòdiques al voltant del punt L4 del problema restringit circular i la de 3 cossos, per a valors del paràmetre de massa superiors a la massa crítica de Routh.

(Aquest report ha sigut enviat a "Celestial Mechanics").

1.- Introduction.

Let m_1, m_2 be the masses of the primaries normalized in such a way that $m_1 = \mu$, $m_2 = 1 - \mu$, $\mu \in [0,1]$. Units of length and time are chosen in order to have one unit of distance between the primaries and a mean motion equal to one.

In a synodical system of coordinates (x,y) the two primaries are fixed at $(1-\mu, 0)$ and $(-\mu, 0)$, respectively. The equations of motion are (see |12|)

$$\ddot{x} - 2\dot{y} = \Omega_x \qquad \qquad (1)$$
$$\ddot{y} + 2\dot{x} = \Omega_y$$

where

$$\Omega(x,y) = \frac{1}{2}(\mu r_1^2 + (1-\mu) r_2^2) + \frac{\mu}{r_1} + \frac{1-\mu}{r_2}$$

and

$$r_1^2 = (x - 1 + \mu)^2 + y^2$$
$$r_2^2 = (x + \mu)^2 + y^2$$

System (1) has the Jacobian integral

$$C = 2\,\Omega(x,y) - (x^2 + y^2)$$

We consider the equilibrium point L_4, i.e.

$$x = 1/2 - \mu \qquad y = \sqrt{3}/2$$

In connection with this equilateral point, there exists a critical mass ratio

$$\mu_1 = \frac{1}{2}\left[1 - \frac{1}{9}(69)^{1/2}\right] = 0.03852\ldots$$

which has the following properties (see |12|)

(i) All four characteristic exponents of the equilibrium are distinct if and only if $\mu \neq \mu_1$, $1-\mu_1$

(ii) For $\mu < \mu_1$ they are of the form $\pm i\, n_s$, $\pm i\, n_1$, where the real numbers n_s and n_1 satisfy the inequalities

$$0 < n_1 < 1/\sqrt{2} < n_s < 1 \qquad (2)$$

they are thus of the linearly stable type. Similar for $\mu > 1 - \mu_1$

(iii) For $1-\mu_1>\mu>\mu_1$ the characteristic exponents are of the form $\alpha \pm i\beta$, where the real numbers α and β are both different from zero, they are thus of the unstable type.

Lyapunov's theorem (see |10|) establishes that:

(a) For any $\mu \in (0,\mu_1)$, there emanates from L_4 a family $\mathcal{L}_4^s(\mu)$ of short period orbits which depends analytically on a real parameter ε; when ε goes to zero, the periodic orbit vanishes at the point L_4 itself, and its period tens to $2\pi/n_s$.

(b) For any $\mu \in (0, \mu_1)$ except at the critical mass ratios

$$\mu_k = \frac{1}{2}\left[1 - \left(1 - \frac{16\,k^2}{27\,(k^2+1)^2}\right)^{1/2}\right] \quad k=1,2,..$$

there exists another family of periodic orbits around L_4 which depends analytically on a real parameter ε; when ε goes to zero, the periodic orbit vanishes at the point L_4 itself, but this time, its period tends to $2\pi/n_1$. Hence in view of (2), the family is called the family of long period orbits at L_4, $\mathcal{L}_4^1(\mu)$.

For μ slightly greater than μ_1, Brown (|1|) analyzed the equilibrium configuration up to the third order. His conclusions where the following:

(i) There still exist two families of periodic orbits at L_4 and both families depend analytically on a real parameter ε.

(ii) This orbital parameter admits a strictly positive lower bound α the same for both families. α is an analytical function of the mass ratio μ; when ε goes to α, both families close to a common periodic orbit, which is called the limiting orbit.

(iii) The limiting orbit is properly periodic, the period being equal to $2\pi\sqrt{2}$ whatever the mass ratio may be. For $\alpha > 0$, the limiting orbit stays at finite distance from L_4. When μ tends to the critical value μ_1, α goes to zero and the limiting orbit vanishes at L_4.

It is known too (|8|) that:

There exists a neighbourhood of μ_1 such that for every $\mu>\mu_1$ in the neigbourhood, there is one family of periodic solutions around L_4, which depends analytically on a real parameter, formed by periodic solutions of the stable type and now this family does not contain the equilibrium point itself.

Pedersen (|9|) using Fourier series depending on time for the coordinates found some of these limiting orbits for $\mu=0.03907, 0.04072$.

Deprit (|3|) using d'Alembert series and a fourteen order theory computed limiting orbits for values of μ up to $\mu=0.044$.

In this paper we have computed, using numerical continuation, an isoperiodic family of limiting orbits for values of μ up to $\mu_M = 0.864496...$, where a maximum for the variable μ is reached for the family.

For some values of μ, i.e., $\mu=0.04, 0.04542, 0.051, 0.3, 0.5, ...$ and starting at the limiting orbit, we have computed the natural families $\mathcal{L}^s(\mu)$ and $\mathcal{L}^1(\mu)$, as an analytical continuation of it. The behaviour of these families is the following:

For any mass ratio, of the ones that we have explored, the family $\mathcal{L}^s(\mu)$ of asymetric periodic orbits, emanating from the corresponding limiting one, terminates on a symmetric periodic orbit $B_{4,5}(\mu)$ that belongs to the family $\mathcal{L}_3(\mu)$ of periodic orbits emanating from the collinear equilibrium point L_3.

In fact we have computed the family of bifurcation periodic orbits $B_{4,5}(\mu)$ for values of $\mu \in (0.02, 0.92)$ so this behaviour seems to be quite general.

For the other branch, $\mathcal{L}^1(\mu)$ the situation is the following:

(i) If $\mu \lessapprox 0.0454299697...$ the family $\mathcal{L}^1(\mu)$ maintains its termination at a bifurcation periodic orbit B_s^1 of $\mathcal{L}^s(\mu)$ travelled twice. So the evolution for this family is the same that the one we have for $\mu \in (\mu^{**}, \mu_1)$. For $\mu^{**}=0.02072...$ it exists a bifurcation periodic orbit which belongs to the family $\mathcal{L}^1$ and to the bridge B(2S,3S). See |6| for more details.

(ii) If $\mu > 0.0454299697...$, $\mathcal{L}^1(\mu)$ seems to be an "open family" ending at some homoclinic orbit at L_4. In fact we have observed the typical transitions from stability to instability and viceversa leading to this ending which appear in other families of periodic orbits of hamiltonian systems with two degrees of freedom such as the Hénon-Heiles, But in fact the situation here is different. For the Hénon-Heiles hamiltonian the equilibrium point considered is a saddle-center one and the amplitude of the oscillations tends to be constant. For L_4 this amplitude grows exponentially. For more details see |7|.

For values of μ near μ_M the behaviour of the long and short period families that we have followed is different, they are no longer connected at the limiting orbit. For values of μ near μ_M this behaviour is shown qualitatively in Fig. 1

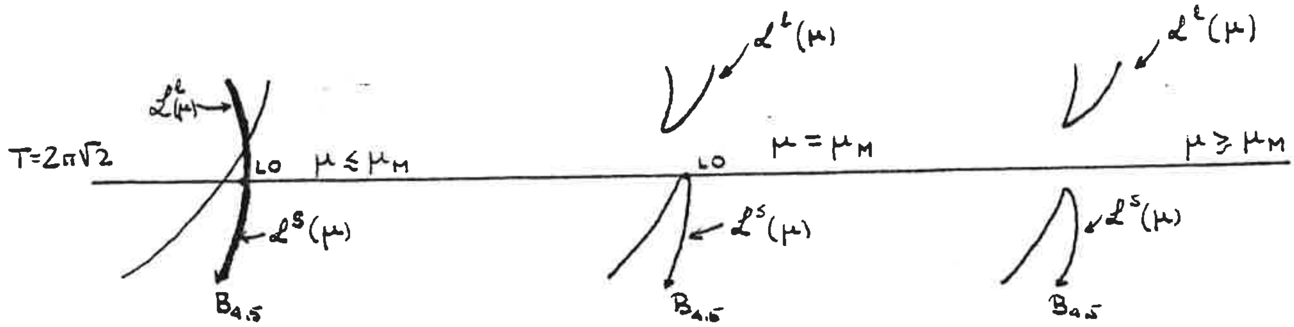


Fig. 1 Qualitative behaviour of the short and long period families of periodic orbits near the limiting orbit (LO).

For the value of μ =0.04 we have looked for the chain of two-lane bridges, B(qL,pS), connecting the short period orbit $B^s_{p/q}$ traveled p times to the long period orbit $B^l_{q/p}$ traveled q times, (p,q) been relatively prime integers such that $1 < q < p < 2q$. We have computed the families B(2L,3S) and B(3L,4S) and the situation found for these bridges is similar to the one corresponding to $\mu = \mu_1$. (See |3| for more details).

2.- Limiting orbits

Following the idea given by Pedersen and Deprit we have computed an isoperiodic family of limiting orbits with period $T = 2\pi\sqrt{2}$.

With a predictor-corrector algorithm based on the integration of the first variational equations we have obtained the evolution of this family for $\mu \varepsilon [\mu_1, \mu_M]$ where $\mu_M = 0.864496...$. Linear stability is studied by analyzing the variational matrix at the end of the fundamental period.

TABLE 1. LIMITING PERIODIC ORBITS. (Y=0.86602540)

N	MU	X	XD	YD	C	TR-2
1	0.0386000	0.49805429	0.02164473	-0.02403870	3.00000001	1.960996
2	0.0387000	0.51547405	0.03170394	-0.03632130	3.00000005	1.911423
3	0.0388000	0.52793858	0.03877749	-0.04544212	3.00000013	1.861555
4	0.0389000	0.53803172	0.04441995	-0.05303142	3.00000024	1.811395
5	0.0390000	0.54665980	0.04917759	-0.05966404	3.00000039	1.760946
6	0.0400000	0.60096095	0.07746053	-0.10449917	3.00000415	1.240963
7	0.0410000	0.63349328	0.09272841	-0.13396200	3.00001270	0.694279
8	0.0440000	0.69159780	0.11586101	-0.19173143	3.00007413	-1.086685
9	0.0450000	0.70466748	0.12016388	-0.20572106	3.00010804	-1.721464
10	0.0451000	0.70586618	0.12053905	-0.20702439	3.00011183	-1.785955
11	0.0452000	0.70704715	0.12090533	-0.20831188	3.00011568	-1.850621
12	0.0453000	0.70821082	0.12126298	-0.20958387	3.00011961	-1.915464
13	0.0454000	0.70935760	0.12161224	-0.21084069	3.00012361	-1.980482
14	0.0454100	0.70947136	0.12164671	-0.21096555	3.00012401	-1.986992
15	0.0454200	0.70958496	0.12168110	-0.21109026	3.00012441	-1.993505
16	0.0454300	0.70969840	0.12171541	-0.21121483	3.00012482	-2.000019
17	0.0455000	0.71048789	0.12195333	-0.21208266	3.00012768	-2.045673
18	0.0510000	0.75488971	0.13243762	-0.26380827	3.00046413	-5.861172
19	0.0850000	0.81120822	0.12013397	-0.35843849	3.00620929	-33.517553
20	0.1150000	0.79589159	0.09926896	-0.36650891	3.01348126	-58.233117
21	0.1450000	0.76784105	0.08035096	-0.35838235	3.02083433	-81.083954
22	0.1750000	0.73520363	0.06357188	-0.34434862	3.02757733	-101.092563
23	0.2050000	0.70070548	0.04859201	-0.32805648	3.03346250	-117.448107
24	0.2350000	0.66549672	0.03509729	-0.31110730	3.03843283	-129.460327
25	0.2650000	0.63014324	0.02284431	-0.29430093	3.04251939	-136.610288
26	0.2950000	0.59496150	0.01164583	-0.27807374	3.04579484	-138.606555
27	0.3200000	0.56592154	0.00301103	-0.26518098	3.04796999	-136.317644
28	0.3450000	0.53722746	-0.00506820	-0.25296436	3.04970061	-130.599313
29	0.3750000	0.50335401	-0.01412225	-0.23929173	3.05126865	-119.678132
30	0.3900000	0.48668080	-0.01841846	-0.23288582	3.05187153	-112.806696
31	0.4200000	0.45393695	-0.02661629	-0.22098263	3.05276942	-96.941402
32	0.4500000	0.42210658	-0.03438172	-0.21034979	3.05332352	-79.443347
33	0.4800000	0.39134200	-0.04183793	-0.20105815	3.05360634	-61.982225
34	0.5000000	0.37151488	-0.04670599	-0.19564592	3.05367582	-51.292510
35	0.5400000	0.33380955	-0.05641574	-0.18680866	3.05360190	-34.923363
36	0.5700000	0.30759177	-0.06390545	-0.18202523	3.05341240	-29.447747
37	0.6000000	0.28359667	-0.07183556	-0.17893185	3.05314581	-31.974073
38	0.6300000	0.26240563	-0.08048967	-0.17763393	3.05282278	-44.055945
39	0.6600000	0.24485210	-0.09019545	-0.17824462	3.05244813	-66.548109
40	0.6900000	0.23218380	-0.10131247	-0.18089157	3.05200428	-99.350149
41	0.7200000	0.22637675	-0.11418174	-0.18573847	3.05143863	-140.578757
42	0.7500000	0.23080295	-0.12894699	-0.19305466	3.05063701	-185.887952
43	0.7800000	0.25184186	-0.14490350	-0.20348732	3.04936254	-226.866917
44	0.8100000	0.30353951	-0.15783516	-0.21945052	3.04708549	-247.801458
45	0.8400000	0.42593388	-0.14648191	-0.25522219	3.04228920	-215.084153
46	0.8500000	0.50401015	-0.12285668	-0.28436183	3.03926266	-177.105861
47	0.8600000	0.63603746	-0.06626292	-0.34924706	3.03396262	-101.688332
48	0.8620000	0.68058185	-0.04414967	-0.37559156	3.03210514	-74.198072
49	0.8640000	0.75270479	-0.00642903	-0.42264923	3.02903266	-27.949053
50	0.8643000	0.77341488	0.00471266	-0.43710127	3.02813985	-14.304989
51	0.8644000	0.78328114	0.01005621	-0.44412542	3.02771338	-7.771915
52	0.8644500	0.78968328	0.01353466	-0.44872994	3.02743671	-3.492304
53	0.8644800	0.79452569	0.01617113	-0.45223648	3.02722775	-0.247707
54	0.8644900	0.79643629	0.01721233	-0.45362624	3.02714479	1.029880
55	0.8644930	0.79705165	0.01754768	-0.45407480	3.02711782	1.423887
56	0.8644950	0.79747780	0.01778031	-0.45438487	3.02709984	1.720020
57	0.8644960	0.79769441	0.01789844	-0.45454282	3.02709045	1.868408

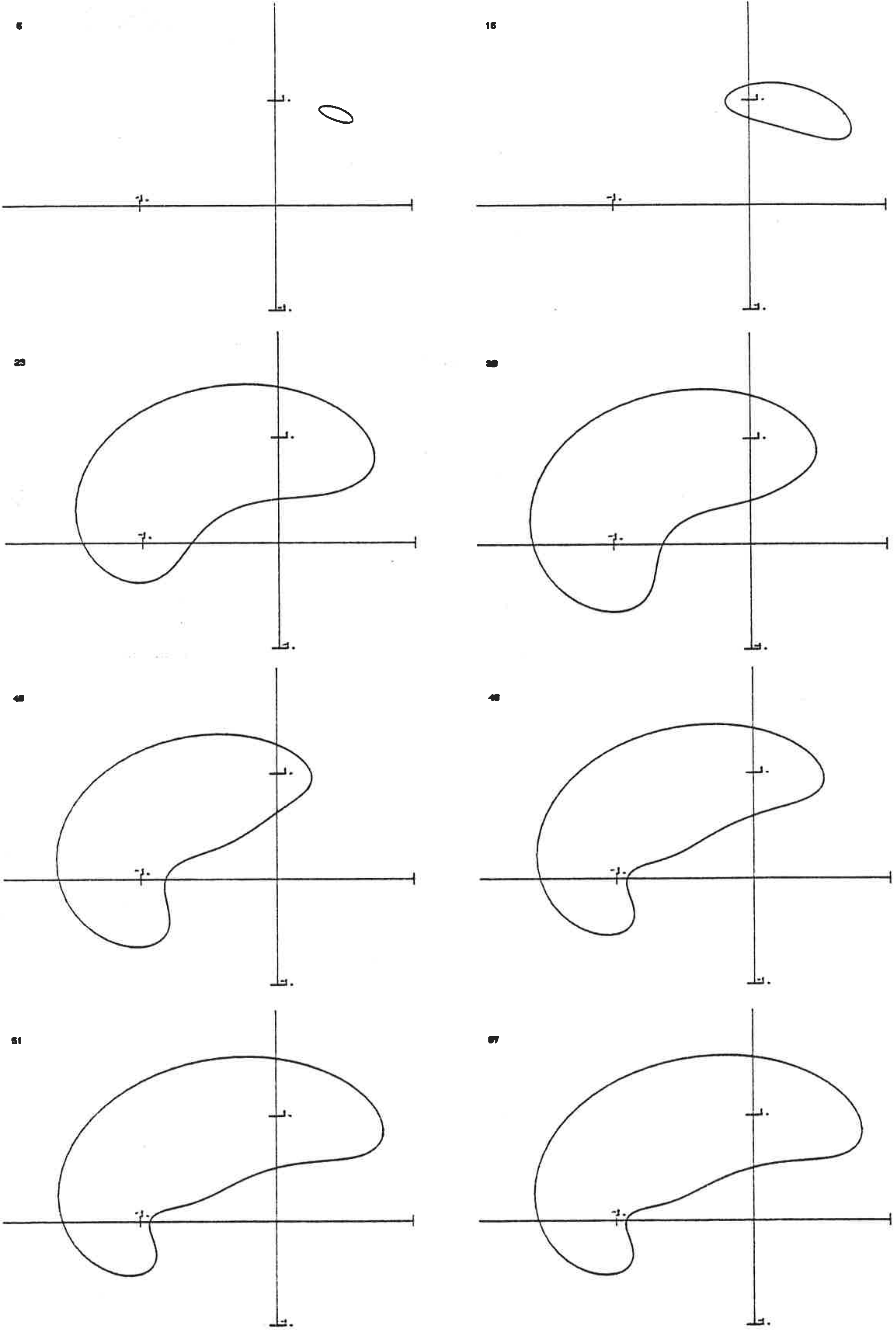


FIG. 2. SOME LIMITING PERIODIC ORBITS

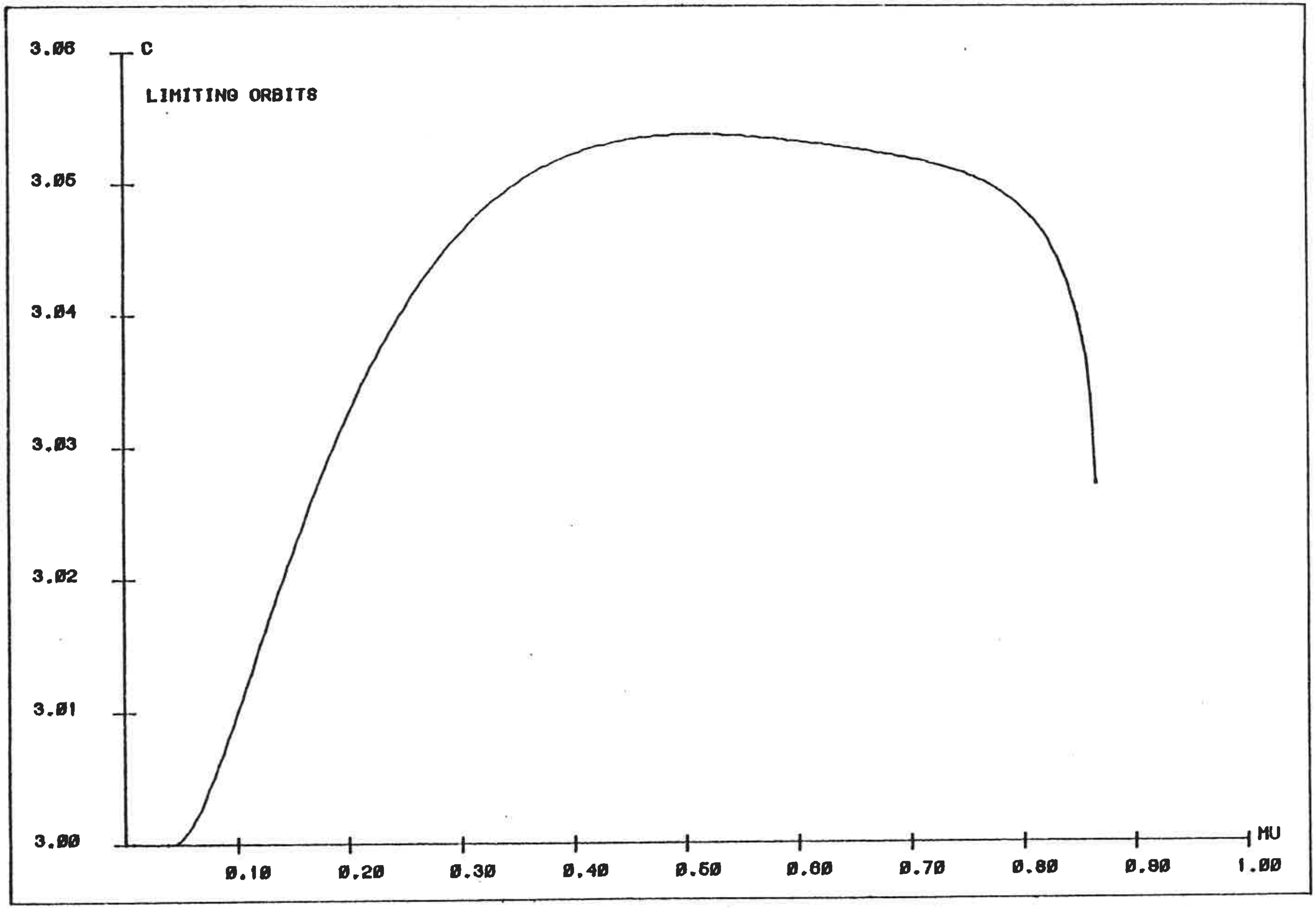


FIG. 3. MASS RATIO-JACOBI CONSTANT OF LIMITING PERIODIC ORBITS

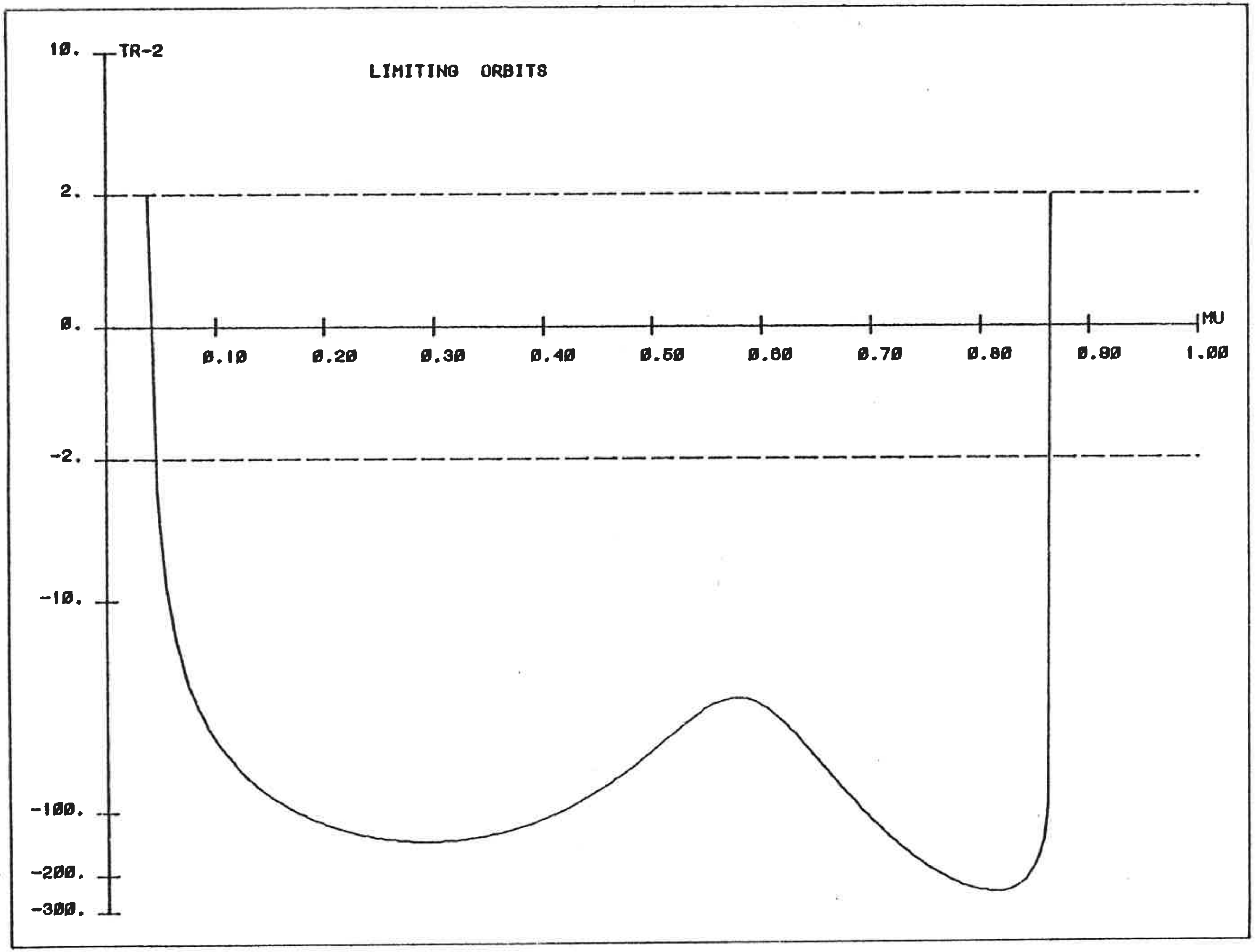


FIG. 4. MASS RATIO-STABILITY INDEX OF LIMITING PERIODIC ORBITS.

The stability of a periodic orbit is characterized quite simply by its trace (Tr). In fact we have that if $|Tr - 2| < 2$ the orbit is linearly stable and if $|Tr - 2| > 2$ the orbit is unstable. Tr = 0,4 corresponds to parabolic orbits.

Using the RKF78 routine with relative local truncation error less than 10^{-11} ensured a good conservation of Jacobi's constant (to 10^{-10}). Synodical coordinates (x,y) where used except when regularization maked the use of the Levi-Civita variables more convenient. Periodicity conditions guaranteed that the Euclidean distance between the initial and final conditions was smaller than 10^{-8}.

In Table 1 we give the mass ratio (m), initial conditions (X,Y,XD,YD), Jacobi constant (C), and stability index (Tr - 2) of some of the orbits computed, and in Fig. 2 some of these orbits have been plotted.

__________ Table 1 here __________

__________ Fig. 2 here __________

If we look at the mass ratio - Jacobi constant and to the mass ratio - stability index curves (see Fig. 3 and Fig. 4) we see that this isoperiodic family starts at the equilibrium point L_4 when $\mu = \mu_1$ with a Jacobi constant equal to 3 and an stability index equal to 2. For a value of $\mu = \mu_M = 0.864496...$, this family of isoperiodic orbits can be continued for decreasing values of μ, but the orbits of the family in which the isoperiodic orbit is imbedded are no longer of the manifolds $\mathcal{L}^1(L_4)$ and $\mathcal{L}^s(L_4)$ in which we are interested. See Fig. 1.

We can see also that this limiting orbits are stable in a narrow band of values of μ comprised between μ_1 and $\mu' = 0.0454299697...$ and another one comprised between $\mu = 0.8644644498...$ and μ_M.

__________ Fig. 3 and Fig. 4 here __________

TABLE 2. NATURAL FAMILY FOR MU = 0.040 (Y=0.86602540)

N	X	XD	YD	T	C	TR-2
1	0.49765074	0.07370185	-1.08066702	6.22414017	1.82783175	1.968331
2	0.63005525	0.06620125	-1.07281888	6.23386431	1.86983180	1.510059
3	0.68876684	0.06765977	-1.06436396	6.24370861	1.90983176	1.050096
4	0.74018759	0.07097874	-1.05267429	6.25687218	1.95983183	0.443038
5	0.83758432	0.08290749	-1.01091814	6.30293608	2.10983181	-1.584854
6	0.87965924	0.09144612	-0.97769600	6.34027576	2.20983171	-3.092475
7	0.94871718	0.12044031	-0.84692007	6.50204086	2.50983167	-7.980841
8	0.94399667	0.14837822	-0.67436928	6.76128817	2.75983167	-10.907977
9	0.89305609	0.15929990	-0.52063459	7.04883146	2.88983178	-9.955327
10	0.84485328	0.15729459	-0.42414606	7.26784325	2.93983173	-8.018297
11	0.79051483	0.14798436	-0.33522382	7.50929546	2.96983171	-5.632309
12	0.74137783	0.13475843	-0.26583543	7.73851871	2.98483181	-3.536224
13	0.71590066	0.12636836	-0.23300996	7.86623859	2.98983169	-2.511306
14	0.67858052	0.11238198	-0.18824621	8.07280731	2.99483180	-1.111908
15	0.65633827	0.10315628	-0.16326720	8.21462250	2.99683189	-0.344311
16	0.62387025	0.08858038	-0.12875931	8.47974396	2.99883175	0.671347
17	0.60316300	0.07858511	-0.10713830	8.79806805	2.99983168	1.210995
		LIMITING	ORBIT			
18	0.60040712	0.07705295	-0.10254615	9.07005692	3.00033188	1.156335
19	0.60437298	0.07876534	-0.10434864	9.27005672	3.00070620	0.868017
20	0.60848123	0.08055119	-0.10681692	9.40005684	3.00098515	0.591879
21	0.62341762	0.08663587	-0.11639177	9.80005646	3.00206876	-0.522253
22	0.63039374	0.08911364	-0.12081014	10.00005722	3.00272465	-1.118222
23	0.63764369	0.09120858	-0.12515523	10.25005722	3.00361085	-1.770850
24	0.64288402	0.09203286	-0.12790465	10.50005722	3.00452018	-2.203565
25	0.64595491	0.09152270	-0.12898378	10.75005817	3.00539303	-2.301135
26	0.64683682	0.08965893	-0.12847492	11.00005627	3.00616884	-1.969083
27	0.64560157	0.08644047	-0.12658498	11.25005722	3.00678778	-1.143607
28	0.64241242	0.08188728	-0.12365325	11.50005627	3.00719357	0.199196
29	0.63756120	0.07607033	-0.12018575	11.75005627	3.00733519	2.035302
30	0.63153666	0.06917294	-0.11689790	12.00005627	3.00717139	4.286652
31	0.62510186	0.06158125	-0.11472457	12.25005627	3.00667477	6.818425
32	0.61931747	0.05396827	-0.11472206	12.50005627	3.00583816	9.441961
33	0.61540633	0.04727662	-0.11779743	12.75005722	3.00467944	11.926235
34	0.61441463	0.04412767	-0.12131380	12.90005684	3.00384712	13.245338
35	0.61776054	0.04029570	-0.13631035	13.30005646	3.00125742	15.683204
36	0.62258786	0.04110874	-0.14607251	13.50005722	2.99984670	16.129368
37	0.63094229	0.04446299	-0.15931147	13.75005722	2.99807644	15.854236
38	0.64105237	0.04992785	-0.17276169	14.00005722	2.99638987	14.669168
39	0.65204453	0.05691650	-0.18560351	14.25005913	2.99487305	12.706528
40	0.66316438	0.06489305	-0.19722962	14.50005817	2.99359393	10.210152
41	0.67381442	0.07343442	-0.20721024	14.75005817	2.99259853	7.507573
42	0.68354279	0.08225582	-0.21523575	15.00005722	2.99191141	4.968553
43	0.69200122	0.09123495	-0.22103813	15.25005817	2.99153852	2.956817
44	0.69886017	0.10050680	-0.22423331	15.50005817	2.99146986	1.780811
45	0.70317870	0.12031372	-0.21800540	15.87005806	2.99187660	1.990160

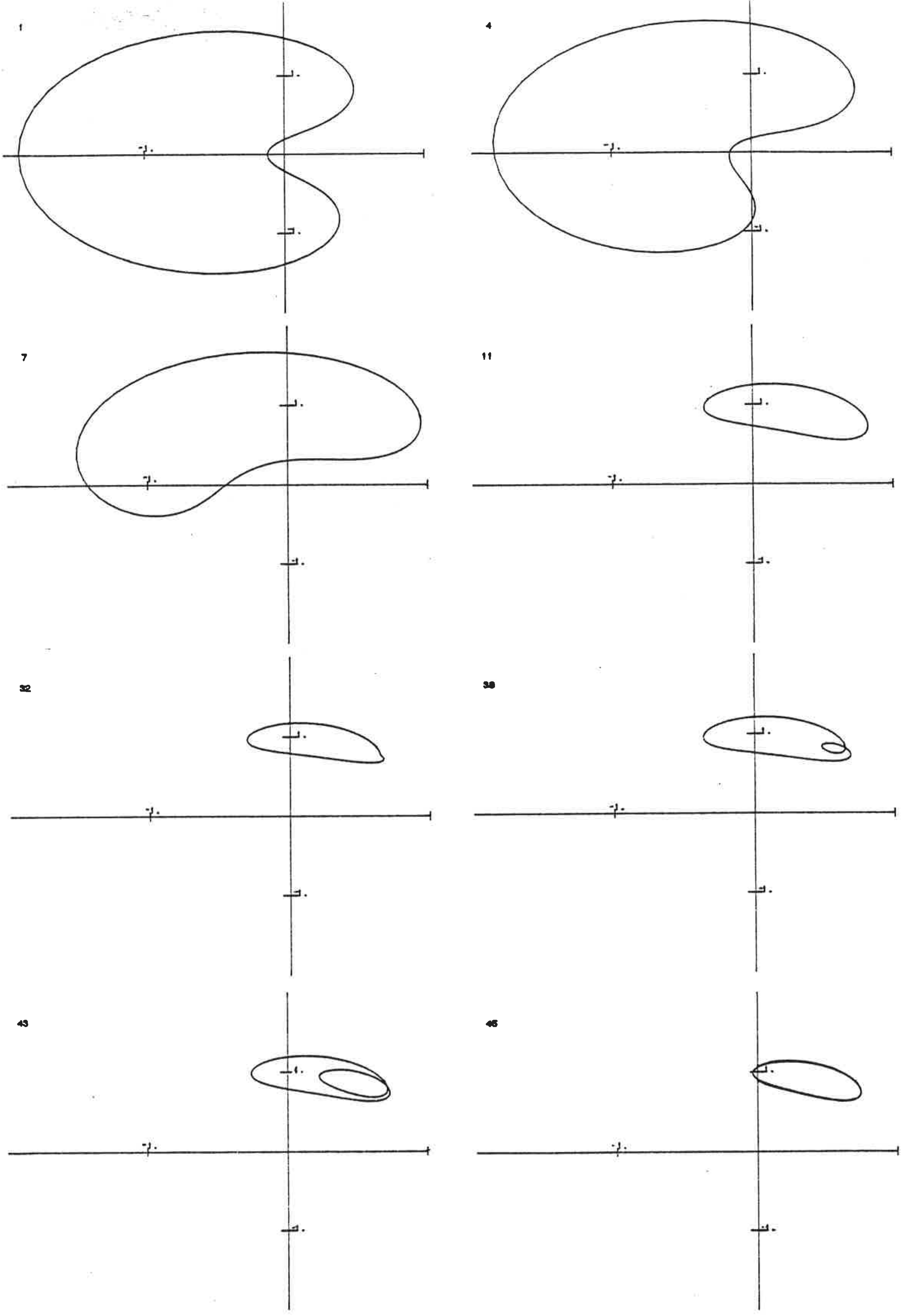


FIG. 5. SOME PERIODIC ORBITS FOR $\mu = 0.040$

TABLE 3. NATURAL FAMILY FOR MU = 0.045420. (Y=0.86602540)

N	X	XD	YD	T	C	TR-2
1	0.44355991	0.08487936	-1.08084214	6.21643639	1.82466578	1.973559
2	0.66328084	0.06728309	-1.06501460	6.23418808	1.89999998	1.099881
3	0.71960855	0.06958312	-1.05345488	6.24845934	1.95000005	0.416335
4	0.76142669	0.07271775	-1.04075706	6.26382923	2.00000000	-0.307681
5	0.79490876	0.07618249	-1.02698863	6.28039837	2.04999995	-1.071347
6	0.82276994	0.07983713	-1.01217985	6.29828453	2.09999990	-1.873323
7	0.84647006	0.08364215	-0.99633265	6.31762409	2.15000010	-2.711640
8	0.94102734	0.11486362	-0.85146689	6.51273346	2.50000000	-9.221385
9	0.91690779	0.15257993	-0.58332056	7.00601196	2.84999990	-12.848172
10	0.84129417	0.15407185	-0.41438767	7.47847319	2.95000005	-9.285885
11	0.71006942	0.12188255	-0.21183152	8.87094498	3.00000405	-2.000050
12	0.71006536	0.12188088	-0.21182534	8.87106705	3.00000501	-1.999992
		LIMITING		ORBIT		
13	0.70744818	0.12076745	-0.20776787	8.95644283	3.00067592	-1.979131
14	0.70716792	0.12064415	-0.20732471	8.96644211	3.00075126	-1.979267
15	0.70689291	0.12052225	-0.20688789	8.97644234	3.00082564	-1.979923
16	0.70662302	0.12040174	-0.20645729	8.98644257	3.00089979	-1.981096
17	0.70635813	0.12028257	-0.20603281	8.99644279	3.00097322	-1.982772
18	0.70609826	0.12016474	-0.20561437	9.00644207	3.00104594	-1.984949
19	0.70584321	0.12004821	-0.20520183	9.01644230	3.00111818	-1.987620
20	0.70559299	0.11993296	-0.20479509	9.02644253	3.00118995	-1.990779
21	0.70534748	0.11981896	-0.20439406	9.03644276	3.00126100	-1.994421
22	0.70510662	0.11970618	-0.20399861	9.04644299	3.00133157	-1.998535
23	0.70487028	0.11959461	-0.20360868	9.05644226	3.00140166	-2.003114
24	0.70181686	0.11804719	-0.19835265	9.20644283	3.00240088	-2.123075
25	0.69886011	0.11624278	-0.19270036	9.40644264	3.00361991	-2.400392
26	0.69667822	0.11455379	-0.18804032	9.60644245	3.00475287	-2.750250
27	0.69389540	0.11186683	-0.18190408	9.90644264	3.00633216	-3.248756
28	0.69072556	0.10859526	-0.17585564	10.20644283	3.00775838	-3.481019
29	0.68642759	0.10439913	-0.16927952	10.50644302	3.00897980	-3.188884
30	0.67569411	0.09469859	-0.15667002	11.00644302	3.01035166	-0.956908
31	0.66037196	0.08108806	-0.14316729	11.50644302	3.01056910	3.893218
32	0.64283764	0.06418084	-0.13296552	12.00644112	3.00934982	11.025950
33	0.62937790	0.04766792	-0.13293992	12.50644207	3.00666904	18.940222
34	0.62676716	0.03920401	-0.14329059	12.90644073	3.00371051	24.172188
35	0.63558429	0.03810574	-0.16616181	13.40644264	2.99959540	27.112606
36	0.65751958	0.04833014	-0.19761604	14.00644302	2.99515176	23.458759
37	0.67401356	0.05879291	-0.21606414	14.40644360	2.99301672	17.177191
38	0.68930548	0.07005114	-0.23061305	14.80644417	2.99174070	9.398559
39	0.70212322	0.08097189	-0.24069992	15.20644379	2.99133468	1.958461
40	0.71185923	0.09091262	-0.24628954	15.60644436	2.99170756	-3.502992
41	0.71831810	0.09955270	-0.24762569	16.00644302	2.99270773	-6.047252
42	0.72184724	0.10833697	-0.24388362	16.50644302	2.99456739	-5.248095
43	0.72061348	0.11486502	-0.23487148	17.00644302	2.99677610	-1.885384
44	0.71495348	0.11932558	-0.22124299	17.50644112	2.99901605	1.249258
45	0.71027523	0.12163329	-0.21223567	17.73999977	2.99999642	1.995956
46	0.71007526	0.12186796	-0.21184482	17.74209976	3.00000477	1.999958

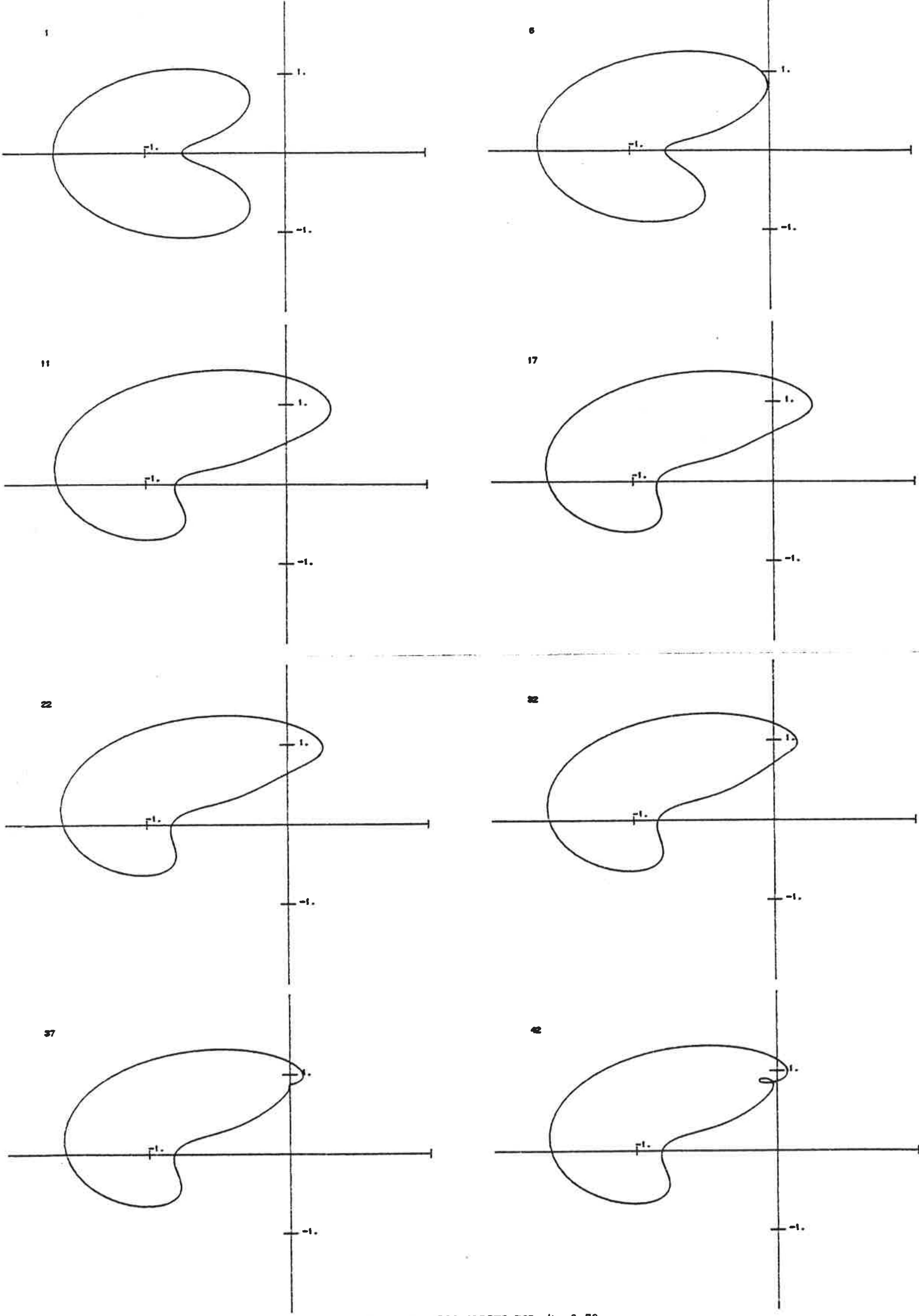


FIG. 14. SOME PERIODIC ORBITS FOR $\mu = 0.70$

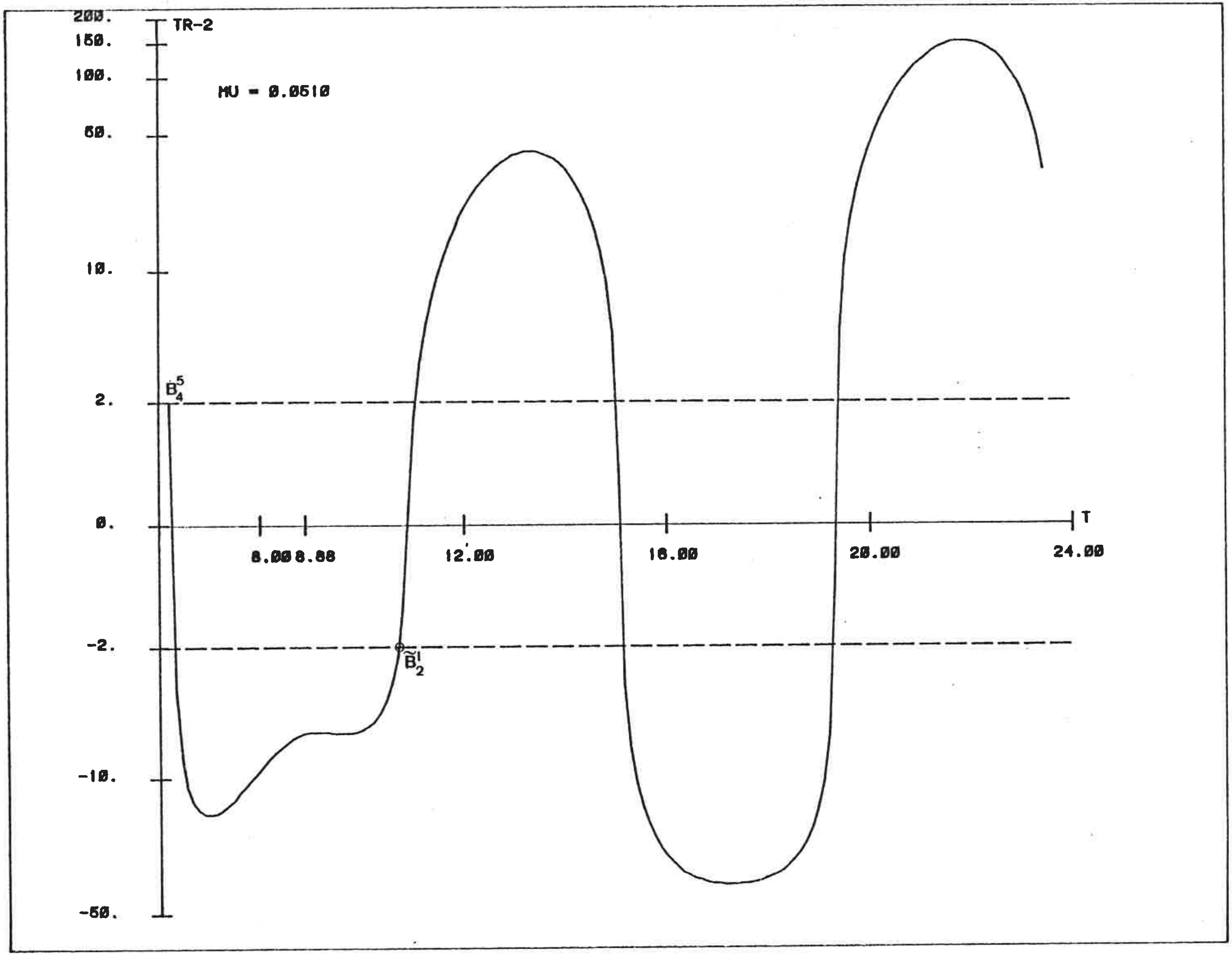


FIG. 15. PERIOD-STABILITY INDEX OF NATURAL FAMILY FOR $\mu = 0.0510$

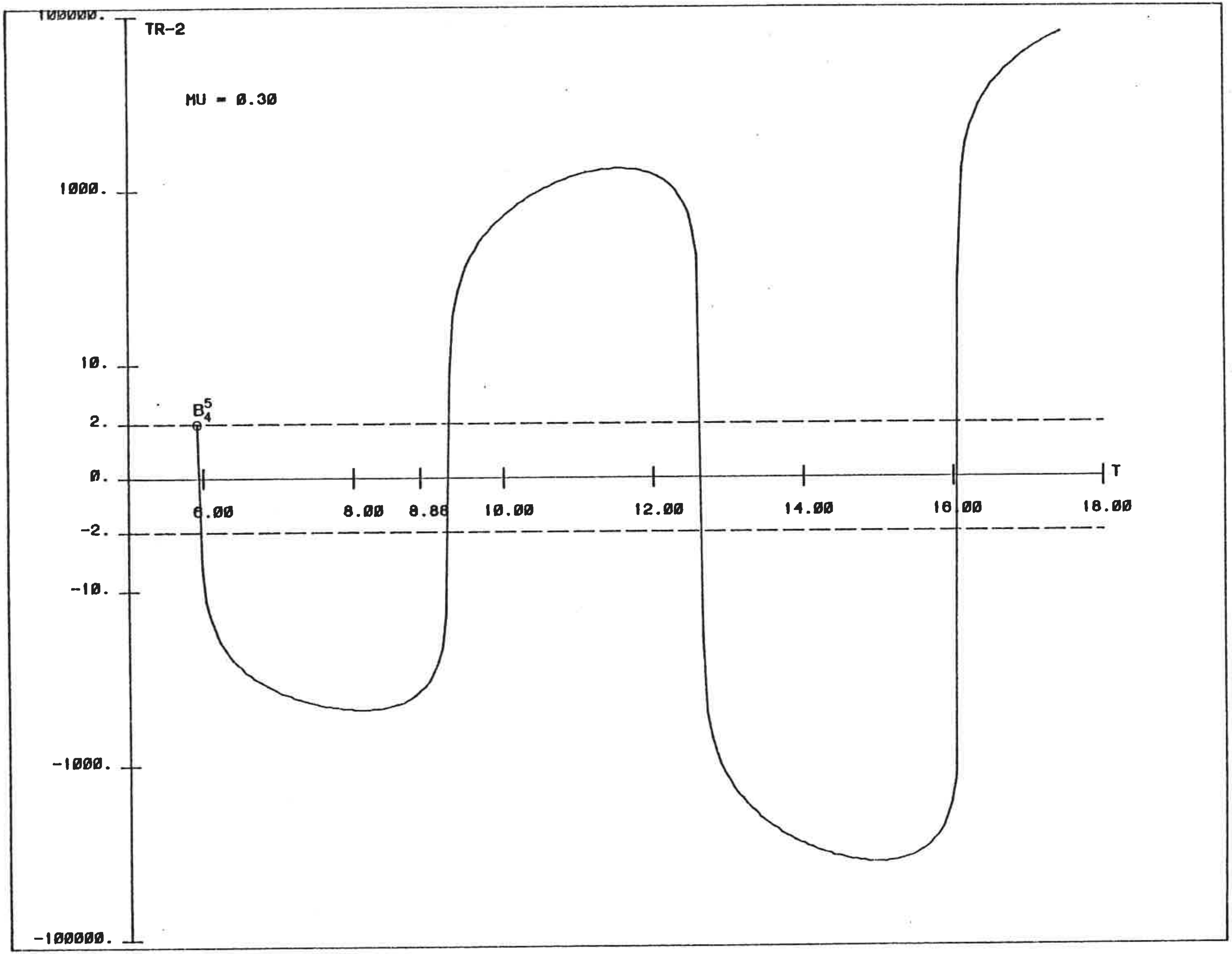


FIG. 16. PERIOD-STABILITY INDEX OF NATURAL FAMILY FOR $\mu = 0.30$

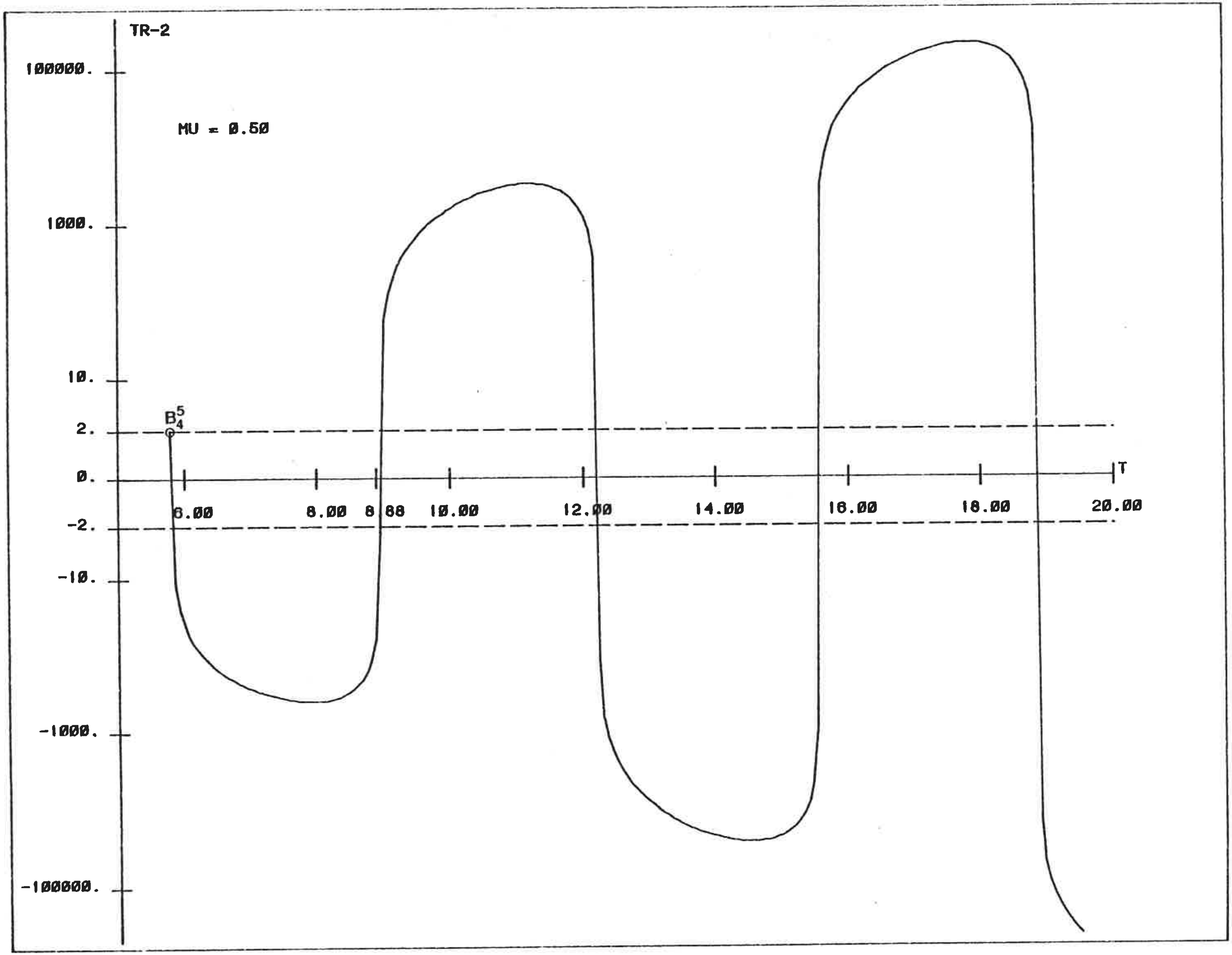


FIG. 17. PERIOD-STABILITY INDEX OF NATURAL FAMILY FOR $\mu = 0.50$

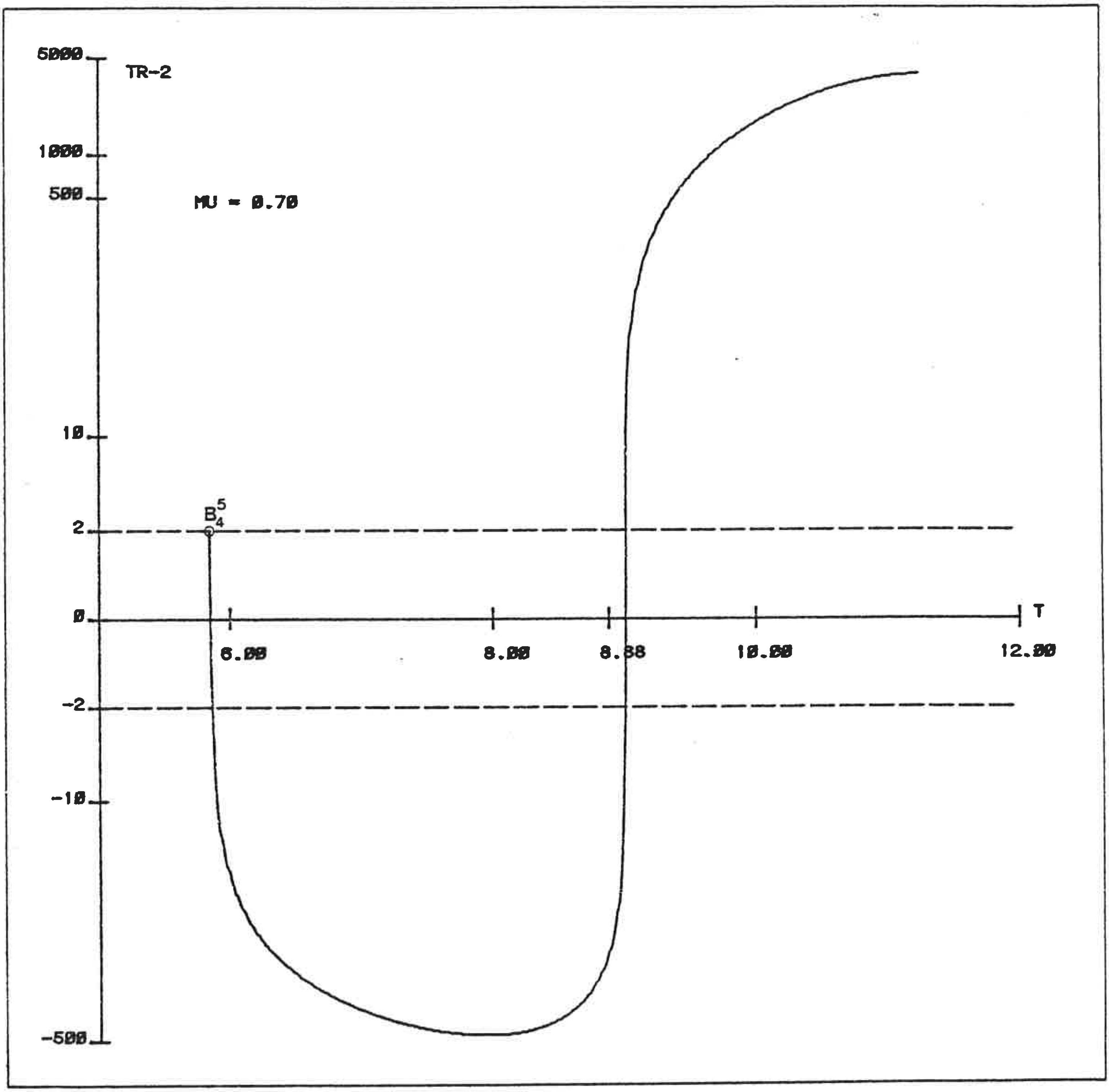


FIG. 18. PERIOD-STABILITY INDEX OF NATURAL FAMILY FOR μ = 0.70

Table 4 here

Fig. 11 here

Table 5 here

Fig. 12 here

Table 6 here

Fig. 13 here

Table 7 here

Fig. 14 here

In Figs. 15,16,17 and 18 we have plotted the period stability index curves for these four values of μ.

Figs. 15,16,17,18 here

The behaviour of the short period branch is the same that the one we have already described for $\mu < \mu'$. For the long period ones we have the transitions from stability to instability and viceversa which appear in families of periodic orbits of hamiltonian systems with two degrees of freedom, for which their end is some homoclinic periodic orbit associated to a saddle point. We must say too, that associated to the extrema of the stability index, inner loops around L_4 appear in the shape of the orbit.

Devaney |2| has shown that for a family of periodic orbits of a hamiltonian system of the type considered, the asymptotic relation between the stability parameter (s = Tr - 2) and the period (T) is:

$$s = \exp(f_1(s)\, T) \cdot (f_2(s) \cos(\beta T) + f_3(s) \sin(\beta T))$$

where f_i are adecuate functions. In fact this expresion can be inproved |11| to the following one:

$$s = A \exp(\alpha T) \cdot (\cos(\beta T) + B \sin(\beta T))$$

where A, B are adecuate constants and $\alpha \pm i\beta$ are the eigenvalues of the equilibrium point L_4. The values of the constants A, B found for these four mass ratios are:

μ	A	B
0.051	-2.210	0.0406
0.3	-1.894	0.7353
0.5	-2.791	0.6830
0.7	-5.320	0.4216

TABLE 8. SHORT PERIOD FAMILY FOR MU = 0.86449. (Y=0.0)

N	X	XD	YD	T	C	TR-2
1	-1.59632194	-0.01119085	1.15553617	6.34868717	2.69868588	1.661488
2	-1.59621704	-0.04337494	1.15251184	6.37813807	2.70368862	-8.294412
3	-1.59511042	-0.12008745	1.12845683	6.61754036	2.74368548	-51.253460
4	-1.58944416	-0.19045013	1.06756771	7.26006699	2.84368539	-149.485184
5	-1.57910740	-0.21876681	1.00339985	8.02865696	2.94368553	-168.044174
6	-1.57184649	-0.22514945	0.96833909	8.51681232	2.99528551	-108.110641
7	-1.56875479	-0.22749051	0.95273316	8.76533604	3.01801515	-54.534077
8	-1.56912065	-0.23094410	0.94758534	8.87985229	3.02694488	-4.059661
		LIMITING		ORBIT		
9	-1.56958842	-0.23178400	0.94776684	8.88602352	3.02714682	3.437662
10	-1.56981277	-0.23215005	0.94792330	8.88757515	3.02712917	4.450204
11	-1.57012475	-0.23263448	0.94818825	8.88879681	3.02702594	8.261208
12	-1.57130992	-0.23431461	0.94949895	8.88760090	3.02612925	20.257936
13	-1.57150745	-0.23457822	0.94974804	8.88684559	3.02592897	21.999819
14	-1.57264113	-0.23603463	0.95128375	8.88071918	3.02460289	31.175884
15	-1.58671749	-0.25117683	0.97527206	8.75560379	2.99975061	107.579475
16	-1.59249902	-0.25662428	0.98611903	8.70663643	2.98775077	131.093369
17	-1.59983194	-0.26307231	1.00027430	8.65092087	2.97175050	155.418823
18	-1.60688245	-0.26882184	1.01418269	8.60381794	2.95575166	181.754257
19	-1.61629725	-0.27585116	1.03304410	8.55053425	2.93375039	194.282883
20	-1.62717474	-0.28307277	1.05507874	8.50103855	2.90775204	218.855194
21	-1.64128816	-0.29105327	1.08374023	8.45295811	2.87374997	234.188019
22	-1.67022216	-0.30266705	1.14161766	8.40544891	2.80575109	242.355133
23	-1.67110860	-0.30292875	1.14335299	8.40436935	2.80374956	242.236618
24	-1.67736578	-0.30460089	1.15555239	8.40227413	2.78975105	240.820923
25	-1.68100929	-0.30545214	1.16259098	8.40179634	2.78175116	239.630463
26	-1.68470705	-0.30622500	1.16968393	8.40205956	2.77375102	238.168579
27	-1.69228542	-0.30752990	1.18405592	8.40483284	2.75775099	234.455811
28	-1.71258152	-0.30929178	1.22138596	8.42541695	2.71775103	220.756454
29	-1.78996718	-0.29809982	1.34807551	8.62368202	2.60575104	145.160858
30	-1.80305469	-0.29415989	1.36730349	8.66949558	2.59275103	131.089554
31	-1.81141162	-0.29141349	1.37929547	8.70005703	2.58525109	122.067795
32	-1.82713926	-0.28580162	1.40129542	8.76006985	2.57275105	105.082214
33	-1.83279812	-0.28365058	1.40903759	8.78241348	2.56875110	98.983467
34	-1.87756896	-0.26447460	1.46734762	8.97188377	2.54575109	51.510281

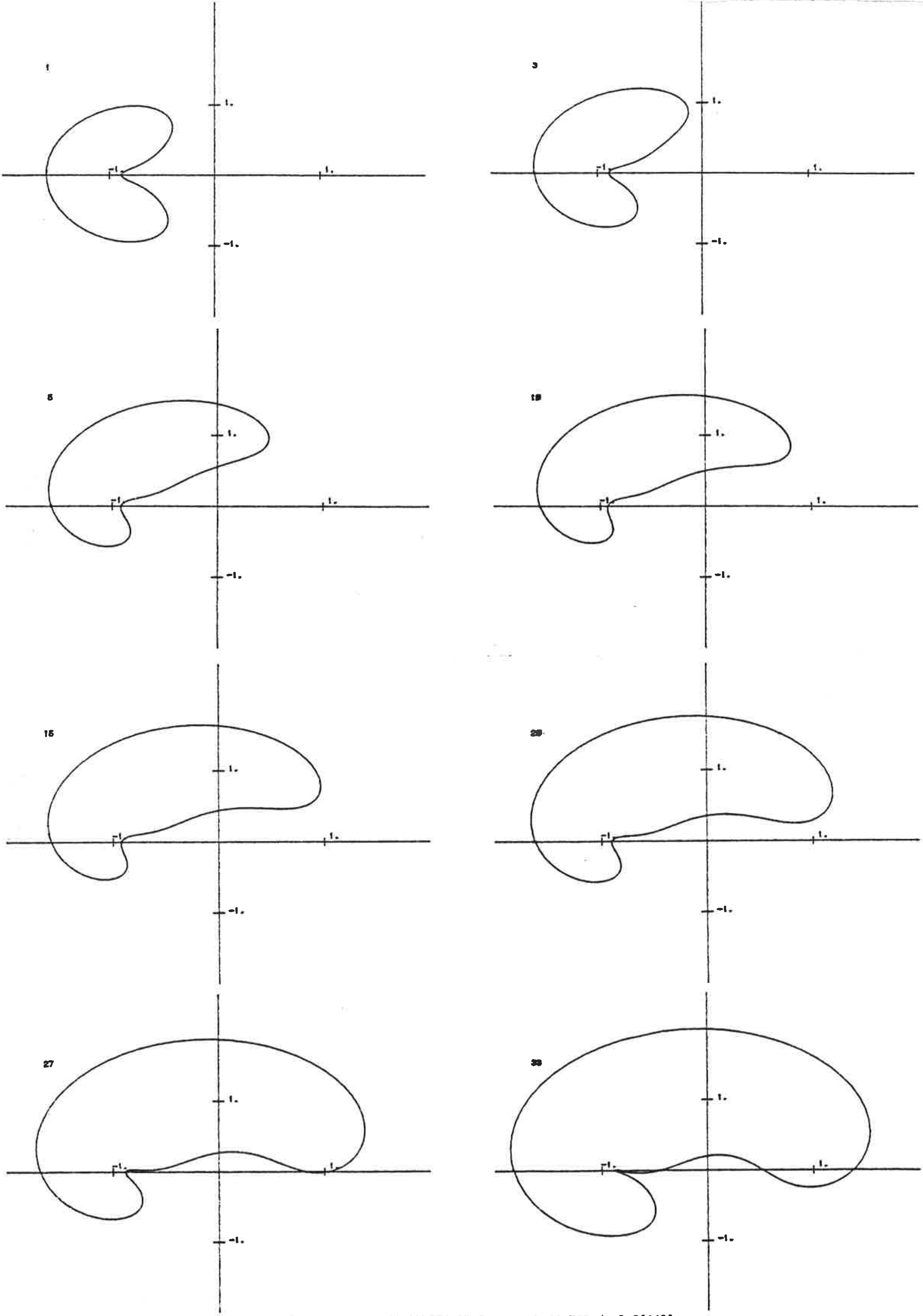


FIG. 19. SOME PERIODIC ORBITS OF SHORT PERIOD FOR μ= 0.864490

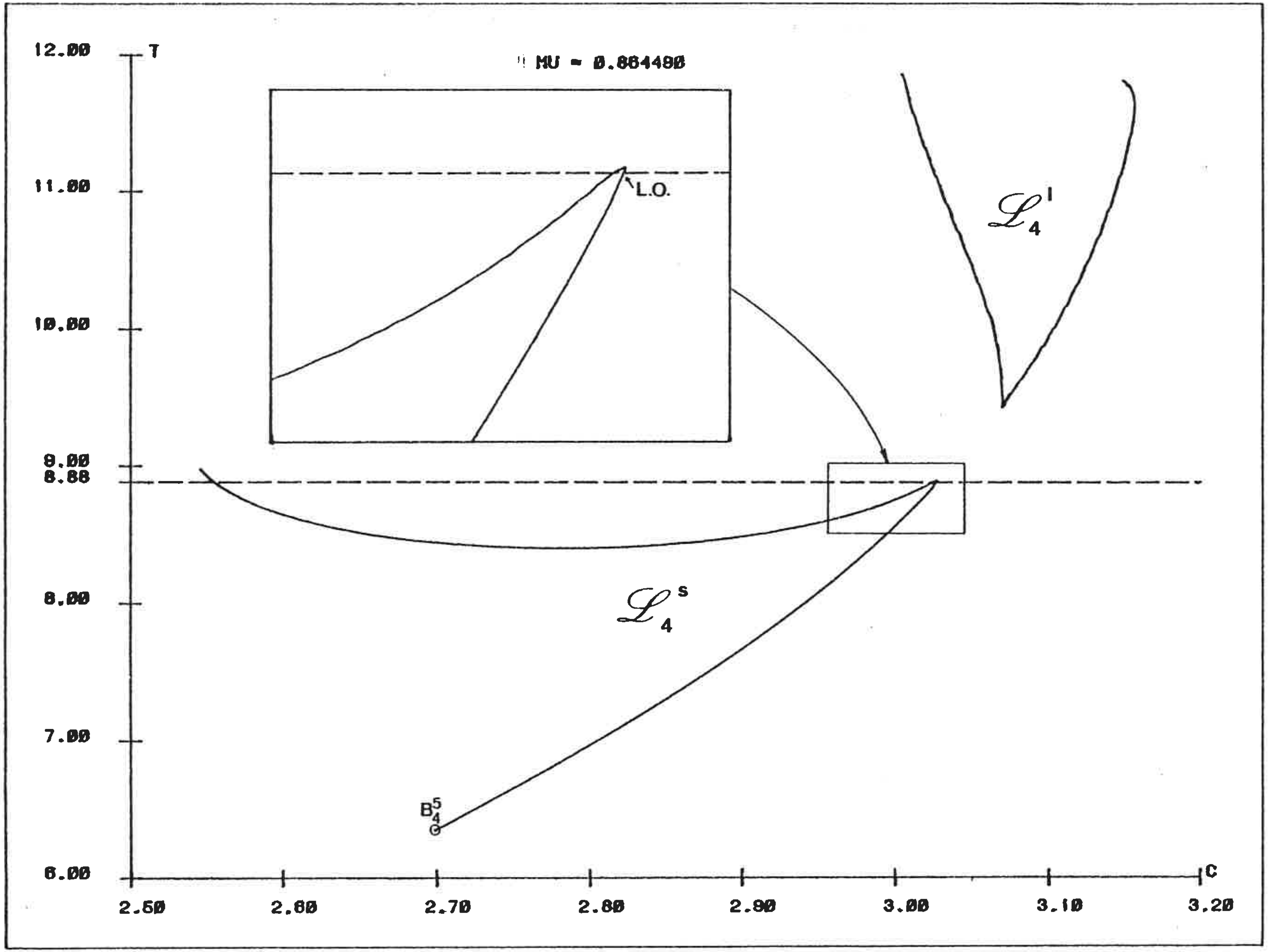


FIG. 20. JACOBI CONSTANT-PERIOD OF NATURAL FAMILY FOR $\mu = 0.864490$

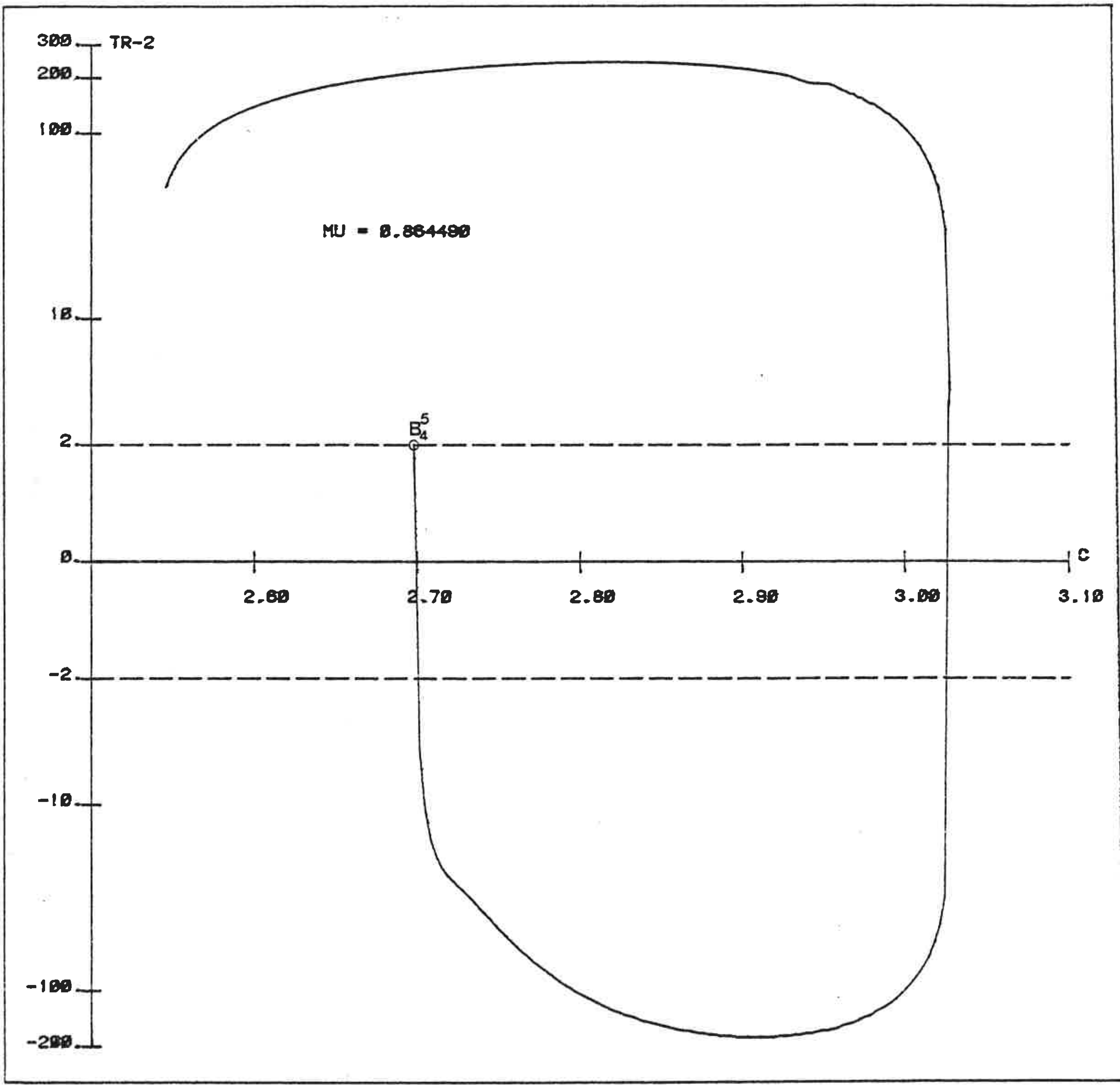
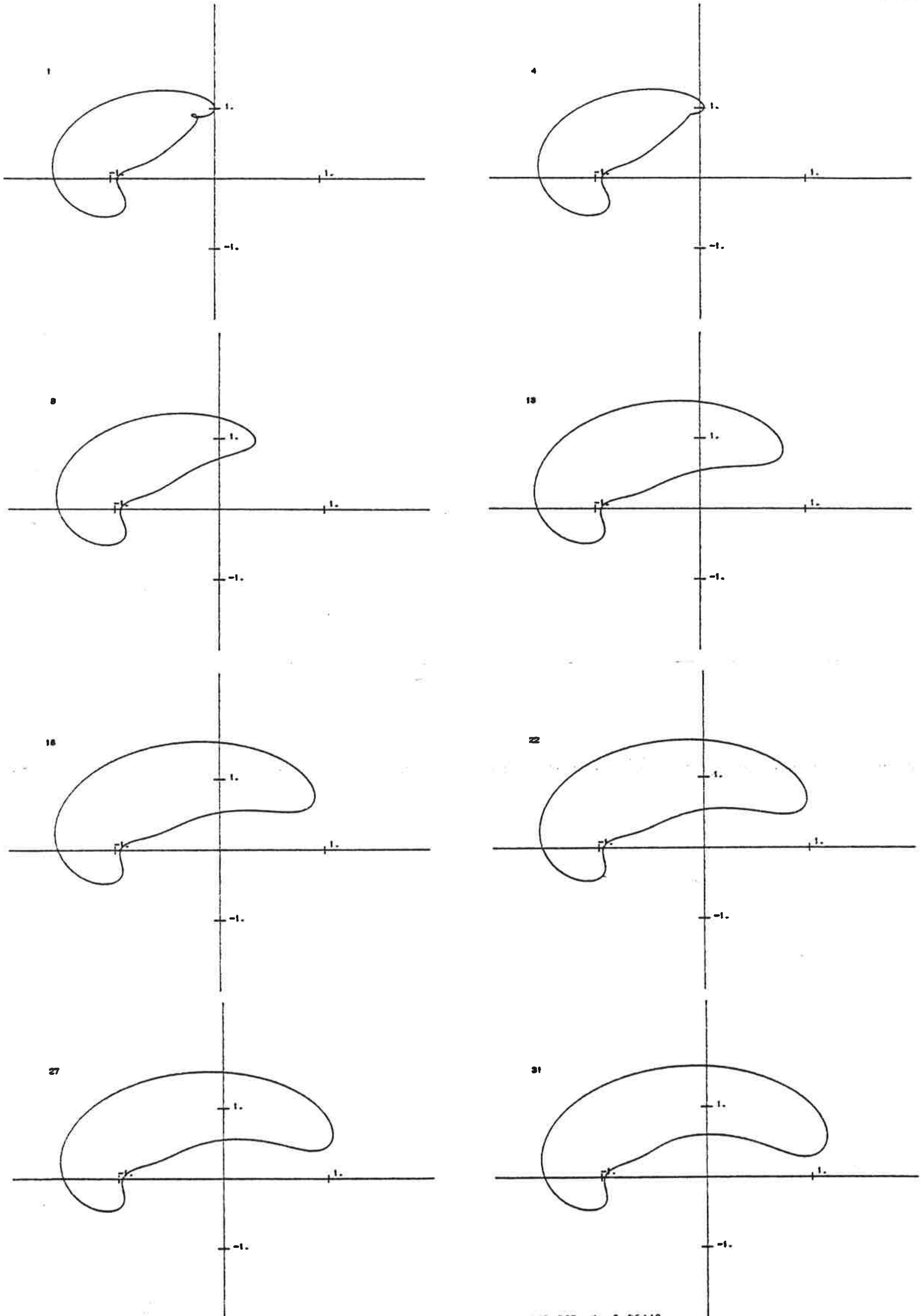


FIG. 21. JACOBI CONSTANT-STABILITY INDEX OF SHORT PERIOD FAMILY
FOR μ = 0.864490

TABLE 9. LONG PERIOD FAMILY FOR MU = 0.86449. (Y=0.0)

N	X	XD	YD	T	C	TR-2
1	-0.93252307	-0.34512374	-1.86154187	11.86170959	3.00479198	1822.796143
2	-0.93289733	-0.34454054	-1.85536492	11.78731251	3.00649214	1796.854736
3	-0.93439329	-0.34295970	-1.83090460	11.51700115	3.01349235	1629.703125
4	-0.93642312	-0.34260711	-1.79827249	11.19289112	3.02349210	1318.278320
5	-0.93832862	-0.34431869	-1.76807559	10.90656662	3.03349209	1006.775330
6	-0.94007987	-0.34836894	-1.74037838	10.63596344	3.04349232	703.934326
7	-0.94219261	-0.36161795	-1.70524168	10.21089268	3.05849218	297.823059
8	-0.94238240	-0.36463112	-1.70153940	10.14581108	3.06049204	248.172409
9	-0.94252861	-0.36823493	-1.69827366	10.07589436	3.06249237	195.944138
10	-0.94258118	-0.37872043	-1.69410324	9.90698147	3.06649208	96.352036
11	-0.94227213	-0.38860884	-1.69530642	9.78019142	3.06849217	41.108974
12	-0.93918699	-0.43902883	-1.72425270	9.45678139	3.07049203	30.470842
13	-0.94047886	-0.44047281	-1.70399189	9.55476856	3.07749224	68.802773
14	-0.94182158	-0.43837532	-1.68464541	9.64900112	3.08349228	94.703026
15	-0.94276607	-0.43637794	-1.67148542	9.71485615	3.08749199	110.711983
16	-0.94499314	-0.43083727	-1.64158010	9.87153244	3.09649205	144.245010
17	-0.94681054	-0.42577013	-1.61819422	10.00220108	3.10349202	167.844971
18	-0.94842672	-0.42096233	-1.59809637	10.12119865	3.10949206	185.881424
19	-0.95038509	-0.41480577	-1.57457411	10.26953697	3.11649203	203.116241
20	-0.95183724	-0.41001692	-1.55769479	10.38276386	3.12149215	212.478455
21	-0.95334047	-0.40485606	-1.54071140	10.50319004	3.12649226	218.011871
22	-0.95522499	-0.39807460	-1.52011108	10.65905857	3.13249207	218.402954
23	-0.95654196	-0.39310756	-1.50616729	10.77135944	3.13649201	213.682632
24	-0.95757020	-0.38908231	-1.49554157	10.86094761	3.13949203	206.994476
25	-0.95938826	-0.38159353	-1.47732568	11.02317142	3.14449215	188.163132
26	-0.96057016	-0.37641782	-1.46589553	11.13075161	3.14749217	170.484634
27	-0.96186632	-0.37038538	-1.45376682	11.24967575	3.15049195	145.508759
28	-0.96477103	-0.35476187	-1.42849028	11.50935650	3.15549231	64.303429
29	-0.96781099	-0.32802811	-1.40843177	11.76657391	3.15267873	-203.665970
30	-0.96796989	-0.32404804	-1.40866232	11.79256153	3.15067887	-275.655914
31	-0.96801686	-0.32087213	-1.40968585	11.81284046	3.14867854	-344.842072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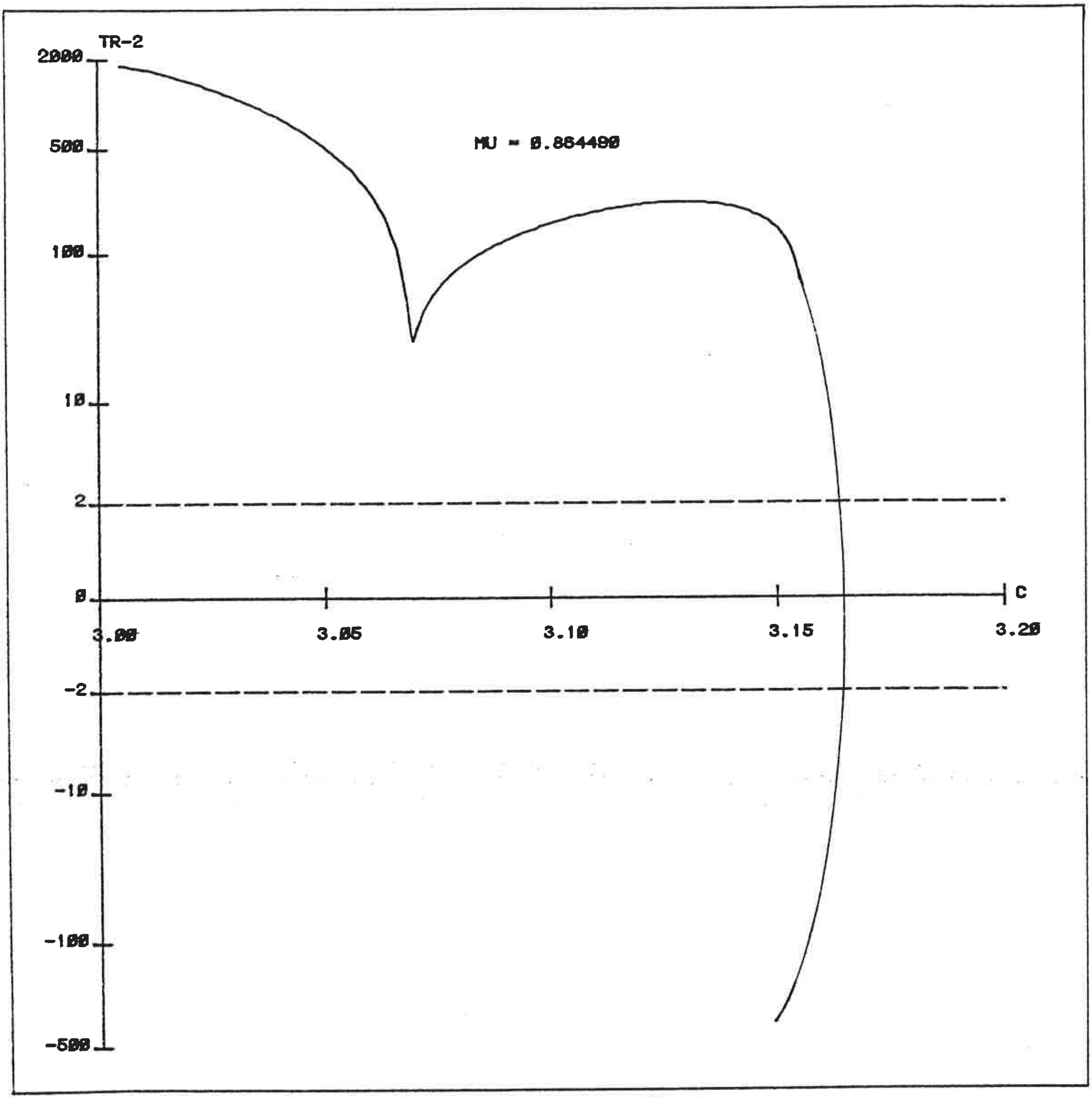


FIG. 23. JACOBI CONSTANT-STABILITY INDEX OF LONG PERIOD FAMILY FOR $\mu = 0.864490$

The evolution of the long and short period families is completely different for values of μ near μ_M. For the value $\mu = \mu_M$ the families have been studied with some detail. For this value of μ the short and long period families are not connected at the limiting orbit and they are in fact two different families. The short period one starting at the bifurcation orbit $B_{4,5}$ and the long period one at some homoclinic orbit of L_4. We have not computed these two families until their natural end and the partial results obtained are given in Table 8 and Figs. 19,20,21 for the short period familiy and in Table 9 and Figs. 20,22,23 for the long period one. The situation is similar to the one found by Henrard for the families $\mathcal{L}^1(\mu^{**})$ and B(2S,3S) where $\mu^{**} = 0.02072..$ (See |6| for more details).

Table 8 here

Figs. 19,20,21 here

Table 9 here

Figs. 22, 23 here

4.- Bridges between $\mathcal{L}^s(0.04)$ and $\mathcal{L}^1(0.04)$.

In |3| Deprit and Henrard computed a fine structure of bridges connecting the manifolds $\mathcal{L}^s_4(\mu_1)$ and $\mathcal{L}^1_4(\mu_1)$. This bridges in fact connect one resonant orbit in $\mathcal{L}^s_4$ with another one in $\mathcal{L}^1_4$. It was found convenient to denote them by B(pL, qS) to recall that the natural family starts at a long period orbit travelled p times and ends at a short period orbit travelled q times. In |3| the following three ones where computed: B(2L, 3S), B(3L, 4S) and B(4L, 5S).

We have asked for the existence of these bridges for values of μ greater than μ_1. Because of their situation in the case $\mu = \mu_1$, it is reasonable to ask for them in the case $\mu < \mu'$. We have studied with some detail the case $\mu = 0.04$.

We where able to compute the following two ones: B(2L, 3S) and B(3L, 4S). Both bridges heve two different tines. For the B(2L, 3S),

TABLE 10. LOWER TINE IN THE BRIDGE B(2L,3S). (Y=0.86602540)

N	X	XD	YD	T	C	TR-2
1	0.64245649	0.09042184	-0.13049188	20.73655411	3.00400001	2.081970
2	0.64725999	0.08025438	-0.14635744	21.20059172	3.00300001	6.555364
3	0.64869708	0.07668389	-0.15325228	21.58206294	3.00200001	10.774408
4	0.65116131	0.07555095	-0.15979917	21.93827706	3.00100001	13.842477
5	0.65211924	0.07557209	-0.16179331	22.04423227	3.00070001	14.454848
6	0.65393851	0.07594151	-0.16516910	22.22235059	3.00020001	15.105649
7	0.65603601	0.07671892	-0.16861704	22.40456320	2.99970001	15.252628
8	0.65841547	0.07791138	-0.17213927	22.59365500	2.99920001	14.855207
9	0.66406685	0.08169751	-0.17938297	23.00718082	2.99820001	12.333685
10	0.67258802	0.08989251	-0.18800933	23.62599159	2.99700001	6.660897
11	0.67874233	0.10243387	-0.19097967	24.22874475	2.99600000	2.763796
12	0.67870063	0.10500580	-0.19005851	24.28734284	2.99580000	2.446704
13	0.67813221	0.10723143	-0.18871144	24.31113282	2.99560000	2.217653
14	0.67714230	0.10913051	-0.18705238	24.30758636	2.99540000	2.067775
15	0.67579469	0.11076447	-0.18512484	24.28345488	2.99520000	2.002458

TABLE 11. UPPER TINE IN THE BRIDGE B(2L,3S). (Y=0.86602540)

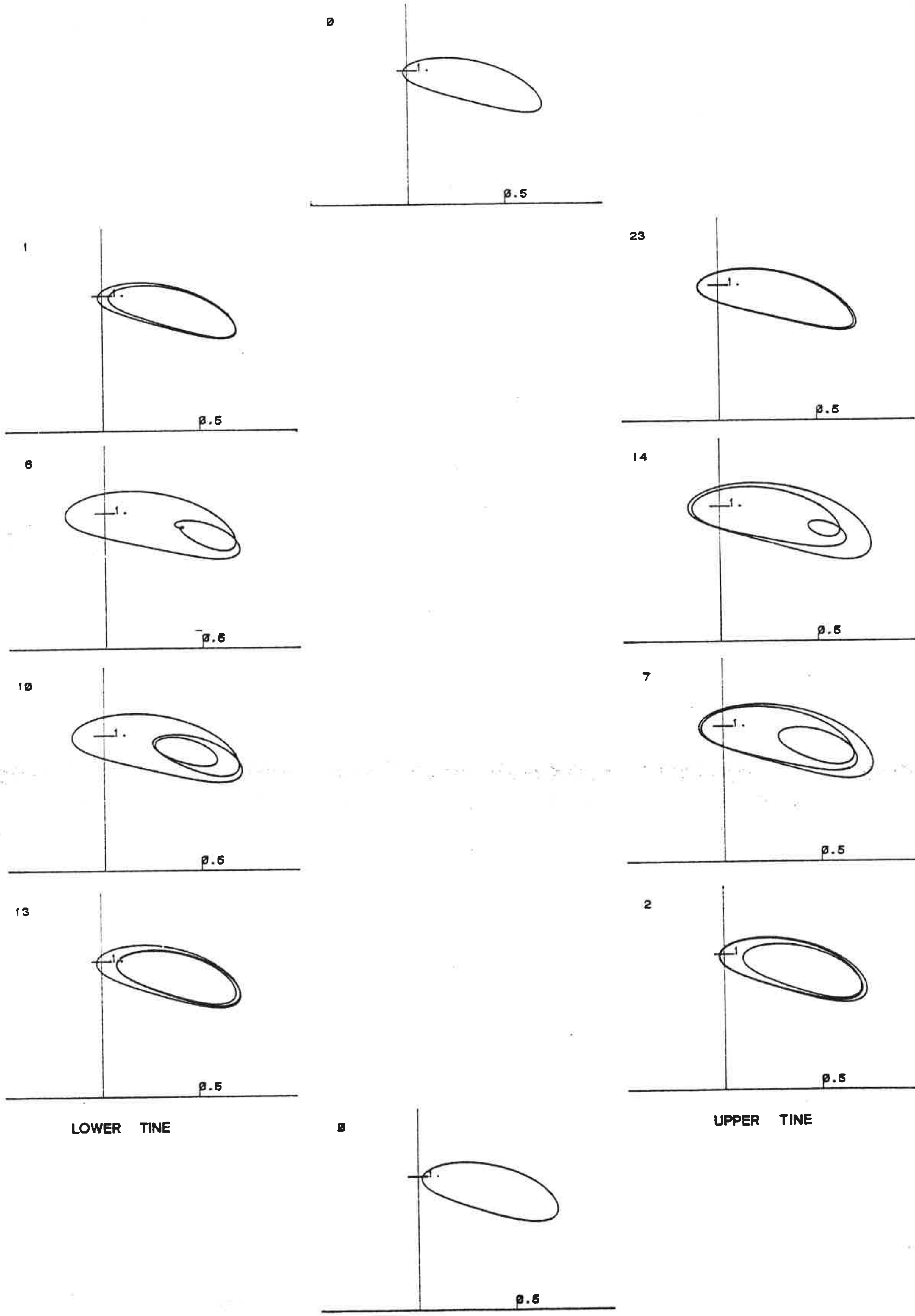
N	X	XD	YD	T	C	TR-2
1	0.67410064	0.11219160	-0.18291302	24.24301201	2.99500000	2.026428
2	0.66943593	0.11459221	-0.17728374	24.11842164	2.99460000	2.324741
3	0.66609277	0.11562063	-0.17341463	24.02975345	2.99440000	2.553955
4	0.66214112	0.11641958	-0.16890273	23.92975191	2.99423040	2.712716
5	0.65793367	0.11693150	-0.16411963	23.82975305	2.99411377	2.683009
6	0.64856656	0.11724238	-0.15341888	23.62975163	2.99403747	1.792314
7	0.63763210	0.11664618	-0.14070396	23.42975325	2.99418597	-0.490930
8	0.62466074	0.11501381	-0.12520682	23.22975090	2.99459034	-4.237465
9	0.61972038	0.11417364	-0.11917719	23.16257205	2.99479034	-5.782034
10	0.61301188	0.11285319	-0.11087758	23.07845619	2.99509034	-7.868248
11	0.60275294	0.11043348	-0.09795301	22.96456787	2.99559034	-10.859346
12	0.59375098	0.10789003	-0.08642301	22.87845661	2.99604522	-13.139828
13	0.58112993	0.10358661	-0.07010836	22.77845740	2.99666279	-15.653975
14	0.57160964	0.09970441	-0.05785408	22.71845391	2.99708224	-17.015077
15	0.53852226	0.08165302	-0.01801422	22.60296123	2.99799999	-19.107292
16	0.58556737	0.03404152	-0.11006257	22.39982581	2.99999999	-19.963875
17	0.58168759	0.03361510	-0.10147534	22.31461921	3.00099999	-18.715429
18	0.58063806	0.03552198	-0.09456143	22.23708676	3.00199999	-16.467459
19	0.58621497	0.04374693	-0.08945850	22.13208261	3.00350000	-11.375378
20	0.60416821	0.06006075	-0.09559506	22.03774546	3.00500001	-4.396199
21	0.61480481	0.06824693	-0.10229094	22.00826097	3.00550001	-1.680449
22	0.62374634	0.07466221	-0.10883743	21.99099146	3.00580001	0.038200
23	0.64007896	0.08555041	-0.12240738	21.97402208	3.00610001	1.822345
24	0.64305157	0.08743401	-0.12506846	21.97290123	3.00612001	1.943591

TABLE 12. BRIDGE B(3L,4S). (Y=0.86602540)

N	X	XD	YD	T	C	TR-2
1	0.62967425	0.08887419	-0.12036181	29.93448747	3.00265000	1.990849
2	0.61833189	0.08697834	-0.10705319	29.98716273	3.00260070	1.960790
3	0.60917996	0.08518319	-0.09651564	30.10433622	3.00250000	1.628968
4	0.59182823	0.08203179	-0.07727204	30.39374051	3.00200000	-0.776741
5	0.58441817	0.08200610	-0.06923918	30.60678641	3.00150000	-3.484942
6	0.58226695	0.08399365	-0.06705683	30.80115032	3.00100000	-5.913360
7	0.58253961	0.08518115	-0.06748679	30.87733941	3.00080000	-6.722431
8	0.58391204	0.08722052	-0.06928797	30.99191872	3.00050000	-7.703101
9	0.58818010	0.09092838	-0.07477176	31.18790806	3.00000000	-8.587724
10	0.59420257	0.09458650	-0.08255648	31.39729020	2.99950000	-8.369832
11	0.60153963	0.09792685	-0.09209416	31.63091273	2.99900000	-6.931016
12	0.61023905	0.10083581	-0.10339418	31.90839840	2.99850000	-4.279625
13	0.62127606	0.10317047	-0.11764845	32.28240761	2.99800000	-0.704807
14	0.62407341	0.10351786	-0.12125319	32.38283188	2.99790000	0.045941
15	0.62729583	0.10377843	-0.12541242	32.50168902	2.99780000	0.770429
16	0.63131244	0.10385955	-0.13062775	32.65447593	2.99770000	1.431275
17	0.63801431	0.10314869	-0.13955764	32.91676697	2.99760000	1.942656
18	0.64078290	0.10234639	-0.14345661	33.02255510	2.99758400	1.992241
19	0.64226086	0.10169810	-0.14565135	33.07533560	2.99758100	1.999110

TABLE 13. BRIDGE B(3L,4S). (Y=0.86602540)

N	X	XD	YD	T	C	TR-2
1	0.63214396	0.08899308	-0.12386932	29.85642944	3.00255053	2.008544
2	0.63500025	0.08920448	-0.12756272	29.84256315	3.00250053	2.044586
3	0.64364818	0.08901114	-0.13959653	29.87626405	3.00220053	2.656326
4	0.64968630	0.08766278	-0.14943858	30.08621781	3.00170053	4.669563
5	0.65395813	0.08637046	-0.15746338	30.48572101	3.00100053	8.383315
6	0.65629578	0.08620964	-0.16180497	30.80279951	3.00050053	10.621820
7	0.65856202	0.08671712	-0.16564002	31.14209864	3.00000053	11.708261
8	0.66088888	0.08794161	-0.16911684	31.51046495	2.99950053	11.153928
9	0.66521557	0.09324769	-0.17407697	32.39376241	2.99850053	5.442077
10	0.65125485	0.09958166	-0.15735390	33.15426647	2.99760053	2.001231
11	0.64583829	0.09858331	-0.15187648	33.15941490	2.99758053	1.999994



[illegible] ORBITS IN THE BRIDGE B(2L,3S) FOR $\mu = 0.040$

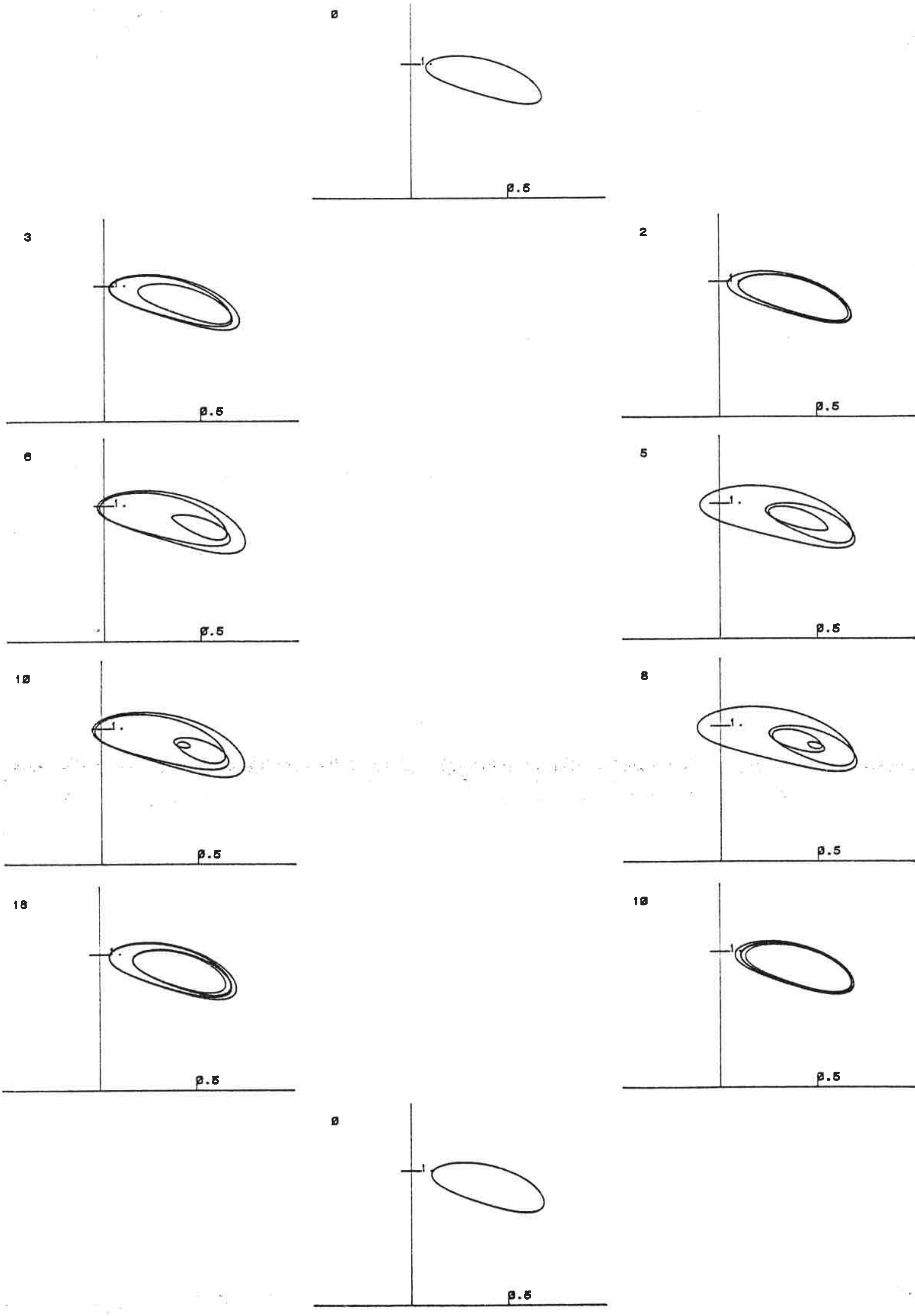


FIG. 25 SOME PERIODIC ORBITS IN THE BRIDGE B(3L,4S) FOR μ= 0.040

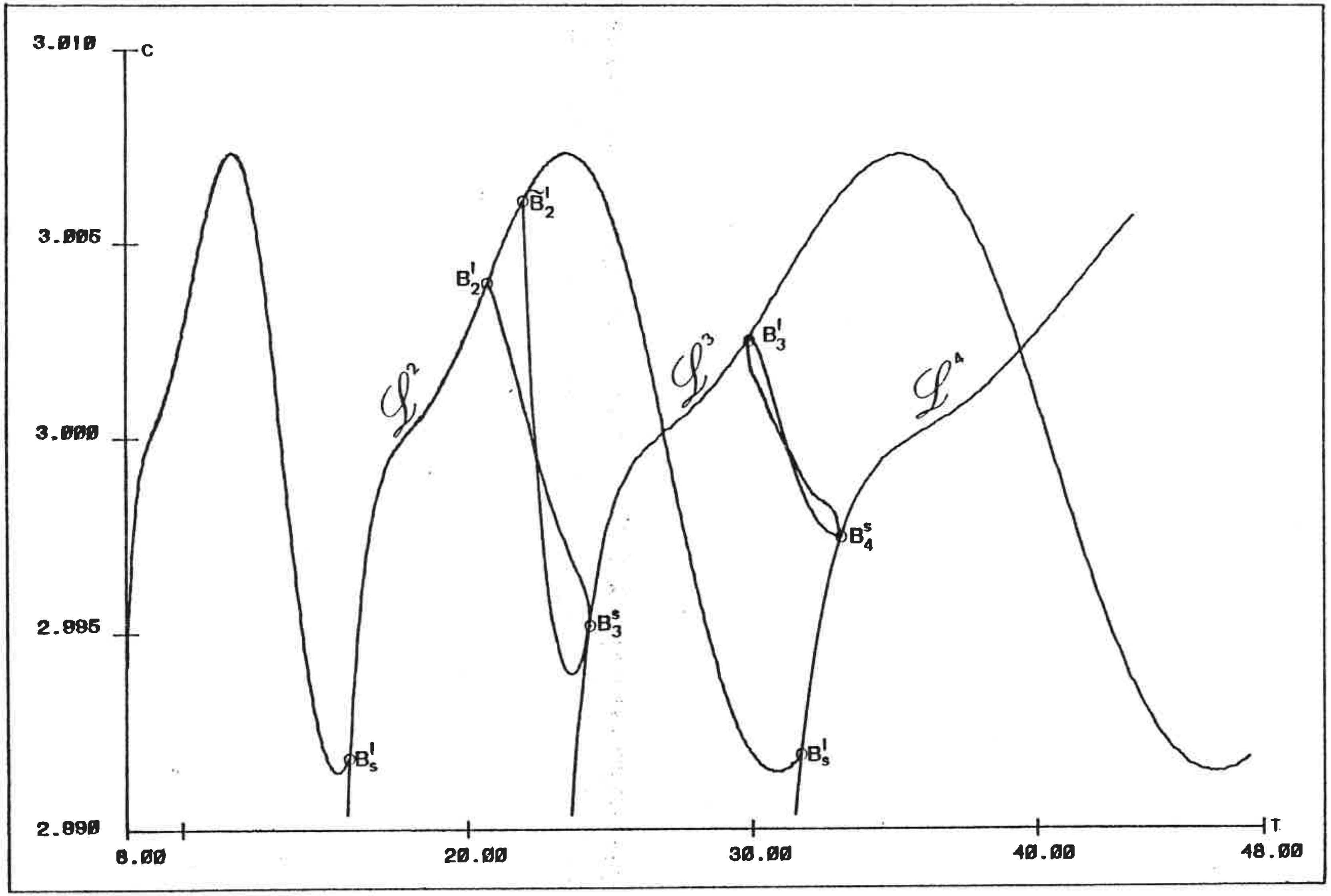


FIG. 26. PERIOD-JACOBI CONSTANT OF NATURAL FAMILY AND BRIDGES FOR $\mu = 0.045420$
The curves denoted by $\mathcal{L}^i$ i=2,3,4 represents the natural family of periodic orbits travelled i times.

TABLE 14. UPPER TINE IN THE BRIDGE B(2L,3S) FOR MU = 0.0510. (Y=0.8660540)

N	X	XD	YD	T	C	TR-2
1	0.69797927	0.09993656	-0.18134694	21.51116180	3.01349998	0.607895
2	0.69889361	0.10059272	-0.18234158	21.46094131	3.01344991	1.449764
3	0.71037000	0.10693139	-0.19444282	21.43490982	3.01329994	-2.051808
4	0.73558521	0.11942563	-0.22340871	21.48430634	3.01200008	-45.569553
5	0.74560875	0.12394474	-0.23587303	21.52398109	3.01099992	-77.525276
6	0.77824050	0.13694987	-0.28075084	21.80187035	3.00500011	-235.101455
7	0.79075760	0.14123379	-0.30030686	22.04549217	3.00099993	-295.042846
8	0.80274087	0.14497973	-0.32194433	22.63897133	2.99499989	-248.832962
9	0.80417264	0.14549097	-0.32640809	23.08809471	2.99300003	-128.248962
10	0.80204970	0.14507654	-0.32506374	23.60000038	2.99245214	17.648571
11	0.79880524	0.14433494	-0.32113376	24.00000000	2.99286699	110.638313
12	0.79478025	0.14336517	-0.31569171	24.39999771	2.99374390	177.269744
13	0.78811169	0.14165686	-0.30609852	25.00000191	2.99552917	232.143218
14	0.78367120	0.14044200	-0.29945952	25.40000343	2.99684262	246.909103
15	0.77759016	0.13865159	-0.29006752	26.00000000	2.99877834	250.342681
16	0.77418619	0.13756260	-0.28460753	26.40000153	2.99995947	244.881011
17	0.77041626	0.13623132	-0.27820438	27.00000000	3.00145912	224.337112
18	0.76896459	0.13563202	-0.27541411	27.39999962	3.00222492	195.865005
19	0.76859963	0.13530341	-0.27385107	28.00000191	3.00293660	107.873023
20	0.76960099	0.13551143	-0.27467781	28.40000534	3.00307488	-0.008089
21	0.77269197	0.13640283	-0.27846348	29.00000763	3.00275421	-280.942993
22	0.77609754	0.13746396	-0.28314999	29.50001907	3.00207829	-552.611083
23	0.78009611	0.13873868	-0.28906265	30.09998322	3.00101566	-885.587280

the upper tine starts at a long period orbit $\tilde{B}_2^1$ travelled twice, and ends at a short period orbit B_3^s travelled three times. The lower tine starts at a different long period orbit B_2^1 travelled twice and ends at the same short period orbit B_3^s travelled three times. For the B(3L, 4S), both bridges start at a B_3^1 travelled three times and end at a B_4^s travelled four times.

Their initial conditions and shape are given in Tables 10,11,12 and 13 and in Figs. 24,25 and 26.

——— Tables 10,11,12 and 13 here ———

——— Figs. 24, 25 and 26 here ———

For the case $\mu < \mu'$ we have computed the B(2L, 3S) bridge for $\mu = 0.051$. The lower tine does not exist and the upper one shows the typical transitions from stability to instability so its seems to end at some homoclinic orbit al L_4 .

——— Table 14 here ———

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