# Some mathematical questions related to the MHD equations

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Dedicated to Harold GRAD
on his 60th anniversary
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ABSTRACT

The purpose of this article is to study some questions related to the magnetohydrodynamic equations (MHD equations) for a viscous incompressible resistive fluid. Most of the questions which we investigate are related to the large time behaviour of the solutions and the results tend to show that, for large times (i.e. after some transient period), if the dimension of space if N=2, the flow is totally determined by a finite number of parameters. The same results hold also if the dimension of space is three, unless singularities in the sense of J. LERAY do develop in three dimensional flows.

RESUME

Le but de cet article est l'étude de quelques questions concernant les équations de la Magnétohydrodynamique (MHD) pour un fluide visqueux, incompressible et résistif. La plupart des questions abordées sont liées au comportement des solutions lorsque le temps est grand et les résultats tendent à montrer que, pour des temps grands (c.à d. après une période de transition), si la dimension d'espace est N=2, l'écoulement est totalement déterminé par un nombre fini de paramètres. Le même résultat a lieu si la dimension d'espace est trois, à moins que des singularités au sens de J. LERAY ne se soient développées.

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Introduction

The purpose of this article is to study some questions related to the magnetohydrodynamic equations (MHD equations) for a viscous incompressible resistive fluid. Most of the questions which we investigate are related to the large time behaviour of the solutions and the results tend to show that, for large times (i.e. after some transient period), if the dimension of space is N = 2, the flow is totally determined by a finite number of parameters. The same results hold also if the dimension of space is three, unless singularities in the sense of J. LERAY [18] [20] do develop in three dimensional flows (see also V. SCHEFFER [22] [24],L. CAFARELLI, R. KOHN, L. NIRENBERG [3]). Similar results were established recently for the Navier Stokes equations (see in particular C. FOIAS, R. TEMAM [13] [11], C. FOIAS, O. MANLEY, R. TEMAM, Y. TREVE [10]).

After a brief description of the physical system in § 1, we introduce in § 2 the functional setting of the equations ; we restrict ourselves to the flow in a bounded domain with appropriate boundary conditions, or to the flow in the whole space, assuming a space periodicity property in all the directions. In § 3 we recall the main existence and uniqueness results for weak and strong solutions of the MHD equations ; these results are well known and can be found for instance in G. DUVAUT, J.L. LIONS [8].

The following sections (§ 4 to 6) contain the main results. In § 4 we establish regularity properties and bound on the solutions to the MHD equations which are valid for all time. We then introduce (when the driving forces are independent of time), the concept of functional invariant sets, which can be attracting sets, and we show that, if the data are sufficiently regular then the functional invariant set is contained in the space of smooth functions. In § 5.1 we state the so-called squeezing property of the trajectories (see [13][14]) ; this property which will be used in § 5.2 is also interesting by itself : it shows that all the trajectories are squeezed in a small vicinity of a finite dimensional like manifold.In § 5.2 we show that any functional invariant set for the MHD equations, and in particular any attractor, has a finite Hausdorff dimension. Finally in § 6 we derive another result which shows that the flow is totally determined for large time by a finite number of parameters : actually it is totally determined if we know the values of the velocity vector and the magnetic field at every point of a finite set, provided this set is sufficiently dense.

1. - The MHD equations.

Let us assume that a viscous incompressible and resistive fluid fills a region $\Omega$ of the space $\mathbb{R}^N$ (N=2 or 3). The macroscopic state of the fluid can be described by the functions :

$\rho = \rho(x,t)$ the density of the fluid

$p = p(x,t)$ the pressure of the fluid

$u = (u_1(x,t), u_2(x,t), u_3(x,t))$ the velocity of the particule of fluid which is at point x at time t

$B = (B_1(x,t),\ B_2(x,t),\ B_3(x,t))$ the magnetic field at point x at time t.

We shall suppose that the fluid is homogéneous at time t=0 (i.e. $\rho(x,0)=\rho_o$ $\forall x \in \Omega$) ; the incompressibility of the fluid yields that

$$\rho(x,t) = \rho_o \text{ for every } x \in \Omega, \text{ and every } t \geq 0.$$

The nondimensional form of the Magnetohydrodynamic (MHD) equations is (cf. COWLING [ 5 ], LANDAU-LIFCHITZ [17]) :

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \dfrac{1}{Re}\Delta u + \nabla p + S\nabla(\dfrac{B^2}{2}) - S(B\cdot\nabla)B = f, \\ \dfrac{\partial B}{\partial t} + (u\cdot\nabla)B - (B\cdot\nabla)u + \dfrac{1}{Rm}\,\text{curl}\,(\text{curl}\ B) = 0, \\ \text{div}\ u = 0, \\ \text{div}\ B = 0. \end{cases}$$

Here p,u,B, are nondimensional quantities corresponding to the normalization by reference units denoted $L_*$, $T_*$, $U_* = \frac{L_*}{T_*}$, $B_*$, for lengths, times, velocities, and magnetic fields and we have set for simplicity $\rho(x,t) = \rho_o = 1$ ; $f(x,t)$ represents a (non-dimensional) volume density forces. Three non-dimensional numbers appear in (1.1), they are :

- the Reynolds number Re ($= \dfrac{L_* u_*}{\nu}$ , where $\nu$ is the kinematic viscosity),
- the magnetic Reynolds number Rm ($= L_* u_* \sigma\mu$, where $\mu$ is the magnetic permeability and $\sigma$ the resistivity of the fluid, which we assume to be constant),
- the number $S = \dfrac{M^2}{ReRm}$ ( $= \dfrac{B_*^2}{\mu\rho_* u_*^2}$ ), where M is the Hartman number.

When the region $\Omega$ is a bounded domain of the space, we supplement the system (1.1) with the following initial and boundary conditions

(1.2) $u(x,0) = u_o(x)$ , $B(x,0) = B_o(x)$ $\forall x \in \Omega$

(1.3) $\begin{cases} u = 0 \text{ on } \Gamma \text{ (non slip condition)}, \\ B \cdot n = 0 \text{ and } \operatorname{curl} B \times n = 0 \text{ on } \Gamma \text{ (perfectly conducting wall)}, \end{cases}$

where $\Gamma = \partial\Omega$ is the boundary of $\Omega$ and n is the unit outward normal on $\Gamma$.

We will also treat the space periodic case, i.e. the case where $\Omega$ is the whole space and (1.3) is replaced by

(1.4) $u(x+Le_i,t) = u(x,t)$, $B(x+Le_i,t) = B(x,t)$

where L is the period and $(e_i)_{i=1}^{N}$ an orthonormal basis of the space. Although this case does not correspond to a physical configuration, its mathematical treatment is technically easier, while it retains the main mathematical difficulties of the problem of the flow in a bounded region.

When the dimension of space is N=2, we classically define, the operators curl and $\tilde{\text{curl}}$

$$\operatorname{curl} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \text{ for every vector function } u = (u_1,u_2),$$

$$\tilde{\text{curl}}\, \phi = \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}\right) \text{ for every scalar function } \phi \text{ ;}$$

we recall that we have the 2-d formula

(1.5) $\tilde{\text{curl}} \operatorname{curl} u = \operatorname{grad} \operatorname{div} u - \Delta u$

which corresponds to the 3-d formula

(1.6) $\operatorname{curl} \operatorname{curl} u = \operatorname{grad} \operatorname{div} u - \Delta u$

The 2-d MHD equations are (1.1)-(1.3) (or (1.4)) with the term

$\tilde{\text{curl}}$ (curl B) instead of curl (curl B) and the boundary condition curl B=0 instead of curl $B \times n = 0$. Physically, the 2-d case means that the region is a cylinder $\Omega \times \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, and all the quantities are independent of $x_3$, u and B being parallel to the $Ox_1\ x_2$ plane.

## 2. - FUNCTIONAL SETTING OF THE EQUATIONS.

### 2.1. Function spaces (I) (case of a bounded domain)

In the case of a bounded domain, we assume from now on that

$$(2.1)\quad \begin{cases} \Omega \text{ is a bounded simply-connected}^{(1)} \text{ domain } \Omega \text{ of } \mathbb{R}^N, \\ N=2 \text{ or } 3, \text{ with boundary } \Gamma \text{ which is a } N-1 \text{ manifold of class } \mathcal{C}^\infty \\ \text{and } \Omega \text{ is located locally on one side of } \Gamma. \end{cases}$$

We denote by $L^2(\Omega)$ the space of real valued functions on $\Omega$ which are square integrable for the Lebesgue measure $dx = dx_1 \ldots dx_N$ ; this is a Hilbert space for the scalar product

$$(u,v) = \int_\Omega u(x)v(x)dx.$$

We denote by $H^m(\Omega)$ the Sobolev space of functions which are in $L^2(\Omega)$ together with their weak derivatives of order $\le m$ (cf. R.A. ADAMS [1], J.L. LIONS-E. MAGENES [21] ) ; $H^m_o(\Omega)$ is the Hilbert subspace of $H^m(\Omega)$ made of functions vanishing on $\Gamma$ (we shall also use the notations $\mathbb{L}^2(\Omega) = (L^2(\Omega))^N$, $\mathbb{H}^m(\Omega) = (H^m(\Omega))^N$, $\mathbb{H}^m_o(\Omega) = (H^m_o(\Omega))^N$).

Let $T > 0$ and let X be a Banach space. We shall consider $L^p(0,T;X)$ $(1 \le p \le \infty)$ which is the space of (classes of) functions from $[0,T]$ into X, which are $L^p$ for the Lebesgue measure dt. This is a Banach space for the norm

$$\left(\int_o^T \|u(t)\|^p_X\ dt\right)^{\frac{1}{p}} \text{ if } 1 \le p < \infty, \ \underset{0 \le t \le T}{\text{ess sup}} \|u(t)\|_X , \text{ for } p=\infty .$$

The spaces used in the theory of the MHD equations are a combination of spaces used for the Navier-Stokes equations (spaces denoted with the index 1)

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(1) this assumption is not essential ; without this assumption, problem (1.1)-(1.3) becomes well-posed by prescribing the values of a finite number of integrals of B (cf. DOMINGUEZ [ 6 ] , and a similar situation in hydrodynamics in C. FOIAS-R. TEMAM [12]).

and spaces used in the theory of Maxwell equations (spaces denoted with the index 2). They are

$$\mathcal{V}_1 = \{v \in \mathcal{C}_c^\infty(\Omega))^N \quad , \quad \operatorname{div} v = 0\}$$

$$V_1 = \text{the closure of } \mathcal{V}_1 \text{ in } \mathbb{H}_0^1(\Omega)$$

$$= \{v \in \mathbb{H}_0^1(\Omega) \quad , \quad \operatorname{div} v = 0\}$$

$$H_1 = \text{the closure of } \mathcal{V}_1 \text{ in } \mathbb{L}^2(\Omega)$$

$$= \{v \in \mathbb{L}^2(\Omega) \quad , \quad \operatorname{div} \mathbf{v} = 0 \text{ and } v \cdot n|_\Gamma = 0\}$$

$$\mathcal{V}_2 = \{C \in (\mathcal{C}^\infty(\bar{\Omega}))^N \quad , \quad \operatorname{div} C = 0 \text{ and } C \cdot n|_\Gamma = 0\}$$

$$V_2 = \text{the closure of } \mathcal{V}_2 \text{ in } \mathbb{H}^1(\Omega)$$

$$= \{C \in \mathbb{H}^1(\Omega) \quad , \quad \operatorname{div} C = 0 \text{ and } C \cdot n|_\Gamma = 0\}$$

$$H_2 = \text{the closure of } \mathcal{V}_2 \text{ in } \mathbb{L}^2(\Omega) = H_1$$

The dual space of $V_1$ is characterized by

$$V_1' = \{v \in \mathbb{H}^{-1}(\Omega) \quad , \quad \operatorname{div} v = 0\} .$$

For more details on these spaces, in particular for the previous characterization of $V_1$, $H_1$, $V_2$, $H_2$, $V_1'$, we refer to R. TEMAM [25].

We equip $V_1$ with the scalar product

$$((u,v))_1 = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right),$$

which is a scalar product on $\mathbb{H}_0^1(\Omega)$ thanks to Poincaré inequality. The dual norm on $V_1'$ is denoted by $\|\ \|_*$.

We equip $V_2$ with the scalar product

$$((u,v))_2 = (\operatorname{curl} u, \operatorname{curl} v).$$

Since the domain $\Omega$ is simply-connected, the above bilinear form is actually a scalar product on $V_2$ ; it defines a norm which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on $V_2$ (cf. DUVAUT-LIONS [7] and also [6] [12]).

Finally, we introduce

$$V = V_1 \times V_2 \ , \ H = H_1 \times H_2, \ V' \text{ the dual space of } V,$$

which satisfy $V \subset H \subset V'$, where the injections are continuous and each space is dense in the following one.

We equip H with the following scalar products,

$$(\Phi,\Psi) = (u,v) + (B,C) \qquad \forall\, \Phi = (u,B),\ \Psi = (v,C) \in H$$

$$[\Phi,\Psi] = (u,v) + S(B,C)$$

providing the (equivalent) norms on H $(0 < S < \infty)$ :

$$|\Phi| = \{(\Phi,\Phi)\}^{1/2}\ ,\ [\Phi] = \{[\Phi,\Phi]\}^{1/2}\ ;$$

we also equip V with the two scalar products

$$((\Phi,\Psi)) = \frac{1}{Re}\,((u,v))_1 + \frac{1}{Rm}\,((B,C))_2$$

$$[\![\Phi,\Psi]\!] = \frac{1}{Re}\,((u,v))_1 + \frac{S}{Rm}\,((B,C))_2$$

providing the equivalent norms on V

$$\|\Phi\| = \{((\Phi,\Phi))\}^{1/2},\ [\![\Phi]\!] = \{[\![\Phi,\Phi]\!]\}^{1/2}.$$

2.2. Function spaces (II) (space-periodic case).

Let us very briefly describe the functions spaces used in the periodic case (cf. R. TEMAM [26] for more details).

We denote by L the period, by $H^m_p(Q)$, $m \in \mathbb{R}_+$, $Q = (0,L)^N$, the space of functions which are locally in $H^m(\mathbb{R}^N)$ (i.e. $\in H^m_{loc}(\mathbb{R}^N)$) and are periodic with period L :

$$\text{(2.2)} \qquad u(x+Le_i) = u(x),\ i=1,\ldots,N.$$

For m= 0, $H_p^o(Q)$ coincides with $L^2(Q)$.

We also introduce

$$\dot{H}_p^m(Q) = \{u \in H_p^m(Q) \quad , \quad \int_\Omega u(x)dx = 0\} ,$$

and $\dot{H}_p^{-m}(Q)$, the dual space of $\dot{H}_p^m(Q)$. Of course the functions in $\dot{H}_p^m(Q)$, $m \in \mathbb{R}$, can be characterized by their Fourier series expansion.

In the periodic case, we set

$$V_1 = V_2 = \{u \in \mathbb{H}_p^1(Q), \text{ div } u = 0 \text{ in } \mathbb{R}^N\} , \ V = V_1 \times V_2,$$

$$H_1 = H_2 = \{u \in \mathbb{H}_p^o(Q), \text{ div } u = 0 \text{ in } \mathbb{R}^N\} , \ H = H_1 \times H_2,$$

$$V', \text{ the dual space of } V, = \{(u,B) \in (\mathbb{H}_p^{-1}(\Omega))^2, \text{ div } u = 0, \text{ div } B = 0 \text{ in } \mathbb{R}^N\}$$

The spaces V, $V_1$, $V_2$ are equipped with the same scalar products as in § 2.1. We have

$$V \subset H \subset V'$$

where the injections are continuous and each space is dense in the following one. In particular, we remark that, for the periodic case, we do not need to distinguish between u and B, neither for the boundary conditions, nor for the function spaces.

2.3. <u>The operator $\mathcal{A}$.</u>

We define three operators $\mathcal{A}_1 \in \mathcal{L}(V_1,V_1')$, $\mathcal{A}_2 \in \mathcal{L}(V_2,V_2')$, $\mathcal{A} \in \mathcal{L}(V,V')$ by setting

$$\langle\mathcal{A}_1 u,v\rangle = ((u,v))_1 \qquad \forall\ u,v \in V_1$$

$$\langle\mathcal{A}_2 B,C\rangle = ((B,C))_2 \qquad \forall\ B,C \in V_2$$

$$\langle\mathcal{A}\Phi,\Psi\rangle = ((\Phi,\Psi)) \qquad \forall\ \Phi,\Psi \in V.$$

We can also consider $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}$ as unbounded operators on $H_1,H_2,H$ whose domains are

$$D(\mathcal{A}_1) = \{u \in V_1 \quad , \quad \mathcal{A}_1 u \in H_1\} ,$$

$$D(\mathcal{A}_2) = \{B \in V_2 \quad , \quad \mathcal{A}_2 B \in H_2\} ,$$

$$D(\mathcal{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_2).$$

These spaces are well-known and these characterizations are available in the literature. Let us recall the main points.

i) Characterization of $D(\mathcal{A}_1)$ (bounded domain case)

It is classical (cf. R. TEMAM [25]), and this is the motivation for the introduction of $\mathcal{A}_1$, that if $u \in V_1$ is a solution of $\mathcal{A}_1 u = f \in V_1'$, then there exists $p \in L^2(\Omega)$ such that u and p are solutions to the linearized Stokes problem :

$$(2.3) \qquad \begin{cases} -\Delta u + \text{grad } p = f \text{ in } \Omega, \\ \text{div } u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma. \end{cases}$$

The solution of the Stokes problem has the following regularity properties :

$$(2.4) \qquad \begin{cases} \text{If } f \in \mathbb{H}^s(\Omega) \ (s \geq 0) \text{ then } u \in \mathbb{H}^{s+2}(\Omega),\ p \in H^{s+1}(\Omega), \\ \text{and there exists a constant } C = C(\Omega,s) \text{ such that} \\ \|u\|_{H^{s+2}} + \|p\|_{H^{s+1}/\mathbb{R}} \leq C(\Omega,s)\,\|f\|_{H^s} \end{cases}$$

(this result is proved by L. CATTABRIGA [4] for n=3 ; see R. TEMAM [25] for N=2). Thus we have $D(\mathcal{A}_1) = \mathbb{H}^2(\Omega) \cap V_1$.

ii) Characterization of $D(\mathcal{A}_2)$ (bounded domain case)

Let $B \in V_2$ be a solution of $\mathcal{A}_2 B = f \in H_2$. The first step is to interprete the relation $\mathcal{A}_2 B = f$.

Lemma 2.1 : Let $f \in H_2$. The following conditions are equivalent

(i) $B \in V_2$ satisfies $\mathcal{A}_2 B = f$

(ii) $B \in \mathbb{H}^1(\Omega)$ satisfies

$$(2.5) \quad \begin{cases} \text{curl (curl } B) = f \text{ in } \Omega, \\ \text{div } B = 0 \text{ in } \Omega, \\ B \cdot n = 0 \text{ on } \Gamma, \\ \text{curl } B \times n = 0 \text{ on } \Gamma. \end{cases}$$

Proof of Lemma 2.1 : For any $C \in \mathbb{H}^1(\Omega)$ such that $C \cdot n = 0$ on $\Gamma$, let us introduce a function $\phi \in H^1(\Omega)$ satisfying

$$\begin{cases} \Delta\phi = \text{div } C \text{ in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0 \text{ on } \Gamma. \end{cases}$$

Thanks to the regularity results of AGMON-DOUGLIS-NIRENBERG [ 2 ], $\phi \in H^2(\Omega)$, so that $\tilde{C} = C - \text{grad } \phi \in V_2$. We have

$$<\mathcal{A}_2 B, \tilde{C}> \; = \; (\text{ curl } B, \text{ curl } C) \text{ since curl grad } \equiv 0,$$

$$(f, \tilde{C}) = (f, C) \text{ since } f \in H_2 \;;$$

hence

$$(\text{curl } B, \text{ curl } C) = (f, C) \qquad \forall\, C \in \mathbb{H}^1(\Omega),$$

and we obtain (2.5) by using the formula

$$(2.6) \qquad \int_\Omega \text{curl } B \cdot C \, dx = \int_\Omega B \text{ curl } C \, dx + \int_\Gamma (B \times C) \cdot n \, d\sigma \;. \;\blacksquare$$

Using the formula (1.5), we see that a solution $B \in \mathbb{H}^1(\Omega)$ of (2.5) also satisfies

$$(2.7)\qquad \begin{cases} -\ \Delta B = f \text{ in } \Omega, \\ B\cdot n = 0 \text{ on } \Gamma, \\ \text{curl } B\times n = 0 \text{ on } \Gamma. \end{cases}$$

Concerning problem (2.7), we have the following regularity result (due to V. GEORGESCU [15], th. 3.2.3).

$$(2.8)\qquad \begin{cases} \text{If } f\in \mathbb{H}^s(\Omega) \quad (s\geq 0) \text{ then } u\in \mathbb{H}^{s+2}(\Omega), \\ \text{and there exists a constant } C = C(\Omega,s) \text{ such that} \\ \|B\|_{H^{s+2}} \leq C(\Omega,s)\,\|f\|_{H^s} \end{cases}$$

Thus, we have $D(\mathcal{A}_2) = \mathbb{H}^2(\Omega)\cap V_2$.

Combining i),ii), we obtain $D(\mathcal{A}) = (\mathbb{H}^2(\Omega))^2\cap V$. It is a trivial matter to obtain the same result in the space periodic case.

2.4. <u>The form $\mathcal{B}_o$ and the operator $\mathcal{B}$</u>.

We define now a trilinear form on $\mathbb{L}^1(\Omega)\times \mathbb{W}^{1,1}(\Omega)\times \mathbb{L}^1(\Omega)$ by setting

$$b(u,v,w) = \sum_{i,j=1}^{N} \int_\Omega u_i D_i v_j w_j \,dx \quad (\text{where } D_i = \frac{\partial}{\partial x_i})$$

whenever the integrals make sense.

We recall (cf. R.TEMAM [26]) that if the $m_i$'s are $\geq 0$ and satisfy :

$$(2.9)\qquad \begin{cases} m_1 + m_2 + m_3 > \frac{N}{2} \\ \text{or} \\ m_1 + m_2 + m_3 = \frac{N}{2} \quad \text{and at least two } m_i\text{'s are} \neq 0, \end{cases}$$

then b is a trilinear continuous form on $\mathbb{H}^{m_1}(\Omega)\times \mathbb{H}^{m_2+1}(\Omega)\times \mathbb{H}^{m_3}(\Omega)$, and

$$(2.10)\qquad |b(u,v,w)| \leq c_1 |u|_{H^{m_1}} |v|_{H^{m_2+1}} |w|_{H^{m_3}}, \forall (u,v,w)\in \mathbb{H}^{m_1}(\Omega)\times \mathbb{H}^{m_2+1}(\Omega)\times \mathbb{H}^{m_3}(\Omega).$$

In particular we have $(m_1 = m_3 = 1,\ m_2 = 0)$,

(2.11) the trilinear form b is continuous on $(\mathbb{H}^1(\Omega))^3$.

We have also (due to Stokes formula and (2.10))

$$(2.12)\qquad \begin{cases} b(u,v,v) = 0 \quad \forall u \in V_2,\ \forall v \in \mathbb{H}^1(\Omega), \\ b(u,v,w) = -b(u,w,v) \quad \forall u \in V_2,\ \forall v,w \in \mathbb{H}^1(\Omega). \end{cases}$$

Thanks to (2.11), we can define a continuous trilinear form $\mathcal{B}_o$ on $V \times V \times V$ by setting :

$$\begin{cases} \mathcal{B}_o(\Phi_1,\Phi_2,\Phi_3) = b(u_1,u_2,u_3) - Sb(B_1,B_2,u_3) + b(u_1,B_2,B_3) - b(B_1,u_2,B_3) \\ \forall\ \Phi_i = (u_i,B_i) \in V, \end{cases}$$

and we define a continuous bilinear operator $\mathcal{B}$ from $V \times V$ into $V'$ with

$$< \mathcal{B}(\Phi_1,\Phi_2),\Phi_3> = \mathcal{B}_o(\Phi_1,\Phi_2,\Phi_3) \qquad \forall \Phi_i \in V.$$

Taking $m_i \geq 0$ satisfying (2.9), we get from (2.10) and the discrete Hölder's inequality that

$$(2.13)\qquad \begin{cases} |\mathcal{B}_o(\Phi_1,\Phi_2,\Phi_3)| \leq c_2 \max(1,S) |\Phi_1|_{H^{m_1}} |\Phi_2|_{H^{m_2+1}} |\Phi_3|_{H^{m_3}} \\ \forall (\Phi_1,\Phi_2,\Phi_3) \in \mathbb{H}^{m_1}(\Omega) \times \mathbb{H}^{m_2+1}(\Omega) \times \mathbb{H}^{m_3}(\Omega). \end{cases}$$

Combining now (2.13) with the regularity results (2.4),(2.8), we get (take $m_1 = m_2 = 0.5$, $m_3 = 0$ for N=2, $m_1 = 1$, $m_2 = 0.5$, $m_3 = 0$ for N=3)

$$(2.14)\qquad (\text{case } N=2)\quad |\mathcal{B}_o(\Phi_1,\Phi_2,\Phi_3)| \leq c_3 |\Phi_1|^{1/2} \|\Phi_1\|^{1/2} \|\Phi_2\|^{1/2} |\mathcal{A}\Phi_2|^{1/2} |\Phi_3|$$

$$\forall\ \Phi_1 \in V,\ \Phi_2 \in D(\mathcal{A}),\ \Phi_3 \in H,$$

$$(2.15)\qquad (\text{case } N=3)\quad |\mathcal{B}_o(\Phi_1,\Phi_2,\Phi_3)| \leq c_4 \|\Phi_1\| \|\Phi_2\|^{1/2} |\mathcal{A}\Phi_2|^{1/2} |\Phi_3|$$

$$\forall \Phi_1 \in V,\ \Phi_2 \in D(\mathcal{A}),\ \Phi_3 \in H,$$

where $c_3$, $c_4$ depend only on $\Omega$,S,Re,Rm.

Finally, we introduce a diagonal matrix $M \in M_6(\mathbb{R})$ :

$$(2.16)\qquad m_{ii} = 1 \text{ if } 1 \leq i \leq 3,\quad m_{ii} = S \text{ if } 4 \leq i \leq 6,$$

From the relation

$$\mathcal{B}_o(\Phi_1,\Phi_2, M\Phi_2) = b(u_1,u_2,u_2) + Sb(u_1,B_2,B_2) - S[b(B_1,B_2,u_2)+b(B_1,u_2,B_2)],$$

and from (2.12), we get

$$(2.17)\quad \begin{cases} \mathcal{B}_o(\Phi_1,\Phi_2,M\Phi_2) = 0 & \forall\Phi_1,\Phi_2 \in V, \\ \mathcal{B}_o(\Phi_1,\Phi_2,M\Phi_3) = -\mathcal{B}_o(\Phi_1,\Phi_3,M\Phi_2) & \forall\Phi_i \in V. \end{cases}$$

2.5. Weak formulation of the equations.

Let $T > 0$ be given and let us assume that p,u,B is a smooth solution of (1.1)-(1.3).

We multiply the first equation of (1.1) by a test function $v \in \mathcal{V}_1$, we integrate over $\Omega$, use Stokes formula and the relation div u = 0 and we find that

$$\frac{d}{dt}(u,v) + \frac{1}{Re}((u,v))_1 + b(u,u,v) - Sb(B,B,v) = (f,v)$$

We multiply the second equation of (1.1) by a function $C \in \mathcal{V}_2$, we integrate over $\Omega$ and we use formula (2.6) ; we obtain

$$\frac{d}{dt}(B,C) + \frac{1}{Rm}((B,C))_2 + b(u,B,C) - b(B,u,C) = 0$$

This suggests the following weak formulation of problem (1.1)-(1.3) (cf. G. DUVAUT - J.L. LIONS [ 8 ]).

Problem 2.1 (weak solutions)

For $\Omega \subset \mathbb{R}^N$, N=2 or 3, satisfying (2.1) (resp. for the space periodic condition), for f given in $L^2(0,T;V_1')$ and $\Phi_o = (u_o,B_o)$ given in H, to find $\Phi = (u,B)$ satisfying

$$(2.18)\quad \Phi \in L^2(0,T;V)$$

$$(2.19)\quad \frac{d}{dt}(\Phi,\Psi) + ((\Phi,\Psi)) + \mathcal{B}_o(\Phi,\Phi,\Psi) = \langle f,v\rangle \quad \forall\ \Psi = (v,C) \in V$$

$$(2.20)\quad \Phi(0) = \Phi_o.$$

When $f \in L^2(0,T;H)$, $\Phi_0 \in V$, we call <u>strong solutions</u> of Problem 2.1 the solutions which satisfy $\Phi \in L^2(0,T;D(\mathcal{A})) \cap L^\infty(0,T;V)$.

Using the operators $\mathcal{A}, \mathcal{B}$ previously defined, Equation (2.19) can equivalently be written as

$$\text{(2.21)} \qquad \frac{d}{dt}\Phi + \mathcal{A}\Phi + \mathcal{B}(\Phi,\Phi) = F$$

where $F = (f,0)$.

Let us assume that $(u,B)$ is a smooth solution of Problem 2.1. We can see that there exists a function $p$ such that (1.1)-(1.3) is satisfied. In order to get the first equation in (1.1), we use the same technics as in § 2.3 i). For the second equation, we observe that

$$\text{(2.22)} \qquad b(u,B,C) - b(B,u,C) = \int_\Omega \text{curl } (B \times u)\cdot C \, dx = \int_\Omega B \times u \cdot \text{curl } C \, dx.$$

As in § 2.3, ii), we can take

$$C = \tilde{C} + \text{grad } \phi$$

so that, by (2.19)

$$\frac{d}{dt}(B,C) + \frac{1}{Rm}((B,C))_2 + b(u,B,C) - b(B,u,C) = 0, \quad \forall C \in \mathbb{H}^1(\Omega)$$

with $C\cdot n = 0$ on $\Gamma$.

We obtain finally the second equation in (1.1).

<u>Remark 2.1</u> : If $\Phi$ is a strong solution of Problem 2.1, we obtain by (2.15) that $\Phi' \in L^2(0,T;H)$. The two conditions $\Phi \in L^2(0,T;D(A))$ and $\Phi' \in L^2(0,T;H)$ imply with an interpolation result given in J.L. LIONS - E. MAGENES [21] that $\Phi$ is almost everywhere equal to a continuous function from $(0,T)$ into $V$. Thus, a strong solution of Problem 2.1 also satisfies $\Phi \in \mathcal{C}([0,T];V)$.

## 3. - EXISTENCE AND UNIQUENESS RESULTS.

We recall and complete some well-known results on the existence and uniqueness of weak (and strong) solutions to Problem 2.1. We start with various a priori estimates on the norms of the solutions of Problem 2.1.

### 3.1. A priori estimates.

We assume that $\Phi$ is a smooth solution of Problem (2.1). We can rewrite (2.19) as

$$(3.1)\qquad (\frac{d\Phi}{dt},\Psi) + ((\Phi,\Psi)) + \mathcal{B}_o(\Phi,\Phi,\Psi) = (f,v)\ ,\quad \forall\,\Psi = (v,C) \in V.$$

i) A priori estimate in $L^\infty(0,T;H) \cap L^2(0,T;V)$ (T = a finite number).

Let us take $\Psi = M\Phi(t)$ in (3.1). Thanks to (2.17), we have

$$(3.2)\qquad \frac{1}{2}\frac{d}{dt}\,(|u(t)|^2 + S|B(t)|^2) + \frac{1}{Re}\|u(t)\|_1^2 + \frac{S}{Rm}\|B(t)\|_2^2 = (f(t),u(t)).$$

By integration, we obtain the energy equality

$$(3.3)\qquad |u(t)|^2 + S|B(t)|^2 + \frac{2}{Re}\int_o^t \|u(s)\|_1^2\,ds + \frac{2S}{Rm}\int_o^t \|B(s)\|_2^2\,ds =$$

$$= |u_o|^2 + S|B_o|^2 + 2\int_o^t (f(s),u(s))ds.$$

We obtain classically from (3.2) that

$$(3.4)\qquad \int_o^T \|u(t)\|_1^2\,dt \le Re\,K_1,\ \int_o^T \|B(t)\|_2^2\,dt \le \frac{Rm}{2S}K_1,$$

$$(3.5)\qquad \operatorname*{Sup}_{t\in[0,T]}\,(|u(t)|^2 + S|B(t)|^2) \le K_1,$$

where

$$(3.6)\qquad K_1 = |u_o|^2 + S|B_o|^2 + Re\int_o^T \|f(t)\|_*^2\,dt.$$

Like $K_1$, the constants $K_j$ which will appear hereafter depend only on the data, $\Phi_o,f,\Omega,T$ and the non-dimensional numbers Re,Rm,S .

When $f \in L^\infty(0,\infty;V_1')$, one can obtain an a priori estimate in $L^\infty(0,\infty;H)$. From the continuity of the injection of $V_i$ in $H_i$, there exist constants $c_i$ such that

(3.7) $$|u| \le c_1 \|u\|_1 \quad \forall u \in V_1, \quad |B| \le c_2 \|B\|_2 \quad \forall B \in V_2 ;$$

the best constant $c_i$ is equal to $\frac{1}{\sqrt{\lambda_1^i}}$, where $\lambda_1^i$ is the first eigenvalue of the compact operator $\mathcal{A}_i^{-1}$ from $H_i$ into itself.

From (3.2), we obtain

(3.8) $$\frac{d}{dt}(|u(t)|^2 + S|B(t)|^2) + \min\left(\frac{1}{Re\, c_1^2}, \frac{2}{R_m\, c_2^2}\right)(|u(t)|^2 + S|B(t)|^2) \le Re\, \|f(t)\|^2$$

Hence, by Gronwall's lemma technics

(3.9) $$\sup_{t \in [0,\infty)} (|u(t)|^2 + S|B(t)|^2) \le K_2$$

where

(3.10) $$K_2 = |u_0|^2 + S|B_0|^2 + \max\,(Re^2\, c_1^2,\ ReRmc_2^2) \sup_{t \in [0,\infty)} \|f(t)\|_*^2 .$$

ii) <u>A priori estimate in</u> $L^\infty(0,T;V) \cap L^2(0,T;D(\mathcal{A}))$.

Taking $\Psi = \mathcal{A}\Phi$ in (3.1), we obtain

(3.11) $$\frac{1}{2}\frac{d}{dt}\|\Phi\|^2 + |\mathcal{A}\Phi|^2 = (f, \mathcal{A}_1 u) - \mathcal{B}_0(\Phi, \Phi, \mathcal{A}\Phi).$$

The bounds on $\mathcal{B}_0$ are different, depending on the dimension (cf. (2.14)-(2.15)).

a) <u>Case N=2</u>

Using (2.14), the right-hand side of (3.11) can be majorized by

$$\{|f|\ |\mathcal{A}\Phi| + c_3|\Phi|^{1/2}\|\Phi\|\ \|\mathcal{A}\Phi\|^{3/2}\}$$

where $c_3 = c_3(\Omega, S, Re, Rm)$.

We will use Young's inequality in the form :

(3.12) $$ab \le \varepsilon a^p + c_\varepsilon b^q, \quad \forall \varepsilon, a, b > 0,\ 1 < p < \infty,\ q = \frac{p}{p-1},\ c_\varepsilon = \frac{(p-1)}{p^q \varepsilon^{1/(p-1)}}$$

Using Young's inequality with the exponents $p = \frac{4}{3}$ , $q = 4$ and with an $\varepsilon$ sufficiently small, we obtain from (3.11)

$$(3.13) \qquad \frac{d}{dt} \|\Phi(t)\|^2 + |\mathcal{A}\Phi(t)|^2 \le 2|f| + c_3'|\Phi|^2 \|\Phi\|^4 ;$$

hence, by (3.5)

$$(3.14) \qquad \frac{d}{dt} \|\Phi(t)\|^2 + |\mathcal{A}\Phi(t)|^2 \le 2|f(t)| + c_3' K_1 \|\Phi(t)\|^4 .$$

From (3.4), we get

$$(3.15) \qquad \int_0^T \|\Phi(t)\|^2 \, dt \le c_3' K_1$$

which, together with (3.14) and Gronwall's lemma technics , gives

$$(3.16) \qquad \sup_{t \in [0,T]} \|\Phi(t)\|^2 \le K_2 ;$$

using (3.14) again, we obtain

$$(3.17) \qquad \int_0^T |\mathcal{A}\Phi(t)|^2 \, dt \le K_3 .$$

b) <u>Case N=3</u>.

Using (2.15), the right-hand side of (3.11) can be majorized by

$$\{|f|\,|\mathcal{A}\Phi| + c_4 \|\Phi\|^{3/2} \|\mathcal{A}\Phi\|^{3/2}\}.$$

Using Young's inequality with the exponents $p = \frac{4}{3}$ , $q = 4$ and $\varepsilon$ sufficiently small, we obtain

$$(3.18) \qquad \frac{d}{dt} \|\Phi(t)\|^2 + |\mathcal{A}\Phi|^2 \le 2|f(t)| + c_4' \|\Phi\|^6 .$$

By lack of a priori bound in $L^4(0,T;V)$, we cannot obtain an a priori estimate in $L^\infty(0,T;V)$. But by comparison with the differential equation $\phi'(t) \le c\,\phi(t)^3$, we obtain the existence of two numbers $T_*$, $K_4$ depending on $\|\Phi_0\|$ (and the data $f,\Omega,T$, the nondimensional numbers Re,Rm,S) and such that

(3.19) $$\sup_{t\in[0,T_*]} \|\Phi(t)\|^2 \le 2(1+\|\Phi_o\|^2)$$

(3.20) $$\int_o^{T_*} |\mathcal{A}\Phi(t)|^2\, dt \le K_4.$$

It is interesting to make explicit the form of $T_* = T_*(\|\Phi_o\|)$ (see C. FOIAS-R. TEMAM [13])

(3.21) $$T_*(\|\Phi_o\|) = \frac{K_5}{(1+\|\Phi_o\|^2)^2} ,$$

where $K_5$ depends only on $f,\Omega$ and the nondimensional numbers Re, Rm,S.

3.2. Existence and uniqueness results.

Theorem 3.1 : For $f,u_o,B_o$ given with

(3.22) $$f\in L^2(0,T,V_1'),\ \Phi_o = (u_o,B_o)\in H,$$

there exists a weak solution $\Phi = (u,B)$ to Problem 2.1 satisfying

(3.23) $$\Phi\in L^2(0,T,V)\cap L^\infty(0,T,H).$$

Furthermore

i) if N=2, $\Phi$ is unique and satisfies

(3.24) $$\Phi'\in L^2(0,T;V'),\ \Phi\in \mathcal{C}([0,T];H),$$

ii) if N=3, there is at most one solution to Problem 2.1 satisfying

(3.25) $$\Phi\in L^4(0,T;V).$$

Theorem 3.2 : Let $f,u_o,B_o$ be given with

(3.26) $$f\in L^\infty(0,T;H),\ \Phi_o = (u_o,B_o)\in V,$$

i) if N=2, the solution $\Phi = (u,B)$ of Problem 2.1 satisfies

(3.27) $$\Phi\in L^2(0,T;D(\mathcal{A}))\cap L^\infty(0,T,V)\ ;$$

ii) <u>if</u> N=3, <u>there exists</u> $T_* > 0$ (<u>depending on</u> $\Omega, f, \|\Phi_o\|$) <u>and</u>, <u>on</u> $[0,T_*]$, <u>there exists a unique solution</u> $\Phi$ <u>to Problem 2.1 which satisfies</u> (3.27) <u>with</u> T <u>replaced by</u> $T_*$.

The proofs of Theorem 3.1 can be found in G. DUVAUT - J.L. LIONS [8]. Let us give a sketch of the proof which will be useful in the sequel.

We implement a Galerkin method using as a basis of H the eigenfunctions $w_i$ of the operator $\mathcal{A}$. For every integer m, $W_m$ is the space spanned by $w_1,\ldots,w_m$ and $P_m$ the orthogonal projector in H onto $W_m$ ([1]).

We associate to (2.21) the ordinary differential system

$$(3.28)\quad \begin{cases} \dfrac{d\Phi_m}{dt} + \mathcal{A}\Phi_m + P_m \mathcal{B}(\Phi_m,\Phi_m) = P_m \;, \\ \Phi_m(0) = P_m\Phi_o , \end{cases}$$

which has a unique solution on some interval $(0,T_m)$, $T_m > 0$ ; in fact the following a priori estimate shows that $T_m = T$.

Using the same technics as in Sec. 3.1 i), we obtain that $\Phi_m$ remains in a bounded set of $L^2(0,T;V) \cap L^\infty(0,T;H)$ ; from the a priori bound in $L^2(0,T;V)$ and from $\|P_m\|_{\mathcal{L}(V';V')} \leq 1$, we get that $\frac{d}{dt}\Phi_m$ is bounded in $L^1(0,T;V')$ ; thus, we can use a compactness argument (R. TEMAM [25], Ch. III, th. 2.3) to get that a subsequence $\Phi_{m'}$ satisfies

$$(3.29)\qquad \Phi_{m'} \to \Phi \begin{cases} \text{in } L^2(0,T;V) \text{ weakly,} \\ \text{in } L^\infty(0,T;H) \text{ weak star,} \\ \text{in } L^2(0,T;H) \text{ strongly.} \end{cases}$$

The passage to the limit in (3.29) allows us to conclude that $\Phi = (u,B)$ is a solution of Problem 2.1.

For the uniqueness of a solution $\Phi \in L^4(0,T;V)$ (case N=3), we use essentially the same technique as in R. TEMAM [25], Ch. III, th. 3.4.

Using the same technics as in Sec. 3.1, we can also obtain that $\Phi_m$ remains in a bounded set of $L^2(0,T,D(\mathcal{A})) \cap L^\infty(0,T;V)$ ($T = T_*$ if N=3) and we obtain th. 3.2 (for more details in a similar situation for the Navier-Stokes equations, see R. TEMAM [25]).

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([1]) $P_m$ is also the orthogonal projector in $D(\mathcal{A}),V,V'$ onto $W_m$.

## 4. - SMOOTHNESS OF STRONG SOLUTIONS FOR LARGE TIME.

### 4.1. Estimates on the time derivatives.

From now on we will restrict ourselves to strong solutions of Problem 2.1. According to Theorem 3.2, if the dimension of space is N=2, and the initial data satisfy (3.26), then the unique solution to Problem 2.1 is a strong one, i.e. it satisfies (3.27). When the dimension of space is N=3, then the existence of a strong solution is not provided by Theorem 3.2 for an arbitrary interval of time $[0,T]$ ($T > T_*(\|\Phi_0\|)$). We will then only consider solutions of Problem 2.1 which belong to $L^\infty(0,T;V)$ ; it is then an easy matter to check that such a solution (which is unique, Theorem 3.1, ii)), actually belongs to $L^2(0,T;D(\mathcal{A}))$ (we use techniques similar to those leading to (3.18)). Thus this solution is a strong one. Further regularity results are given below.

Finally we will be interested in the large time behaviour of the solutions and therefore we will allow T to be infinite, $0 < T \le \infty$.

By use of technics slightly different than those used by C. GUILLOPE [16] for the Navier-Stokes equations, we prove

Theorem 4.1 : Let there be given f and $\Phi_0$ satisfying

(4.1) $\qquad \Phi_0 \in H$

(4.2) $\qquad f^{(j)} \in L^\infty(0,T;H_1)$ for $j=0,\dots,j_0$

where $0 < T \le +\infty$ and $j_0 \in \mathbb{N}$.

Then, in the case N=2, the solution $\Phi$ of Problem 2.1 satisfies, for every $\alpha_0 > 0$,

(4.3)
$$\begin{cases} \Phi^{(j)} \in L^\infty(\alpha_0,T;D(\mathcal{A})) \text{ for } j=0,\dots,j_0-1 \\ \text{and } \Phi^{(j_0)} \in L^\infty(\alpha,T;V). \end{cases}$$

In the case N=3, if the strong solution $\Phi$ of Problem 2.1 is such that

(4.4) $\qquad \Phi \in L^\infty(0,T;V)$,

then $\Phi$ satisfies (4.3) as well.

Before proving theorem 4.1, let us state

Lemma 4.1 : Let $t_o < t_1$ be two reals numbers and $\alpha, \alpha_1, \alpha_2, \alpha_3$ be four positive numbers with $\alpha \leq t_1 - t_o$. Let $\theta_i : (t_o, t_1) \to \mathbb{R}_+$, $i=1,2$, be two measurable functions which satisfy

$$(4.5) \qquad \int_{t-\alpha}^{t} \theta_i(s)ds \leq \alpha_i \quad \forall t \in [t_o+\alpha, t_1] \text{ , } i=1,2.$$

If an absolutely continuous function $y : [t_o, t_1] \to \mathbb{R}_+$ satisfies

$$(4.6) \qquad \int_{t-\alpha}^{t} y(s)ds \leq \alpha_3 \quad \forall t \in [t_o+\alpha, t_1],$$

and

$$(4.7) \qquad y'(t) \leq \theta_1(t) + \theta_2(t)y(t) \quad \forall t \in [t_o+\alpha, t_1],$$

then

$$(4.8) \qquad y(t) \leq (\alpha_1 + \frac{\alpha_3}{\alpha})e^{\alpha_2} \quad \forall t \in [t_o+\alpha, t_1].$$

Proof of lemma 4.1 : The Gronwall's lemma technics applied to (4.7) gives

$$(4.9) \qquad \frac{d}{d\tau}[y(\tau)\ \exp(-\int_{t_o}^{\tau} \theta_2(\sigma)d\sigma)] \leq \theta_1(\tau)\ \exp\ (-\int_{t_o}^{\tau} \theta_2(\sigma)d\sigma).$$

For some fixed number $t \in [t_o+\alpha, t_1]$, we consider a number $s \in [t-\alpha, t]$ and we integrate (4.9) over the interval $[s,t]$ :

$$y(t) \leq y(s)\ \exp\ (\int_{s}^{t} \theta_2(\sigma)d\sigma) + \int_{s}^{t} \theta_1(\tau)\exp(\int_{\tau}^{t} \theta_2(\sigma)d\sigma)d\tau.$$

From (4.5), we obtain

$$(4.10) \qquad y(t) \leq y(s)e^{\alpha_2} + \alpha_1 e^{\alpha_2}.$$

We integrate (4.10) with respect to s over $[t-\alpha, t]$ and we obtain (4.8) thanks to (4.6). ∎

Proof of Theorem 4.1 : For a given strong solution of Problem 2.1 on $[0,T)$ (T can be infinite), we consider a Galerkin method (cf. Sec. 3.2) starting from some fixed time $t_o$ not necessarily equal to 0, $0 \le t_o < T$ :

$$\frac{d\Phi_m}{dt} + \mathcal{A}\Phi_m + P_m \mathcal{B}(\Phi_m, \Phi_m) = P_m F, \tag{4.11}$$

$$\Phi_m(t_o) = P_m \Phi(t_o). \tag{4.12}$$

From the a priori estimate (3.9), $\Phi_m$ is defined on $[t_o, T)$ and

$$\Phi_m \in L^\infty(t_o, T; H) \cap \mathcal{C}^\infty(t_o, T; V_m). \tag{4.13}$$

For the case N=2, we shall see (step i)) that

$$\Phi_m \in L^\infty(t_o+\alpha, T; V) \text{ for every } \alpha > 0. \tag{4.14}$$

For the case N=3, we don't know if (4.14) is satisfied. Nevertheless, thanks to (3.19),(3.21),

$$\|\Phi_m(t)\| \le 2K_o \text{ for } t_o \le t \le t_o + \eta \tag{4.15}$$

where $K_o = \sup\limits_{0 \le t \le T} \|\Phi(t)\|$ and $\eta = \dfrac{K_5}{(1+K_o)^2}$.

For any fixed $j_o \ge 0$ and any fixed $\alpha_o$ such that $0 < \alpha_o < \eta$, we take

$$\alpha = \frac{\alpha_o}{2j_o+1}. \tag{4.16}$$

For every fixed $t_o \ge 0$ ($t_o \le T-\eta$ if T is finite), we shall prove by a finite induction on j ($0 \le j \le j_o$) that

$$(4.17)_j \quad \begin{cases} |\Phi_m^{(j)}(t)| \le K_1^j \text{ for } t_o+2j\alpha \le t \le t_o+\eta \\ \displaystyle\int_{t-\alpha}^t \|\Phi_m^{(j)}(s)\|^2 \, ds \le K_2^j \text{ for } t_o+(2j+1)\alpha \le t \le t_o+\eta \end{cases}$$

$$(4.18)_j \quad \begin{cases} \|\Phi_m^{(j)}(t)\| \le K_3^j \text{ for } t_o+(2j+1)\alpha \le t \le t_o+\eta \\ \displaystyle\int_{t-\alpha}^t |\mathcal{A}\Phi_m^{(j)}(s)|^2 \, ds \le K_4^j \text{ for } t_o+(2j+2)\alpha \le t \le t_o+\eta \end{cases}$$

$$(4.19)_j \quad \begin{cases} |\mathcal{A}\Phi_m^{(j-1)}| \le K_5^j \quad (j \ge 1) \text{ for } t_o+(2j+1)\alpha \le t \le t_o+\eta \\ \displaystyle\int_{t-\alpha}^t |\Phi_m^{(j+1)}(s)|^2 \, ds \le K_6^j \text{ for } t_o+(2j+2)\alpha \le t \le t_o+\eta \end{cases}$$

where all the constants $K_i^j$ are independent of $t_o$ and m, and depend only on $\Omega$, S,Re,Rm,f,$\Phi_o$,j, and, in the case N=3, on $K_o = \sup_{0\le t\le T} \|\Phi(t)\|$.

i) j=0

At first, let us prove (4.14) when N=2. From the first a priori estimate (§ 3.1 i)),

$$(4.20) \qquad \int_{t-\alpha}^{t} \|\Phi_m(s)\|^2 \, ds \le c \qquad \forall t,\ t_o+\alpha \le t < T$$

and from (4.13) and the second a priori estimate (§ 3.1, ii)),

$$(4.21) \qquad \frac{d}{dt} \|\Phi_m(t)\|^2 \le c + c\|\Phi_m(t)\|^4 \qquad \forall t,\ t_o \le t < T.$$

Thus, (4.14) is a consequence of lemma 4.1.

From now on, we treat simultaneously the cases N=2 or 3 and we observe that (4.14) or (4.15) yields the first estimate in $(4.18)_o$ (with possibly $t_o+\eta=T$). The second estimate in $(4.18)_o$ is a consequence of (3.14). The continuity property (2.15) gives

$$(4.22) \qquad |\mathcal{B}(\Phi,\Psi)| \le c\|\Phi\| \ \|\Psi\|^{1/2} \ |\mathcal{A}\Psi|^{1/2},$$

which gives the second inequality in $(4.19)_o$.

ii) $1 \le j \le j_o$.

Because of the regularity of $\Phi_m$, we can differentiate (4.11) j times. We obtain

$$(4.23) \qquad \frac{d}{dt} \Phi_m^j + \mathcal{A}\Phi_m^j = F_m^j - \beta_m^j$$

where

$$\Phi_m^j = \Phi_m^{(j)} = \frac{d^j\Phi_m}{dt^j},$$

$$F_m^j = \frac{d^j P_m F}{dt^j},$$

$$\beta_m^j = \frac{d^j}{dt^j} P_m \mathcal{B}(\Phi_m,\Phi_m) = \sum_{i=0}^{j} C_j^i \ P_m \ \mathcal{B}(\Phi_m^i,\Phi_m^{j-i}).$$

We recall that we introduced in Sec. 2.1 the scalar products

$$[\Phi,\Psi] = (M\Phi,\Psi), \quad [\![\Phi,\Psi]\!] = ((M\Phi,\Psi))$$

and that $|\Phi|$ and $[\Phi]$ (resp. $\|\Phi\|$ and $[\![\Phi]\!]$) are equivalent norms on H (resp. V).

<u>First a priori estimate.</u>

We take the scalar product in H of (4.23) with $M\Phi_m^j$ :

$$(4.24) \qquad \frac{1}{2}\frac{d}{dt}[\Phi_m^j]^2 + [\![\Phi_m^j]\!]^2 = [F_m^j,\Phi_m^j] - \sum_{i=0}^{j} C_j^i P_m \mathcal{B}_o(\Phi_m^i,\Phi_m^{j-i},M\Phi_m^j).$$

From (2.17), we know that

$$(4.25) \qquad \mathcal{B}_o(\Phi_m,\Phi_m^j,M\Phi_m^j) = 0.$$

The continuity of $\mathcal{B}_o$ on $V\times V\times V$ gives

$$(4.26) \qquad |\mathcal{B}_o(\Phi_m^i,\Phi_m^{j-i},M\Phi_m^j)| \le c\|\Phi_m^i\|\,\|\Phi_m^{j-i}\|\,\|\Phi_m^j\| \quad ;$$

writting $K_o^j = \max\limits_{0\le i\le j-1}(K_3^i)$, we obtain from the induction hypothesis

$$(4.27) \qquad |\mathcal{B}_o(\Phi_m^i,\Phi_m^{j-i},M\Phi_m^j)| \le c(K_o^j)^2\|\Phi_m^j\| \quad \text{for } 1\le i\le j-1.$$

The inequality (2.13) with $m_1=1$, $m_2=0$, $m_3=0.5$ gives

$$(4.28) \qquad |\mathcal{B}_o(\Phi_m^j,\Phi_m,M\Phi_m^j)| \le c\|\Phi_m\|\,\|\Phi_m^j\|^{3/2}\,|\Phi_m^j|^{1/2}$$

$$\le \text{(by Young's inequality and } (4.18)_o)$$

$$\le \varepsilon[\![\Phi_m^j]\!]^2 + c_\varepsilon[\Phi_m^j]^2 .$$

We conclude from (4.24)-(4.28) that

$$(4.29) \qquad \frac{d}{dt}[\Phi_m^j]^2 + [\![\Phi_m^j]\!]^2 \le c'(K_o^j)^2(1+[\Phi_m^j]^2).$$

Thanks to (4.29) and $(4.19)_{j-1}$, we can use lemma 4.1 and we obtain $(4.17)_j$ with $K_1^j$, $K_2^j$ depending on the constant $K_6^{j-1}$ and $K_o^j$.

Second a priori estimate

We take the scalar product in H of (4.23) with $\mathcal{A}\Phi_m^j$ :

(4.30) $$\frac{1}{2}\frac{d}{dt}||\Phi_m^j||^2 + |\mathcal{A}\Phi_m^j|^2 = (F_m^j, \mathcal{A}\Phi_m^j) - \sum_{i=0}^{j} C_j^i \mathcal{B}_0(\Phi_m^i, \Phi_m^{j-i}, \mathcal{A}\Phi_m^j).$$

The inequality (2.15) gives :

(4.31) $$|\mathcal{B}_0(\Phi_m^i, \Phi_m^{j-i}, \mathcal{A}\Phi_m^j)| \le c\, ||\Phi_m^i||\ ||\Phi_m^{j-i}||^{1/2}\ |\mathcal{A}\Phi_m^{j-i}|^{1/2}\ |\mathcal{A}\Phi_m^j| \quad ;$$

we obtain from the induction hypothesis :

(4.32) $$|\mathcal{B}_0(\Phi_m^i, \Phi_m^{j-i}, \mathcal{A}\Phi_m^j)| \le c(K_0^j)^{3/2}\ |\mathcal{A}\Phi_m^{j-i}|^{1/2}\ |\mathcal{A}\Phi_m^j| \quad \text{for } 1 \le i \le j-1$$

$$\le \text{(by Young's inequality)}$$

$$\le \varepsilon\ |\mathcal{A}\Phi_m^j|^2 + c_\varepsilon (K_0^j)^{3/2}\ |\mathcal{A}\Phi_m^{j-i}|.$$

For i = 0, (4.31) becomes :

(4.33) $$|\mathcal{B}_0(\Phi_m, \Phi_m^j, \mathcal{A}\Phi_m^j)| \le c\, ||\Phi_m||\ ||\Phi_m^j||^{1/2}\ |\mathcal{A}\Phi_m^j|^{3/2}$$

$$\le \text{(by Young's inequality and (4.18))}$$

$$\le \varepsilon\ |\mathcal{A}\Phi_m^j|^2 + c_\varepsilon\ K_0\ ||\Phi_m^j||^2.$$

For i = j, (4.31) yields :

(4.34) $$|\mathcal{B}_0(\Phi_m^j, \Phi_m, \mathcal{A}\Phi_m^j)| \le c\, ||\Phi_m^j||\ ||\Phi_m||^{1/2}\ |\mathcal{A}\Phi_m|^{1/2}\ |\mathcal{A}\Phi_m^j|$$

$$\le \text{(by interpolation)}$$

$$\le c\, ||\Phi_m||^{1/2}\ |\mathcal{A}\Phi_m|^{1/2}\ |\Phi_m^j|^{1/2}\ |\mathcal{A}\Phi_m^j|^{3/2}$$

$$\le \text{(by (4.17))}$$

$$\le \varepsilon\ |\mathcal{A}\Phi_m^j|^2 + c_\varepsilon\ K_0^{1/2}(K_1^j)^{1/2}\ |\mathcal{A}\Phi_m|^2.$$

Combining the inequalities (4.30) to (4.34), we obtain :

$$\frac{d}{dt} ||\Phi_m^j||^2 + |\mathcal{A}\Phi_m^j|^2 \le K\,(1 + |\mathcal{A}\Phi_m|^2 + \sum_{i=1}^{j-1} |\mathcal{A}\Phi_m^i| + ||\Phi_m^j||^2) \tag{4.35}$$

where K depends on $K_o$, $K_1^j$, $K_o^j$.

By useof lemma 4.1, we obtain $(4.18)_j$ with $K_3^j$, $K_1^j$ depending on $K_o^j$, $K_1^j$, $K_2^j$.

Finally, by use of (4.22), we observe that :

$$|\beta_m^{j-1}| \le c \sum_{i=0}^{j-1} ||\Phi_m^i||\ ||\Phi_m^{j-1-i}||^{1/2}\ |\mathcal{A}\Phi_m^{j-1-i}|^{1/2} \tag{4.36}$$

which yields :

$$|\beta_m^{j-1}| \le c(K_o^j)^{3/2}\ |\mathcal{A}\Phi_m^{j-1}|^{1/2} \tag{4.37}$$

By making use of (4.23) for j-1 and (4.17) for j, we get :

$$|\mathcal{A}\Phi_m^{j-1}| \le K(1 + |\mathcal{A}\Phi_m^{j-1}|^{1/2}), \tag{4.38}$$

which gives the first estimate in $(4.19)_j$ with $K_5^j$ depending on $K_o^j$ and $K_1^j$. The same kind of estimate for j instead of j-1 gives :

$$|\Phi_m^{j+1}| \le K\,(1 + |\mathcal{A}\Phi_m^j|^{1/2} + |\mathcal{A}\Phi_m^j|) \tag{4.39}$$

which gives the second estimate in $(4.19)_j$ with $K_6^j$ depending on $K_o^j$ and $K_4^j$!

iii) Passage to the limit.

From (3.28) and from the uniqueness of a strong solution (cf. theorem 3.1), we have :

$$\Phi_m \to \Phi \quad \text{in} \quad L^2(t_o, t_o+\eta\ ;\ V) \quad \text{weakly.} \tag{4.40}$$

We get from $(4.19)_j$, $1 \le j \le j_o$, and $(4.18)_{j_o}$ that :

$$\left\{\begin{array}{ll} |\mathcal{A}\Phi_m^{(j)}(t)| \le K_5^j & \forall\, t \in [t_o+\alpha_o, t_o+\eta],\ \ o \le j \le j_o-1, \\ \text{and} & \\ ||\Phi_m^{(jo)}(t)|| \le K_3^{jo} & \forall\, t \in [t_o+\alpha_o, t_o+\eta]. \end{array}\right. \tag{4.41}$$

Since the unit ball in $L^\infty(t_o+\alpha_o, t_o+\eta ; H)$ (resp. in $L^\infty(t_o+\alpha_o, t_o+\eta ; V)$) is weakly star closed, we obtain :

$$(4.42) \quad \begin{cases} |\mathcal{A}\Phi^{(j)}(t)| \le K_5^j & \forall t \in [t_o+\alpha_o, t_o+\eta], \quad o \le j \le j_o-1 \\ \text{and} & \\ ||\Phi^{(jo)}(t)|| \le K_3^{jo} & \forall t \in [t_o+\alpha_o, t_o+\eta], \end{cases}$$

which gives (4.3) since $K_3^j$ and $K_5^j$ are independent of $t_o$. ∎

Remark 4.1. : We can pass to the limit in (4.23), (4.29), (4.35). We then conclude that $\Phi^j$ satisfies the following equation :

$$(4.43) \qquad \frac{d}{dt}\Phi^j + \mathcal{A}\Phi^j = F^j - \sum_{i=0}^{j} C_j^i \mathcal{B}(\Phi^i, \Phi^{j-i})$$

and the following differential inequalities

$$(4.44) \qquad \frac{d}{dt}[\Phi^j]^2 + [\![ \Phi^j ]\!]^2 \le K\,(1 + [\Phi^j]^2)$$

$$(4.45) \qquad \frac{d}{dt}||\Phi^j||^2 + |\mathcal{A}\Phi^j|^2 \le K(1 + ||\Phi^j||^2)$$

where K depends on the data $\Omega$, S, Re, Rm, f, $\Phi_o$ (and on $K_o$ in the case N = 3).

Remark 4.2. : When T is finite, the assumptions that $\Phi \in L^\infty(0, T ; V)$ means that singularities in the sense of J. LERAY [18][19][20] do not develop in the flow. When T = +∞ and the driving force f satisfies a mild regularity assumptions which is easily verified when f is independent of time, then it is shown in [14], that the assumption (4.4) is equivalent to the fact that singularities can not develop in a finite time : if the norm of $\Phi(t)$ in V becomes infinite for large time, $\Phi$ being a weak solution, then there exists another weak solution (corresponding to another initial value $\Phi_o$ and an appropriate driving force) which develops singularities on a finite interval of time. Thus for such forces the assumption (4.4) fails to be true only if the analogue for MHD of the conjecture of LERAY for the Navier-Stokes equations is verified.

Theorem 4.2. : Let be given f and $\Phi_o$ satisfying

(4.46) $\Phi_o \in H,$

(4.47) $f^{(j)} \in L^\infty(0,T\ ;\ H_1 \cap \mathbb{H}^k) \quad \forall j,\ k \geq 0,$

where $0 < T \leq +\infty$.
Then, in the case $N = 2$, the solution $\Phi$ of problem 2.1 satisfies :

(4.48) $\Phi^{(j)} \in L^\infty(\alpha,T\ ;\ V \cap \mathbb{H}^k) \quad \forall j,\ k \geq 0\ , \quad \forall\ \alpha > 0.$

In the case $N = 3$, if the strong solution $\Phi$ of problem 2.1 satisfies :

(4.49) $\Phi \in L^\infty(0,T\ ;\ V)$

then $\Phi$ satisfies (4.48) as well.

Proof : For $k = 2$, the result is a consequence of theorem 4.1 and of the imbedding $D(\mathcal{A}) \subset \mathbb{H}^2(\Omega)$. Let us assume that (4.48) is true for some $k \geq 2$. Using the same methods as in R. TEMAM [26], we obtain that :

(4.50) $\mathcal{B}$ is continuous from $(V \cap \mathbb{H}^{m+1}(\Omega))^2 \to H \cap \mathbb{H}^m(\Omega)$ for $m \geq 1$.

On the other hand, as we recalled in sec. 2,

(4.51) $\mathcal{A}^{-1}$ is continuous from $H \cap \mathbb{H}^m(\Omega) \to V \cap \mathbb{H}^{m+2}(\Omega)$ for $m \geq 0$.

We then rewrite (4.43) as :

(4.52) $\Phi^j = \mathcal{A}^{-1}\left[- \Phi^{j+1} + F^j - \sum_{i=0}^{j} C_j^i\, \mathcal{B}(\Phi^i, \Phi^{j-i})\right]$

and we obtain from (4.50) (4.51) that $\Phi^j \in L^\infty(\alpha,\ T\ ;\mathbb{H}^{k+1}(\Omega))$ for every $j \geq 0$. ∎

Remark 4.3 : By using the conclusions of the theorems 4.1, 4.2 and the interpolation results of J.L. LIONS - E. MAGENES [21],one can complete (4.3) as follows : for every $\alpha$, $0 < \alpha \leq T$, $\Phi^{(j)} \in \mathcal{C}([\alpha,T]\ ;\ D(\mathcal{A}))$, $j = 0,\dots,j_o-1$ and $\Phi^{(j_o)} \in \mathcal{C}([\alpha,T]\ ;\ V)$. Thus :

$$\Phi^{(j)} \in \mathcal{C}((0,T] \; ; \; D(\mathcal{A})) \; , \; j = 0,\ldots,j_o-1,$$
$$\Phi^{(j_o)} \in \mathcal{C}((0,T] \; ; \; V).$$

4.2. Application to functional invariant sets.

Throughout this section, we assume that $f(t) = f$ is independent of $t$ and belongs to $H_1$.

Let $S(t)$ be the semi-group associated to the strong solution of Problem 2.1, i.e., for $\Phi_o \in V$ and $t > 0$, $S(t)\,\Phi_o = \Phi(t) \in V$ where $\Phi \in \mathcal{C}([0,t] \; ; \; V)$ is the solution of Problem 2.1 (if such a solution exists).

Definition : A functional invariant set for Problem 2.1 is a subset $X$ of $V$ which satisfies the following properties :

(i) for every $\Phi_o \in X$, Problem 2.1 has a strong solution on $[0,\infty)$.

(ii) $S(t)\,X = X$, $\forall t > 0$.

We recall that an attractor (in $V$ or $H$) is a functional invariant set $X$ which satisfies furthermore the condition :

(iii) $X$ possesses an open neighborhood $\omega$ (in $V$ or $H$), and for every $u_o \in \omega$, $S(t)\,u_o$ tends to $X$, in $V$ or $H$, as $t \to \infty$.

Examples : These are two trivial examples of functional invariant sets :

(i) If $\Phi$ is a stationnary solution, $\{\Phi\}$ is a functional invariant set,

(ii) If $\Phi$ is a periodic solution, of period $T$, then the set :

$$X = \{\Phi(t), \quad o \le t \le T\}$$

is functional invariant. Of course functional invariant sets may be much more complicated and may correspond to much less smooth behavior. ∎

Concerning functional invariant sets for the MHD, we have the following regularity result.

Theorem 4.3 : For $N = 2$ or $3$, let us assume that $f \in (\mathcal{C}^\infty(\bar{\Omega}))^N \cap H_1$. Then any functional invariant set for Problem 2.1 is contained in $(\mathcal{C}^\infty(\bar{\Omega}))^{2N}$.

Proof : If $\Phi_1 \in V$ is contained in some functional invariant set X, then, for every $t_1 > 0$, there exists $\Phi_0 \in V$ and $\Phi \in \mathcal{C}([0,t_1] ; V)$ solution of Problem 2.1 with $\Phi(t_1) = \Phi_1$. By the interpolation of J.L. LIONS, E. MAGENES [21] (see also R. TEMAM [25]), we infer from theorem 4.2 that :

$$(4.53) \qquad \Phi \in \mathcal{C}([t_1/2,t_1]; \mathbb{H}^k(\Omega)) \quad \forall k \geq 0 ;$$

hence :

$$(4.51) \qquad \Phi_1 \in \mathbb{H}^k(\Omega) \quad \forall k \geq 0.$$

Using now the Sobolev injection property ($N = 2$ or $3$) :

$$H^2(\Omega) \subset \mathcal{C}^0(\bar{\Omega}),$$

we obtain $\Phi_1 \in (\mathcal{C}^\infty(\bar{\Omega}))^{2N}$. ∎

Remark 4.4.i) : Of course if we assume partial regularity results for f, we can obtain partial regularity results for the corresponding functional invariant sets.

ii) : One can show exactly like for Lemma 12.1 in [26] that if $\Phi$ is a strong solution to the MHD equations for $t \geq 0$ ($N = 2$ or $N = 3$ and (4.4) is satisfied) then there exists a functional invariant set X bounded in V and the distance in H of $\Phi(t)$ to X tends to 0 as $t \to \infty$. For such a set X, the properties established in Theorem 4.3 above and in § 5 apply ($f(t) \equiv f \in H$ is assumed to be independent of time).

## 5. - Hausdorff dimension of a functional invariant set

Our aim in this section is to show that a functional invariant set (which can be in particular an attractor, see § 4 for the precise definition), has a finite Hausdorff dimension. In Section 5.1 we describe a property of the trajectories which is interesting by itself and which will be used in the proof of the main result : this is the so-called squeezing property, which shows that, the trajectories of the fluid are squeezeed in a small neighborhood of a finite dimensional like manifold. In Section 5.2 we recall a few facts on the Hausdorff measure and dimension and we then state and prove the main result.

5.1. The squeezing property.

We assume that N = 2 or 3. Let $\Phi_o$ and $\Psi_o$ be given in V with $||\Phi_o|| \le R$, $||\Psi_o|| \le R$, and let f be given satisfying :

(5.1) f and f' are bounded from $[0,\infty)$ into $H_1$.

Let $\Phi$ and $\Psi$ be two strong solutions of Problem 2.1, corresponding respectively to $(\Phi_o,f)$ and to $(\Psi_o,f)$, defined and bounded in V on some finite interval [0,T].

We recall that $w_m$, $\lambda_m$ denote the eigenfunctions and eigenvalues of $\mathcal{A}$ and that $P_m$ is the projector in H (or V, V', $D(\mathcal{A})$) onto $W_m$.

We will prove elsewhere the following result (see C. FOIAS - R. TEMAM [13][14] and R. TEMAM [26] for a similar situation in the Navier-Stokes equations).

Theorem 5.1 : Under the above hypotheses, for every $\alpha > 0$, there exist two positive constants $\beta_1$, $\beta_2$ which depend only on $\Omega$, T, S, Re, Rm, R, $\alpha$ and the norms of f and f' in $L^\infty(0,T ; H_1)$ and such that for every t, $\alpha \le t \le T$, and for every m sufficiently large, i.e. satisfying :

(5.2) $$\lambda_{m+1} \ge \beta_1$$

we have, either :

(5.3) $$|\Phi(t) - \Psi(t)| \le \sqrt{2}\,|P_m(\Phi(t) - \Psi(t))|,$$

or :

(5.4) $$|\Phi(t) - \Psi(t)| \le e^{-\beta_2 \lambda_{m+1}^{1/2}} |\Phi_o - \Psi_o|.$$

5.2. Hausdorff dimension of a functional invariant set.

For a metric space X and a real number $\gamma > 0$, let us recall the definition of the $\gamma$ dimensional Hausdorff measure (cf. H. FEDERER [9]).

At first, we define a constant $\alpha(\gamma)$ which, when $\gamma$ is an integer, is the volume in $R^\gamma$ of a ball of radius 1/2. More generally, we put :

(5.5) $$\alpha(\gamma) = 2^{-\gamma}\, \Gamma(\tfrac{1}{2})^\gamma \,/\, \Gamma(\tfrac{\gamma}{2} + 1).$$

The $\gamma$-dimensional measure of a subset Y of X is :

(5.6) $$\mu_\gamma(Y) = \lim_{\delta\to 0} \mu_{\gamma,\delta}(Y)$$

where :

(5.7) $$\mu_{\gamma,\delta}(Y) = \alpha(\gamma) \text{ Inf } \sum_{i=1}^{\infty} (\text{diam } G_i)^\gamma,$$

the infimum being taken over all the countable coverings of Y by subsets $G_i$ in X such that diam $G_i \le \delta$.

If $\mu_\gamma(Y)$ is finite for some $\gamma \in \mathbb{R}_+$, we say that Y has a finite Hausdorff dimension and we call Hausdorff dimension of Y the number :

$$\text{Inf } \{\gamma \in \mathbb{R}_+,\ \mu_\gamma(Y) < \infty\}.$$

It is known that a set with finite Hausdorff dimension is homeomorphic to a subset of a finite dimensional euclidian space.

As in Section 4.2, we consider the Problem 2.1 associated with $f(t) = f \in H_1$ independent of t.

<u>Theorem 5.2</u> (N = 2 or 3) : <u>Every functional invariant set for Problem 2.1 which is bounded in V has a finite Hausdorff dimension in H.</u>

<u>Proof</u> : We are going to make use of the squeezing property. Let R > 0 be the radius of a ball in V which contains X. For some fixed t > 0, theorem 5.1. shows that there exist $0 < \eta < 1$, $c_1 > 0$, $m \in \mathbb{N}$ such that (S = S(t)) :

(5.8) $$|S\varphi - S\psi| \le \eta\ |\varphi - \psi| + c_1\ |P_m(S\varphi - S\psi)| \quad \forall \varphi,\psi \quad ||\varphi||,||\psi|| \le R\ ;$$

there exists also $c_2 > 0$ (cf. lemma 11.2 in R. TEMAM [26]) such that :

(5.9) $$|S\varphi - S\psi| \le c_2\ |\varphi - \psi| \qquad \forall \varphi,\psi \qquad ||\varphi||,\ ||\psi|| \le R.$$

Let there be given D, $\delta > 0$. We consider a countable covering of X by subsets $G_i$ of X such that diam $G_i \le \delta$. Since X is functional invariant, we have :

(5.10) $$X \subset \bigcup_{i=1}^{\infty} S(G_i \cap X).$$

It follows from (5.8), (5.9) that, for $\varphi,\psi \in S(G_i \cap X)$,

$$(5.11) \qquad |\varphi - \psi| \le \eta\,\delta + c_1\,|P_m(\varphi - \psi)|,$$

$$(5.12) \qquad |\varphi - \psi| \le c_2\,\delta.$$

We now fix i and we define a reiterated covering of $S(G_i \cap X)$. From (5.12), the set $P_m(S(G_i \cap X))$ is contained in a ball of $P_m H$ of radius $2c_2\,\delta$. Let $\tilde{G}_{i,1},\ldots,\tilde{G}_{i,p}$ be a covering of $P_m(S(G_i \cap X))$ by balls of $P_m H$ of radius $r_1 = \frac{1-\eta}{4c_1}\delta$ ; the number p can be majorized by

$$(5.13) \qquad p \le \ell_m\left(\frac{r_1}{2c_2\,\delta}\right) = \ell_m\left(\frac{1-\eta}{8c_1c_2}\right),$$

where $\ell_m(\sigma)$ is the minimum number of balls of radius $\le \sigma$ which is necessary to cover a ball of radius 1 in $\mathbb{R}^m$. We now define a covering $\{G_{i,k}\}$ of $S(G_i \cap X)$ as follows :

$$(5.14) \qquad G_{i,k} = S(G_i \cap X) \cap P_m^{-1}(\tilde{G}_{i,k}).$$

Thanks to (5.11) and to the choice of $r_1$, we have :

$$(5.15) \qquad \operatorname{diam}(G_{i,k}) \le \eta\,\delta + 2\,c_1\,r_1 = \delta_1$$

with $\delta_1 = \left(\frac{1+\eta}{2}\right)\delta < \delta$.

From this new covering of X, we deduce that :

$$\mu_{D,\delta_1}(X) \le \sum_{i,j} (\operatorname{diam} G_{ij})^D \le p\left(\frac{1+\eta}{2}\right)^D \sum_{i=1}^{\infty} (\operatorname{diam} G_i)^D.$$

Since the covering of X by $\{G_i\}_{i=1}^{\infty}$ with $\operatorname{diam}(G_i) \le \delta$ is arbitrary, we obtain :

$$(5.16) \qquad \mu_{D,\delta_1}(X) \le p\left(\frac{1+\eta}{2}\right)^D \mu_{D,\delta}(X) \quad \text{with } \delta_1 < \delta.$$

If $p.\left(\frac{1+\eta}{2}\right)^D < 1$, i.e. $D > D_o$, with $D_o$ given by :

$$(5.17) \qquad D_o = \frac{\operatorname{Log}\left\{\ell_m\left(\frac{1-\eta}{8c_1c_2}\right)\right\}}{\left|\operatorname{Log}\left(\frac{1+\eta}{2}\right)\right|},$$

we use the fact that the function $\delta \to \mu_{D,\delta}(X)$ is non increasing and we obtain that either $\mu_{D,\delta}(X) = 0$ or $\mu_{D,\delta}(X) = +\infty$. Since X is compact in H, we can cover it by a finite family of balls of H of radius $\leq \frac{\delta}{2}$. Thus, $\mu_{D,\delta}(X)$ is finite and we have $\mu_{D,\delta}(X) = 0$ for $D > D_o$. Since $\delta$ is arbitrary, we obtain $\mu_D(X) = 0$ for $D > D_o$. Hence, X has a finite Hausdorff dimension $\gamma$ in H and $\gamma \leq D_o$, where $D_o$ is given by (5.17). ∎

6. - Determination of solutions by a set of nodal values.

In this section, we show that if the large time behavior of the solution to the MHD equations is known on a set of points in the domain which is sufficiently dense but finite, then the large time behavior of the solution itself is totally determined. The same property was previously obtained by C. FOIAS-R. TEMAM [11] for the Navier-Stokes equations.

At first, let us introduce some new notations. To any given finite set of points in $\Omega$, $\mathcal{E}$, we associate the number :

$$d_{\mathcal{E}} = \sup_{x \in \Omega} (\min_{y \in \mathcal{E}} |x-y|),$$

which "measures" the density of the set $\mathcal{E}$ in $\Omega$ ; for every function $w \in \mathcal{C}(\bar{\Omega})$, we also introduce :

$$\eta_{\mathcal{E}}(w) = \max_{x \in \mathcal{E}} |w(x)|.$$

Let us now state the main result of this section.

The dimension of the space is N = 2 or 3 ; we are given two forces f, g which are bounded for all time in the $H_1$-norm (= $\mathbb{L}^2$ norm), together with their first time derivatives ($f, g, f', g' \in L^\infty(o,\infty ; H_1)$). We assume that :

$$(6.1) \qquad f(t) - g(t) \to 0 \text{ in } H \text{ as } t \to \infty.$$

We denote $\Phi$ and $\Psi$ the strong solutions of the initial value problems for the MHD equations :

$$(6.2) \qquad \begin{cases} \dfrac{d\Phi}{dt} + \mathcal{A}\Phi + \mathcal{B}(\Phi,\Phi) = F \\ \Phi(o) = \Phi_o \end{cases}$$

$$(6.3)\quad \begin{cases} \dfrac{d\Psi}{dt} + \mathcal{A}\,\Psi + \mathcal{B}(\Psi,\Psi) = G \\ \Psi(o) = \Psi_o \end{cases}$$

where $F = (f,o)$, $G = (g,o)$ and where $\Phi_o$, $\Psi_o$ are given in V.

In the case $N = 3$, we suppose that $\Phi$ and $\Psi$ are in $L^\infty(o,\infty\ ;\ V)$ while this property is automatically satisfied if $N = 2$. Because of the assumptions made on f and g, it follows from theorem 4.1 and remark 4.3 that $\Phi$ is continuous from $(o,\infty)$ into $D(\mathcal{A})$ ; $\Phi$ is (at least) a continuous functions on $(o,\infty) \times \bar{\Omega}$. We are also given a finite subset $\mathcal{E}$ of $\Omega$.

Theorem 6.1 : The assumptions are these above. We assume also that, as $t \to \infty$ :

$$(6.4)\qquad \Phi(x,t) - \Psi(x,t) \to 0 \qquad \forall x \in \mathcal{E}^{(1)}.$$

Then there exists a constant $\beta$ which depends only on the data i.e. $\Omega$, S, Re, Rm and the norms in $L^\infty(o,\infty\ ;\ H_1)$ of f,g and f',g' such that if :

$$(6.5)\qquad d_{\mathcal{E}} \leq \beta$$

then, as $t \to \infty$,

$$(6.6)\qquad \Phi(t) - \Psi(t) \to 0$$

in the norm of V and in the norm of uniform convergence in $\bar{\Omega}$.

The proof follows the following lemma.

Lemma 6.1 : There exist constants $c_j$ depending only on $\Omega$ such that, for every finite subset $\mathcal{E}$ of $\Omega$ and every $w \in H^2(\Omega)$, we have :

$$(6.7)\qquad \sup_{x\in\bar{\Omega}} |w(x)| \leq \eta_{\mathcal{E}}(w) + c_1\, d_{\mathcal{E}}^{1/2}\, |w|_{H^2},$$

---

(1) $\Phi(x,t)$, $\Psi(x,t)$ make sense since, as we pointed out before, $\Phi$ and $\Psi$ are continuous functions on $(o,\infty) \times \bar{\Omega}$ .

(6.8) $$|w|_{L^2} \leq c_2 \ \eta_{\mathcal{E}}(w) + c_3 \ d_{\mathcal{E}}^{1/2} \ |w|_{H^2},$$

(6.9) $$|w|_{H^1} \leq c_4 \ \eta_{\mathcal{E}}(w)^{1/2} \ |w|_{H^2}^{1/2} + c_5 \ d_{\mathcal{E}}^{1/4} \ |w|_{H^2}.$$

For the convenience of the reader, we recall the proof of lemma 6.1 given in C. FOIAS - R. TEMAM [11]. We know that for $N \leq 3$, the Sobolev space $H^2(\Omega)$ is continuously imbedded in the space $\mathcal{C}^{1/2}(\bar{\Omega})$ of functions which are Hölder continuous on $\bar{\Omega}$ with exponent 1/2. Hence there exists a constant c which only depends on $\Omega$ and such that :

$$|w(x)-w(y)| \leq c|x-y|^{1/2}|w|_{H^2} \ , \forall x,y \in \bar{\Omega}, \forall w \in H^2(\Omega).$$

and thus :

(6.10) $$|w(x)| \leq |w(y)| + c \ |x-y|^{1/2}|w|_{H^2}, \ \forall x,y \in \bar{\Omega}, \ \forall w \in H^2(\Omega).$$

For every $x \in \bar{\Omega}$, there exists $y \in \mathcal{E}$ such that $|x-y| \leq d_{\mathcal{E}}$ and (6.7) follows. In order to obtain (6.8), we just write $|w|_{L^2} \leq \text{meas}(\Omega)^{1/2} \ |w|_{L^\infty}$. For (6.9), we use an interpolation inequality :

(6.11) $$|w|_{H^1} \leq c(\Omega) \ |w|_{L^2}^{1/2} \ |w|_{H^2}^{1/2}.$$

Together with (6.8) and by using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, this inequality implies (6.9). ∎

We now give the :

<u>Proof of theorem 6.1</u> : Let $w = \Phi - \Psi$ ; substracting (6.3) from (6.2), we get :

(6.12) $$\frac{dw}{dt} + \mathcal{A} w + \mathcal{B}(\Phi,w) + \mathcal{B}(w,\Psi) = F - G.$$

We take the scalar product of this equation with $\mathcal{A}w$ and we get :

$$(6.13)\quad \frac{1}{2}\frac{d}{dt}||w||^2 + |\mathcal{A}w|^2 = -\mathcal{B}_o(\Phi,w,\mathcal{A}w) - \mathcal{B}_o(w,\Psi,\mathcal{A}w) + (F-G,\mathcal{A}w)$$

$$\leq \text{(with (2.13) for } m_1=2,\ m_2=m_3=0 \text{ and } m_1=m_2=1,\ m_2=0)$$

$$\leq c_6\ (|\mathcal{A}\Phi| + |\mathcal{A}\Psi|)\ ||w||\ |\mathcal{A}w| + |f-g|\ |\mathcal{A}w|$$

$$\leq c_6\ \alpha_1\ ||w||\ |\mathcal{A}w| + |f-g|\ |\mathcal{A}w| \quad .$$

where $\alpha_1$ depends on $|f|_{L^\infty(o,\ \infty;\ H_1)}$, $|f'|_{L^\infty(o,\ \infty;\ H_1)}$, $|g|_{L^\infty(o,\ \infty;\ H_1)}$, $|g'|_{L^\infty(o,\infty\ ;\ H_1)}$ (cf. theorem 4.1).

We now use inequality (6.9) :

$$||w||\ |\mathcal{A}w| \leq c_4\ \eta_{\mathcal{E}}^{1/2}(w)\ |\mathcal{A}w|^{3/2} + c_5\ d_{\mathcal{E}}^{1/4}\ |\mathcal{A}w|^2$$

$$\leq \text{(by Young inequality)}$$

$$\leq \left(\frac{1}{2c_6\alpha_1} + c_5\ d_{\mathcal{E}}^{1/4}\right)\ |\mathcal{A}w|^2 + \alpha_1'\ \eta_{\mathcal{E}}^2(w)\ ;$$

we write also :

$$|f-g|\ |\mathcal{A}w| \leq \frac{1}{2}\ |\mathcal{A}w|^2 + \frac{1}{2}\ |f-g|^2$$

and we finally obtain from (7.13) :

$$(6.14)\quad \frac{d}{dt}||w||^2 + (1 - c_5\ c_6\ \alpha_1\ d_{\mathcal{E}}^{1/4})\ |\mathcal{A}w|^2 \leq \frac{1}{2}\ |f-g|^2 + \alpha_1'\ \eta_{\mathcal{E}}^2(w)\ .$$

If :

$$(6.15)\quad d_{\mathcal{E}} \leq \beta = \left(\frac{1}{2c_5c_6\alpha_1}\right)^4,$$

we obtain :

$$\frac{d}{dt}||w||^2 + \frac{1}{2}\ |\mathcal{A}w|^2 \leq h$$

with $h = h(t)$ = the right-hand side of (6.14).

Since the injection of $H^2(\Omega)$ into $H^1(\Omega)$ is continuous, there exists a constant $c_7$ which depends on $\Omega$, S, Re, Rm, such that :

$$(6.16)\quad ||w||^2 \leq c_7\ \ |\mathcal{A}w|^2,\ \forall w \in D(\mathcal{A})\ \ ;$$

therefore :

$$(6.17) \qquad \frac{d}{dt} ||w||^2 + \frac{1}{2c_7} ||w||^2 \leq h.$$

By the assumptions (6.1), (6.4), h(t) goes to 0 as t converges to infinity. Hence, for every $\varepsilon > 0$, there exists $t_o = t_o(\varepsilon)$ such that $|h(t)| \leq \varepsilon$ for $t \geq t_o(\varepsilon)$. By the Gronwall lemma we then obtain for $t \geq t_o$ :

$$||w(t)||^2 \leq ||w(t_o)||^2 \exp\left(-\frac{t-t_o}{2c_7}\right) + \frac{\varepsilon}{2c_7} .$$

We conclude that, as $t \to \infty$,

$$w(t) \to 0 \text{ in } V.$$

The families $\Phi(t)$ and $\Psi(t)$ being bounded in $H^2(\Omega)$, so is w(t). Since $H^{7/4}(\Omega)$ is continuously imbedded in $\mathcal{C}(\bar{\Omega})$ and the injection of $H^2(\Omega)$ into $H^{7/4}(\Omega)$ is compact, the family w(t) is relatively compact in $\mathcal{C}(\bar{\Omega})^{2N}$ ; since w(t) converges in V to 0, as $t \to \infty$, the convergence is also uniform. The proof of theorem 6.1. is complete. ∎

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