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Some Matrix Power and Karcher Means Inequalities Involving Positive Linear Maps

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Abstract. In this paper, we generalize some matrix inequalities involving the matrix power means and Karcher mean of positive definite matrices. Among other inequalities, it is shown that if $\mathbb{A} = (A_1, \cdots, A_n)$ is an n-tuple of positive definite matrices such that $0 < m \leq A_i \leq M$ $(i = 1, \cdots, n)$ for some scalars $m < M$ and $\omega = (w_1, \cdots, w_n)$ is a weight vector with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, then

$$\Phi^p\Big(\sum_{i=1}^{n} w_i A_i\Big) \leq \alpha^p \Phi^p(P_t(\omega; \mathbb{A})) \qquad \text{and} \qquad \Phi^p\Big(\sum_{i=1}^{n} w_i A_i\Big) \leq \alpha^p \Phi^p(\Lambda(\omega; \mathbb{A})),$$

where $p > 0$, $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\right\}$, Φ is a positive unital linear map and $t \in [-1, 1]\backslash\{0\}$.

1. Introduction and preliminaries

Let $\mathcal{M}_k$ be the C^*-algebra of all $k \times k$ complex matrices with the identity I, and $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{C}^k$. For Hermitian matrices $A, B \in \mathcal{M}_k$, we write $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geq B$ if $A - B \geq 0$. If m, M are real scalars, then we mean $m \leq A \leq M$ whenever $mI \leq A \leq MI$.

The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a Hermitian matrix A and the C^*-algebra generated by A and I. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$. A linear map Φ on $\mathcal{M}_k$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. Let $A, B \in \mathcal{M}_k$ be two positive definite and $t \in [0, 1]$. The operator t-weighted arithmetic, geometric, and harmonic means of A, B are defined by $A \nabla_t B = (1 - t)A + tB$, $A \sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ and $A !_t B = ((1 - t)A^{-1} + tB^{-1})^{-1}$, respectively, in which $A !_t B \leq A \sharp_t B \leq A \nabla_t B$. In particular, for $t = \frac{1}{2}$ we get the usual operator arithmetic mean ∇, the geometric mean $\sharp$ and the harmonic mean $!$.

Throughout the paper, let $\mathbb{A} = (A_1, \cdots, A_n)$ be an n-tuple of positive definite matrices A_i $(i = 1, \cdots, n)$ and $\omega = (w_1, \cdots, w_n)$ be a positive probability weight vector (we simply write the weight vector), where $w_i \geq 0$ $(i = 1, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. In [15], Lim and Palfia introduced matrix power mean of positive

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definite matrices of some fixed dimension. The matrix power mean $P_t(\omega; \mathbb{A})$ is defined to be the unique positive definite solution of the non-linear equation:

$$X = \sum_{i=1}^{n} w_i(X\sharp_t A_i), \ \ t \in (0, 1].$$

For $t \in [-1, 0)$, it is defined by $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1}, \cdots, A_n^{-1})$. For further information we refer the reader to [14, 20] and references therein. We denote by $P_1(\omega; \mathbb{A}) = \sum_{i=1}^{n} w_iA_i$ and $P_{-1}(\omega; \mathbb{A}) = (\sum_{i=1}^{n} w_iA_i^{-1})^{-1}$, the weighted arithmetic and harmonic means of $A_1, \cdots, A_n$, respectively.
There is one of the important properties of matrix power mean $P_t(\omega; \mathbb{A})$, that $P_t(\omega; \mathbb{A})$ interpolates between the weighted harmonic and arithmetic means:

$$\left(\sum_{i=1}^{n} w_iA_i^{-1}\right)^{-1} \leq P_t(\omega; \mathbb{A}) \leq \sum_{i=1}^{n} w_iA_i \tag{1}$$

for all $t \in [-1, 1]\backslash\{0\}$. The Karcher mean of n positive definite matrices $A_1, \cdots, A_n$ is defined as the unique minimizer of the sum of squares of the Riemannian trace metric distances to each of the A_i, i.e., $\Lambda(\omega; A_1, \cdots, A_n) = \arg\min_{X\in\mathbb{P}} \sum_{i=1}^{n} w_i\delta^2(X, A_i)$ (Recall that the trace metric distance between two positive definite matrices is given by $\delta(A, B) = (\sum_{i=1}^{n} \log(\lambda_i(A^{-1}B))^{1/2}$, where $\lambda_i(X)$ denotes the i-th eigenvalue of X in ascending order.). In fact, the Karcher mean coincides with the unique positive definite solution of the Karcher equation:

$$\sum_{i=1}^{n} w_i \log\left(X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}}\right) = 0. \tag{2}$$

The Karcher mean satisfies from (2) that $\Lambda(\omega; \mathbb{A}^{-1})^{-1} = \Lambda(\omega; \mathbb{A})$. It is well known that (see [15])

$$\lim_{t\to 0} P_t(\omega; \mathbb{A}) = \Lambda(\omega; \mathbb{A}) \tag{3}$$

and

$$\left(\sum_{i=1}^{n} w_iA_i^{-1}\right)^{-1} \leq \Lambda(\omega; \mathbb{A}) \leq \sum_{i=1}^{n} w_iA_i.$$

For further information about the matrix power mean, Karcher mean, operator mean and their properties, we refer the readers to [4, 5, 14–16] and references therein.
It is well known that for two positive definite matrices A and B, if $A \geq B$, then

$$A^p \geq B^p \quad (0 \leq p \leq 1). \tag{4}$$

In general (4) is not true for $p > 1$. Let Φ be a unital positive linear map. The following inequality is known as Choi's inequality(see [7, 12]):

$$\Phi(A)^{-1} \leq \Phi(A^{-1}). \tag{5}$$

Marshal and Olkin [19] proved a counterpart of Choi's inequality (5) as follows:

$$\Phi(A^{-1}) \leq \frac{(M + m)^2}{4Mm} \Phi(A)^{-1} \tag{6}$$

for positive definite A with $0 < m \leq A \leq M$. In addition, Lin [17] and Fu [9] improved inequality (6) for $p \geq 2$ to the form $\Phi^p(A^{-1}) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right)^p \Phi(A)^{-p}$.
The matrix power means satisfy the following inequality(see [8, 15]): For each $t \in (0, 1]$

$$\Phi(P_t(\omega; \mathbb{A})) \leq P_t(\omega; \Phi(\mathbb{A})), \tag{7}$$

where Φ is a unital positive linear map, $\mathbb{A} = (A_1, \cdots, A_n)$ is a n-tuple of positive definite matrices and $\Phi(\mathbb{A}) = (\Phi(A_1), \cdots, \Phi(A_n))$. Ando [1] proved that if Φ is a positive linear map, then for positive definite matrices $A, B \in \mathcal{M}_k$, we have

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B). \tag{8}$$

A reverse of Ando's inequality (8) states that [12, Remark 5.3]: If $A, B \in \mathcal{M}_k$ and $0 < m \leq A, B \leq M$, then

$$\Phi(A)\sharp\Phi(B) \leq \frac{M+m}{2\sqrt{Mm}}\Phi(A\sharp B).$$

By inequality (4) we get

$$(\Phi(A)\sharp\Phi(B))^p \leq \Big(\frac{M+m}{2\sqrt{Mm}}\Big)^p \Phi^p(A\sharp B), \quad (0 < p \leq 1). \tag{9}$$

In [10], Fujii et al. obtained a reverse of inequality (1) as following:

$$\sum_{i=1}^{n} w_i A_i \leq \frac{(M+m)^2}{4Mm} P_t(\omega; \mathbb{A}).$$

Applying (7), we can obtain the following operator inequality:

$$\begin{aligned}\Phi\Big(\sum_{i=1}^{n} w_i A_i\Big) &\leq \frac{(M+m)^2}{4Mm}\Phi(P_t(\omega; \mathbb{A}))\\ &\leq \frac{(M+m)^2}{4Mm} P_t(\omega; \Phi(\mathbb{A})).\end{aligned}$$

Now, using inequality (4), we get

$$\Phi^p\Big(\sum_{i=1}^{n} w_i A_i\Big) \leq \Big(\frac{(M+m)^2}{4Mm}\Big)^p P_t^p(\omega; \Phi(\mathbb{A})) \qquad (0 \leq p \leq 1). \tag{10}$$

Dehghani et al. [8] established counterparts of (7) involving matrix power means as follows:

$$P_t^2(\omega; \Phi(\mathbb{A})) \leq \Big(\frac{(m+M)^2}{4mM}\Big)^2 \Phi^2(P_t(\omega; \mathbb{A}))$$

for all $t \in [-1, 1]\backslash\{0\}$ and $0 < m \leq A_i \leq M$ $(1 \leq i \leq n)$. Applying inequality (4), we get

$$P_t^p(\omega; \Phi(\mathbb{A})) \leq \Big(\frac{(m+M)^2}{4mM}\Big)^p \Phi^p(P_t(\omega; \mathbb{A})), \quad (0 \leq p \leq 2). \tag{11}$$

It is interesting to ask whether inequality (11) is true for $p \geq 2$. This is the first motivation of this paper. Moreover, we improve inequality (9) for $p \geq 2$. We also obtain some reverses of (1). In the last section, we establish several refinements of obtained inequalities.

2. Main results

To prove our first result, we need the following lemmas.

Lemma 2.1. *[2, 3, 6, 11] Let $A, B \in \mathcal{M}_k$ be positive definite matrices and $\alpha > 0$. Then*
(i) $\|AB\| \leq \frac{1}{4}\|A + B\|^2$.
(ii) *For* $\beta \geq 1, \|A^\beta + B^\beta\| \leq \|(A + B)^\beta\|$.
(iii) $A \leq \alpha B$ *if and only if* $\|A^{\frac{1}{2}} B^{-\frac{1}{2}}\| \leq \alpha^{\frac{1}{2}}$.
(iv) *If* $0 \leq A \leq B$ *and* $0 < m \leq A \leq M$, *then* $A^2 \leq \frac{(M+m)^2}{4Mm} B^2$.

Lemma 2.2. *[13] Let $A \in \mathcal{M}_k$ and t be a positive number. Then $|A| \leq tI$ if and only if $\|A\| \leq t$ if and only if* $\begin{bmatrix} tI & A \\ A^* & tI \end{bmatrix}$ *is positive.*

Theorem 2.3. *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be an n-tuple of positive definite matrices with $0 < m \leq A_i \leq M$ $(i = 1, \cdots, n)$ for some scalars $m < M$ and $\omega = (w_1, \cdots, w_n)$ be a weight vector. If Φ is a unital positive linear map, then*

$$P_t^p(\omega; \Phi(\mathbb{A})) \leq \Big(\frac{(m + M)^2}{4^{\frac{2}{p}} mM}\Big)^p \Phi^p(P_t(\omega; \mathbb{A})) \tag{12}$$

for every $p \geq 2$ and $t \in [-1, 1]\backslash\{0\}$.

Proof. *Applying Lemma 2.1(iii), inequality* (12) *is equivalent to*

$$\Big\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\Big\| \leq \frac{(m + M)^p}{4M^{\frac{p}{2}} m^{\frac{p}{2}}}. \tag{13}$$

Hence, it is enough to prove inequality (13). *So*

$$\begin{aligned}
M^{\frac{p}{2}} m^{\frac{p}{2}} \Big\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\Big\| &= \Big\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A})) M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\Big\| \\
&\leq \frac{1}{4}\Big\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A})) + M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\Big\|^2 \\
&\qquad \textit{(by Lemma 2.1(i))} \\
&\leq \frac{1}{4}\Big\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi^{-1}(P_t(\omega; \mathbb{A})))^{\frac{p}{2}}\Big\|^2 \\
&\qquad \textit{(by Lemma 2.1(ii))} \\
&= \frac{1}{4}\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi^{-1}(P_t(\omega; \mathbb{A})))\|^p \\
&\leq \frac{1}{4}\Big\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi(P_t(\omega; \mathbb{A})^{-1})\Big\|^p \\
&\qquad \textit{(by (5))} \\
&\leq \frac{1}{4}\Big\|\sum_{i=1}^n w_i\Phi(A_i) + Mm\Phi(\sum_{i=1}^n w_i A_i^{-1})\Big\|^p \\
&\qquad \textit{(by (1))} \\
&= \frac{1}{4}\Big\|\sum_{i=1}^n w_i\Big(\Phi(A_i) + Mm\Phi(A_i^{-1})\Big)\Big\|^p.
\end{aligned} \tag{14}$$

It follows from $0 < m \leq A_i \leq M$ that $(M - A_i)(m - A_i)A_i^{-1} \leq 0$ $(i = 1, 2, \cdots, n)$. Hence

$$Mm\Phi(A_i^{-1}) + \Phi(A_i) \leq M + m \qquad (i = 1, 2, \cdots, n). \tag{15}$$

Applying inequalities (14) *and* (15), *we get*

$$\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\| \leq \frac{(m + M)^p}{4M^{\frac{p}{2}} m^{\frac{p}{2}}}.$$

This completes the proof. □

Corollary 2.4. *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *be an n-tuple of positive definite matrices with* $0 < m \le A_i \le M$ $(i = 1, \cdots, n)$ *for some scalars* $m < M$ *and* $\omega = (w_1, \cdots, w_n)$ *be a weight vector. If* Φ *is a unital positive linear map, then*

$$\Lambda^p(\omega; \Phi(\mathbb{A})) \le \Big(\frac{(m+M)^2}{4^{\frac{2}{p}} mM}\Big)^p \Phi^p(\Lambda(\omega; \mathbb{A})) \tag{16}$$

for every $p \ge 2$.

Proof. *The proof follows from Theorem 2.3 and relation* (3). □

We would like to state the following lemma which use in the next result (see [21, page 582]).

Lemma 2.5. *Let* $A, B \in \mathcal{M}_k$ *be positive definite. Then*

$$\Lambda(1-\alpha, \alpha; A, B) = A \sharp_\alpha B$$

for $\alpha \in (0, 1)$.

Proof. Using the definition of Karcher mean for two positive definite matrices A, B and $\omega = (1-\alpha, \alpha)$ we have

$$(1-\alpha)\log(X^{\frac{-1}{2}} A X^{\frac{-1}{2}}) + \alpha \log(X^{\frac{-1}{2}} B X^{\frac{-1}{2}}) = 0. \tag{17}$$

Let X be the positive solution of (17). We assert that $X = A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$. First, we shall show that the Karcher mean of two matrices I and B is the operator B^α. Let X be the solution of $(1-\alpha)\log X^{-1} + \alpha \log(X^{\frac{-1}{2}} B X^{\frac{-1}{2}}) = 0$, which is equivalent to $X^{\frac{1-\alpha}{\alpha}} = X^{\frac{-1}{2}} B X^{\frac{-1}{2}}$. Hence $X = B^\alpha$ or equivalently $\Lambda(\omega; I, B) = B^\alpha$. Hence by the properties of Karcher mean (see [15, Corollary 4.5]), we have

$$\begin{aligned}\Lambda(\omega; A, B) &= A^{\frac{1}{2}} \Lambda(\omega; I, A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}} = A\sharp_\alpha B.\end{aligned}$$

□

Corollary 2.6. *Let* $A, B \in \mathcal{M}_n$ *be positive definite matrices such that* $0 < m \le A, B \le M$ *for some scalars* $m < M$ *and* $\alpha \in [0, 1]$. *Then*

$$(\Phi(A)\sharp_\alpha \Phi(B))^p \le \Big(\frac{(m+M)^2}{4^{\frac{2}{p}} mM}\Big)^p \Phi^p(A\sharp_\alpha B)$$

for any $p \ge 2$ *and unital positive linear map* Φ.

Proof. Applying Lemma 2.5, we have $\Lambda(1-\alpha, \alpha; A, B) = A\sharp_\alpha B$, $(\alpha \in [0, 1])$. If we put $n = 2, w_1 = 1-\alpha$ and $w_2 = \alpha$ in inequality (16), then we get the desired result. □

In the next theorem, we show an extension of inequality (10) for $p > 1$.

Theorem 2.7. *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *be an n-tuple of positive definite matrices with* $0 < m \le A_i \le M$ $(i = 1, \cdots, n)$ *for some scalars* $m < M$ *and* $\omega = (w_1, \cdots, w_n)$ *be a weight vector. Then*

$$\Phi^p\Big(\sum_{i=1}^n w_i A_i\Big) \le \alpha^p \Phi^p(P_t(\omega; \mathbb{A})), \tag{18}$$

where $t \in [-1, 1]\backslash\{0\}$, $p > 1$ *and* $\alpha = \max\Big\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\Big\}$.

Proof. First we show inequality (18) for $p = 2$. We have

$$\begin{aligned} Mm\left\|\Phi\Big(\sum_{i=1}^n w_iA_i\Big)\Phi^{-1}(P_t(\omega;\mathbb{A}))\right\| &= \left\|\Phi\Big(\sum_{i=1}^n w_iA_i\Big)Mm\Phi^{-1}(P_t(\omega;\mathbb{A}))\right\| \\ &\le \frac{1}{4}\left\|\Phi\Big(\sum_{i=1}^n w_iA_i\Big)+Mm\Phi^{-1}(P_t(\omega;\mathbb{A}))\right\|^2 \\ &\qquad \text{(by Lemma 2.1)} \\ &\le \frac{1}{4}\left\|\Phi\Big(\sum_{i=1}^n w_iA_i\Big)+Mm\Phi(\sum_{i=1}^n w_iA_i^{-1})\right\|^2 \\ &\le \frac{1}{4}(M+m)^2, \end{aligned}$$

whence

$$\left\|\Phi\Big(\sum_{i=1}^n w_iA_i\Big)\Phi^{-1}(P_t(\omega;\mathbb{A}))\right\| \le \frac{(M+m)^2}{4Mm}.$$

Hence

$$\Phi^2\Big(\sum_{i=1}^n w_iA_i\Big) \le \Big(\frac{(M+m)^2}{4Mm}\Big)^2\Phi^2(P_t(\omega;\mathbb{A})).$$

Therefore

$$\Phi^p\Big(\sum_{i=1}^n w_iA_i\Big) \le \Big(\frac{(M+m)^2}{4Mm}\Big)^p\Phi^p(P_t(\omega;\mathbb{A})) \qquad (0\le p\le 2). \tag{19}$$

Now, we prove inequality (18) for $p > 2$. In this case we have

$$\begin{aligned} \left\|\Phi^{\frac{p}{2}}\Big(\sum_{i=1}^n w_iA_i\Big)M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(P_t(\omega;\mathbb{A}))\right\| &\le \frac{1}{4}\left\|\Phi^{\frac{p}{2}}\Big(\sum_{i=1}^n w_iA_i\Big)+M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(P_t(\omega;\mathbb{A}))\right\|^2 \\ &\qquad \text{(by Lemma 2.1(i))} \\ &\le \frac{1}{4}\left\|\Big(\Phi\Big(\sum_{i=1}^n w_iA_i\Big)+Mm\Phi^{-1}(P_t(\omega;\mathbb{A}))\Big)^{\frac{p}{2}}\right\|^2 \\ &\qquad \text{(by Lemma 2.1(ii))} \\ &= \frac{1}{4}\left\|\Phi\Big(\sum_{i=1}^n w_iA_i\Big)+Mm\Phi^{-1}(P_t(\omega;\mathbb{A}))\right\|^p \\ &\le \frac{(M+m)^p}{4}. \end{aligned}$$

Hence

$$\left\|\Phi^{\frac{p}{2}}\Big(\sum_{i=1}^n w_iA_i\Big)\Phi^{-\frac{p}{2}}(P_t(\omega;\mathbb{A}))\right\| \le \frac{1}{4}\Big(\frac{(M+m)^p}{M^{\frac{p}{2}}m^{\frac{p}{2}}}\Big).$$

Thus

$$\Phi^p\Big(\sum_{i=1}^n w_iA_i\Big) \le \Big(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\Big)^p\Phi^p(P_t(\omega;\mathbb{A})). \tag{20}$$

Now, if we take $\alpha = \max\Big\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}}Mm}\Big\}$, then applying (19) and (20) we get the desired result. □

Corollary 2.8. *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be an n-tuple of positive definite matrices with $0 < m \le A_i \le M$ $(i = 1, \cdots, n)$ for some scalars $m \le M$ and $\omega = (w_1, \cdots, w_n)$ be a weight vector. Then*

$$\Phi^p\Big(\sum_{i=1}^{n} w_i A_i\Big) \le \alpha^p \Phi^p(\Lambda(\omega; \mathbb{A})),$$

where $p \ge 1$ and $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right\}$.

Remark 2.9. *By letting $\mathbb{A} = (A, B)$ and $\omega = (w_1, w_2)$ with $w_1 = w_2 = \frac{1}{2}$ in Theorem 2.7, the following inequality holds:*

$$\Phi^p\left(\frac{A+B}{2}\right) \le \alpha^p \Phi^p(A \sharp B)$$

for $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right\}$, which is appeared in [12, Theorem 4].

In the next result we extend inequalities (12) and (18) to the following form:

Theorem 2.10. *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be an n-tuple of positive definite matrices with $0 < m \le A_i \le M$ $(i = 1, \cdots, n)$ for some scalars $m \le M$ and $\omega = (w_1, \cdots, w_n)$ be a weight vector, let $t \in [-1, 1]\backslash\{0\}$ and also Φ be a positive unital linear map. Then*

$$P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \le 2\alpha^p$$

and

$$\Phi^p\Big(\sum_{i=1}^{n} w_i A_i\Big)\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))\Phi^p\Big(\sum_{i=1}^{n} w_i A_i\Big) \le 2\alpha^p, \tag{21}$$

where $p > 0$ and $\alpha = \max\left\{\frac{(m+M)^2}{4mM}, \frac{(m+M)^2}{4^{\frac{1}{p}} mM}\right\}$.

Proof. Applying inequality (11) and Lemma 2.1(iii) for $0 < p \le 1$, we have

$$\|P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A}))\| \le \Big(\frac{(m+M)^2}{4mM}\Big)^p.$$

We put $\alpha = \frac{(m+M)^2}{4mM}$. Applying Lemma 2.2,

$$\begin{bmatrix} \alpha^p I & P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) \\ \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) & \alpha^p I \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha^p I & \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \\ P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) & \alpha^p I \end{bmatrix}$$

are positive. Hence

$$\begin{bmatrix} 2\alpha^p I & P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \Phi\mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \\ \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) + P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) & 2\alpha^p I \end{bmatrix}$$

is positive. Applying Lemma 2.2, we get

$$P_t^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \Phi(\mathbb{A})) \le 2\alpha^p.$$

For $p > 1$, applying inequality (12) with the same argument, we get the desired inequality.
Applying Theorem 2.7 and a similar method we have inequality (21). □

Corollary 2.11. *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be a n-tuple of positive definite matrices with $0 < m \leq A_i \leq M$ $(i = 1, \cdots, n)$ for some scalars $m \leq M$ and $\omega = (w_1, \cdots, w_n)$ be a weight vector, and also Φ be a positive unital linear map. Then*

$$\Lambda^p(\omega; \Phi(\mathbb{A}))\Phi^{-p}(\Lambda(\omega; \mathbb{A})) + \Phi^{-p}(\Lambda(\omega; \mathbb{A}))\Lambda^p(\omega; \Phi(\mathbb{A})) \leq 2\alpha^p$$

and

$$\Phi^p\Big(\sum_{i=1}^n w_i A_i\Big)\Phi^{-p}(\Lambda(\omega; \mathbb{A})) + \Phi^{-p}(\Lambda(\omega; \mathbb{A}))\Phi^p\Big(\sum_{i=1}^n w_i A_i\Big) \leq 2\alpha^p,$$

where $p > 0$ and $\alpha = \max\left\{\frac{(m+M)^2}{4mM}, \frac{(m+M)^2}{4^{\frac{1}{p}}mM}\right\}$.

3. Some refinements

In this section, we give a refinement of inequality (18). This inequality can be refined by a similar method that known in [22].

Theorem 3.1. *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be an n-tuple of positive definite matrices with $0 < m \leq A_i \leq M$ $(i = 1, \cdots, n)$ for some scalars $m \leq M$ and $\omega = (w_1, \cdots, w_n)$ be a weight vector, and also let $t \in [-1, 1]\backslash\{0\}$. Then for every positive unital linear map Φ*

$$\Phi^{2p}\Big(\sum_{i=1}^n w_i A_i\Big) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(P_t(\omega; \mathbb{A})), \tag{22}$$

where $p \geq 2$ and $K = \frac{(M+m)^2}{4mM}$.

Proof. For $p \geq 2$, we have

$$\begin{aligned}\Big\|\Phi^p\Big(\sum_{i=1}^n w_i A_i\Big)M^p m^p \Phi^{-p}(P_t(\omega; \mathbb{A}))\Big\| &\leq \frac{1}{4}\Big\|K^{\frac{p}{2}}\Phi^p\Big(\sum_{i=1}^n w_i A_i\Big) + (\frac{M^2m^2}{K})^{\frac{p}{2}}\Phi^{-p}(P_t(\omega; \mathbb{A}))\Big\|^2 \\ &\qquad \text{(by Lemma 2.1(i))} \\ &\leq \frac{1}{4}\Big\|\Big(K\Phi^2\Big(\sum_{i=1}^n w_i A_i\Big) + \frac{M^2m^2}{K}\Phi^{-2}(P_t(\omega; \mathbb{A}))\Big)^{\frac{p}{2}}\Big\|^2 \\ &\qquad \text{(by Lemma 2.1(ii))} \\ &= \frac{1}{4}\Big\|\Big(K\Phi^2\Big(\sum_{i=1}^n w_i A_i\Big) + \frac{M^2m^2}{K}\Phi^{-2}(P_t(\omega; \mathbb{A}))\Big)\Big\|^p.\end{aligned}$$

Now, It follows from inequality (18) and operator reverse monotonicity of the inverse that

$$\Phi^{-2}(P_t(\omega; \mathbb{A})) \leq K^2\Phi^{-2}\Big(\sum_{i=1}^n w_i A_i\Big).$$

So

$$\begin{aligned}\Big\|\Phi^p\Big(\sum_{i=1}^n w_i A_i\Big)M^p m^p \Phi^{-p}(P_t(\omega; \mathbb{A}))\Big\| &\leq \frac{1}{4}\Big\|\Big(K\Phi^2\Big(\sum_{i=1}^n w_i A_i\Big) + KM^2m^2\Phi^{-2}\Big(\sum_{i=1}^n w_i A_i\Big)\Big\| \\ &\leq \frac{1}{4}(K(M^2 + m^2))^p \qquad \text{(by [18, (4.7)])}.\end{aligned}$$

Hence

$$\left\|\Phi^p\Big(\sum_{i=1}^{n} w_iA_i\Big)\Phi^{-p}(P_t(\omega;\mathbb{A}))\right\| \le \frac{1}{4}\Big(\frac{K(M^2+m^2)}{Mm}\Big)^p. \tag{23}$$

Since (23) is equivalent to (22), thus inequality (22) holds. □

Corollary 3.2. *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *be an n-tuple of positive definite matrices with* $0 < m \le A_i \le M$ $(i = 1, \cdots, n)$ *for some scalars* $m \le M$ *and* $\omega = (w_1, \cdots, w_n)$ *be a weight vector. Then for every positive unital linear map* Φ

$$\Phi^{2p}\Big(\sum_{i=1}^{n} w_iA_i\Big) \le \frac{(K(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(\Lambda(\omega;\mathbb{A})),$$

where $p \ge 2$ *and* $K = \frac{(M+m)^2}{4mM}$.

Remark 3.3. *If we put* $\mathbb{A} = (A, B)$ *and* $\omega = (w_1, w_2)$ *with* $w_1 = w_2 = \frac{1}{2}$ *in Corollary 3.2, then we get [22, Theorem 2.6] as follows:*

$$\Phi^{2p}\left(\frac{A+B}{2}\right) \le \frac{(K(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(A\sharp B).$$

Theorem 3.4. *Let* $\mathbb{A} = (A_1, \cdots, A_n)$ *be an n-tuple of positive definite matrices with* $0 < m \le A_i \le M$ $(i = 1, \cdots, n)$ *for some scalars* $m \le M$ *and* $\omega = (w_1, \cdots, w_n)$ *be a weight vector, and also let* $t \in [-1, 1]\backslash\{0\}$. *Then for every positive unital linear map* Φ

$$P_t^{2p}(\omega;\Phi(\mathbb{A})) \le \frac{(K(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(P_t(\omega;\mathbb{A})), \tag{24}$$

where $p \ge 2$ *and* $K = \frac{(M+m)^2}{4mM}$.

Proof. For $p \ge 2$, we have

$$\begin{aligned}
\left\|P_t^p(\omega;\Phi(\mathbb{A}))M^pm^p\Phi^{-p}(P_t(\omega;\mathbb{A}))\right\| &\le \frac{1}{4}\left\|\frac{1}{K^{\frac{p}{2}}}P_t^p(\omega;\Phi(\mathbb{A})) + (KM^2m^2)^{\frac{p}{2}}\Phi^{-p}(P_t(\omega;\mathbb{A}))\right\|^2 \\
&\qquad \text{(by Lemma 2.1(i))} \\
&\le \frac{1}{4}\left\|\Big(\frac{1}{K}P_t^2(\omega;\Phi(\mathbb{A})) + KM^2m^2\Phi^{-2}(P_t(\omega;\mathbb{A}))\Big)^{\frac{p}{2}}\right\|^2 \\
&\qquad \text{(by Lemma 2.1(ii))} \\
&= \frac{1}{4}\left\|\Big(\frac{1}{K}P_t^2(\omega;\Phi(\mathbb{A})) + KM^2m^2\Phi^{-2}(P_t(\omega;\mathbb{A}))\Big)\right\|^p \\
&\le \frac{1}{4}\left\|\Big(K\Phi^2(P_t(\omega;\mathbb{A})) + KM^2m^2\Phi^{-2}(P_t(\omega;\mathbb{A}))\Big)\right\|^p \\
&\qquad \text{(by (12))} \\
&\le \frac{1}{4}(K(M^2+m^2))^p. \qquad \text{(by [18, (4.7)])}
\end{aligned}$$

Therefore

$$\left\|P_t^p(\omega;\Phi(\mathbb{A}))\Phi^{-p}(P_t(\omega;\mathbb{A}))\right\| \le \frac{1}{4}\Big(\frac{K(M^2+m^2)}{Mm}\Big)^p.$$

Since the last inequality is equivalent to (24), this completes the proof. □

Corollary 3.5. *Let $\mathbb{A} = (A_1, \cdots, A_n)$ be an n-tuple of positive definite matrices with $0 < m \leq A_i \leq M$ $(i = 1, \cdots, n)$ for some scalars $m \leq M$ and $\omega = (w_1, \cdots, w_n)$ be a weight vector. Then for every positive unital linear map Φ*

$$\Lambda^{2p}(\omega; \Phi(\mathbb{A})) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}} \Phi^{2p}(\Lambda(\omega; \mathbb{A})),$$

where $p \geq 2$ and $K = \frac{(M+m)^2}{4mM}$.

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