

Some Methods for Generating Topologies by Relations

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Abstract. Relation on a set is a simple mathematical model to which many real-life data can be connected. In fact, topological structures are generalized methods for measuring similarity and dissimilarity between objects in the universes. In this work, some methods for generating topologies are obtained using binary relations. The relationship between these methods are discussed. We also investigate some properties of these topologies. Moreover, we obtain a quasi-discrete topology from a symmetric relation instead of an equivalence relation. Finally, several examples are given to indicate counter connections.

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1. Introduction

Relations are used in the construction of topological structures in many fields such as dynamics [4], rough set theory and approximation space [8, 9], digital topology [11, 12], biochemistry [13] and biology [14]. In fact, topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. It should be noted that the generation of topology by relations and the representation of topological concepts via relations will narrow the gap between topology and its applications.

The aim of this paper is to study some methods which are used to generate topologies by relations. We introduce new methods and we study the relationship between them and the other methods. In addition, we show that the topology generated by aftersets is the dual to the topology generated by forsets when the relation is pre-order. Moreover, we obtain a quasi-discrete topology using a symmetric relation, which is considered as a generalization for the equivalence relation in [8, 9].

2. The aftersets and the forsets

A relation R from a universe X to a universe X (relation on X) is a subset of $X \times X$, i.e., $R \subseteq X \times X$. The formula $(x, y) \in R$ is abbreviated as xRy and means that x is in relation R with y .

Definition 2.1. *If R is a relation on X , then the aftersets of $x \in X$ is xR where $xR = \{y : xRy\}$ and the forsets of $x \in X$ is Rx , where $Rx = \{y : yRx\}$.*

Definition 2.2. [6] *If R is a relation on X , then the class $S_1 = \{xR : x \in X\}$ (resp. $S_2 = \{Rx : x \in X\}$) is a subbase for the topology τ_1 (resp. τ_2) on X .*

We illustrate the last definition by the following example.

Example 2.1. Let $X = \{a, b, c, d\}$ and $R = \{(a, a), (a, b), (b, d), (c, d), (d, a)\}$ be a relation on X , then

$$\begin{aligned} S_1 &= \{\{a, b\}, \{d\}, \{a\}\}, \\ S_2 &= \{\{a, d\}, \{a\}, \{b, c\}\}, \\ \beta_1 &= \{\{a\}, \{d\}, \{a, d\}, \phi, X\}, \text{ and} \\ \beta_2 &= \{\{a\}, \{a, d\}, \{b, c\}, \phi, X\}. \end{aligned}$$

Note that $X \in \beta$, since by definition X is the empty intersection of members of S [8]. So that,

$$\tau_1 = \{\phi, X, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$$

and

$$\tau_2 = \{\phi, X, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}\}.$$

We shall denote the complement of any subset A of X by A^c .

Definition 2.3. *If τ is the topology on a finite set X and the class $\tau^c = \{G^c : G \in \tau\}$ is also the topology on X , then τ^c is the dual of τ .*

Remark 2.1. Generally for any binary relation R , then τ_1 is not the dual of τ_2 .

Now, we introduce some examples about special cases of the relation R to study the duality of τ_1 and τ_2 .

Example 2.2. Let $X = \{a, b, c, d\}$ and R be a reflexive relation,

$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, d), (c, a)\}$, then

$\tau_1 = \{\phi, X, \{a\}, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, d\}, \{a, c\}, \{a, b, d\}, \{a, b, c\}, \{a, c, d\}\}$, and

$\tau_2 = \{\phi, X, \{a\}, \{c\}, \{b\}, \{a, c\}, \{a, b\}, \{b, d\}, \{b, c\}, \{a, b, d\}, \{a, b, c\}, \{b, c, d\}\}$.

Note that τ_1 is not the dual of τ_2 .

Example 2.3. Let $X = \{a, b, c, d\}$ and R be a symmetric relation,

$R = \{(a, a), (b, b), (a, b), (b, a), (c, d), (d, c), (a, c), (c, a)\}$, then

$\tau_1 = \tau_2 = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, d\}, \{a, c\}, \{a, b, d\}, \{a, b, c\}, \{a, c, d\}\}$.

Note that τ_1 is not the dual of τ_2 .

Example 2.4. Let $X = \{a, b, c, d\}$ and R be a transitive relation,

$$R = \{(a, a), (a, b), (b, a), (b, b), (c, d), (d, a), (c, a), (d, b), (c, b)\},$$

$$\tau_1 = \{\phi, X, \{a, b\}, \{a, b, d\}\} \text{ and}$$

$$\tau_2 = \{\phi, X, \{c\}\}.$$

Note that τ_1 is not the dual of τ_2 .

Lemma 2.1. *If R is symmetric, then $\tau_1 = \tau_2$.*

Proof. Let R be a symmetric relation, i.e., if $(a, b) \in R$ then $(b, a) \in R$. Hence if $b \in aR$ then $b \in Ra$ and so $Ra = aR$ for all $a \in X$, then $\tau_1 = \tau_2$. \blacksquare

Definition 2.4. [3] *A space is called a quasi-discrete if every open set is closed set and vice versa.*

Remark 2.2. If $\tau_1 = \tau_2$, then τ_1 is the dual of τ_2 if τ_1 and τ_2 are quasi-discrete spaces.

Example 2.5. Let $X = \{a, b, c, d\}$ and R be a reflexive and transitive relation,

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (a, c), (c, d), (a, d), (b, c), (b, d)\},$$

$$\text{then } \tau_1 = \{\phi, X, \{d\}, \{c, d\}\} \text{ and } \tau_2 = \{\phi, X, \{a, b\}, \{a, b, c\}\}.$$

Note that τ_1 is the dual of τ_2 .

Definition 2.5. *The relation R is preorder if and only if R is reflexive and transitive.*

Lemma 2.2. *If R is a preorder relation, then*

- (i) $A \in \tau_1$ if and only if $A = \cup_{x \in A} xR$.
- (ii) $A \in \tau_2$ if and only if $A = \cup_{x \in A} Rx$.

Proof.

- (i) We assume that $A \in \tau_1$ (i.e., A is τ_1 -open). If $y \in A$, then by reflexivity we have $y \in yR$ and so $y \in \cup_{x \in A} xR$, therefore $A \subseteq \cup_{x \in A} xR$. Now, if $y \in \cup_{x \in A} xR$ then there exist $x \in A$ such that $y \in xR$. In addition, if $z \in yR$ then by transitivity we get $z \in xR$, i.e., yR is the smallest open set contains y . Thus $y \in yR \subseteq A \in \tau_1$ and hence $\cup_{x \in A} xR \subseteq A$, therefore $A = \cup_{x \in A} xR$. Conversely, we suppose that R is preorder and $A = \cup_{x \in A} xR$ then for every $y \in A$ there is $yR \in \tau_1$ such that $y \in yR \subseteq \cup_{x \in A} xR = A$, hence y is an interior point of A , i.e., A is τ_1 -open.
- (ii) The proof is the same for (i). \blacksquare

Proposition 2.1. τ_1 is the dual of τ_2 if and only if R is a preorder relation.

Proof. Let τ_1 be the dual of τ_2 (i.e., $A \in \tau_1$ if $A^c \in \tau_2$) and R be not preorder (i.e., $A \neq \cup_{x \in A} xR$ and $A^c \neq \cup_{x \in A^c} Rx$). If $y \in \cup_{x \in A} xR$ and $y \notin A$, then there is $x \in A$ such that $y \in xR$ and $y \in A^c$. Therefore, there is no τ_2 -open set which contains y and is contained in A^c (i.e., y is not an interior point of A^c), which is a contradiction to $A^c \in \tau_2$. We have another contradiction if we use $A^c \neq \cup_{x \in A^c} Rx$, then R is preorder.

Conversely, let R be a preorder relation and $A \in \tau_1$. We will show that $A^c \in \tau_2$ (i.e., $A^c = \cup_{x \in A^c} Rx$). Suppose that $A^c \neq \cup_{x \in A^c} Rx$, then there exist $y \in \cup_{x \in A^c} Rx$ and $y \notin A^c$. Hence there is $z \in A^c$ such that yRz and $y \in A$. Thus $z \in yR \subseteq \cup_{x \in A} xR = A$, which is a contradiction. Therefore, $A^c = \cup_{x \in A^c} Rx$ and so $A^c \in \tau_2$, then τ_1 is the dual of τ_2 . \blacksquare

Definition 2.6. [1] Let R be any binary relation on X , a set $\langle p \rangle R$ is the intersection of all aftersets containing p , i.e.,

$$\langle p \rangle R = \begin{cases} \bigcap_{p \in xR} (xR) & \text{if } \exists x : p \in xR, \\ \phi & \text{otherwise.} \end{cases}$$

Also, $R\langle p \rangle$ is the intersection of all forsets containing p , i.e.,

$$R\langle p \rangle = \begin{cases} \bigcap_{p \in Rx} (Rx) & \text{if } \exists x : p \in Rx, \\ \phi & \text{otherwise.} \end{cases}$$

Definition 2.7. Let R be a reflexive relation on X , then the class $\{\langle x \rangle R : x \in X\}$ is a base for the topology τ_3 on X .

The following example is an illustrating for the previous definition.

Example 2.6. Let $X = \{a, b, c, d\}$ and R be a reflexive relation on X such that

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, d), (c, d)\}, \text{ then} \\ \langle a \rangle R = \{a, b\}, \langle b \rangle R = \{b\}, \langle c \rangle R = \{c, d\}, \langle d \rangle R = \{d\}.$$

The corresponding topology of this relation is

$$\tau_3 = \{\phi, X, \{b\}, \{d\}, \{a, b\}, \{c, d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}.$$

3. Rough sets in topological spaces

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space (X, R) , where X is a set called the universe and R is an equivalence relation [10]. In the approximation space, we consider two operators, the upper and lower approximations of subsets: Let $A \subseteq X$.

$$(3.1) \quad \underline{R}(A) = \{x \in X : [x]_R \subset A\} \\ \overline{R}(A) = \{x \in X : [x]_R \cap A \neq \phi\}$$

where $[x]_R$ is the equivalence class containing x . The reference space in rough set theory is the approximation space, whose topology generated by the equivalence classes of R . In this topology, the closure and interior operators are the same of the upper and lower approximation operators. Moreover, this topology belongs to a special class known by Clopen topology, in which every open set is closed. Clopen topology is called the quasi-discrete topology.

Yao [16] introduced and investigated the notion of generalized approximation space by using the aftersets concepts as follows:

$$(3.2) \quad \begin{aligned} \underline{R}(A) &= \{x \in X : xR \subset A\} \\ \overline{R}(A) &= \{x \in X : xR \cap A \neq \phi\} \end{aligned}$$

Obviously, if R is an equivalence relation, then $xR = [x]_R$. In addition, these definitions are equivalent to the original Pawlak's definitions (3.1).

The following theorem states that a reflexive and transitive relation is sufficient for the approximation operators (3.2) to be interior and closure operators [5].

Theorem 3.1. *Suppose R is a reflexive and transitive relation on X . The pair of lower and upper approximations (3.2) is a pair of interior and closure operators satisfying Kuratowski axioms (topological space).*

Remark 3.1. Suppose R is a reflexive and transitive relation on X . The topology generated by the lower and upper approximation operators (3.2) is the same of the topology τ_1 generated by the aftersets as a subbase.

The authors in [2] introduced and investigated another notion of generalized approximation space by using the aftersets concepts as follows:

$$(3.3) \quad \begin{aligned} \underline{R}(A) &= \{x \in X : \langle x \rangle R \subset A\} \\ \overline{R}(A) &= \{x \in X : \langle x \rangle R \cap A \neq \phi\}. \end{aligned}$$

Obviously, if R is an equivalence relation, then $\langle x \rangle R = [x]_R$. In addition, these definitions are equivalent to the original Pawlak's definitions (3.1).

The following theorem states that a reflexive relation is sufficient for the approximation operators (3.3) to be interior and closure operators [2].

Theorem 3.2. *Suppose R is a reflexive relation on X . The pair of lower and upper approximations (3.3) is a pair of interior and closure operators satisfying Kuratowski axioms.*

Remark 3.2. Suppose R is a reflexive relation on X . The topology generated by the lower and upper approximation operators (3.3) is the same of the topology τ_3 generated by the base in the definition 2.7.

4. Closure operator

In Section 2, topologies were generated on a set using aftersets (forsets) as a subbase. The purpose of this section is to introduce topologies by defining closure operator using binary relations.

Definition 4.1. [3] *A closure space is a pair (X, cl) , where X is any set, and $cl : P(X) \rightarrow P(X)$ is a mapping associating with each subset $A \subseteq X$ a subset $cl(A) \subseteq X$, called the closure of A , such that*

- (i) $cl(\phi) = \phi$,
- (ii) $A \subseteq cl(A)$,
- (iii) $cl(A \cup B) = cl(A) \cup cl(B)$.

Definition 4.2. [10] *Let R be any binary relation on X , then the relation R gives rise to a closure operator on X as follows:*

$$cl_1(A) = A \cup \{y \in X : \exists x \in A, yRx\}.$$

Lemma 4.1. *Let R be any binary relation on X , then*

$$\{y \in X : \exists x \in A, yRx\} = \{y \in X : yR \cap A \neq \phi\}.$$

Proof. Let $z \in \{y \in X : \exists x \in A, yRx\}$, then there exist $x \in A$ such that zRx , i.e., $x \in zR$ so that $zR \cap A \neq \phi$. Hence, $z \in \{y \in X : yR \cap A \neq \phi\}$.

Conversely, assume that $z \in \{y \in X : yR \cap A \neq \phi\}$, then there is $x \in A$ such that $x \in zR$, so zRx . Hence, $z \in \{y \in X : \exists x \in A, yRx\}$. ■

Now, from Lemma 4.1 we can write the closure operator in Definition 4.2 as follows:

$$cl_1(A) = A \cup \{y \in X : yR \cap A \neq \phi\}.$$

The following result gives an equivalent definition for cl_1 by forsets.

Lemma 4.2. *Let R be any binary relation on X , then*

$$\{y \in X : \exists x \in A, yRx\} = \cup_{x \in A} Rx.$$

Proof. Suppose that $z \in \{y \in X : \exists x \in A, yRx\}$, then there exist $x \in A$ such that zRx , i.e., $z \in Rx$, hence $z \in \cup_{x \in A} Rx$, so that $\{y \in X : \exists x \in A, yRx\} \subseteq \cup_{x \in A} Rx$.

Conversely, assume that $z \in \cup_{x \in A} Rx$, then there is $x \in A$ such that $z \in Rx$, i.e., zRx , and hence $z \in \{y \in X : \exists x \in A, yRx\}$, therefore, $\cup_{x \in A} Rx \subseteq \{y \in X : \exists x \in A, yRx\}$. ■

As a result of Lemma 4.2, we can write the closure operator in Definition 4.2 as follows:

$$cl_1(A) = A \cup (\cup_{x \in A} Rx).$$

On the other hand, we can define another closure operator on X as follows:

Definition 4.3. *Let R be any binary relation on X , then the relation R gives rise to a closure operator on X as follows:*

$$cl_2(A) = A \cup \{y \in X : \exists x \in A, xRy\}.$$

Lemma 4.3. *Let R be any binary relation on X , then*

$$\{y \in X : \exists x \in A, xRy\} = \{y \in X : Ry \cap A \neq \phi\}.$$

Proof. The proof is the same for Lemma 4.1. ■

By the following result we can give an equivalent definition to cl_2 using aftersets.

Lemma 4.4. *Let R be any binary relation on X , then*

$$\{y \in X : \exists x \in A, xRy\} = \cup_{x \in A} xR.$$

Proof. The proof is similar to that of Lemma 4.2. ■

Now, from Lemma 4.3 and Lemma 4.4 we can write cl_2 as follows:

$$cl_2(A) = A \cup \{y \in X : Ry \cap A \neq \phi\} = A \cup (\cup_{x \in A} xR).$$

It is very well known that, for $i \in \{1, 2, 3\}$, τ_i^* will always denote the topology generated by the closure cl_i , i.e., the topology consisting of complements of cl_i -closed sets.

Proposition 4.1. *For any binary relation R , τ_1^* is the dual of τ_2^* .*

Proof. We want to show that if $A \in \tau_1^*$ then $A^c \in \tau_2^*$, or if $cl_1(A) = A$, then $cl_2(A^c) = A^c$. We assume that $cl_1(A) = A$, then $\{y \in X : \exists x \in A, yRx\} \subseteq A$. We will show that $\{y \in X : \exists x \in A^c, xRy\} \subseteq A^c$, i.e., $A \cap \{y \in X : \exists x \in A^c, xRy\} = \phi$. Now, if $A \cap \{y \in X : \exists x \in A^c, xRy\} \neq \phi$, then there are $z \in A$ and $x \in A^c$ such that xRz . In addition, since $cl_1(A) = A$, we have $x \in A$, which it is a contradiction, so that $z \notin A$ and $A \cap \{y \in X : \exists x \in A^c, xRy\} = \phi$, i.e., $\{y \in X : \exists x \in A^c, xRy\} \subseteq A^c$. Therefore, $cl_2(A^c) = A^c$. ■

Proposition 4.2. *If R is transitive the following hold,*

- (i) $cl_1(cl_1(A)) = cl_1(A)$,
- (ii) $cl_2(cl_2(A)) = cl_2(A)$.

Proof.

- (i) The proof in [4].
- (ii) We want to show that $cl_2(cl_2(A)) \subseteq cl_2(A)$. Suppose that $z \in cl_2(cl_2(A))$, then $z \in cl_2(A)$ or there is $y \in cl_2(A)$ such that yRz , also there is $x \in A$ such that xRy . Hence by transitivity of R , we get xRz , i.e., $z \in cl_2(A)$. Thus $cl_2(cl_2(A)) \subseteq cl_2(A)$. ■

Remark 4.1. If R is a reflexive relation then the following hold,

- (i) $cl_1(A) = \cup_{x \in A} Rx$,
- (ii) $cl_2(A) = \cup_{x \in A} xR$.

Lemma 4.5. *Let R be any binary relation then the following hold,*

- (i) $A \in \tau_1^*$ if and only if $\cup_{x \in A} xR \subseteq A$,
- (ii) $A \in \tau_2^*$ if and only if $\cup_{x \in A} Rx \subseteq A$.

Proof.

- (i) If $A \in \tau_1^*$, then A is τ_2^* -closed, i.e., $cl_2(A) = A$, hence $A = A \cup (\cup_{x \in A} xR)$ so that $\cup_{x \in A} xR \subseteq A$. Conversely, we assume that $\cup_{x \in A} xR \subseteq A$, then $cl_2(A) = A$, i.e., A is τ_2^* -closed, hence $A \in \tau_1^*$.
- (ii) The proof is similar to that of (i). ■

Remark 4.2. Suppose R is a transitive relation on X . The topology τ_1^* generated by the closure operator in Definition 4.2 is the same of the topology τ_1 generated by the aftersets as a subbase.

On the other hand, we will introduce another closure operator, which is an idempotent operator for any binary relation.

Definition 4.4. [1] *Let X be any set and $R \subseteq X \times X$ be any binary relation on X , then the relation R gives rise to a closure operator cl_R on X as follows:*

$$cl_3(A) = A \cup \{x \in X : \langle x \rangle R \cap A \neq \phi\}.$$

In the following example we can find the topologies τ_i^* for all $i \in \{1, 2, 3\}$ from the closure operators (cl_i) , respectively, as follows:

Example 4.1. Let $X = \{a, b, c, d\}$ and $R = \{(a, a), (a, b), (b, d), (c, d), (d, a)\}$ be a binary relation on X , then

$$\begin{aligned} aR &= \{a, b\}, bR = \{d\}, cR = \{d\}, dR = \{a\}, \\ Ra &= \{a, d\}, Rb = \{a\}, Rc = \phi, Rd = \{b, c\}, \text{ and} \\ \langle a \rangle R &= \{a\}, \langle b \rangle R = \{a, b\}, \langle c \rangle R = \phi, \langle d \rangle R = \{d\}. \end{aligned}$$

Hence,

$$\begin{aligned} \tau_1^* &= \{\phi, X, \{a, b, d\}\}, \\ \tau_2^* &= \{\phi, X, \{c\}\}, \text{ and} \\ \tau_3^* &= \{\phi, X, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}. \end{aligned}$$

Note that, τ_1^* is the dual of τ_2^* .

Lemma 4.6. For any binary relation R , cl_3 is idempotent, i.e., $cl_3(cl_3(A)) = cl_3(A)$.

Proof. We want to show that $cl_3(cl_3(A)) \subseteq cl_3(A)$. Suppose $y \in cl_3(cl_3(A))$. Then since $cl_3(cl_3(A)) = cl_3(A) \cup \{x \in X : \langle x \rangle R \cap cl_3(A) \neq \phi\}$, we have either

$$(4.1) \quad y \in cl_3(A)$$

or $y \in \{x \in X : \langle x \rangle R \cap cl_3(A) \neq \phi\}$. In the latter case we have $\langle y \rangle R \cap cl_3(A) \neq \phi$, i.e., $\langle y \rangle R \cap (A \cup \{x \in X : \langle x \rangle R \cap A \neq \phi\}) \neq \phi$, and hence $(\langle y \rangle R \cap A) \cup (\langle y \rangle R \cap \{x \in X : \langle x \rangle R \cap A \neq \phi\}) \neq \phi$. It follows that either $\langle y \rangle R \cap A \neq \phi$ or $\langle y \rangle R \cap \{x \in X : \langle x \rangle R \cap A \neq \phi\} \neq \phi$. In the former case we have

$$(4.2) \quad y \in cl_3(A)$$

and in the latter, there is a z such that $z \in \langle y \rangle R$ and $z \in \{x \in X : \langle x \rangle R \cap A \neq \phi\}$, i.e., $\langle z \rangle R \cap A \neq \phi$; in this case, since $z \in \langle y \rangle R$, we have $\langle z \rangle R \subseteq \langle y \rangle R$, and hence $\langle y \rangle R \cap A \neq \phi$, so

$$(4.3) \quad y \in cl_3(A)$$

From (4.1), (4.2) and (4.3), therefore, we have $y \in cl_3(A)$, so $cl_3(cl_3(A)) \subseteq cl_3(A)$. Since cl_3 is a closure space, the reverse inclusion also hold, so $cl_3(cl_3(A)) = cl_3(A)$. ■

Remark 4.3. If R is a reflexive relation on X . then the topology τ_3^* generated by the closure operator in Definition 4.4 is the same of the topology τ_3 generated by the base in Definition 2.7.

Remark 4.4. If R is an equivalence relation on X , then

$$\tau_1 = \tau_2 = \tau_3 = \tau_1^* = \tau_2^* = \tau_3^*.$$

5. Interior operator

In the previous section, topologies were generated on a set by defining a closure operator using binary relation. In this section, we introduce topologies by defining interior operator using binary relation.

Let R be any binary relation on X , then the relation gives rise to an interior operator on X (corresponding to cl_1) as follows [4].

$$Int_1(A) = \{x \in A : xR \subseteq A\}$$

Thus, the $Int_1(A)$ consists of those elements of A which are not R -related to any elements outside A .

The important idea is that, we present another interior operator on X corresponding to cl_2 as follows:

Since $Int_2(A) = (cl_2(A^c))^c$, then

$$\begin{aligned} Int_2(A) &= (A^c \cup \{x \in X : Rx \cap A^c \neq \phi\})^c \\ &= A \cap (\{x \in X : Rx \cap A^c \neq \phi\})^c \\ &= A \cap \{x \in X : Rx \cap A^c = \phi\} \\ &= A \cap \{x \in X : Rx \subseteq A\} \\ &= \{x \in A : Rx \subseteq A\}. \end{aligned}$$

Lemma 5.1. *If R is a symmetric relation then $\tau_1^* = \tau_2^*$.*

Proof. Let R be a symmetric relation, i.e., $xR = Rx \forall x \in X$, then $cl_1(A) = A \cup (\cup_{x \in A} Rx) = A \cup (\cup_{x \in A} xR) = cl_2(A)$. Moreover, $Int_1(A) = \{x \in A : xR \subseteq A\} = \{x \in A : Rx \subseteq A\} = Int_2(A)$, i.e., $\tau_1^* = \tau_2^*$. \blacksquare

Lemma 5.2. *If R is a symmetric relation then the following conditions are equivalent.*

- (i) $A = Int_1(A) = Int_2(A)$,
- (ii) $A = cl_1(A) = cl_2(A)$.

Proof. Let A be τ_1^* -open, then $\cup_{x \in A} xR \subseteq A$. Since R is symmetric, we have $\cup_{x \in A} Rx \subseteq A$ and $cl_1(A) = A \cup (\cup_{x \in A} Rx) = A$, i.e., A is τ_1^* -closed. Conversely, let A be τ_1^* -closed, so $A = A \cup (\cup_{x \in A} Rx)$, i.e., $\cup_{x \in A} Rx \subseteq A$. Also, since R is symmetric, we get $\cup_{x \in A} xR \subseteq A$, therefore, A is τ_1^* -open. \blacksquare

Now, from Definition 2.4 and Lemma 5.2 we have the following remark.

Remark 5.1. If R is a symmetric relation, then (X, cl_1) and (X, cl_2) are quasi-discrete space.

Finally, we will introduce another interior operator on X corresponding to cl_3 as follows:

$$\begin{aligned} Int_3(A) &= (A^c \cup \{x \in X : \langle x \rangle R \cap A^c \neq \phi\})^c \\ &= A \cap (\{x \in X : \langle x \rangle R \cap A^c \neq \phi\})^c \\ &= A \cap \{x \in X : \langle x \rangle R \cap A^c = \phi\} \\ &= A \cap \{x \in X : \langle x \rangle R \subseteq A\} \\ &= \{x \in A : \langle x \rangle R \subseteq A\}. \end{aligned}$$

6. Neighborhood operator

Let $cl : P(X) \rightarrow P(X)$ be the closure operator. Its conjugate is the interior operator. The associated neighborhood operator $\mathbf{N}(x) = \{N \subseteq X : x \in Int(N)\}$. It is not hard to show that closure, interior and neighborhood can be used to define each other. For instance, we have $x \in cl(A)$ iff $A^c \notin \mathbf{N}(x)$.

Remark 6.1. If we take $cl_1(A) = A \cup \{y \in X : \exists x \in A, yRx\}$, then we can take the minimal neighborhood of A in the form $N_1(A) = A \cup \{y \in X : \exists x \in A, xRy\}$, also if we take $cl_2(A) = A \cup \{y \in X : \exists x \in A, xRy\}$, then we can take the minimal neighborhood of A in the form $N_2(A) = A \cup \{y \in X : \exists x \in A, yRx\}$.

Lemma 6.1. [15] *Let R be any binary relation on X , then $cl(cl(A)) = cl(A)$ if and only if $A \in \mathbf{N}(x) \Leftrightarrow Int(A) \in \mathbf{N}(x)$.*

Now, from Proposition 4.2 and Lemma 6.1, we get the following proposition.

Proposition 6.1. *In a closure space (X, cl_1) (resp. (X, cl_2)) if R is a transitive relation on X , then $A \in \mathbf{N}(x) \Leftrightarrow Int_1(A)$ (resp. $Int_2(A)$) $\in \mathbf{N}(x)$.*

Proposition 6.2. *In a closure space (X, cl_1) (resp. (X, cl_2)) if R is a symmetric relation on X , then $cl_1(A)$ ($cl_2(A)$) is the minimal neighborhood of a set A .*

Proof. If R be a symmetric relation, then $N_1(A) = A \cup \{y \in X : \exists x \in A, xRy\} = A \cup \{y \in X : \exists x \in A, yRx\} = cl_1(A)$. Moreover, we have $N_2(A) = A \cup \{y \in X : \exists x \in A, yRx\} = A \cup \{y \in X : \exists x \in A, xRy\} = cl_2(A)$. ■

The important idea is that, we give another definition of the minimal neighborhood of a point x in a closure space (X, cl_3) as follows:

$$N_3(x) = \begin{cases} \langle x \rangle R & \text{if } \langle x \rangle R \neq \phi, \\ \{x\} & \text{if } \langle x \rangle R = \phi. \end{cases}$$

Also, from Lemma 4.6 and Lemma 6.1 we get.

Proposition 6.3. *In a closure space (X, cl_3) $A \in \mathbf{N}(x) \Leftrightarrow Int_3(A) \in \mathbf{N}(x)$ for any binary relation R on X .*

Lemma 6.2. *In a closure space (X, cl_3) the open sets are precisely the unions $\cup_{x \in A} (N_3(x))$ for all $A \subseteq X$.*

Proof. Let A be an open set in (X, cl_3) , then

$$A = Int_3(A) = \{x \in A : \langle x \rangle R \subseteq A\}.$$

Hence A is a neighborhood of each of its elements, so for each $y \in A$, we have $N_3(y) \subseteq A$, then $\cup_{x \in A} (N_3(x)) \subseteq A$. But since $y \in N_3(y)$ for all $y \in X$, we have

$$A \subseteq \cup_{x \in A} (N_3(x)).$$

And so A is the union of the minimal neighborhoods of its elements. Conversely, consider any subset $A \subseteq X$. We want to show that $\cup_{x \in A} (N_3(x))$ is an open set. We will show that $N_3(x)$ is open. First if $\langle x \rangle R \neq \phi$, then for any point $y \in N_3(x) = \langle x \rangle R$ we have $\langle y \rangle R \subseteq \langle x \rangle R$, and hence $y \in Int_3(\langle x \rangle R) = Int_3(N_3(x))$, thus $N_3(x)$ is open. Second if $\langle x \rangle R = \phi$ then $N_3(x) = \{x\} = \{x \in \{x\} : \langle x \rangle R \subseteq \{x\}\} = Int_3(\{x\})$, i.e., $N_3(x) = Int_3(N_3(x))$, then $N_3(x)$ is an open set. ■

References

- [1] A. A. Allam, M. Y. Bakeir and E. A. Abo-Tabl, New approach for closure spaces by relations, *Acta Mathematica Academiae Paedagogicae Nyregyhziensis*, **22**(2006), 285–304.

- [2] A. A. Allam, M. Y. Bakeir and E. A. Abo-Tabl, New approach for basic rough set concepts. *In: Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing*, Lecture Notes in Artificial Intelligence 3641, D. Slezak, G. Wang, M. Szczuka, I. Dntsch, Y. Yao (Eds.), Springer-Verlag GmbH, Regina, (2005), 64–73.
- [3] E. Čech, *Topological Spaces*, Wiley, London, 1966.
- [4] A. Galton, A generalized topological view of motion in discrete space, *Theoretical Computer Sciences* **305**(1-3), (2003), 111–134.
- [5] J. Kortelainen, On relationship between modified sets, topological spaces and rough sets, *Fuzzy Sets and System* **61**(1994), 91–95.
- [6] E. F. Lashin, A. M. Kozae, A. A. Abo Khadra and T. Medhat, Rough set theory for topological spaces, *International Journal of Approximate Reasoning* **40**(2005), 35–43.
- [7] S. Lipschutz, Theory and problems of general topology, *Schaum's Series*, McGraw-Hill Book Company, Singapore, 1965.
- [8] Z. Pawlak, Rough sets, *International Journal of Information and Computer Sciences* **11**(1982), 341–356.
- [9] Z. Pawlak, Rough sets. *Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishers, 1991.
- [10] J. Slapal, Relations and topologies, *Czech. Math. J.* **43**(1993), 141–150.
- [11] J. Slapal, A Jordan curve theorem with respect to certain closure operations on the digital plane, *Electronic Notes in Theoretical Computer Science* **46**(2001), 1–20.
- [12] J. Slapal, Closure operation for digital topology, *Theoretical Computer Science* **5**(1–3)(2003), 457–471.
- [13] B. M. R. Stadler and P. F. Stadler, Generalized topological space in evolutionary theory and combinatorial chemistry, *J. Chem. Inf. Comput. Sci.* **42**(2002), 577–585.
- [14] B. M. R. Stadler, P. F. Stadler, M. Shpak and G. P. Wagner, Recombination spaces, metrics, and pretopologies, *Z. Phys. Chem.* **216**(2002), 155–173.
- [15] P. F. Stadler and B. M. R. Stadler, Genotype-Phenotype maps , *Biological Theory* **1**(3)(2006), 268–279 .
- [16] Y. Y. Yao, Two views of the theory of rough sets in finite universes, *International Journal of Approximation Reasoning* **15**(1996), 291–317.