Some Modal Operators on Intuitionistic Fuzzy Multisets

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Abstract

In this paper, we extend the idea of modal operators to intuitionistic fuzzy multisets (IFMSs) since it is the extension of intuitionistic fuzzy sets. We defined two modal operators,□ and ◊, which are analogous to the modal logic operators n*ecessity* and *possibility* on intuitionistic fuzzy multisets A and B. We deduce and prove some theorems based on these operators.

Keywords: fuzzy multisets, fuzzy sets, intuitionistic fuzzy sets, intuitionistic fuzzy multisets, modal operators.

1. Introduction

Intuitionistic fuzzy sets (IFSs) introduced by [2,3] as an extension of fuzzy sets proposed earlier by [1] received much attentions in fuzzy community due to its flexibility and resourcefulness in tackling the issue of vagueness in standard set theory. The main advantage of IFSs is the ability to cope with the hesitancy that may exist. This is achieved by incorporating a second function along with the membership function of the traditional fuzzy sets called non-membership function.

Subsequently, Shinoj and Sunil [5] proposed IFMSs by combining IFSs [2] and fuzzy multisets(FMSs) introduced by [4] because there are times that each element has different membership values with a corresponding non-membership values. In this article, we extend the idea of modal operators in [3] to IFMS.

2. Preliminaries

Definition 1: Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$, where $\mu_A(x): X \longrightarrow [0,1]$ is the membership function of the fuzzy set A.

Definition 2: Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A(x), \nu_A(x): X \longrightarrow [0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of non-determinacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ i.e.,$\pi_A(x): X \longrightarrow [0,1]$ for every $x \in X$. The hesitation margin expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 3: Let X be a nonempty set. A fuzzy multiset (FMS) A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A: X \longrightarrow Q$ where Q is the set of all crisp multisets drawn from the unit interval [0,1]. Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by $(\mu_A^I(x), \mu_A^2(x), \ldots, \mu_A^n(x))$, where $\mu_A^I(x) \geq \mu_A^2(x) \geq . \quad . \quad . \quad \geq \mu_A^n(x)$.

3. Concept of Intuitionistic Fuzzy Multisets (IFMSs)

Definition 4: Let X be a nonempty set. An IFMS A drawn from X is characterized by two functions: "count membership" of A denoted as CM_A and "count non-membership" of A denoted as CN_A given respectively by $CM_A: X \longrightarrow Q$ and $CN_A: X \longrightarrow Q$ where Q is the set of all crisp multisets drawn from the unit interval $[0,1]$ such that for each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CM_A(x)$ and it is denoted as $(\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^n(x))$, where $\mu_A^1(x) \geq \mu_A^2(x) \geq . \; . \; . \geq \mu_A^n(x)$ but the corresponding non-membership sequence of elements in $CN_A(x)$ is denoted by $(\nu_A^1(x), \nu_A^2(x), \ldots, \nu_A^n(x))$ such that

$0 \leq \mu_A^i(x) + \nu_A^i(x) \leq 1$ for every $x \in X$. This means, an IFMS A is defined as; $A = \{\langle x, CM_A(x), CN_A(x)\rangle : \in X\}$ or $A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\}$, for $i = 1, . . ., n$. $\pi_A^i(x) = 1 - \mu_A^i(x) - \nu_A^i(x)$ is the intuitionistic fuzzy multisets index or hesitation margin of x in A. The hesitation margin $\pi_A^i(x)$ for each $i = 1, . . ., n$ is the degree of non-determinacy of $x \in X$, to the set A and $\pi_A^i(x) \in [0,1]$.The function $\pi_A^i(x)$ expresses lack of knowledge of whether $x \in X$ or not. Note: $\mu_A^i(x) + \nu_A^i(x) + \pi_A^i(x) = 1$.

4. Operations in Intuitionistic Fuzzy Multisets

[Complement] $A^c = \{\langle x, \nu_A^i(x), \mu_A^i(x)\rangle : x \in X\}$

[Union] $A \cup B = \{\langle x, max\ (\mu_A^i(x), \mu_B^i(x)), min\ (\nu_A^i(x), \nu_B^i(x))\rangle : x \in X\}$

[Intersection] $A \cap B = \{\langle x, min\ (\mu_A^i(x), \mu_B^i(x)), max\ (\nu_A^i(x), \nu_B^i(x))\rangle : x \in X\}$

[Addition] $A \oplus B = \{\langle x, (\mu_A^i(x) + \mu_B^i(x) - \mu_A^i(x)\mu_B^i(x)), (\nu_A^i(x)\ \nu_B^i(x))\rangle : x \in X\}$

[Multiplication] $A \otimes B = \{\langle x, (\mu_A^i(x)\mu_B^i(x)), (\nu_A^i(x) + \nu_B^i(x) - \nu_A^i(x)\ \nu_B^i(x))\rangle : x \in X\}$.

5. Modal Operators in Intuitionistic Fuzzy Multisets

We define over the set of all IFMSs, two modal operators, □ and ◊, which transform every IFMS to FMS.These modal operators are similar to the operators; *necessity* and *possibility* defined in some modal logics. The concept of modal operators in IFS was introduced in [3] but we extend the concept to IFMS.

Definition 5: Let X be nonempty. If A is an IFMS drawn from X , then;

(i) $\Box A = \{\langle x, \mu_A^i(x)\rangle : x \in X\} = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$

(ii) $\Diamond A = \{\langle x, 1-\nu_A^i(x)\rangle : x \in X\} = \{\langle x, 1-\nu_A(x), \nu_A(x), \rangle : x \in X\}$, for each $i = 1,2,\dots,n$.

Theorem 1: *Let X be nonempty. For every IFMS A in X*;

a. $\Box\Box A = \Box A$

b. $\Box\Diamond A = \Diamond A$

c. $\Diamond\Box A = \Box A$

d. $\Diamond\Diamond A = \Diamond A$

Proof

$a.\ A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\}$

$\Box A = \{\langle x, \mu_A^i(x)\rangle : x \in \mathrm{X}\} = \{\langle x, \mu_A^i(x), 1 - \mu_A^i(x)\rangle : x \in \mathrm{X}\}$; $\Box A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\}$

$\Box\Box A = \{\langle x, \mu_A^i(x)\rangle : x \in \mathrm{X}\} = \{\langle x, \mu_A^i(x), 1 - \mu_A^i(x)\rangle : x \in \mathrm{X}\} = \Box A$

$b.\ A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\} = \{\langle x, 1 - \nu_A^i(x), 1 - \mu_A^i(x)\rangle : x \in X\}$

$\Diamond A = \{\langle x, 1 - \nu_A^i(x)\rangle : x \in X\} = \{\langle x, 1 - (1 - \mu_A^i(x))\rangle : x \in X\}$

$= \{\langle x, \mu_A^i(x)\rangle : x \in X\} = \{\langle x, \mu_A^i(x), 1 - \mu_A^i(x)\rangle : x \in \mathrm{X}\} = \Box\Diamond A$

Then $\Box\Diamond A = \{\langle x, \mu_A^i(x), 1 - \mu_A^i(x)\rangle : x \in \mathrm{X}\} = \{\langle x, 1 - \nu_A^i(x), \nu_A^i(x)\rangle : x \in \mathrm{X}\} = \Diamond A$

$c.\ A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\}$

$\Box A = \{\langle x, \mu_A^i(x)\rangle : x \in X\} = \{\langle x, \mu_A^i(x), 1 - \mu_A^i(x)\rangle : x \in X\}$

$= \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\} = \{\langle x, 1 - \nu_A^i(x), 1 - \mu_A^i(x)\rangle : x \in X\}$

$\Diamond\Box A = \{\langle x, 1 - \nu_A^i(x)\rangle : x \in X\} = \{\langle x, \mu_A^i(x)\rangle : x \in X\} = \Box A$

$d. A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\} = \{\langle x, 1-\nu_A^i(x), 1-\mu_A^i(x)\rangle : x \in X\}$

$\lozenge A = \{\langle x, 1-\nu_A^i(x)\rangle : x \in X\} = \{\langle x, 1-\nu_A^i(x), \nu_A^i(x)\rangle : x \in X\} = \{\langle x, 1-\nu_A^i(x), 1-\mu_A^i(x)\rangle : x \in X\}$

$\lozenge\lozenge A = \{\langle x, 1-\nu_A^i(x)\rangle : x \in X\} = \lozenge A.$ □

Theorem 2: *Let X be nonempty. For every two IFMSs A and B in X*;

a. $\square(A \cap B) = \square A \cap \square B$

b. $\lozenge(A \cap B) = \lozenge A \cap \lozenge B$

c. $\square(A \cup B) = \square A \cup \square B$

d. $\lozenge(A \cup B) = \lozenge A \cup \lozenge B$

e. $\square(A \oplus B) = \square A \oplus \square B$

f. $\square(A \otimes B) = \square A \otimes \square B$

g. $\lozenge(A \otimes B) = \lozenge A \otimes \lozenge B$

h. $\lozenge(A \oplus B) = \lozenge A \oplus \lozenge B$

Proof

$a. A \cap B = \{\langle x, min(\mu_A^i(x), \mu_B^i(x)), max(\nu_A^i(x), \nu_B^i(x))\rangle : x \in X\}$

$\square(A \cap B) = \{\langle x, min(\mu_A^i(x), \mu_B^i(x))\rangle : x \in X\} = \{\langle x, \mu_A^i(x)\rangle : x \in X\} \cap \{\langle x, \mu_B^i(x)\rangle : x \in X\} = \square A \cap \square B$

$\square(A \cap B) = \square A \cap \square B$

$b. A \cap B = \{\langle x, min(\mu_A^i(x), \mu_B^i(x)), max(\nu_A^i(x), \nu_B^i(x))\rangle : x \in X\}$

$A \cap B = \{\langle x, min(1-\nu_A^i(x), 1-\nu_B^i(x)), max(1-\mu_A^i(x), 1-\mu_B^i(x))\rangle : x \in X\}$

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$$\Diamond\ (A \cap B) = \{\langle x, min\ (1-\nu_A^i(x), 1-\nu_B^i(x))\rangle : x \in X\}$$

$$\Diamond(A \cap B) = \{\langle x,\ 1-\nu_A^i(x)\rangle : x \in X\} \cap \{\langle x,\ 1-\nu_B^i(x))\rangle : x \in X\} = \Diamond A \cap \Diamond B$$

$$\Diamond(A \cap B) = \Diamond A \cap \Diamond B$$

$$c.\ A \cup B = \{\langle x, max\ (\mu_A^i(x), \mu_B^i(x)), min\ (\nu_A^i(x), \nu_B^i(x))\rangle : x \in X\}$$

$$\Box(A \cup B) = \{\langle x, max\ (\mu_A^i(x), \mu_B^i(x))\rangle : x \in X\} = \{\langle x, \mu_A^i(x)\rangle : x \in X\} \cup \{\langle x, \mu_B^i(x)\rangle : x \in X\} = \Box A \cup \Box B$$

$$\Box(A \cup B) = \Box A \cup \Box B$$

$$d.\ A \cup B = \{\langle x, max\ (\mu_A^i(x), \mu_B^i(x)), min\ (\nu_A^i(x), \nu_B^i(x))\rangle : x \in X\}$$

$$A \cup B = \{\langle x,\ max\ (1-\nu_A^i(x), 1-\nu_B^i(x)), min\ (1-\mu_A^i\ (x), 1-\mu_B^i(x))\rangle : x \in X\}$$

$$\Diamond(A \cup B) = \{\langle x, max(1-\nu_A^i(x), 1-\nu_B^i(x))\rangle : x \in X\} = \{\langle x, (1-\nu_A^i(x)\rangle : x \in X\} \cup \{\langle x, (1-\nu_B^i(x)\rangle : x \in X\} = \Diamond A \cup \Diamond B$$

$$\Diamond(A \cup B) = \Diamond A \cup \Diamond B$$

$$e.\ A \oplus B = \{\langle x, (\mu_A^i(x) + \mu_B^i(x) - \mu_A^i(x)\mu_B^i(x)), (\nu_A^i(x)\ \nu_B^i(x))\rangle : x \in X\}$$

$$A \oplus B = \{\langle x, (\mu_A^i(x) + \mu_B^i(x) - (1-\nu_A^i(x)\ )(1-\nu_B^i(x)), \nu_A^i(x)\ \nu_B^i(x)\rangle : x \in X\}$$

$$\Box\ (A \oplus B) = \{\langle x, (\mu_A^i(x) + \mu_B^i(x)\rangle : x \in X\} = \{\langle x, (\mu_A^i(x)\rangle : x \in X\} \oplus \{\langle x, (\mu_B^i(x)\rangle : x \in X\} = \Box A \oplus \Box B$$

$$\Box\ (A \oplus B) = \Box A \oplus \Box B$$

$$f.\ A \otimes B = \{\langle x, (\mu_A^i(x)\mu_B^i(x)), (\nu_A^i(x) +\ \nu_B^i(x) -\ \nu_A^i(x)\ \nu_B^i(x)\ )\ \rangle : x \in X\}$$

$$\Box(A \otimes B) = \{\langle x, \mu_A^i(x)\mu_B^i(x)\rangle : x \in X\} = \{\langle x, \mu_A^i(x)\rangle : x \in X\} \otimes \{\langle x, \mu_B^i(x)\rangle : x \in X\} = \Box A \otimes \Box B$$

$\Box(A\otimes B) = \Box A\otimes\Box B$

$g. A\otimes B = \{\langle x, (\mu_A^i(x)\mu_B^i(x)), (\nu_A^i(x) + \nu_B^i(x) - \nu_A^i(x)\,\nu_B^i(x))\rangle : x \in X\}$

$= \{\langle x, (1 - \nu_A^i(x))(1 - \nu_B^i(x)), (1 - \mu_A^i(x) + 1 - \mu_B^i(x) - (1 - \mu_A^i(x))(1 - \mu_B^i(x)))\rangle : x \in X\}$

$\Diamond(A\otimes B) = \{\langle x, (1 - \nu_A^i(x))(1 - \nu_B^i(x))\rangle : x \in X\}$

$= \{\langle x, (1 - \nu_A^i(x))\rangle : x \in X\} \otimes \{\langle x, (1 - \nu_B^i(x))\rangle : x \in X\} = \Diamond A\otimes \Diamond B$

$\Diamond(A\otimes B) = \Diamond A\otimes \Diamond B$

$h. A\oplus B = \{\langle x, (\mu_A^i(x) + \mu_B^i(x) - \mu_A^i(x)\mu_B^i(x)), (\nu_A^i(x)\,\nu_B^i(x))\rangle : x \in X\}$

$A\oplus B = \{\langle x, (1 - \nu_A^i(x) + 1 - \nu_B^i(x)) - \mu_A^i(x)\mu_B^i(x)), (1 - \mu_A^i(x))(1 - \mu_B^i(x))\rangle : x \in X\}$

$\Diamond(A\oplus B) = \{\langle x, (1 - \nu_A^i(x) + 1 - \nu_B^i(x))\rangle : x \in X\} = \{\langle x, 1 - \nu_A^i(x)\rangle : x \in X\} \oplus \{\langle x, 1 - \nu_B^i(x)\rangle : x \in X\} = \Diamond A\oplus\Diamond B$

$\Diamond(A\oplus B) = \Diamond A\oplus\Diamond B.$ □

Theorem 3*: Let X be a nonempty IFMS. Let* $A, B \in X$*, then*

$a. A \subseteq \Box B$ *if and only if* $\Box A \subseteq \Box B$

$b. A \subseteq \Diamond B$ *if and only if* $\Diamond A \subseteq \Diamond B$

Proof: Given that $A = \{\langle x, \mu_A^i(x), \nu_A^i(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B^i(x), \nu_B^i(x)\rangle : x \in X\}$.

$a. \Box B = \{\langle x, \mu_B^i(x)\rangle : x \in X\} = \{\langle x, \mu_B^i(x), 1 - \mu_B^i(x)\rangle : x \in X\} = \{\langle x, \mu_B^i(x), \nu_B^i(x)\rangle : x \in X\} = B$

i.e. $\Box B = B$ and consequently $\Box A = A$ whenever π^i tend to zero in IFMS.

$A \subseteq \square B \Rightarrow \mu_A^i(x) \leq \mu_B^i(x)$ and $\nu_A^i(x) \geq \nu_B^i(x) \forall x \in X$. Since $\square A = A \Rightarrow \square A \subseteq \square B$.

Conversely, if $\square A \subseteq \square B \Rightarrow A \subseteq \square B$ for the same reason as in above.

$b. \lozenge B = \{\langle x, 1 - \nu_B^i(x)\rangle : x \in X\} = \{\langle x, 1 - \nu_B^i(x), \nu_B^i(x)\rangle : x \in X\} = \{\langle x, \mu_B^i(x), \nu_B^i(x)\rangle : x \in X\} = B$

$\Rightarrow \lozenge B = B$ and also $\lozenge A = A$. Then if $A \subseteq \lozenge B \Rightarrow \mu_A^i(x) \leq \mu_B^i(x)$ and $\nu_A^i(x) \geq \nu_B^i(x) \forall x \in X$. Obviously, $\lozenge A \subseteq \lozenge B$ since $\lozenge A = A$.

Conversely, if $\lozenge A \subseteq \lozenge B \Rightarrow A \subseteq \lozenge B$ is true for the same reason as in above.□

Conclusions

We have presented and proved some basic theorems using the modal operators on IFMS. Extending the work to Cartesian product of two IFMSs thereby defining relations and functions more theorems could be developed.

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