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Some multivariable orthogonal polynomials of the Askey tableau-discrete families

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A multivariable generalization is presented for all the discrete families of the Askey tableau. This significantly extends the multivariable Hahn polynomials introduced by Karlin and McGregor. The latter are recovered as a limit case from a family of multivariable Racah polynomials.

I. INTRODUCTION

At the head of the Askey tableau[1] of classical orthogonal polynomials lie the Wilson family[2,3] and their discrete analogs the Racah polynomials.[2] The latter can be expressed as the following hypergeometric series

$$r_n(\alpha,\beta,\delta,\gamma|x) = (\alpha+1)_n(\gamma+1)_n(\beta+\delta+1)_n \times {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix};1\right), \quad (1.1)$$

where $\alpha,\beta,\delta,\gamma$ are complex parameters, n is a non-negative integer, and $(\alpha+1)_n \equiv \Gamma(n+\alpha+1)/\Gamma(\alpha+1)$ denotes the Pochhammer symbol. These are polynomials of degree $2n$ in x and they satisfy the following discrete orthogonality relation:

$$\sum_{x=0}^{\Delta} \frac{(\gamma+\delta+1)_x(\gamma/2+\delta/2+3/2)_x(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x}{(\gamma/2+\delta/2+1/2)_x(\gamma+\delta-\alpha+1)_x(\gamma-\beta+1)_x(\delta+1)_x x!} r_n(x)r_m(x) = \lambda_n\delta_{nm}, \quad (1.2)$$

$$\lambda_n = n!(\alpha+1)_n(\beta+1)_n(\gamma+1)_n(\alpha-\delta+1)_n(\alpha+\beta-\gamma+1)_n(\beta+\delta+1)_n \frac{(n+\alpha+\beta+1)_n}{(\alpha+\beta+2)_{2n}} \times \frac{\Gamma(\gamma+\delta-\alpha+1)\Gamma(-\alpha-\beta-1)\Gamma(\gamma-\beta+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+2)\Gamma(-\beta)\Gamma(\gamma-\alpha-\beta)\Gamma(\delta-\alpha)},$$

with $\alpha+1, \beta+\delta+1$, or $\gamma+1=-\Delta$, where Δ is a non-negative integer. The x sum is over the positive integers $x=0,1,2,\ldots,\Delta$ and the indices are confined by $0\leqslant n,m\leqslant\Delta$.

The limit $\delta\to\infty$ with $\alpha+1$ or $\gamma+1=-\Delta$ gives the Hahn orthogonality relation

$$\sum_{x=0}^{\Delta} \frac{(\alpha+1)_x(\gamma+1)_x}{(\gamma-\beta+1)_x x!} h_n(x)h_m(x) = \lambda_n\delta_{nm}, \quad (1.3)$$

where

$$h_n(\alpha,\beta,\gamma|x) = (\alpha+1)_n(\gamma+1)_n \times {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, \gamma+1 \end{matrix};1\right),$$

$$0\leqslant n\leqslant\Delta,$$

$$\lambda_n = (-1)^n n!(\alpha+1)_n(\beta+1)_n(\gamma+1)_n \times(\alpha+\beta-\gamma+1)_n \frac{(n+\alpha+\beta+1)_n}{(\alpha+\beta+2)_{2n}} \times \frac{\Gamma(-\alpha-\beta-1)\Gamma(\gamma-\beta+1)}{\Gamma(-\beta)\Gamma(\gamma-\alpha-\beta)}. \quad (1.4)$$

Letting $\beta\to\infty$ in (1.2) with $\alpha+1=-\Delta$ gives the dual Hahn orthogonality

$$\sum_{x=0}^{\Delta} \frac{(\gamma+\delta+1)_x(\gamma/2+\delta/2+3/2)_x(\alpha+1)_x(\gamma+1)_x}{(\gamma/2+\delta/2+1/2)_x(\gamma+\delta-\alpha+1)_x(\delta+1)_x x!} \times(-1)^x d_n(x)d_m(x) = \lambda_n\delta_{nm}, \quad (1.5)$$

with

$$d_n(\alpha,\delta,\gamma|x) = (\alpha+1)_n(\gamma+1)_n \times {}_3F_2\left(\begin{matrix} -n, -x, x+\gamma+\delta+1 \\ \alpha+1, \gamma+1 \end{matrix};1\right),$$

$$0\leqslant n\leqslant\Delta,$$

$$\lambda_n = n!(\alpha+1)_n(\gamma+1)_n(\alpha-\delta+1)_n \times \frac{\Gamma(\gamma+\delta-\alpha+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+2)\Gamma(\delta-\alpha)}. \quad (1.6)$$

The Meixner polynomials limit from the Hahn family by transforming $\alpha\to\xi c$, $\gamma+1\to\beta$, $\beta\to-\xi$, and taking $\xi\to\infty$. These are given by

$$\sum_{x=0}^{\infty} \frac{c^x}{x!}(\beta)_x m_n(x)m_{n'}(x) = (\beta)_n n! c^{-n}(1-c)^{-\beta}\delta_{nn'},$$

$$m_n(\beta,c|x) = (\beta)_n\, {}_2F_1(-n,-x;\beta;1-c^{-1}), \quad (1.7)$$

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where the sum is over all non-negative integers $x=0,1,2,...,\infty$. The Krawtchouk polynomials are obtained from the Meixner family in the special case when

$$\beta = -\Delta, \quad c = q/(q-1), \tag{1.8}$$

where Δ is again a non-negative integer, and these satisfy

$$\sum_{x=0}^{\Delta} \binom{\Delta}{x} q^x (1-q)^{\Delta - x} k_n(x) k_m(x) = \frac{\Delta! n!}{(\Delta - n)!} q^{-n} (1-q)^n \delta_{nm},$$

$$k_n(\Delta, q|x) = (-\Delta)_n \, {}_2F_1(-n, -x; -\Delta; q^{-1}). \tag{1.9}$$

The Charlier polynomials result upon setting $c = a/\xi$ and $\beta = \xi$ in the Meixner family and taking $\xi \to \infty$. These are given by

$$\sum_{x=0}^{\infty} \frac{a^x}{x!} c_n(x) c_m(x) = n! a^{-n} \exp(a) \delta_{nm},$$

$$c_n(a|x) = {}_2F_0(-n, -x; -a^{-1}). \tag{1.10}$$

A family of multivariable Hahn polynomials was introduced by Karlin and McGregor[4] in the context of linear growth models with many types. In this paper we extend those polynomials to the remaining discrete families of the Askey tableau. That is, to the Racah, dual Hahn, Meixner, Krawtchouk, and Charlier polynomials. The Hahn family is recovered as a limit case of the Racah polynomials. The analogous generalizations of the continuous families of the Askey tableau are discussed in a companion paper.[5]

II. MULTIVARIABLE RACAH, HAHN, AND DUAL HAHN, POLYNOMIALS

The multivariable Racah polynomials of p variables $x_1, x_2 ... x_p$ are given by the following expression

$$R\begin{pmatrix} \alpha_1, \alpha_2, ..., \alpha_{p+1} & x_1, x_2, ..., x_p \\ \eta, \gamma & n_1, n_2, ..., n_p \end{pmatrix} = \Big[\prod_{k=1}^{p-1} r_{n_k}(2N_1^{k-1} + \eta + \alpha_2^k, \alpha_{k+1} - 1, N_1^{k-1} + \alpha_1^k + x_{k+1}, N_1^{k-1} - x_{k+1} - 1 | -N_1^{k-1} + x_k)\Big]$$

$$\times r_{n_p}(2N_1^{p-1} + \eta + \alpha_2^p, \alpha_{p+1} - 1, N_1^{p-1} + \alpha_1^p - \gamma - 1, N_1^{p-1} + \gamma | -N_1^{p-1} + x_p), \tag{2.1}$$

where $r_n(x)$ are the single variable Racah polynomials (1.1) and we are using the following shorthand notation

$$N_j^i \equiv \sum_{k=j}^{i} n_k, \quad \alpha_j^i \equiv \sum_{k=j}^{i} \alpha_k \quad (N_i^{i-1} = \alpha_i^{i-1} \equiv 0). \tag{2.2}$$

Although these polynomials are products of single variable orthogonal polynomials they are nontrivial in that the parameter arguments also depend on the variables. The parameters $\alpha_1, \alpha_2, ..., \alpha_{p+1}, \eta, \gamma$ are in general complex and $x_1, x_2, ..., x_p, n_1, n_2, ..., n_p$ are non-negative integers. These polynomials are of total degree $2N_1^p$ in the variables $x_1, x_2, ..., x_p$ and are associated with the following weight function

$$\rho(\alpha_1, ..., \alpha_{p+1}, \eta, \gamma | x_1, x_2, ..., x_p) = \frac{(\alpha_1)_{x_1}}{x_1!} \frac{(\eta+1)_{x_1}}{(\alpha_1 - \eta)_{x_1}} \Big[\prod_{k=1}^{p-1} \frac{\Gamma(\alpha_{k+1} + x_{k+1} - x_k)}{(x_{k+1} - x_k)!} \times \frac{\Gamma(\alpha_1^{k+1} + x_{k+1} + x_k)}{\Gamma(\alpha_1^k + 1 + x_{k+1} + x_k)} \frac{(\alpha_1^k/2 + 1)_{x_k}}{(\alpha_1^k/2)_{x_k}}\Big]$$

$$\times \frac{(\alpha_1^p/2+1)_{x_p}}{(\alpha_1^p/2)_{x_p}} \frac{(\alpha_1^{p+1} - \gamma - 1)_{x_p}}{(\gamma - \alpha_{p+1} + 2)_{x_p}} \frac{(\gamma+1)_{x_p}}{(\alpha_1^p - \gamma)_{x_p}}, \tag{2.3}$$

which is nonvanishing over the domain $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_p \leqslant \Delta$ where Δ is a non-negative integer and $\gamma + 1 = -\Delta$. The parameters are assumed to be such that this weight function is finite. In the case $p=1$ the empty products are to be interpreted as unity in which case (2.1) and (2.3) reduce to the respective single variable expressions (1.1) and (1.2) apart from a redefinition of the parameters.

These polynomials satisfy the following orthogonality relation:

$$\sum_{x_p=0}^{\Delta} \sum_{x_{p-1}=0}^{x_p} \cdots \sum_{x_2=0}^{x_3} \sum_{x_1=0}^{x_2} \rho(x) R_n(x) R_m(x) = \lambda_n \prod_{k=1}^{p} \delta_{n_k m_k},$$

$$\lambda_n = \Big[\prod_{k=1}^{p} (\alpha_1^k)^{-1} n_k! \Gamma(n_k + \alpha_{k+1})(N_1^k + N_1^{k-1} + \eta + \alpha_2^{k+1})_{n_k} \frac{\Gamma(N_1^k + N_1^{k-1} + \eta + \alpha_2^k + 1)}{\Gamma(2N_1^k + \eta + \alpha_2^{k+1} + 1)}\Big]$$

$$\times \frac{\Gamma(\alpha_1 - \eta)\Gamma(\alpha_1^p - \gamma)\Gamma(N_1^p + \eta + \alpha_2^{p+1} - \gamma)}{\Gamma(\alpha_1)\Gamma(\eta+1)\Gamma(\alpha_{p+1} - \gamma - 1)\Gamma(\alpha_1 - \eta - \gamma - 1)} (\alpha_1^{p+1} - \gamma - 1)_{N_1^p} (\gamma+1)_{N_1^p} (\eta - \alpha_1 + \gamma + 2)_{N_1^p}, \tag{2.4}$$

which is proven as follows. The x_1 summation

$$\sum_{x_1=0}^{x_2} \frac{(\alpha_1)_{x_1}(\eta+1)_{x_1}\Gamma(\alpha_2 + x_2 - x_1)\Gamma(\alpha_1^2 + x_2 + x_1)(\alpha_1/2+1)_{x_1}}{x_1!(\alpha_1 - \eta)_{x_1}(x_2 - x_1)!\Gamma(\alpha_1 + 1 + x_2 + x_1)(\alpha_1/2)_{x_1}} \times r_{n_1}(\eta, \alpha_2 - 1, \alpha_1 + x_2, -x_2 - 1 | x_1) r_{m_1}(\eta, \alpha_2 - 1, \alpha_1 + x_2, -x_2 - 1 | x_1) \tag{2.5}$$

is evaluated by using the single variable results (1.1) and (1.2) with the parameters depending on the variable x_2. This gives

$$\delta_{n_1 m_1}(\alpha_1)^{-1} n_1!\Gamma(n_1+\alpha_2)(n_1+\eta+\alpha_2)_{n_1}\frac{\Gamma(n_1+\eta+1)}{\Gamma(2n_1+\eta+\alpha_2+1)}\frac{\Gamma(\alpha_1-\eta)}{\Gamma(\alpha_1)\Gamma(\eta+1)}$$
$$\times\frac{\Gamma(n_1+\eta+\alpha_2+1+x_2)\Gamma(n_1+\alpha_1^2+x_2)}{(x_2-n_1)!\Gamma(\alpha_1-\eta+x_2-n_1)}, \tag{2.6}$$

which suggests that after the $x_1, x_2, \ldots, x_j$ summations we obtain

$$\left[\prod_{k=1}^{j}\delta_{n_k m_k}(\alpha_1^k)^{-1} n_k!\Gamma(n_k+\alpha_{k+1})(N_1^k+N_1^{k-1}+\eta+\alpha_2^{k+1})_{n_k}\frac{\Gamma(N_1^k+N_1^{k-1}+\eta+\alpha_2^k+1)}{\Gamma(2N_1^k+\eta+\alpha_2^{k+1}+1)}\right]$$
$$\times\frac{\Gamma(\alpha_1-\eta)}{\Gamma(\alpha_1)\Gamma(\eta+1)}\frac{\Gamma(N_1^j+\eta+\alpha_2^{j+1}+1+x_{j+1})\Gamma(N_1^j+\alpha_1^{j+1}+x_{j+1})}{(x_{j+1}-N_1^j)!\Gamma(\alpha_1-\eta+x_{j+1}-N_1^j)}, \quad j=1,2,\ldots,p-1, \tag{2.7}$$

where by convention $1/(x_{j+1}-N_1^j)!=0$ if $x_{j+1}-N_1^j=0,-1,-2,\ldots$. The postulate (2.7) is then proven by induction on j. Multiplying (2.7) by the remaining x_{j+1} dependent part of the weight function and polynomials and summing over x_{j+1} gives

$$\left[\prod_{k=1}^{j}\delta_{n_k m_k}(\alpha_1^k)^{-1} n_k!\Gamma(n_k+\alpha_{k+1})(N_1^k+N_1^{k-1}+\eta+\alpha_2^{k+1})_{n_k}\frac{\Gamma(N_1^k+N_1^{k-1}+\eta+\alpha_2^k+1)}{\Gamma(2N_1^k+\eta+\alpha_2^{k+1}+1)}\right]\frac{\Gamma(\alpha_1-\eta)}{\Gamma(\alpha_1)\Gamma(\eta+1)}$$
$$\times\sum_{x_{j+1}=0}^{x_{j+2}}\frac{\Gamma(N_1^j+\eta+\alpha_2^{j+1}+1+x_{j+1})\Gamma(N_1^j+\alpha_1^{j+1}+x_{j+1})}{(x_{j+1}-N_1^j)!\Gamma(\alpha_1-\eta+x_{j+1}-N_1^j)}$$
$$\times\frac{\Gamma(\alpha_{j+2}+x_{j+2}-x_{j+1})\Gamma(\alpha_1^{j+2}+x_{j+2}+x_{j+1})(\alpha_1^{j+1}/2+1)_{x_{j+1}}}{(x_{j+2}-x_{j+1})!\Gamma(\alpha_1^{j+1}+1+x_{j+2}+x_{j+1})(\alpha_1^{j+1}/2)_{x_{j+1}}}$$
$$\times r_{n_{j+1}}(2N_1^j+\eta+\alpha_2^{j+1},\alpha_{j+2}-1,N_1^j+\alpha_1^{j+1}+x_{j+2},N_1^j-x_{j+2}-1|-N_1^j+x_{j+1})$$
$$\times r_{m_{j+1}}(2M_1^j+\eta+\alpha_2^{j+1},\alpha_{j+2}-1,M_1^j+\alpha_1^{j+1}+x_{j+2},M_1^j-x_{j+2}-1|-M_1^j+x_{j+1}), \tag{2.8}$$

and noting that this expression vanishes unless $M_1^j=N_1^j$ due to the Kronecker deltas, the summation is then evaluated by (1.1) and (1.2) yielding

$$\left[\prod_{k=1}^{j+1}\delta_{n_k m_k}(\alpha_1^k)^{-1} n_k!\Gamma(n_k+\alpha_{k+1})(N_1^k+N_1^{k-1}+\eta+\alpha_2^{k+1})_{n_k}\frac{\Gamma(N_1^k+N_1^{k-1}+\eta+\alpha_2^k+1)}{\Gamma(2N_1^k+\eta+\alpha_2^{k+1}+1)}\right]$$
$$\times\frac{\Gamma(\alpha_1-\eta)}{\Gamma(\alpha_1)\Gamma(\eta+1)}\frac{\Gamma(N_1^{j+1}+\eta+\alpha_2^{j+2}+1+x_{j+2})\Gamma(N_1^{j+1}+\alpha_1^{j+2}+x_{j+2})}{(x_{j+2}-N_1^{j+1})!\Gamma(\alpha_1-\eta+x_{j+2}-N_1^{j+1})}, \tag{2.9}$$

which is (2.7) with $j+1$ replacing j thus proving (2.7). Setting $j=p-1$ in (2.7) and substituting into the left side of (2.4) then gives

$$\sum_{x_p=0}^{\Delta}\sum_{x_{p-1}=0}^{x_p}\cdots\sum_{x_2=0}^{x_3}\sum_{x_1=0}^{x_2}\rho(x)R_n(x)R_m(x)$$
$$=\left[\prod_{k=1}^{p-1}\delta_{n_k m_k}(\alpha_1^k)^{-1} n_k!\Gamma(n_k+\alpha_{k+1})(N_1^k+N_1^{k-1}+\eta+\alpha_2^{k+1})_{n_k}\frac{\Gamma(N_1^k+N_1^{k-1}+\eta+\alpha_2^k+1)}{\Gamma(2N_1^k+\eta+\alpha_2^{k+1}+1)}\right]$$
$$\times\frac{\Gamma(\alpha_1-\eta)}{\Gamma(\alpha_1)\Gamma(\eta+1)}\sum_{x_p=0}^{\Delta}\frac{\Gamma(N_1^{p-1}+\eta+\alpha_2^p+1+x_p)\Gamma(N_1^{p-1}+\alpha_1^p+x_p)}{(x_p-N_1^{p-1})!\Gamma(\alpha_1-\eta+x_p-N_1^{p-1})}\frac{(\alpha_1^p/2+1)_{x_p}}{(\alpha_1^p/2)_{x_p}}$$
$$\times\frac{(\alpha_1^{p+1}-\gamma-1)_{x_p}}{(\gamma-\alpha_{p+1}+2)_{x_p}}\frac{(\gamma+1)_{x_p}}{(\alpha_1^p-\gamma)_{x_p}}r_{n_p}(2N_1^{p-1}+\eta+\alpha_2^p,\alpha_{p+1}-1,N_1^{p-1}+\alpha_1^p-\gamma-1,N_1^{p-1}+\gamma|-N_1^{p-1}+x_p)$$
$$\times r_{m_p}(2M_1^{p-1}+\eta+\alpha_2^p,\alpha_{p+1}-1,M_1^{p-1}+\alpha_1^p-\gamma-1,M_1^{p-1}+\gamma|-M_1^{p-1}+x_p), \tag{2.10}$$

and as before noting that this expression vanishes unless $M_1^{p-1}=N_1^{p-1}$ due to the Kronecker deltas, this summation is then also evaluated by using (1.1) and (1.2) yielding the orthogonality relation (2.4).

Apart from normalization, the weight function (2.3) and summation region are invariant under the following permutation of parameters and variables:

$$\alpha_1\leftrightarrow-\alpha_1^p+2\gamma+2, \quad \alpha_{k+1}\leftrightarrow\alpha_{p-k+1},$$
$$k=1,2,\ldots,p-1,$$
$$\alpha_{p+1}\leftrightarrow\eta+1, \quad x_k\leftrightarrow\Delta-x_{p-k+1}, \quad k=1,2,\ldots,p, \tag{2.11}$$

and if this is applied to the orthogonality relation (2.4) it implies that the transformed polynomials also form an orthogonal family with the same weight function. The polynomials are not invariant under (2.11) so one immediately obtains a second family of multivariable Racah polynomials. After a redefinition of the indices $n_k \leftrightarrow n_{p-k+1}$ this second family is given by

$$\bar{R}\left(\begin{matrix}\alpha_1,\dots,\alpha_{p+1}\\ \eta,\gamma\end{matrix}\middle|\begin{matrix}x_1,\dots,x_p\\ n_1,\dots,n_p\end{matrix}\right)$$
$$= r_{n_1}(2N_2^p + \alpha_2^{p+1} - 1, \eta, N_2^p - \alpha_1 + \gamma + 1, N_2^p + \gamma | - N_2^p + \Delta - x_1)$$
$$\times \left[\prod_{k=2}^{p} r_{n_k}(2N_{k+1}^p + \alpha_{k+1}^{p+1} - 1, \alpha_k - 1, N_{k+1}^p - \alpha_1^k - \Delta - x_{k-1}, N_{k+1}^p + \gamma + x_{k-1} | - N_{k+1}^p + \Delta - x_k)\right], \tag{2.12}$$

and these are also of total degree $2N_1^p$ in $x_1,x_2,\dots,x_p$. The normalization constant is not given explicitly but it can be evaluated from (2.4) after applying (2.11).

Another multivariable generalization of the Racah polynomials has been studied by Gustafson.[6] These are closely related to the so-called U(n) multivariable hypergeometric series introduced by Holman *et al.*[7,8] Those polynomials are associated with a different weight function than (2.3) and apparently are not related to the polynomials discussed here.

Multivariable dual Hahn polynomials follow as limit cases of these Racah families. For example, dividing (2.4) by $\eta^{N_1^p + M_1^p}$ and taking the limit $\eta \to \infty$ yields

$$\sum_{x_p=0}^{\Delta} \sum_{x_{p-1}=0}^{x_p} \cdots \sum_{x_2=0}^{x_3} \sum_{x_1=0}^{x_2} \rho(x) D_n(x) D_m(x) = \lambda_n \prod_{k=1}^{p} \delta_{n_k m_k}, \tag{2.13}$$

$$D\left(\begin{matrix}\alpha_1,\dots,\alpha_{p+1}\\ \gamma\end{matrix}\middle|\begin{matrix}x_1,x_2,\dots,x_p\\ n_1,n_2,\dots,n_p\end{matrix}\right)$$
$$= \left[\prod_{k=1}^{p-1} d_{n_k}(N_1^{k-1} + \alpha_1^{k+1} + x_{k+1} - 1, N_1^{k-1} + \alpha_1^k + x_{k+1}, N_1^{k-1} - x_{k+1} - 1 | - N_1^{k-1} + x_k)\right]$$
$$\times d_{n_p}(N_1^{p-1} + \alpha_1^{p+1} - \gamma - 2, N_1^{p-1} + \alpha_1^p - \gamma - 1, N_1^{p-1} + \gamma | - N_1^{p-1} + x_p),$$

$$\rho(\alpha_1,\dots,\alpha_{p+1},\gamma | x_1,x_2,\dots,x_p)$$
$$= \frac{(\alpha_1)_{x_1}}{x_1!}(-1)^{x_1}\left[\prod_{k=1}^{p-1} \frac{\Gamma(\alpha_{k+1} + x_{k+1} - x_k)}{(x_{k+1} - x_k)!} \times \frac{\Gamma(\alpha_1^{k+1} + x_{k+1} + x_k)}{\Gamma(\alpha_1^k + 1 + x_{k+1} + x_k)} \frac{(\alpha_1^k/2 + 1)_{x_k}}{(\alpha_1^k/2)_{x_k}}\right]$$
$$\times \frac{(\alpha_1^p/2 + 1)_{x_p}}{(\alpha_1^p/2)_{x_p}} \frac{(\alpha_1^{p+1} - \gamma - 1)_{x_p}}{(\gamma - \alpha_{p+1} + 2)_{x_p}} \frac{(\gamma + 1)_{x_p}}{(\alpha_1^p - \gamma)_{x_p}},$$

$$\lambda_n = \left[\prod_{k=1}^{p} (\alpha_1^k)^{-1} n_k! \Gamma(n_k + \alpha_{k+1})\right]$$
$$\times \frac{\Gamma(\alpha_1^p - \gamma)}{\Gamma(\alpha_1)\Gamma(\alpha_{p+1} - \gamma - 1)}$$
$$\times (\alpha_1^{p+1} - \gamma - 1)_{N_1^p} (\gamma + 1)_{N_1^p} (-1)^{\gamma+1},$$

where $d_n(x)$ are the single variable dual Hahn polynomials (1.6). These are also of total degree $2N_1^p$ in $x_1,x_2,\dots,x_p$. After multiplying by appropriate renormalization factors one can obtain further families of multivariable dual Hahn polynomials in the limit $\eta \to \infty$ of the second Racah family, or the limit $\alpha_{p+1} \to \infty$ of either Racah family.

The multivariable Hahn polynomials of Karlin and McGregor[4] are contained as a limit case of the Racah family (2.1). To obtain these divide (2.4) by $\alpha_1^{\alpha_2^p - p + 1 + N_1^p + M_1^p}$, take the limit $\alpha_1 \to \infty$, redefine $\eta \to \alpha_1$, $\alpha_k \to \alpha_k + 1$, $k = 2,3,\dots,p+1$, and make the change of variables $y_1 \equiv x_1$, $y_k \equiv x_k - x_{k-1}$, $k = 2,3,\dots,p$. This gives

$$\sum_{y_p} \cdots \sum_{y_2} \sum_{y_1} \rho(y) H_n(y) H_m(y) = \lambda_n \prod_{k=1}^{p} \delta_{n_k m_k},$$

$$H\left(\begin{matrix}\alpha_1,\dots,\alpha_{p+1}\\ \gamma\end{matrix}\middle|\begin{matrix}y_1,\dots,y_p\\ n_1,\dots,n_p\end{matrix}\right)$$
$$= \left[\prod_{k=1}^{p-1} h_{n_k}(2N_1^{k-1} + \alpha_1^k + k - 1, \alpha_{k+1}, N_1^{k-1} - Y_1^{k+1} - 1 | - N_1^{k-1} + Y_1^k)\right]$$
$$\times h_{n_p}(2N_1^{p-1} + \alpha_1^p + p - 1, \alpha_{p+1}, N_1^{p-1} + \gamma | - N_1^{p-1} + Y_1^p),$$

$$\rho(\alpha_1,\dots,\alpha_{p+1},\gamma | y_1,\dots,y_p) = \left[\prod_{k=1}^{p} \frac{(\alpha_k + 1)_{y_k}}{y_k!}\right. \times \frac{(\gamma + 1)_{Y_1^p}}{(\gamma - \alpha_{p+1} + 1)_{Y_1^p}},$$

$$\lambda_n = \left[\prod_{k=1}^{p} n_k! (N_1^k + N_1^{k-1} + \alpha_1^{k+1} + k)_{n_k} \times \frac{\Gamma(n_k + \alpha_{k+1} + 1)}{\Gamma(\alpha_k + 1)} \times \frac{\Gamma(N_1^k + N_1^{k-1} + \alpha_1^k + k)}{\Gamma(2N_1^k + \alpha_1^{k+1} + k + 1)}\right]$$
$$\times \frac{\Gamma(N_1^p + \alpha_1^{p+1} + p - \gamma)}{\Gamma(\alpha_{p+1} - \gamma)} (\gamma + 1)_{N_1^p} (-1)^{N_1^p}, \tag{2.14}$$

where the summation region is over non-negative integers satisfying $y_k \geqslant 0$, $Y_1^p \leqslant \Delta$, and $h_n(x)$ are the single variable

Hahn polynomials (1.4). These polynomials are of total degree N_1^p in $y_1,y_2,...,y_p$ and form a complete set for polynomials in these variables.

In these variables it is obvious that the summation region and weight function are invariant under an arbitrary permutation of the labels $(1,2,...,p)$. If one defines $y_{p+1}\equiv\Delta - Y_1^p$, then they are also invariant (apart from normalization of the weight function) under an arbitrary permutation of the labels $(1,2,...,p+1)$. That is, under an arbitrary simultaneous permutation of $(\alpha_1,\alpha_2,...,\alpha_{p+1})$ and $(y_1,y_2,...,y_{p+1})$. The polynomials given in (2.14) are, in general, not invariant and thus one generates distinct families satisfying orthogonality relation (2.14) under these permutations.

A family of biorthogonal polynomials[9,10] with the same weight function is already known. These are given by

$$\mathscr{H}_n(y) = F^{1:2;...;2}_{1:1;...;1}\left(\begin{matrix} N_1^p + \alpha_1^{p+1} + p: -n_1, -y_1;...; -n_p, -y_p \\ \gamma+1:\alpha_1+1;...;\alpha_p+1 \end{matrix}\right),$$
$$\bar{\mathscr{H}}_n(y) = (\gamma+1+Y_1^p)_{N_1^p} F^{1:2;...;2}_{1:1;...;1}\left(\begin{matrix} -N_1^p - \alpha_{p+1}: -n_1, -y_1;...; -n_p, -y_p \\ -N_1^p - \gamma - Y_1^p:\alpha_1+1;...;\alpha_p+1 \end{matrix}\right), \tag{2.15}$$
$$\sum_{y_p}\cdots\sum_{y_2}\sum_{y_1}\rho(y)\bar{\mathscr{H}}_n(y)\mathscr{H}_m(y) \sim \prod_{k=1}^{p}\delta_{n_k m_k},$$

where $F^{p:j_1;...;j_p}_{q:i_1;...;i_p}$ is the Srivastava hypergeometric function[11] or more commonly known as the generalized Kampé de Fériet hypergeometric series. These polynomials are invariant under permutations of $(1,2,...,p)$ but are not invariant under permutations involving the $p+1$ label; the latter transformations therefore generate new families of biorthogonal polynomials with respect to the same weight function.

III. MULTIVARIABLE MEIXNER, KRAWTCHOUK, AND CHARLIER POLYNOMIALS

The Hahn polynomials (2.14) contain as a limit case a family of multivariable Meixner polynomials. Set $\gamma+1=\beta$, $\alpha_k=\xi c_k$, $\alpha_{p+1}=-\xi$, divide (2.14) by $\xi^{N_1^p+M_1^p}$, and take the limit $\xi\to\infty$. This yields

$$\sum_{y_p=0}^{\infty}\cdots\sum_{y_2=0}^{\infty}\sum_{y_1=0}^{\infty}\rho(y)M_n(y)M_m(y) = \lambda_n\left[\prod_{k=1}^{p}\delta_{n_k m_k}\right], \tag{3.1}$$

$$M\left(\begin{matrix} c_1,c_2,...,c_p \\ \beta \end{matrix}\middle| \begin{matrix} y_1,y_2,...,y_p \\ n_1,n_2,...,n_p \end{matrix}\right) = \left[\prod_{k=1}^{p-1} m_{n_k}(N_1^{k-1} - Y_1^{k+1}, - C_1^k/c_{k+1} | - N_1^{k-1} + Y_1^k)\right]$$
$$\times m_{n_p}(N_1^{p-1}+\beta, C_1^p | - N_1^{p-1} + Y_1^p),$$

$$\rho(c_1,c_2,...,c_p,\beta|y_1,y_2,...,y_p) = \left[\prod_{k=1}^{p}\frac{c_k^{y_k}}{y_k!}\right](\beta)_{Y_1^p},$$

$$c_k\neq 0,\quad \sum_{k=1}^{p}|c_k|<1,$$

$$\lambda_n = \left[\prod_{k=1}^{p} n_k!(C_1^k)^{-n_k+n_{k-1}}(c_k)^{n_{k-1}}\right](\beta)_{N_1^p}$$
$$\times(1-C_1^p)^{-N_1^{p-1}-\beta}\quad (n_0\equiv 0),$$

where the summation convention is being used for the c_k parameters. These polynomials are also of total degree N_1^p in $y_1,y_2,...,y_p$ and form a complete set. In analogy with the Hahn family, the summation region and weight function are invariant under an arbitrary simultaneous permutation of $(c_1,c_2,...,c_p)$ and $(y_1,y_2,...,y_p)$, and as before one generates new families satisfying (3.1) by applying these permutations to the polynomials.

A family of biorthogonal polynomials[12] with this same weight function are given by

$$\mathscr{M}_n(y) = (Y_1^p+\beta)_{N_1^p} F^{0:2;...;2}_{1:0;...;0}\left(\begin{matrix} -n_1, -y_1;...; -n_p, -y_p \\ -N_1^p - Y_1^p - \beta + 1 \end{matrix}; c_1^{-1}\cdots c_p^{-1}\right),$$
$$\bar{\mathscr{M}}_n(y) = F^{0:2;...;2}_{1:0;...;0}\left(\begin{matrix} -n_1, -y_1;...; -n_p, -y_p \\ \beta \end{matrix}; (C_1^p-1)c_1^{-1}\cdots(C_1^p-1)c_p^{-1}\right),$$
$$\sum_{y_p=0}^{\infty}\cdots\sum_{y_2=0}^{\infty}\sum_{y_1=0}^{\infty}\rho(y)\mathscr{M}_n(y)\bar{\mathscr{M}}_m(y) \sim \prod_{k=1}^{p}\delta_{n_k m_k}, \tag{3.2}$$

and these are invariant under permutations of $(1,2,...,p)$.

A special case of the Meixner family are the multivariable Krawtchouk polynomials. These are obtained for

$$\beta = -\Delta,\quad \Delta=0,1,2...,\quad c_k = q_k/(Q_1^p - 1),$$
$$k=1,2,...,p,$$
$$q_k>0,\quad 0<Q_1^p<1,\quad Q_j^i\equiv\sum_{k=j}^{i} q_k, \tag{3.3}$$

and are given by

$$\sum_{y_p=0}\cdots\sum_{y_1=0}\frac{\Delta!}{(\Delta-Y_1^p)!}\left[\prod_{k=1}^{p}\frac{q_k^{y_k}}{y_k!}\right]\times(1-Q_1^p)^{\Delta-Y_1^p}K_n(y)K_m(y)=\lambda_n\left[\prod_{k=1}^{p}\delta_{n_k m_k}\right],$$

$$K\left(\begin{matrix}q_1,q_2,\ldots,q_p\\ \Delta\end{matrix}\middle|\begin{matrix}y_1,y_2,\ldots,y_p\\ n_1,n_2,\ldots,n_p\end{matrix}\right)=\left[\prod_{j=1}^{p-1}k_{n_j}\left(N_1^{j-1}-Y_1^{j+1},-\frac{Q_1^j}{q_{j+1}}\middle|-N_1^{j-1}+Y_1^j\right)\right]\times k_{n_p}\left(N_1^{p-1}-\Delta,\frac{Q_1^p}{(Q_1^p-1)}\middle|-N_1^{p-1}+Y_1^p\right),$$

$$\lambda_n=\left[\prod_{k=1}^{p}n_k!(Q_1^k)^{-n_k+n_{k-1}}(q_k)^{n_{k-1}}\right]\times\frac{\Delta!}{(\Delta-N_1^p)!}(1-Q_1^p)^{n_p}, \qquad (3.4)$$

where $k_n(y)$ are the single variable Krawtchouk polynomials (1.9) and the sum is over non-negative integers satisfying $y_k \geqslant 0$, $Y_1^p \leqslant \Delta$. This weight function and summation region have additional symmetries compared with the Meixner family. If one defines $y_{p+1}\equiv\Delta-Y_1^p$ and $q_{p+1}\equiv 1-Q_1^p$ then they are invariant under an arbitrary simultaneous permutation of $(y_1,y_2,\ldots,y_{p+1})$ and $(q_1,q_2,\ldots,q_{p+1})$. The polynomials are again not invariant and once more one generates new families of orthogonal polynomials under these permutations.

The biorthogonal families[12] with the same weight function are given by

$$\mathscr{K}_n(y)=(Y_1^p-\Delta)_{N_1^p}F^{0:2;\ldots;2}_{1:0;\ldots;0}\left(\begin{matrix}-n_1,-y_1;\ldots;-n_p,-y_p\\ -N_1^p-Y_1^p+\Delta+1\end{matrix};(Q_1^p-1)q_1^{-1}\cdots(Q_1^p-1)q_p^{-1}\right),$$

$$\overline{\mathscr{K}}_n(y)=F^{0:2;\ldots;2}_{1:0;\ldots;0}\left(\begin{matrix}-n_1,-y_1;\ldots;-n_p,-y_p\\ -\Delta\end{matrix};q_1^{-1}\cdots q_p^{-1}\right),$$

$$\sum_{y_p=0}\cdots\sum_{y_1=0}\rho(y)\mathscr{K}_n(y)\overline{\mathscr{K}}_m(y)\sim\prod_{k=1}^{p}\delta_{n_k m_k}, \qquad (3.5)$$

and these are invariant under permutations of $(1,2,\ldots,p)$ but are not so under permutations involving the label $p+1$.

The Meixner polynomials also contain a family of multivariable Charlier polynomials as a limit case. Set $c_k=a_k/\xi$, $\beta=\xi$, divide (3.1) by $\xi^{n_p+m_p}$, and take $\xi\to\infty$. This yields

$$\sum_{y_p=0}^{\infty}\cdots\sum_{y_1=0}^{\infty}\left[\prod_{k=1}^{p}\frac{a_k^{y_k}}{y_k!}\right]C_n(y)C_m(y)=\lambda_n\left[\prod_{k=1}^{p}\delta_{n_k m_k}\right],$$

$$C\left(a_1,a_2,\ldots,a_p\middle|\begin{matrix}y_1,y_2,\ldots,y_p\\ n_1,n_2,\ldots,n_p\end{matrix}\right)=\left[\prod_{k=1}^{p-1}m_{n_k}\left(N_1^{k-1}-Y_1^{k+1},-\frac{A_1^k}{a_{k+1}}\middle|-N_1^{k-1}+Y_1^k\right)\right]\times c_{n_p}(A_1^p|-N_1^{p-1}+Y_1^p),$$

$$\lambda_n=\left[\prod_{k=1}^{p}n_k!(A_1^k)^{-n_k+n_{k-1}}(a_k)^{n_{k-1}}\right]\exp(A_1^p), \qquad (3.6)$$

where $c_n(y)$ are the single variable Charlier polynomials (1.10). The weight function is simply the product of single variable Charlier weights but the polynomials are a nontrivial multivariable extension. An obvious family of polynomials orthogonal with respect to this weight function are just the products of single variable Charlier polynomials; these are obtained in the above limit with a different renormalization factor from the biorthogonal Meixner family (3.2). They are clearly invariant under the obvious permutation symmetries of the summation region and weight function. The family given in (3.6) however is again not invariant and, as discussed several times, applying these permutations generates new families satisfying (3.6).

IV. DISCUSSION

We have extended the previously known multivariable Hahn[4] polynomials to all of the remaining discrete families of the Askey tableau. The analogous generalizations of the continuous families are discussed in a companion paper.[5]

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[1] R. Askey and J. Wilson, Mem. Am. Math. **54**, 319 (1985).
[2] J. A. Wilson, SIAM J. Math. Anal. **11**, 690 (1980).
[3] R. Askey and J. Wilson, SIAM J. Math. Anal. **13**, 651 (1982).
[4] S. Karlin and J. McGregor, in *Theory and Application of Special Functions*, edited by R. A. Askey (Academic, New York, 1975), p. 261.
[5] M. V. Tratnik, "Some multivariable orthogonal polynomials of the Askey tableau-continuous families," J. Math. Phys. **32**, 2065–2073 (1991).
[6] R. A. Gustafson, SIAM J. Math. Anal. **18**, 495 (1987).
[7] W. J. Holman III, L. C. Biedenharn, and J. D. Louck, SIAM J. Math. Anal. **7**, 529 (1976).
[8] W. J. Holman III, SIAM J. Math. Anal. **11**, 523 (1980).
[9] M. V. Tratnik, J. Math. Phys. **30**, 627 (1989).
[10] M. V. Tratnik, J. Math. Phys. **31**, 1559 (1990).
[11] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series* (Ellis Horwood, West Sussex, 1985), p. 38.
[12] M. V. Tratnik, J. Math. Phys. **30**, 2740 (1989).