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# SOME NATURAL FAMILIES OF M-IDEALS

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**Abstract.**

We characterize the subspaces of $L^1$ and the translation-invariant subspaces of $\mathscr{M}(G)$ which are duals of $M$-ideals, and we describe their $M$-ideal predual. We show that there is a separable dual which is $L$-complemented in its bidual but is not the dual of an $M$-ideal. We show that a separable $\mathscr{L}^\infty$-space which is isomorphic to an $M$-ideal is actually isomorphic to $c_0(N)$.

## 0. Introduction.

Let $X$ be a Banach space. An *L-projection* $p$ is a linear map from $X$ to $X$ such that $p^2 = p$ and

$$\|x\| = \|p(x)\| + \|x - p(x)\| \tag{1}$$

for every $x \in X$. A subspace $Y$ of $X$ is called an *M-ideal* in $X$ if there is an $L$-projection from $X^*$ onto the orthogonal $Y^\perp$ of $X$ in $X^*$. Since these notions were introduced by Alfsen and Effros in 1972 [1], they have attracted a lot of attention; of particular importance is the class of Banach spaces which are $M$-ideals in their bidual; in the present work, such spaces will simply be called *M-ideals*.

These spaces form a very rich family. Although some significative progress has been recently made in the understanding of their structure (see e.g. [2], [13], [6], [9], [10]), it looks hopeless to classify them or to give a complete description of the class. In the present work, we will investigate some natural subfamilies in which positive results are available. We will frequently work in a dual way; that is, we will determine when there exists an $L$-projection from the bidual onto a space whose kernel is $w^*$-closed.

Let us briefly describe the contents of this article. In section I, we characterize the subspaces of $L^1$ which are duals of an $M$-ideal and we describe the $M$-ideal predual; our characterization involves the topology of convergence in measure. Section II deals with the corresponding translation-invariant results; we characterize there the $L^1_\Lambda$ and $\mathscr{M}_\Lambda$-spaces which are duals of $M$-ideals and the quotient

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spaces of $\mathscr{C}(G)$, by translation-invariant subspaces, which are $M$-ideals. Section III is devoted to the construction of an example which uses the results of section II. Through harmonic analysis, we construct a separable dual space $Y$ which is $L$-complemented in its bidual, but whose natural predual is not "what you would expect"; that is, $Y$ is not the dual of an $M$-ideal. This example could be considered as an analogue within $M$-structure theory of a Banach lattice constructed by M. Talagrand [23]. In section IV we use different techniques for showing that a separable $\mathscr{L}^\infty$-space which can be renormed into an $M$-ideal in its bidual is isomorphic to $c_0(\mathbf{N})$; this result is an isomorphic version of a result of A. Lima [16] and implies a non-commutative version of a result of [14].

NOTATION. The closed unit ball of a Banach space $X$ is denoted by $X_1$. The measure spaces $(\Omega, \Sigma, \mu)$ we consider are always standard measurable spaces equipped with a positive finite measure $\mu$. Most of the time, the space $L^1(\Omega, \Sigma, \mu)$ will be denoted simply by $L^1$. The Radon-Nikodym theorem provides us with an $L$-projection from $L^{1**}$ onto $L^1$; this $L$-projection is denoted by $\pi$, and its kernel by $L^1_s$. The topology of convergence in measure is defined on $L^1(\Omega, \Sigma, \mu)$ by the metric

$$d(f, g) = \int_\Omega |f - g| (1 + |f - g|)^{-1} \, d\mu;$$

we denote by $L^0$ the corresponding topology. If $X$ is a subspace of $L^1$, we denote by $X^\#$ the space of linear forms on $X$ whose restriction to the unit ball $X_1$ of $X$ is $L^0$-continuous.

If $Z$ is a subspace of a dual Banach space $Y^*$, $Z^T$ denotes the orthogonal of $Z$ in $Y$.

## I. Subspaces of $L^1$ which are duals of $M$-ideals.

We start with two simple lemmas, which are both special instances of general results about weakly sequentially complete Banach lattices.

LEMMA I.1. *Let $\{f_n \mid n \geqq 1\}$ be a sequence in $L^1(\Omega, \mu)$ which converges to zero $\mu$-almost everywhere. Then every $w^*$-cluster point $z$ to the sequence $\{f_n\}$ belongs to the singular part $L^1_s$ of $L^{1**}$.*

PROOF. We write $z = f + v$, with $f \in L^1$ and $v \in L^1_s$. If $f \neq 0$, there is $\varepsilon > 0$ such that

$$\mu\{|f| > \varepsilon\} \geqq \varepsilon \tag{1}$$

Since $\{f_n\}$ converges to zero $\mu$ a.e. there is $N \geqq 1$ such that

$$\mu(\Omega \backslash A) \leqq \varepsilon/2 \tag{2}$$

where we set

$$(3) \qquad A = \bigcap_{n \geqq N} \{|f_n| \leqq \varepsilon/2\}.$$

By (1) and (2) we have

$$(4) \qquad \mu(A \cap \{|f| > \varepsilon\}) \geqq \varepsilon/2.$$

We define $p_A$: $L^1(\Omega) \to L^1(\Omega)$ by $p_A(g) = g \cdot 1_A$. If $\pi$ denotes as usual the canonical projection from $L^{1**}$ onto $L^1$, one has $\pi p_A^{**} = p_A^{**}\pi$ and in particular

$$p_A^{**}(\pi(z)) = p_A(f) = \pi(p_A^{**}(z)).$$

Since $p_A^{**}$ is $w^*$-continuous, $p_A^{**}(z)$ belongs to the $w^*$-closure of the sequence $\{p_A^{**}(f_n) | n \geqq N\}$. Since the set

$$K = \{g \in L^1(A, \mu) | \ |g| \leqq \varepsilon/2\}$$

is weakly compact, we have by (3) that $p_A^{**}(z)$ belongs to $K$ and thus

$$p_A(f) = \pi(p_A^{**}(z)) = p_A^{**}(z) \in K$$

but this contradicts (4) and concludes the proof.

Before stating our next lemma, let us introduce a useful notation: if $X$ is a subspace of $L^1$, we denote by $X^{\#}$ the vector space of linear forms on $X$ whose restriction to $X_1$ is $L^0$-continuous. $X^{\#}$ is clearly a norm-closed linear subspace of $X^*$. The space $X^{\#}$ can be controlled by the following lemma.

LEMMA I.2. *For every subspace $X$ of $L^1$, one has*

$$X^{\#} = (X^{\perp\perp} \cap L_s^1)^T.$$

PROOF. Let $y$ be in $(X^{\perp\perp} \cap L_s^1)^T$ and suppose that $y \notin X^{\#}$. Then there is a sequence $\{x_n\}$ in $X_1$ which converges to 0 in measure and such that $y(x_n)$ does not converge to 0. Passing to a subsequence if necessary, we may assume that

$$\lim_{n \to \infty} x_n = 0 \quad \mu\text{-a.e.}$$

$$\lim_{n \to \infty} y(x_n) = \lambda \neq 0$$

Since $\{x_n\}$ is bounded we may pick a $w^*$-cluster point $z$ to $\{x_n\}$; by I.1 $z$ belongs to $L_s^1$ and clearly $z \in X^{\perp\perp}$; hence $z \in L_s^1 \cap X^{\perp\perp}$ and $z(y) = 0$; but on the other hand

$$z(y) = \lim_{n \to \infty} y(x_n) = \lambda \neq 0$$

and this contradiction shows that $(X^{\perp\perp} \cap L_s^1)^T \subset X^{\#}$.

Take now $y \in X^{\#}$ and $z \in X^{\perp\perp} \cap L^1_s$. Let $\{x_\alpha \mid \alpha \in I\}$ be a net in $X$ with $\|x_\alpha\| \leqq \|z\|$ for every $\alpha$ and $\mathscr{U}$ an ultrafilter on $I$ such that

$$z = w^*\text{-}\lim_{\alpha \to \mathscr{U}} x_\alpha.$$

We claim that for every $\varepsilon > 0$

$$\lim_{\alpha \to \mathscr{U}} \mu\{|x_\alpha| \geqq \varepsilon\} = 0 \tag{1}$$

Indeed, if not, there exist $P \in \mathscr{U}$ and $\eta > 0$ such that

$$\mu\{|x_\alpha| \geqq \varepsilon\} \geqq \eta \quad \forall \alpha \in P.$$

Since $z \in L^1_s$, there exists ([34], Th. 1.19) a measurable subset $A$ of $\Omega$ such that $\mu(A) < \eta/2$, $|z|(1_A) = \|z\|$.
Since the map $\|p_A^{**}(\cdot)\|$ is $w^*$-l.s.c. there exists $P' \in \mathscr{U}$ such that

$$\int_A |x_\alpha| \, d\mu > \|z\| - \varepsilon\eta/3 \quad \forall \alpha \in P'$$

and then for every $\alpha \in P \cap P'$

$$\|x_\alpha\| = \int_A |x_\alpha| \, d\mu + \int_{\Omega \setminus A} |x_\alpha| \, d\mu \geqq \|z\| - \varepsilon\eta/3 + \varepsilon\eta/2 > \|z\|$$

and this contradiction establishes (1). Now (1) means that

$$\lim_{\alpha \to \mathscr{U}} x_\alpha = 0$$

for the topology $L^0$, and since $y \in X^{\#}$ it follows that

$$z(y) = \lim_{\alpha \to \mathscr{U}} y(x_\alpha) = 0$$

this shows that $X^{\#} \subset (X^{\perp\perp} \cap L^1_s)^T$ and concludes the proof.

REMARK. For every non reflexive subspace $X$ of $L^1$, one has $X^{\#} \neq X^*$ (i.e. $X^{\perp\perp} \cap L^1_s \neq \{0\}$).

Indeed, by Komlós's theorem ([35]; [32], p. 122), every bounded sequence in $X$ has a subsequence whose Césaro-means $\sigma_n$ converge in measure in $L^1$; thus $(\sigma_n)$ is $L^0$-Cauchy in $X_1$. If $X^{\#} = X^*$, $(\sigma_n)$ is then weakly Cauchy also, hence converges weakly. From this point, there are many ways to conclude. For instance, noting that $(\sigma_n)$ is norm convergent ([33], IV.8.12), we get that $X$ has the Banach-Saks property, and thus $X$ is reflexive ([32], p. 212, or by using James' theorem).

We must tell that the result follows also from a deep theorem of B. Maurey ([36]).

We are now ready to state the main result of this section.

THEOREM I.3. *Let $X$ be a subspace of $L^1$. The following statements are equivalent:*
(i) *$X$ is isometric to the dual of an $M$-ideal.*
(ii) *The unit ball $X_1$ of $X$ is $L^0$-closed, and $X^\#$ separates $X$.*

*Moreover if* (i), (ii) *are satisfied, then the $M$-ideal predual of $X$ is the space $X^\#$.*

PROOF. (i) $\Rightarrow$ (ii): if we call $Y$ the $M$-ideal predual of $X$, we have $X^{**} = X \oplus_1 Y^\perp$, and this implies by ([15], th. 1) that $Y^\perp = X^{\perp\perp} \cap L_s^1$; we give for completeness a simplified proof of this special case.

Since $X^{**} = X \oplus_1 Y^\perp$ and $X \cap X^{\perp\perp} \cap L_s^1 = \{0\}$, it is enough to show that $Y^\perp \subset (X^{\perp\perp} \cap L_s^1)$. It is classical and easily seen that two elements $u$ and $t$ of the Banach lattice $L^{1**}$ are orthogonal if and only if

$$\|\alpha u + \beta t\| = |\alpha| + |\beta|$$

for all scalars $\alpha$ and $\beta$. Therefore we have $|x| \wedge |z| = 0$ for every $x \in X$ and every $z \in Y^\perp$; the same relation holds for every $x$ in the band $\tilde{X}$ generated by $X$; and since $z \in X^{\perp\perp} \subset (\tilde{X})^{\perp\perp}$, we may assume without loss of generality that $\tilde{X} = L^1$. But $|x| \wedge |z| = 0$ for every $x \in L^1$ means that $z \in L_s^1$, and we have shown that $Y^\perp \subset (X^{\perp\perp} \cap L_s^1)$.

Therefore we can write

$$X^{**} = X \oplus_1 (X^{\perp\perp} \cap L_s^1) \tag{1}$$

and this implies ([5]) that the unit ball $X_1$ of $X$ is $L^0$-closed; indeed let $\{x_n \mid n \geqq 1\}$ be a sequence in the unit ball $X_1$ of $X$ which converges in measure to $x \in L^1$; taking a subsequence if necessary, we may assume that $x = \lim(x_n)$ $\mu$-a.e. Pick now any $w^*$-cluster point $z \in L^{1**}$ to the sequence $\{x_n\}$. By lemma I.1 we have $x = \pi(z)$; but $z \in X^{\perp\perp}$ and by (1) $\pi(X^{\perp\perp}) = X$. This shows that $x \in X$; clearly, $\|x\|_1 \leqq 1$ hence $x \in X_1$.

Finally since we have $Y^\perp = X^{\perp\perp} \cap L_s^1$ it follows by lemma I.2 that

$$Y = (X^{\perp\perp} \cap L_s^1)^T = X^\#.$$

This shows of course that if (i) is satisfied, $X^\#$ separates $X$, and that the $M$-ideal predual coïncides with $X^\#$.

(ii) $\Rightarrow$ (i): By the main result of [5] (see [11], lemme 1), if $X_1$ is $L^0$-closed we have

$$X^{\perp\perp} = X \oplus_1 (X^{\perp\perp} \cap L_s^1).$$

If $X^\#$ separates $X$, we have

$$(X^\#)^\perp \cap X = \{0\}$$

and by lemma I.2

$$(X^\#)^\perp = \overline{(X^{\perp\perp} \cap L_s^1)}^{\,*}$$

but it follows from these three equalities and linear algebra that

$$(X^{\#})^{\perp} = X^{\perp\perp} \cap L^1_s$$

and therefore we have

$$X^{**} = X \oplus_1 (X^{\#})^{\perp}$$

which means that $X^{\#}$ is an isometric predual of $X$ which is an $M$-ideal in its bidual $X^*$. This concludes the proof.

REMARKS. 1) The condition (i) is obviously independent of the isometric embedding of $X$ into $L^1$. Hence so is the condition (ii).

Moreover, let $K$ be a metrizable compact space and $Y$ be a subspace of $\mathscr{C}(K)$ such that $Y^{\perp}$ is separable; then $\mathscr{C}(K)/Y$ is $M$-ideal iff condition (ii) is true for $X = Y^{\perp} \subset L^1(\mu)$ for some $\mu \in \mathscr{M}_+(K)$, and hence it is true for every $\mu \in \mathscr{M}_+(K)$ such that $Y^{\perp} \subset L^1(\mu)$.

2) It follows from ([12], lemma 1.3) that in the statement of the condition (ii) of theorem I.3 we can substitute to the topology $L^0$ the quasi-norm $L^p$ for any $p < 1$, or the topology $L(1, \infty)$ of the Fréchet space "weak-$L^1$". In particular, if $X = L^1 \cap Z$ where $Z$ is a closed subspace of $L^p$ $(p < 1)$ such that $Z^*$ separates $Z$, then $X$ is the dual of an $M$-ideal (cf. [11], théorème 6). A typical example of this situation is $X = H^1(D) = L^1(\mathrm{T}) \cap H^p(D)$ for any $p \in (0, 1)$.

In the next section we will investigate the translation-invariant version of these results.

## II. $L^1_\Lambda$ and $\mathscr{M}_\Lambda$-spaces which are duals of $M$-ideals.

In this section, $G$ denotes a compact metrizable abelian group and $\Gamma = \hat{G}$ the discrete dual group. The additive notation will be used for $\Gamma$. If $\Lambda \subset \Gamma$, we denote as usual.

$$L^1_\Lambda = \{f \in L^1 \mid \hat{f}(\alpha) = 0 \qquad \forall \alpha \notin \Lambda\}$$

and

$$\mathscr{M}_\Lambda = \{\mu \in \mathscr{M}(G) \mid \hat{\mu}(\alpha) = 0 \quad \forall \alpha \notin \Lambda\}$$

where $L^1 = L^1(G, m)$ with $m =$ Haar measure of $G$, and $\hat{f}$, $\hat{\mu}$ are the Fourier transforms of $f$ and $\mu$. The group $\Gamma$ can be seen as a subset of $L^\infty(G, m) = L^{1*}$ as well as a subset of $L^1$. Our next lemma shows that if $L^{1\#}_\Lambda$ is non trivial, then it intersects $\Gamma$.

LEMMA II.1. *Let $\alpha \in \Gamma$. If $y \in (L^1_\Lambda)^{\#}$ and $\hat{y}(-\alpha) = y(\alpha) \neq 0$, then $\tilde{\alpha} \in (L^1_\Lambda)^{\#}$. Here $\tilde{\alpha}$ denotes the restriction of $\tilde{\alpha} \in L^\infty$ to $L^1_\Lambda$.*

PROOF. Without loss of generality, we may assume that $\alpha = 1_G$. We define $\tilde{y} \in L^{1*}_\Lambda$ by

$$\tilde{y}(f) = \int_G \langle y, f_\tau \rangle \, dm(\tau)$$

where $f_\tau(\tau') = f(\tau\tau')$ is a translate of $f$. We claim that $\tilde{y}$ belongs to $(L^1_\Lambda)^{\#}$. Indeed for any $\eta > 0$ there is a neighborhood $V$ of 0 in the unit ball of $L^1_\Lambda$ such that

$$f \in V \Rightarrow |y(f)| < \eta;$$

without loss of generality we may assume that $V = W \cap L^1_\Lambda$ where $W$ is a translation invariant neighborhood of 0 in the unit ball of $L^1$ and then $V$ is translation invariant; hence

$$f \in V \Rightarrow |y(f_\tau)| < \eta \qquad \forall \tau \in G$$

and thus $|\tilde{y}(f)| < \eta$ for $f \in V$ and $\tilde{y} \in (L^1_\Lambda)^{\#}$: Observe now that for every $\alpha \in \Lambda$

$$\tilde{y}(\alpha) = \int_G \langle y, \alpha \rangle \, \alpha(\tau) \, dm(\tau) = \langle y, \alpha \rangle \int_G \alpha(\tau) \, dm(\tau)$$

hence $\tilde{y}(\alpha) = 0$ except if $\alpha = 1_G$ where we have

$$\tilde{y}(1_G) = y(1_G) \neq 0.$$

It follows that $\tilde{y}$ coïncides on $L^1_\Lambda$ with the restriction of a non-zero constant function, and the result follows.

REMARK. We can observe that $\tilde{y}(f) = \hat{y}(0)\hat{f}(0) = y(1_G)\hat{f}(0)$.

With this lemma we can characterize the spaces $\mathcal{M}_\Lambda$ which are duals of an $M$-ideal. The main result of this section is the following:

THEOREM II.2. *Let $\Lambda$ be a subset of the abelian discrete group $\Gamma$. The following assertions are equivalent:*

(i) *$L^1_\Lambda$ is isometric to the dual of an $M$-ideal $Y$.*

(ii) *$\mathcal{M}_\Lambda$ is isometric to the dual of an $M$-ideal $Z$.*

(iii) *The unit ball of $L^1_\Lambda$ is $L^0$-closed, and the restriction of $\tilde{\alpha}$: $f \to \hat{f}(\alpha)$ to $L^1_\Lambda$ belongs to $(L^1_\Lambda)^{\#}$ for every $\alpha \in \Lambda$.*

*Moreover, if the conditions* (i)–(iii) *are satisfied, then $\mathcal{M}_\Lambda = L^1_\Lambda$ and the $M$-ideal predual of $\mathcal{M}_\Lambda = L^1_\Lambda$ is the space $\mathscr{C}(G)/\mathscr{C}_{\Gamma\setminus(-\Lambda)}(G)$* (i.e. $L^{1\perp\perp}_\Lambda \cap L^1_s = L^{1\perp\perp}_\Lambda \cap \mathscr{C}^\perp$).

In the above statement, $\mathscr{C}_{\Gamma\setminus(-\Lambda)}(G) = \mathscr{C}(G) \cap L^1_{\Gamma\setminus(-\Lambda)}$. Let us recall that the sets $\Lambda$ such that $\mathcal{M}_\Lambda = L^1_\Lambda$ are called Riesz sets; the sets $\Lambda$ which satisfy (iii) are called Shapiro sets in [12].

PROOF. (i)$\Leftrightarrow$(ii): If $L^1_\Lambda$ (resp. $\mathcal{M}_\Lambda$) satisfies (i) (resp. (ii)), it has the Radon-Nikodym property (see [31]) and hence $L^1_\Lambda = \mathcal{M}_\Lambda$ ([20]).

(iii) ⇒ (i): If $\Lambda$ satisfies (iii) then $(L^1_\Lambda)^\#$ separates $L^1_\Lambda$, so theorem I.3 gives (i).

(i) ⇒ (iii) Now by theorem I.3 the unit ball of $L^1_\Lambda$ is $L^0$ closed, and $(L^1_\Lambda)^\#$ separates $L^1_\Lambda$; in particular, for every $\alpha \in \Lambda$, there exists $y \in (L^1_\Lambda)^\#$ such that $y(\alpha) \neq 0$, and this implies by lemma II.1 that the restriction of $\bar{\alpha}$ to $L^1_\Lambda$ belongs to $(L^1_\Lambda)^\#$. Let us observe now that under the condition (ii), ([12], Prop. 4.1) shows that the predual $M$-ideal of $\mathscr{M}_\Lambda = L^1_\Lambda$ is indeed $\mathscr{C}(G)/\mathscr{C}_{\Gamma\setminus(-\Lambda)}(G)$. This can also be seen directly: since $\mathscr{M}_\Lambda = L^1_\Lambda$, the space $\mathscr{C}/\mathscr{C}_{\Gamma\setminus(-\Lambda)}$ is an isometric predual of $L^1_\Lambda$; but the restriction of $\Gamma$ to $L^1_\Lambda$ spans this space and is contained in $(L^1_\Lambda)^\#$ by the above; and two preduals which are contained in each other must coincide. This concludes the proof.

We refer to [12] for examples and for a systematic study of Shapiro sets. In the next section, we will use harmonic analysis, together with theorem II.2, to produce an example in the theory of $L$- and $M$-structure.

We conclude this section by the observation that theorem II.2 provides in particular a characterization of the quotient spaces of $\mathscr{C}(G)$ by translation-invariant subspaces which are $M$-ideals in their bidual, since the dual of such a space in an $\mathscr{M}_\Lambda$-space.

## III. An example.

The main result of this section provides an example of a Banach space which is $L$-complemented in its bidual and behaves in a somehow unexpected way; the construction uses crucially the results of §II. We work in this section within the frame of the "little Fourier analysis", that is, $G = \mathbb{T}$ and $\Gamma = \mathbb{Z}$.

Before stating it, let us recall that the dual $X^*$ of a space $X$ which is $M$-ideal in its bidual has the Radon-Nikodym property (see [31]). If $Y \subset X^*$ is such that there exists an $L$-projection $\pi$ from $Y^{**}$ onto $Y$, then by ([15], th. 1) one has $\operatorname{Ker}\pi = (Y^{\perp\perp} \cap X^\perp)$, and thus $\operatorname{Ker}\pi$ is $w^*$-closed and $Y$ is the dual of an $M$-ideal.

This leads to the question to know whether or not every space $Y$ with the Radon-Nikodym property which is $L$-complemented in $Y^{**}$ is the dual of an $M$-ideal. The next statement provides in particular a negative answer to this question.

THEOREM III.1. *There exists a separable space Y which satisfies the following conditions:*

(i) *$Y$ is isometric to a dual space*

(ii) *There is an $L$-projection $\pi$ from $Y^{**}$ onto $Y$*

(iii) $\operatorname{Ker}(\pi)$ *is not $w^*$-closed in $Y^{**}$.*

PROOF. For every $n \geq 1$, we set

$$D_n = \{k2^n \mid |k| \leq n\}$$

and

$$\Lambda = \bigcup_{n=1}^{\infty} D_n.$$

The properties of such sets are studied in ([12], §3.8). We recall for completeness what we need for the present work.

*Claim 1.* $\Lambda$ is a Riesz set (i.e. $\mathcal{M}_\Lambda = L^1_\Lambda$). For any $j \geqq 0$, we let

$$P_j = \{2^j + k2^{j+1} \mid k \in \mathbf{Z}\}.$$

It is easily seen that $n \in P_j$ if and only if $2^j$ divides $n$ and $2^{j+1}$ does not divide $n$; hence $\{P_j \mid j \geqq 0\}$ is a partition of $\mathbf{Z}\backslash\{0\}$, and $P_k \cap D_n = \emptyset$ if $k < n$. Therefore $(\Lambda \cap P_k)$ is contained in $\cup \{D_n \mid n \leqq k\}$ and in particular it is finite.

Let now $\mu = \mu_a + \mu_s \in \mathcal{M}_\Lambda$; we have to show that $\mu_s = 0$. For every $n \in \mathbf{Z}\backslash\{0\}$, there is $j \in \mathbf{N}$ such that $n \in P_j$. There exists a Radon measure $\nu_j$ on $\mathbf{T}$ with finite support such that $\hat{\nu}_j = 1_{P_j}$; since $\nu_j$ is discrete, we have

$$(\mu * \nu_j)_s = \mu_s * \nu_j$$

and since $(\mu * \nu_j)^\wedge = \hat{\mu} \cdot \hat{\nu}_j$, we have

$$\mu * \nu_j \in \mathcal{M}_{\Lambda \cap P_j}$$

but $(\Lambda \cap P_j)$ is finite and thus $(\mu * \nu_j)$ is a trigonometric polynomial and

$$(\mu * \nu_j)_s = \mu_s * \nu_j = 0$$

in particular, $(\mu * \nu_j)^\wedge(n) = \hat{\mu}_s(n)\hat{\nu}_j(n) = \hat{\mu}_s(n) = 0$. We have shown that $\hat{\mu}_s(n) = 0$ for every $n \neq 0$ and it follows that $\mu_s = 0$ since $\mu_s$ is singular.

*Claim 2.* The unit ball of $L^1_\Lambda$ is $L^0$-closed. Let $\{f_k \mid k \geqq 1\}$ be a sequence in the unit ball of $L^1_\Lambda$ which converges in measure to $g \in L^1$. Let $n \in \mathbf{Z}\backslash\Lambda$; we have to show that $\hat{g}(n) = 0$.

We pick as before $j$ such that $n \in P_j$ and $\nu_j$ such that $\hat{\nu}_j = 1_{P_j}$. Since $\nu_j$ has a finite support, we have

$$\lim_k f_k * \nu_j = g * \nu_j$$

in measure, but also in norm since $(f_k * \nu_j)$ belongs to the finite dimensional space $L^1_{\Lambda \cap P_j}$. In particular, we have

$$\lim_k (f_k * \nu_j)^\wedge(n) = (g * \nu_j)^\wedge(n) = \hat{g}(n)$$

but $(f_k * \nu_j)^\wedge(n) = (f_k)^\wedge(n) = 0$ for every $k$ since $n \notin \Lambda$ and it follows that $\hat{g}(n) = 0$.

*Claim 3.* The restriction of $1 \in L^\infty(\mathbf{T})$ to the unit ball of $L^1_\Lambda$ is not $L^0$-continuous. Indeed it is easy to construct a sequence $\{f_k \mid k \geqq 1\}$ of functions in the unit ball of $L^1(\mathrm{T})$ such that

$$\int f_k = (f_k)^\wedge(0) = 0 \quad \forall k$$

$$\lim_{k\to\infty} f_k = 1/2 \cdot 1_{\mathrm{T}} \quad \text{a.e.}$$

By approximation, we may assume that the $f_k$'s are trigonometric polynomials. Observe now that the functions $(z)$ and $(z^{2^n})$ have the same distribution for any $n$. Now if we substitute $(z^{2^n})$ to $(z)$ in the expression of $(f_k)$, then we obtain, if we choose $n$ big enough, a trigonometric polynomial $(g_k)$ whose Fourier transform is supported by $\Lambda$, and since the distribution is unchanged, we still have $\|g_k\|_1 \leqq 1$ and

$$\int g_k = 0 \quad \forall k$$

$$\lim_{k\to\infty} g_k = 1/2 \cdot 1_{\mathrm{T}} \quad \text{a.e.}$$

This shows that the Fourier coefficient in 0 is not $L^0$-continuous on the unit ball of $L^1_\Lambda$, and proves the claim 3.

We are now ready to complete the proof of theorem III.1. We let $Y = L^1_\Lambda(\mathrm{T})$, where $\Lambda = \cup\{D_n \mid n \geqq 1\}$ is defined above.

By the claim 1, $\Lambda$ is a Riesz set and thus $Y = \mathscr{M}_\Lambda$ is canonically isometric to the dual of the space $\mathscr{C}(\mathrm{T})/\mathscr{C}_{\mathrm{Z}\setminus(-\Lambda)}(\mathrm{T})$.

By [5] – see ([11], lemme 1) – and the claim 2, we have

$$Y^{\perp\perp} = Y \oplus_1 (Y^{\perp\perp} \cap L^1_s)$$

and therefore the restriction to $Y^{\perp\perp}$ of the canonical projection from $L^{1**}$ onto $L^1$ is an $L$-projection $\pi$ from $Y^{**}$ onto $Y$.

Finally, the space $\mathrm{Ker}(\pi) = Y^{\perp\perp} \cap L^1_s$ is $w^*$-closed if and only if $Y$ is isometric to the dual of an $M$-ideal; and by theorem II.2 this would imply that the restriction of every Fourier coefficient to the unit ball of $L^1_\Lambda$ would be $L^0$-continuous; and this contradicts the claim 3.

REMARKS. 1) If we drop the requirement $Y$ *separable*, then very simple examples are available, since for instance the space $\mathscr{C}(\mathrm{T})^*$ itself satisfies the conditions (i), (ii), (iii); but of course this space does not have the Radon-Nikodym property, in contrast with our space $Y$ which has R.N.P. since it is a separable dual.

2) The proof of theorem III.1 gives more information on the structure of $Y^{**}$. Indeed the proof of claim 2 shows that the restriction of every Fourier coefficient

but one to $Y$ is $L^0$-continuous. It follows from this fact and lemma I.2 that the space $M = Y^{\perp\perp} \cap L_s^1 \cap \mathscr{C}(\mathrm{T})^\perp$ is of codimension one in $Y^{\perp\perp} \cap \mathscr{C}(\mathrm{T})^\perp$ and in $(Y^{\perp\perp} \cap L_s^1)$, and is not $w^*$-closed, since $(Y^{\perp\perp} \cap L_s^1)$ is not $w^*$-closed. And since $M$ is a hyperplane in $Y^{\perp\perp} \cap \mathscr{C}(\mathrm{T})^\perp$ it follows that $\overline{M}^* = Y^{\perp\perp} \cap \mathscr{C}(\mathrm{T})^\perp$; a fortiori we have $\overline{Y^{\perp\perp} \cap L_s^1}^* \supset \mathscr{C}(\mathrm{T})^\perp$, and by lemma I.2. $Y^\#$ is contained in the restriction of $\mathscr{C}(\mathrm{T})$ to $Y$. From this latter fact it finally follows that $Y^\#$ is the space

$$Y^\# = \{f\,|\,Y|\, f \in \mathscr{C}(\mathrm{T}),\quad \hat{f}(0) = 0\}.$$

3) Actually, the Alexandrov's set $\Lambda$ has stronger properties. By adapting the proof of [28], Example 2, p. 122–123, it can be shown that if $D$ is the countable dense subgroup of T:

$$D = \{e^{2\pi i k/2^n}\,|\, k \in \mathbf{Z},\quad n \in \mathbf{N}^*\}$$

and $\varphi\colon \mathbf{Z} \to \hat{D} = \hat{\mathbf{Z}}/D^\perp$ is the canonical injection, then $\varphi(\Lambda)$ is closed in $\hat{D}$. Therefore:

a) $\Lambda$ is closed in $\mathbf{Z}$ for the Bohr topology and in particular, the unit ball of $L_\Lambda^1$ is $L^0$-closed ([12], Cor. 2.6 (1)); moreover, since $\Lambda \cap P_j$ is finite for every $j \geqq 0$, 0 is the only accumulation point of $\Lambda$ in $\mathbf{Z}$.

b) $\mathscr{C}_\Lambda = L_\Lambda^\infty$ has the Schur property ([29], Th. 3). Hence $\Lambda$ is a Rosenthal set (see also [26], Th. B and [27]); in particular $\Lambda$ is a Riesz set ([20], Th. 3) and more generally $\mathbf{N} \cup \Lambda$ is a Riesz set ([27], Th. 2).

4) For non-translation invariant subspaces $H$ of $L^1(\mathrm{T})$ which are duals of $M$-ideals, we cannot expect in general that $H^{\perp\perp} \cap L_s^1 = H^{\perp\perp} \cap \mathscr{C}(\mathrm{T})^\perp$; for instance, if $h_n, n \geqq 1$, are disjoint positive functions of $L^1$ of norm 1, $H = [h_n, n \geqq 1]$ is isometric to $\ell^1 = c_0^*$ but $H^{\perp\perp} \cap L_s^1 \not\supset \mathscr{C}(\mathrm{T})^\perp$. This comes from the fact that if we consider non-translation invariant subspaces of $L^1(\mathrm{T})$, the topology of T, and then $\mathscr{C}(\mathrm{T})$, plays no canonical role any more.

5) If $Z$ has the Radon-Nikodym property and $V \subset Z^{**}$ is a subspace such that $Z \cap V = \{0\}$, then the unit ball of $V$ cannot be $w^*$-dense in the unit ball of $Z^{**}$, since $Z_1$ has a strongly exposed point $x$, which would belong to $V \cap Z$. This shows that it is not possible to replace the condition (iii) of theorem III.1 by the stronger condition: the unit ball of $(\operatorname{Ker} \pi)$ is $w^*$-dense in $Y_1^{**}$.

However, it is not clear whether or not $(\operatorname{Ker} \pi)$ can be $w^*$-dense in $Y^{**}$. Within the frame of the $L_\Lambda^1$-spaces, this boils down to the following question, which belongs to harmonic analysis.

QUESTION III.2. Does there exist a *Riesz subset* $\Lambda$ of $\mathbf{Z}$ which satisfies the following conditions:

(i) The unit ball of $L_\Lambda^1$ is $L^0$-closed;

(ii) For every $n \in \Lambda$, the restriction of the Fourier coefficient at $n$ to the unit ball of $L_\Lambda^1$ is *not* $L^0$-continuous?

6) It is shown in [23] that there exists a separable Banach lattice $T$ with the Radon-Nikodym property such that the band $T_s$ orthogonal to $T$ in $T^{**}$ is $w^*$-dense in $T^{**}$. Theorem III.1 is the analogue of M. Talagrand's result for $L$-structure; but we should mention that Talagrand did not stop so early, since he proved in [24] that any separable Banach lattice with the Radon-Nikodym property is a dual Banach lattice. Within the frame of $L$-structure, we do not know the answer to the:

QUESTION III.3. Is every space $Y$ with the R.N.P., and $L$-complemented in its bidual, isometric to a dual space?

Observe that by [20], the answer is yes for translation-invariant subspaces of $L^1$.

## IV. $\mathscr{L}^\infty$-spaces which are isomorphic to $M$-ideals.

In our last section we will investigate isomorphic properties. Let us observe that if $X$ is a separable $\mathscr{L}^\infty$-space (see [18]) which is isomorphic to an $M$-ideal then $X^*$ is isomorphic to $\ell^1(\mathsf{N})$ by [17] since then $X^*$ is a separable dual $\mathscr{L}^1$-space. This does not say much, however, about the space $X$ since $\ell^1(\mathsf{N})$ has a huge supply of isomorphic preduals.

The main result of this section is that $X$ is in fact the *natural* isomorphic predual of $\ell^1(\mathsf{N})$; that is, $X$ is isomorphic to $c_0(\mathsf{N})$. The crucial point of the proof is Zippin's deep characterization of $c_0(\mathsf{N})$ [25]. Let us mention that theorem IV.1 and its proof were obtained independently and almost simultaneously by D. Werner.

We refer to [18] and [4] for properties and examples of $\mathscr{L}^\infty$-spaces. We state now

THEOREM IV.1. *Let $X$ be a separable $\mathscr{L}^\infty$-space which can be renormed into an $M$-ideal in its bidual. Then $X$ is isomorphic to $c_0(\mathsf{N})$.*

We are grateful to an anonymous referee for a simplification of the original argument.

PROOF. Since $X^*$ is separable [13] and is an $\mathscr{L}^1$-space, it is isomorphic to $\ell^1(\mathsf{N})$ [17] and thus $X^{**}$ is isomorphic to $\ell^\infty(\mathsf{N})$. We denote by $i$: $X \to \ell^\infty(\mathsf{N})$ the canonical injection.

By Zippin's theorem [25], it is enough to show that for every isomorphic injection $j$ from $X$ into a separable space $Y$, the space $j(X)$ is complemented in $Y$. Since $\ell^\infty(\mathsf{N})$ is injective, the map $k = i[j^{-1}]$ from $j(X)$ into $\ell^\infty(\mathsf{N})$ has an extension $\tilde{k}$ from $Y$ to $\ell^\infty(\mathsf{N})$. We denote by $Z$ the norm-closed subalgebra of $\ell^\infty(\mathsf{N})$ generated by $\tilde{k}(Y)$; the space $Z$ is isomorphic to a separable $\mathscr{C}(K)$-space.

It is classical and easily checked that $i(X)$ is an $M$-ideal in $Z$ since it is an $M$-ideal in $\ell^\infty(\mathsf{N})$ and $i(X) \subset Z \subset \ell^\infty(\mathsf{N})$. Let us mention at this point that the space $\ell^\infty(\mathsf{N})$ is equipped here with an equivalent norm and not with the canonical one. Moreover, the quotient space $Z/i(X)$ has the bounded approximation property. Indeed we may write

$$\ell^\infty(\mathsf{N})^* = \ell^1(\mathsf{N}) \oplus i(X)^\perp$$

and since $Z^\perp \subset i(X)^\perp$

$$Z^* = \ell^\infty(\mathsf{N})^*/Z^\perp = \ell^1(\mathsf{N}) \oplus i(X)^\perp/Z^\perp.$$

The space $i(X)^\perp/Z^\perp$ is complemented in the space $Z^* = \mathscr{M}(K)$ which has the B.A.P. and therefore $i(X)^\perp/Z^\perp$ has the B.A.P.; and $i(X)^\perp/Z^\perp$ is canonically isomorphic to the orthogonal of $i(X)$ in $Z^*$, hence to the dual of $Z/i(X)$; hence $Z/i(X)$ has the B.A.P. since $(Z/i(X))^*$ has it (see [19], p. 34).

In these circumstances there exists by a result of Ando ([2], th. 5) a linear projection $p$ from $Z$ onto $i(X)$. If we let

$$\tilde{p} = ji^{-1}p\tilde{k}.$$

Then $\tilde{p}$ is a linear projection from $Y$ onto $j(X)$.

Remarks. 1) Theorem IV.1 has a quantitative version, namely: there is a function $\varphi(\lambda)$ such that every $\mathscr{L}_\lambda^\infty$-space which is $M$-ideal in its bidual satisfies $d(X, c_0(\mathsf{N})) \leqq \varphi(\lambda)$. Indeed if not, there is $\lambda_0 \in \mathsf{R}$ and a sequence $\{X_n \mid n \geqq 1\}$ of $\mathscr{L}_{\lambda_0}^\infty$-spaces which are $M$-ideals and such that $d(X_n, c_0(\mathsf{N})) \geqq n$. We consider the space

$$Y = (\textstyle\sum \oplus X_n)_{c_0}$$

which is $M$-ideal in its bidual and is also $\mathscr{L}_{\lambda_0}^\infty$; by theorem IV.1 $Y$ is isomorphic to $c_0(\mathsf{N})$ and since the spaces $X_n$ are uniformly complemented in $Y$, their distance to $c_0(\mathsf{N})$ is bounded; this is a contradiction.

2) It is not clear whether or not the assumption $X$ *separable* is necessary in theorem IV.1. The decomposition result ([7], Th. 3) supports the impression that it is not.

3) Theorem IV.1 shows that there exist separable Asplund spaces – such as $\mathscr{C}(\omega^\omega)$ – which contain hereditarily $c_0(\mathsf{N})$ but which cannot be renormed into $M$-ideals in their bidual. This answers a question of M. Fabian (personal communication).

We should mention however that the following question is open.

Question IV.2. Let $X$ be an isomorphic predual of $\ell^1(\mathsf{N})$ which has the property (u) of A. Pelczynski [21]. Is $X$ isomorphic to $c_0(\mathsf{N})$?

A positive answer would extend [22], and *trivialize* theorem IV.1 since the authors have recently shown that every $M$-ideal in its bidual has property (u)

[10]. On the other hand, a negative answer would give an example of a separable Asplund space with (u) which is not isomorphic to an $M$-ideal (another open question).

Since the class of $M$-ideals in their bidual is hereditary and stable under quotient maps [13], theorem IV.1 applies to $\mathscr{L}^\infty$-spaces which are subspaces of quotients of $M$-ideals. Let us mention for instance the

COROLLARY IV.3. *Let $X$ be a separable $\mathscr{L}^\infty$-space which is a subspace of a quotient of the space $K(\ell^2)$ of compact operators in the Hilbert space. Then $X$ is isomorphic to $c_0(\mathbf{N})$.*

This corollary is a non-commutative version of a result of [14]. We refer to [22], [8] for extensions of [14] in another direction.

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