Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 1023 – 1037
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doi:10.3906/mat-1401-85

Some new associated curves of a Frenet curve in $\mathbb{E}^3$ and $\mathbb{E}^4$

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Received: 29.01.2014 • **Accepted:** 05.06.2014 • **Published Online:** 24.10.2014 • **Printed:** 21.11.2014

Abstract: In this paper, firstly, we define a W-direction curve and W-rectifying curve of a Frenet curve in 3-dimensional Euclidean space $\mathbb{E}^3$ by using the unit Darboux vector field W of the Frenet curve and give some characterizations together with the relationships between the curvatures of each associated curve. We also introduce a V-direction curve, which is associated with a curve lying on an oriented surface in $\mathbb{E}^3$. Later, some new associated curves of a Frenet curve are defined in $\mathbb{E}^4$.

Key words: Frenet curve, associated curve, integral curve, general helices, slant helices, rectifying curve

1. Introduction

In differential geometry, the theory of curves in Euclidean 3-space is one of the main study areas. In the theory of curves, helices, slant helices, and rectifying curves are the most fascinating curves. Associated curves of a given curve are also widely studied. Among these curves, the most studied ones are Bertrand curve couples, Mannheim partner curves, spherical indicatrices, and involute-evolute curve couples (e.g., see [2–6,12,13,16,17,21]).

The Darboux vector field $\omega = \tau\mathsf{T} + \kappa\mathsf{B}$, which determines the instantaneous rotation axis of the Frenet frame along the curve, has an important place for space curves in differential geometry.

The curve whose position vector always lies in its rectifying plane is defined as a rectifying curve and some characterizations of rectifying curves were given by Chen in 2003. According to these characterizations he obtained that the position vector of rectifying curves are on the direction of the Darboux vector and thus rectifying curves are interpreted kinematically as the curves whose position vectors determine the instantaneous rotation axis at each point of the curve [6,7].

In 2008, İlarslan and Nešović defined the rectifying curve in $\mathbb{E}^4$ as a curve whose position vector always lies in the orthogonal complement of its principal normal vector field and characterized such curves by means of their curvatures [14].

Önder et al. defined B_2-slant helices in $\mathbb{E}^4$ as the curves whose second binormal vector (B_2) makes a constant angle with a fixed direction and specified the properties of these curves in 2008 [19].

In a recent paper, Choi and Kim introduced principal (binormal)-direction curves, principal (binormal)-donor curves, and PD-rectifying curves. They gave nice characterizations for the general and slant helices via their associated curves and gave a useful method to obtain general helix and slant helix from a planar curve. They also gave a new characterization for Bertrand curves by using the PD-rectifying curve [8]. Later, Choi

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2010 *AMS Mathematics Subject Classification:* 53A04, 53A05.

et al. introduced the notion of the principal (binormal)-directional curve and the principal (binormal)-donor curve of the Frenet curve in the Minkowski space $\mathbb{E}_1^3$ [9], and Körpınar et al. gave new associated curves by using the Bishop frame in $\mathbb{E}^3$ [15].

In this study, we introduce some new associated curves of a given curve. We define

W-direction curves, W-rectifying curves, V-direction curves in $\mathbb{E}^3$

and

principal-direction curves, B_1-direction curves, B_2-direction curves, and B_2-rectifying curves in $\mathbb{E}^4$.

All these new associated curves are defined as the integral curves of vector fields taken from the Frenet frame or Darboux frame along a curve. Some characterizations of these new curves are also studied.

2. Preliminaries

Let M be an oriented surface and $\beta : I \subset \mathbb{R} \to M$ be a regular curve with arc-length parametrization. If the Frenet frame along the curve is denoted by $\{\mathsf{T}, \mathsf{N}, \mathsf{B}\}$, the Frenet formulas are given by

$$\begin{cases} \mathsf{T}' = \kappa \mathsf{N} \\ \mathsf{N}' = -\kappa \mathsf{T} + \tau \mathsf{B} \\ \mathsf{B}' = -\tau \mathsf{N} \end{cases},$$

where T is unit tangent vector, N is principal normal vector, B is the binormal vector, and κ and τ are the curvature and the torsion of β, respectively.

Since the curve β lies on M, there exists another frame, which is called the Darboux frame and is denoted by $\{\mathsf{T}, \mathsf{V}, \mathsf{U}\}$ along the curve. In this frame, T is the unit tangent of the curve, U is the unit normal of the surface restricted to the curve, and V is the unit vector given by $\mathsf{V} = \mathsf{U} \times \mathsf{T}$. The derivative formula of the Darboux frame is [18]

$$\begin{bmatrix} \mathsf{T}' \\ \mathsf{V}' \\ \mathsf{U}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} . \begin{bmatrix} \mathsf{T} \\ \mathsf{V} \\ \mathsf{U} \end{bmatrix},$$

where κ_g is the geodesic curvature, κ_n is the normal curvature, and τ_g is the geodesic torsion of β. The relations between geodesic curvature, normal curvature, geodesic torsion, and κ and τ are given as follows:

$$\kappa_g = \kappa \sin\varphi, \qquad \kappa_n = \kappa \cos\varphi, \qquad \tau_g = \tau + \frac{d\varphi}{ds},$$

where φ is the angle between the vectors N and U.

In the differential geometry of surfaces, for a curve β lying on a surface M, the following are well known:

i) β is a geodesic curve if and only if $\kappa_g = 0$,

ii) β is an asymptotic line if and only if $\kappa_n = 0$,

iii) β is a principal line if and only if $\tau_g = 0$.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be an arbitrary curve with arc-length parametrization. If $\{\mathsf{T}, \mathsf{N}, \mathsf{B}_1, \mathsf{B}_2\}$ is the moving Frenet frame along α, then the Frenet formulas are given by [10]

$$\mathsf{T}' = k_1 \mathsf{N}, \quad \mathsf{N}' = -k_1 \mathsf{T} + k_2 \mathsf{B}_1, \quad \mathsf{B}_1' = -k_2 \mathsf{N} + k_3 \mathsf{B}_2, \quad \mathsf{B}_2' = -k_3 \mathsf{B}_1, \tag{2.1}$$

where $\mathsf{T}, \mathsf{N}, \mathsf{B}_1$, and B_2 denote the tangent, the principal normal, the first binormal, and the second binormal vector fields; $k_i, (i = 1, 2, 3)$ denotes the ith curvature functions ($k_1, k_2 > 0$) of the curve α.

Definition 1 (Ternary product) *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ *be the standard basis of 4-dimensional Euclidean space* $\mathbb{E}^4$. *The ternary product (or vector product) of the vectors* $\mathbf{x} = \sum_{i=1}^{4} x_i \mathbf{e_i}$, $\mathbf{y} = \sum_{i=1}^{4} y_i \mathbf{e_i}$, *and* $\mathbf{z} = \sum_{i=1}^{4} z_i \mathbf{e_i}$ *is defined by [11, 20]*

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

Theorem 1 *Let* $\alpha : I \to \mathbb{E}^4$ *be a unit-speed curve. Then the Frenet vectors of the curve are given by [1]*

$$\mathsf{T} = \alpha', \quad \mathsf{N} = \frac{\alpha''}{||\alpha''||}, \quad \mathsf{B}_2 = -\frac{\alpha' \otimes \alpha'' \otimes \alpha'''}{||\alpha' \otimes \alpha'' \otimes \alpha'''||}, \quad \mathsf{B}_1 = \mathsf{B}_2 \otimes \mathsf{T} \otimes \mathsf{N}. \tag{2.2}$$

Theorem 2 *Let* $\alpha : I \to \mathbb{E}^4$ *be a unit-speed curve. Then the curvatures of the curve are given by [1]*

$$k_1 = ||\alpha''||, \quad k_2 = \frac{\langle \mathsf{B}_1, \alpha''' \rangle}{k_1}, \quad k_3 = \frac{\langle \mathsf{B}_2, \alpha^{(4)} \rangle}{k_1 k_2}. \tag{2.3}$$

Definition 2 (Slant helix) *A unit speed curve is called slant helix if its unit principal normal vector makes a constant angle with a fixed direction [12].*

Theorem 3 *Let* γ *be a unit speed curve with* $\kappa \neq 0$. *Then* γ *is a slant helix if and only if*

$$\sigma(s) = \left(\frac{\kappa^2}{\left(\kappa^2 + \tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s) \tag{2.4}$$

is a constant function [12].

Definition 3 (Rectifying curve in $\mathbb{E}^3$) *Let* γ *be a curve in* $\mathbb{E}^3$. γ *is called a rectifying curve if the position vector of* γ *always lies in its rectifying plane [6].*

Definition 4 (Rectifying curve in $\mathbb{E}^4$) *Let* γ *be a curve in* $\mathbb{E}^4$. γ *is called a rectifying curve if the position vector of* γ *always lies in the orthogonal complement of its principal normal vector field [14].*

Definition 5 (Frenet curve) *A unit speed curve* $\beta : I \to \mathbb{E}^n$ *of class* C^n *is called a* Frenet curve *if the vectors* $\beta'(s), \beta''(s), ..., \beta^{(n-1)}(s)$ *are linearly independent at each point along the curve.*

For a Frenet curve $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3$ with the Frenet frame $\{\mathsf{T}, \mathsf{N}, \mathsf{B}\}$, consider a vector field V given by

$$\mathsf{V}(s) = u(s)\mathsf{T}(s) + v(s)\mathsf{N}(s) + w(s)\mathsf{B}(s), \tag{2.5}$$

where u, v, w are functions on I satisfying $u^2(s) + v^2(s) + w^2(s) = 1$. Then an integral curve $\overline{\gamma}(s)$ of V defined on I is a unit speed curve in $\mathbb{E}^3$ [8].

Remark 1 *The arc-length parameter* $\overline{s}$ *of an integral curve* $\overline{\gamma}$ *of* $\mathsf{V}(s)$ *is obtained as* $\overline{s} = s + c$ *for some constant* c. *Thus, without loss of generality, one can assume* $\overline{s} = s$. *The integral curve* $\overline{\gamma}$ *is unique up to translation of* $\mathbb{E}^3$. *In fact,* $\overline{\gamma}$ *is determined by the initial point [8].*

3. W-direction curves

In this section we introduce the W-direction curve, second W-direction curve, and W-rectifying curve in $\mathbb{E}^3$ and give some characterizations.

Definition 6 (W-direction curves) *Let γ be a Frenet curve in $\mathbb{E}^3$ and W be the unit Darboux vector field of γ. We call an integral curve of $W(s)$ the* W-direction curve *of γ.*

Namely, if $\overline{\gamma}(s)$ is the W-direction curve of γ, then $W(s) = \overline{\gamma}'(s)$, where $W = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\tau\mathsf{T} + \kappa\mathsf{B})$.

Remark 2 *For planar curves, the W-direction curve corresponds to the binormal direction curve. Additionally, W-direction curves of planar curves are lines that are perpendicular to the plane that the curve lies on.*

Theorem 4 *Let $\overline{\gamma}$ be the W-direction curve of a nonplanar curve γ. Then γ is a general helix if and only if $\overline{\gamma}$ is a straight line.*

Proof ($\Rightarrow$) Let γ be a general helix. Then $\frac{\tau}{\kappa} = c(constant)$. Since $\overline{\gamma}$ is the W-direction curve of γ, we have

$$\overline{\gamma}'(s) = W(s) = \frac{1}{\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}} \left(\frac{\tau}{\kappa}\mathsf{T} + \mathsf{B}\right).$$

Differentiating gives $\overline{\gamma}''(s) = \mathbf{0}$, i.e. $\overline{\kappa} = 0$. Thus, $\overline{\gamma}$ is a straight line.
($\Leftarrow$) Let $\overline{\gamma}$ be a straight line. Then the velocity $\overline{\gamma}'(s) = W(s)$ is constant. Hence,

$$\overline{\gamma}''(s) = W'(s) = \frac{\kappa(\tau'\kappa - \tau\kappa')}{(\kappa^2+\tau^2)^{3/2}}\mathsf{T} + \frac{\tau(\tau\kappa' - \tau'\kappa)}{(\kappa^2+\tau^2)^{3/2}}\mathsf{B} = \mathbf{0}.$$

Since $\kappa \neq 0$ and $\tau \neq 0$, we obtain $\tau'\kappa - \tau\kappa' = 0$, i.e. $\frac{\tau}{\kappa} = constant$. This means that γ is a general helix. □

Theorem 5 *Let γ be a Frenet curve in $\mathbb{E}^3$ with the curvature κ and the torsion τ, and $\overline{\gamma}$ be W-direction curve of γ. If γ is not a general helix, then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\overline{\kappa} = \frac{|\tau\kappa' - \tau'\kappa|}{\kappa^2+\tau^2}, \quad \overline{\tau} = \sqrt{\kappa^2+\tau^2}. \tag{3.1}$$

Proof We can use the same arc-length parameter s for γ and $\overline{\gamma}$. By the definition of the W-direction curve, we have $W(s) = \overline{\gamma}'(s) = \overline{\mathsf{T}}(s)$. Then we have

$$\overline{\mathsf{T}} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\mathsf{T} + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\mathsf{B},$$

and the curvature of $\overline{\gamma}$ is given by

$$\overline{\kappa} = \| \overline{\mathsf{T}}' \| = \sqrt{\frac{\kappa^2(\tau'\kappa - \tau\kappa')^2}{(\kappa^2+\tau^2)^3} + \frac{\tau^2(\tau\kappa' - \tau'\kappa)^2}{(\kappa^2+\tau^2)^3}}$$

or

$$\overline{\kappa} = \frac{|\tau\kappa' - \tau'\kappa|}{\kappa^2+\tau^2}.$$

If we assume $\tau\kappa' - \tau'\kappa > 0$, then the principal normal vector field $\overline{\mathsf{N}}$ and the binormal vector field $\overline{\mathsf{B}}$ of $\overline{\gamma}$ are obtained as

$$\begin{aligned}\overline{\mathsf{N}} &= \frac{-\kappa}{\sqrt{\kappa^2+\tau^2}}\mathsf{T} + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\mathsf{B},\\ \overline{\mathsf{B}} &= \overline{\mathsf{T}}\times\overline{\mathsf{N}} = -\left(\frac{\tau^2}{\kappa^2+\tau^2} + \frac{\kappa^2}{\kappa^2+\tau^2}\right)\mathsf{N} = -\mathsf{N}.\end{aligned}$$

If $\tau\kappa' - \tau'\kappa < 0$, then they have another signature.

Since $\overline{\tau} = -\left\langle \overline{\mathsf{B}}', \overline{\mathsf{N}} \right\rangle$, we get $\overline{\tau} = \sqrt{\kappa^2+\tau^2}$. □

Theorem 6 *Let $\overline{\gamma}$ be the W-direction curve of γ, which is not a general helix. Then $\overline{\gamma}$ is a general helix if and only if γ is a slant helix.*

Proof ($\Rightarrow$) Let $\overline{\gamma}$ be a general helix. Then we have $\frac{\overline{\tau}}{\overline{\kappa}} = c(\text{constant})$. Using Theorem 5, we find

$$\frac{\overline{\tau}}{\overline{\kappa}} = \frac{\sqrt{\kappa^2+\tau^2}}{\frac{\tau\kappa'-\tau'\kappa}{\kappa^2+\tau^2}} = \frac{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}{\tau\kappa'-\tau'\kappa} = c \quad \Rightarrow \quad \frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)' = \frac{1}{c}(\text{constant}).$$

This means that γ is a slant helix.

($\Leftarrow$) Let γ be a slant helix. In this case, from Thereom 3 we have $\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)' = c$ or $\frac{\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}}{|\tau'\kappa-\tau\kappa'|} = \frac{1}{c}(\text{constant})$; that is, $\frac{\overline{\tau}}{\overline{\kappa}} = \text{constant}$. This means that $\overline{\gamma}$ is a general helix. □

Definition 7 (Second W-direction curve) *Let $\overline{\gamma}$ be a W-direction curve of a Frenet curve γ and $\overline{\overline{\gamma}}$ be a W-direction curve of $\overline{\gamma}$ in $\mathbb{E}^3$. In this case we call $\overline{\overline{\gamma}}$ the second W-direction curve of γ.*

Corollary 1 *If γ is a slant helix, then the second W-direction curve of γ is a straight line.*

Definition 8 (W-rectifying curve) *Let γ be a Frenet curve and $\overline{\gamma}$ be its W-direction curve. The curve $\overline{\gamma}$ is called a W-rectifying curve if the position vector of $\overline{\gamma}$ always lies in the rectifying plane of γ.*

Theorem 7 *Let γ be a Frenet curve and $\overline{\gamma}$ its W-direction curve. If $\overline{\gamma}$ is a W-rectifying curve, then γ is a general helix.*

Proof Using the definition of a W-rectifying curve, we can write

$$\overline{\gamma} = \lambda(s)\mathsf{T}(s) + \mu(s)\mathsf{B}(s), \tag{3.2}$$

where $\lambda(s)$ and $\mu(s)$ are nonzero functions and $\{\mathsf{T},\mathsf{N},\mathsf{B}\}$ is the Frenet frame along γ. By differentiating this equation we get

$$\overline{\mathsf{T}} = \lambda'\mathsf{T} + (\lambda\kappa - \mu\tau)\mathsf{N} + \mu'\mathsf{B}. \tag{3.3}$$

On the other hand, we also have $W = \overline{\gamma}' = \overline{\mathsf{T}}$. So, from (3.3), we obtain

$$\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\mathsf{T} + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\mathsf{B} = \lambda'\mathsf{T} + (\lambda\kappa - \mu\tau)\mathsf{N} + \mu'\mathsf{B}$$

or

$$\begin{cases} \lambda\kappa - \mu\tau = 0, \\ \lambda' = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \\ \mu' = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}. \end{cases}$$

Using these equations we obtain $\lambda'\mu - \lambda\mu' = 0$. This means that $\frac{\lambda}{\mu} = c(constant)$. Then $\frac{\lambda}{\mu} = \frac{\tau}{\kappa} = c$, i.e. γ is a general helix. □

Example 1 *The W-direction curve of the circular helix* $\gamma(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$ *is*

$$\overline{\gamma}(s) = (c_1, c_2, s + c_3), \quad c_1, c_2, c_3 = constants,$$

since $W(s) = (0, 0, 1)$ *(Figure 1).*

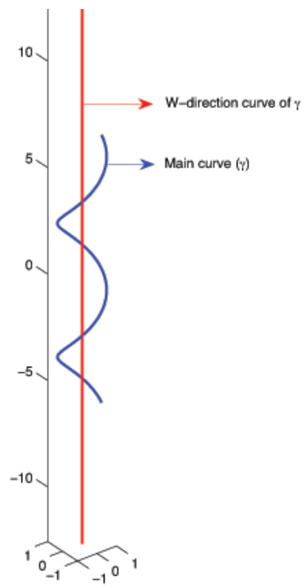

Figure 1. W-direction curve of a circular helix ($c_1 = c_2 = c_3 = 0$).

Example 2 *Let us find the W-direction curve of the slant helix*

$$\gamma(s) = \left(-\frac{3}{2}\cos\left(\frac{s}{2}\right) - \frac{1}{6}\cos\left(\frac{3s}{2}\right), -\frac{3}{2}\sin\left(\frac{s}{2}\right) - \frac{1}{6}\sin\left(\frac{3s}{2}\right), \sqrt{3}\cos\left(\frac{s}{2}\right)\right).$$

We obtain

$$\mathsf{T}(s) = \left(\frac{3}{4}\sin\left(\frac{s}{2}\right) + \frac{1}{4}\sin\left(\frac{3s}{2}\right), -\frac{3}{4}\cos\left(\frac{s}{2}\right) - \frac{1}{4}\cos\left(\frac{3s}{2}\right), -\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right)\right),$$

$$\mathsf{B}(s) = \left(-\frac{1}{2}\cos\left(\frac{s}{2}\right)\left(2\cos^2\left(\frac{s}{2}\right) - 3\right), \sin^3\left(\frac{s}{2}\right), \frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right)\right),$$

$$\kappa(s) = \frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right), \quad \tau(s) = -\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right).$$

Hence,

$$W(s) = \left(-\frac{1}{8}\left(9 + 24\cos\left(\frac{s}{2}\right) + 6\cos(s) + \cos(2s)\right), \frac{1}{2}\sin(s), \frac{\sqrt{3}}{2}\right),$$

and thus the W-direction curve (shown in Figure 2) is obtained as

$$\overline{\gamma}(s) = \left(-\frac{9s}{8} - 6\sin\left(\frac{s}{2}\right) - \frac{3}{4}\sin(s) - \frac{1}{16}\sin(2s), -\frac{1}{2}\cos(s), \frac{\sqrt{3}s}{2}\right) + (c_1, c_2, c_3), \quad c_i = constants.$$

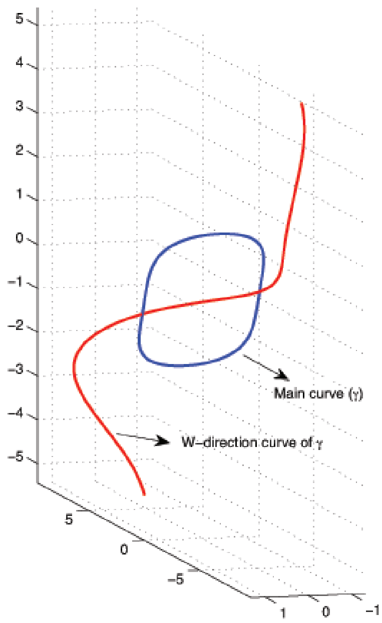

Figure 2. W-direction curve of the slant helix ($c_1 = c_2 = c_3 = 0$).

4. V-direction curves

In this section, we introduce an associated curve of a surface curve in $\mathbb{E}^3$.

Let M be an oriented surface in $\mathbb{E}^3$ and γ be a regular curve lying on M. Let us denote the Darboux frame along γ with $\{\mathsf{T}, \mathsf{V}, \mathsf{U}\}$, where T is the unit tangent vector field of γ, U is the unit normal vector field of the surface, which is restricted to the curve γ, and $\mathsf{V} = \mathsf{U} \times \mathsf{T}$.

Definition 9 (V-direction curve) *Let γ be a unit speed curve on an oriented surface M and $\{\mathsf{T}, \mathsf{V}, \mathsf{U}\}$ be the Darboux frame along γ. The curve $\overline{\gamma}$ lying on M is called the V-direction curve of γ if it is the integral curve of V. In other words, if $\overline{\gamma}$ is the V-direction curve of γ, then $\mathsf{V}(s) = \overline{\gamma}'(s)$.*

Remark 3 *It is easy to see that the V-direction curve of the planar curves coincides with the principal direction curve.*

Remark 4 a) *If the surface M is given with its parametric equation $X = X(u,v)$, then the curve $\overline{\gamma}(s) = X(p(s), q(s))$, which satisfies*

$$\mathsf{V}(s) = X_u(p(s), q(s))\frac{dp}{ds} + X_v(p(s), q(s))\frac{dq}{ds}, \tag{4.1}$$

is the V-direction curve of γ.

b) *If the surface M is given with the implicit form $f(x,y,z) = 0$, then the curve $\overline{\gamma} = (\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$, which satisfies the equations*

$$\mathsf{V}(s) = (\overline{\gamma}'_1, \overline{\gamma}'_2, \overline{\gamma}'_3) \quad \text{and} \quad f(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3) = 0, \tag{4.2}$$

is the V-direction curve of γ.

Theorem 8 *Let γ be a unit speed curve on an oriented surface M and $\overline{\gamma}$ be its V-direction curve. Then the geodesic curvature $(\overline{\kappa}_g)$, the normal curvature $(\overline{\kappa}_n)$, and the geodesic torsion $(\overline{\tau}_g)$ of $\overline{\gamma}$ are given by*

$$\begin{aligned} \overline{\kappa}_n &= \tau_g \sin\theta - \kappa_g \cos\theta, \\ \overline{\tau}_g &= -\theta' - \kappa_n, \\ \overline{\kappa}_g &= \kappa_g \sin\theta + \tau_g \cos\theta, \end{aligned} \tag{4.3}$$

where θ is the angle between T and $\overline{\mathsf{U}}$ in which T is the unit tangent vector field of γ and $\overline{\mathsf{U}}$ is the unit normal vector field of M restricted to $\overline{\gamma}$, and κ_g, κ_n, and τ_g are the geodesic curvature, the normal curvature, and the geodesic torsion of γ, respectively.

Proof Let $\{\overline{\mathsf{T}}, \overline{\mathsf{V}}, \overline{\mathsf{U}}\}$ denote the Darboux frame along $\overline{\gamma}$. We can write $\mathsf{V} = \overline{\gamma}' = \overline{\mathsf{T}}$, since $\overline{\gamma}$ is the V-direction curve of γ. Thus, $\overline{\mathsf{U}}$ is perpendicular to V. If θ denotes the angle between T and $\overline{\mathsf{U}}$, then

$$\overline{\mathsf{U}} = \mathsf{T}\cos\theta + \mathsf{U}\sin\theta, \quad \overline{\mathsf{V}} = \mathsf{U}\cos\theta - \mathsf{T}\sin\theta \tag{4.4}$$

$(-\pi < \theta < \pi$ according to the orientation). Using these equations we get

$$\overline{\kappa}_n = \langle \overline{\mathsf{T}}', \overline{\mathsf{U}} \rangle = \tau_g \sin\theta - \kappa_g \cos\theta,$$

$$\overline{\tau}_g = \langle \overline{\mathsf{V}}', \overline{\mathsf{U}} \rangle = -\theta' - \kappa_n,$$

$$\overline{\kappa}_g = \langle \overline{\mathsf{T}}', \overline{\mathsf{V}} \rangle = \kappa_g \sin\theta + \tau_g \cos\theta.$$

□

Corollary 2 a) *Let γ be a geodesic curve on M. Then:*

$\overline{\gamma}$ is also a geodesic curve $\Leftrightarrow$ γ is a principal line or $\theta = \frac{\pi}{2}$.

$\overline{\gamma}$ is an asimptotic curve $\Leftrightarrow$ γ is a principal line or $\theta = 0$.

b) *Let γ be an asymptotic curve on M. Then:*

$\overline{\gamma}$ is a principal line $\Leftrightarrow$ $\theta = constant$.

c) *Let γ be a principal line on M. Then:*

$\overline{\gamma}$ is an asymptotic curve $\Leftrightarrow$ γ is a geodesic curve or $\theta = \frac{\pi}{2}$.

$\overline{\gamma}$ is a geodesic curve $\Leftrightarrow$ γ is a geodesic curve or $\theta = 0$.

Corollary 3 *Let $\overline{\gamma}$ be the* V*-direction curve of* γ*. The geodesic curvature, normal curvature, and geodesic torsion of* $\overline{\gamma}$ *can be given by means of the curvature and the torsion of* γ *as*

$$\overline{\kappa}_n = \tau \sin\theta + \varphi' \sin\theta - \kappa \sin\varphi \cos\theta,$$

$$\overline{\tau}_g = -\theta' - \kappa \cos\varphi,$$

$$\overline{\kappa}_g = \kappa \sin\varphi \sin\theta + \tau \cos\theta + \varphi' \cos\theta,$$

where $\cos\varphi = \langle \mathsf{N}, \mathsf{U} \rangle$ *and* N *is the principal normal vector field of* γ*.*

Theorem 9 *Let* M *be an oriented surface in* $\mathbb{E}^3$*,* γ *be a unit speed curve on* M*, and* $\overline{\gamma}$ *be the* V*-direction curve of* γ*. The relations between the curvature* $(\overline{\kappa})$ *and the torsion* $(\overline{\tau})$ *of* $\overline{\gamma}$ *and the geodesic curvature, normal curvature, and geodesic torsion of* γ *are given by*

$$\overline{\kappa} = \sqrt{\kappa_g^2 + \tau_g^2}, \quad \overline{\tau} = -\kappa_n + \frac{\kappa_g \tau_g' - \kappa_g' \tau_g}{\kappa_g^2 + \tau_g^2}. \tag{4.5}$$

Proof Let us denote the Darboux frame of γ with $\{\mathsf{T}, \mathsf{V}, \mathsf{U}\}$. Since we can write $\mathsf{V} = \overline{\gamma}' = \overline{\mathsf{T}}$, we obtain $\overline{\kappa} = \|\overline{\mathsf{T}}'\| = \|\mathsf{V}'\| = \sqrt{\kappa_g^2 + \tau_g^2}$.

On the other hand, differentiating $\overline{\gamma}' = \mathsf{V}$ yields

$$\overline{\gamma}'' = -\kappa_g \mathsf{T} + \tau_g \mathsf{U}, \quad \overline{\gamma}''' = (-\kappa_g' - \tau_g \kappa_n)\mathsf{T} + (-\tau_g^2 - \kappa_g^2)\mathsf{V} + (\tau_g' - \kappa_g \kappa_n)\mathsf{U}.$$

Using these equations we get $\overline{\tau} = -\kappa_n + \frac{\kappa_g \tau_g' - \kappa_g' \tau_g}{\kappa_g^2 + \tau_g^2}$. □

Example 3 *Let us find the* V*-direction curve of* $\gamma(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$ *which lies on the circular cylinder* $X(u,v) = (\cos u, \sin u, v)$*. We have* $X\left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) = \gamma(s)$*.*

Since $X_u(u,v) = (-\sin u, \cos u, 0)$ *and* $X_v(u,v) = (0,0,1)$*, the unit normal vector field of the surface is* $\mathsf{U} = \frac{X_u \times X_v}{\|X_u \times X_v\|} = (\cos u, \sin u, 0)$*. Hence, the unit normal vector field restricted to* γ *is* $\mathsf{U}(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), 0\right)$*. Then* $\mathsf{V}(s) = \mathsf{U}(s) \times \mathsf{T}(s) = \left(\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)$*.*

Now let us look for the curve $\overline{\gamma} = X(p(s), q(s))$*, which satisfies*

$$\mathsf{V}(s) = \overline{\gamma}'(s) = X_u(p(s), q(s))\frac{dp}{ds} + X_v(p(s), q(s))\frac{dq}{ds}.$$

Since $X_u(p(s), q(s)) = (-\sin p, \cos p, 0)$ *and* $X_v(p(s), q(s)) = (0,0,1)$*, we get*

$$\mathsf{V}(s) = \left(\frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) = (-p'\sin p, p'\cos p, q')$$

or

$$\begin{cases} -p'\sin p = \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right), \\ p'\cos p = -\frac{1}{\sqrt{2}}\cos\left(\frac{s}{\sqrt{2}}\right), \\ q' = \frac{1}{\sqrt{2}}. \end{cases}$$

If we solve these differential equations, we find

$$p(s) = \frac{s}{\sqrt{2}} + (2k-1)\pi, \quad (k \in \mathbb{Z}), \quad q(s) = \frac{s}{\sqrt{2}} + c_2, \quad c_2 = constant.$$

Thus, the V*-direction curve (shown in Figure 3) of* γ *is obtained as*

$$\overline{\gamma}(s) = \left(-\cos\left(\frac{s}{\sqrt{2}}\right), -\sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} + c_2\right).$$

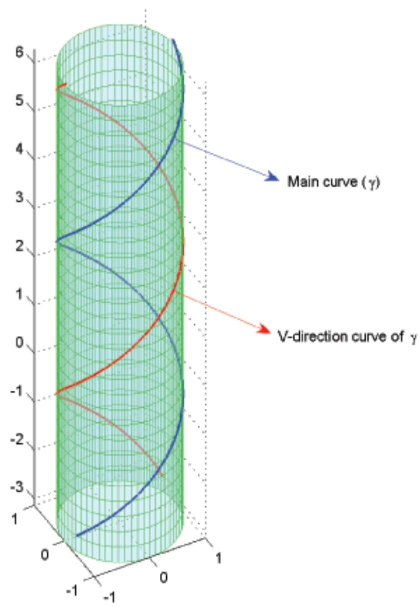

Figure 3. V-direction curve of a cylindrical helix ($c_2 = 0$).

Example 4 *Let us find the* V*-direction curve of*

$$\gamma(s) = \left(\frac{s}{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{s}{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{s}{2}\right), \quad s > 0,$$

which is lying on the circular cone $x^2 + y^2 = z^2$*.*

Let $f(x,y,z) = x^2 + y^2 - z^2$*. Then the unit normal vector field of the surface is*

$$\mathsf{U} = \frac{\nabla f}{\|\nabla f\|} = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{-z}{\sqrt{x^2+y^2+z^2}}\right).$$

Thus,

$$\mathsf{U}(s) = \left(\frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right), -\frac{1}{\sqrt{2}}\right),$$

$$\mathsf{V}(s) = \left(\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{1}{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{1}{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{1}{2}\right).$$

Therefore, we obtain the V*-direction curve (shown in Figure 4) of* γ *as*

$$\overline{\gamma}(s)=\left(-\frac{1}{6}s\left(\cos\left(\sqrt{2}\ln\frac{s}{2}\right)-2\sqrt{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right)\right),-\frac{1}{6}s\left(2\sqrt{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)+\sin\left(\sqrt{2}\ln\frac{s}{2}\right)\right),\frac{s}{2}\right).$$

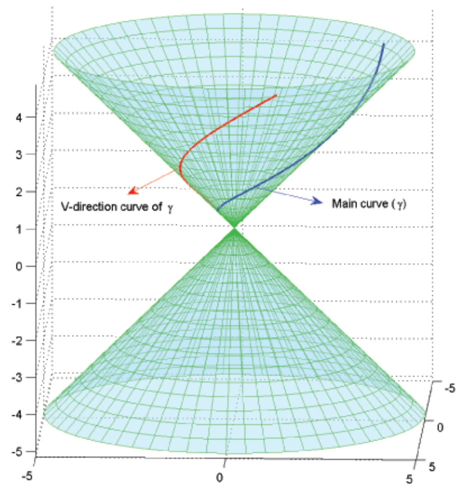

Figure 4. V-direction curve of a curve that is on a cone.

5. Associated curves of a Frenet curve in 4-dimensional Euclidean space

In this section, we define new associated curves in $\mathbb{E}^4$.

Definition 10 *Let* γ *be a Frenet curve and* $\{\mathsf{T},\mathsf{N},\mathsf{B}_1,\mathsf{B}_2\}$ *be its Frenet frame in* $\mathbb{E}^4$*.*

- *An integral curve of the principal normal vector field of* γ *is called the principal-direction curve of* γ*,*
- *An integral curve of the first binormal vector field of* γ *is called the* B_1*-direction curve of* γ*,*
- *An integral curve of the second binormal vector field of* γ *is called the* B_2*-direction curve of* γ*.*

Theorem 10 *Let* γ *be a Frenet curve whose curvatures are* k_1,k_2,k_3 *and let* $\overline{\gamma}$ *be the principal-direction curve of* γ*. The curvatures of* $\overline{\gamma}$ *are given by*

$$\overline{k}_1=\sqrt{k_1^2+k_2^2},$$

$$\overline{k}_2=\frac{\sqrt{\left(k_2'k_1-k_1'k_2\right)^2+k_2^2k_3^2\left(k_1^2+k_2^2\right)}}{\left(k_1^2+k_2^2\right)},$$

$$\overline{k}_3=\frac{\sqrt{k_1^2+k_2^2}\left[2\left(k_2'\right)^2k_1k_3+k_2'k_2\left(k_3'k_1-2k_1'k_3\right)+k_2\left(-k_1'k_3'k_2+k_3\left(-k_2''k_1+k_2k_1''+k_1k_2k_3^2\right)\right)\right]}{\left(k_2'k_1-k_1'k_2\right)^2+k_2^2k_3^2\left(k_1^2+k_2^2\right)}.$$

Proof Let $\{\overline{\mathsf{T}}, \overline{\mathsf{N}}, \overline{\mathsf{B}}_1, \overline{\mathsf{B}}_2, \overline{k}_1, \overline{k}_2, \overline{k}_3\}$ be the Frenet apparatus of $\overline{\gamma}$. By the definition of the principal-direction curve, we may write $\mathsf{N}(s) = \overline{\gamma}'(s) = \overline{\mathsf{T}}(s)$. Hence, $\overline{\mathsf{T}}'(s) = \mathsf{N}'(s) = -k_1 T + k_2 B_1$. The first curvature of $\overline{\gamma}$ is then given by $\overline{k}_1 = \sqrt{k_1^2 + k_2^2}$.

If we use Theorem 1, we find the principal normal vector field and the first and second binormal vector fields of $\overline{\gamma}$ as

$$\overline{\mathsf{N}} = \frac{-k_1}{\sqrt{k_1^2 + k_2^2}} \mathsf{T} + \frac{k_2}{\sqrt{k_1^2 + k_2^2}} \mathsf{B}_1,$$

$$\overline{\mathsf{B}}_1 = \frac{1}{\sqrt{k_2^4 k_3^2 + k_1^2 k_2^2 k_3^2 + (k_1' k_2 - k_2' k_1)^2}} \left[\frac{-k_1' k_2^2 + k_2' k_1 k_2}{\sqrt{k_1^2 + k_2^2}} \mathsf{T} + \frac{k_1^2 k_2' - k_1 k_2 k_1'}{\sqrt{k_1^2 + k_2^2}} \mathsf{B}_1 + \frac{k_2^3 k_3 + k_1^2 k_2 k_3}{\sqrt{k_1^2 + k_2^2}} \mathsf{B}_2 \right],$$

$$\overline{\mathsf{B}}_2 = -\frac{k_2^2 k_3 \mathsf{T} + k_1 k_2 k_3 \mathsf{B}_1 + (k_1' k_2 - k_2' k_1) \mathsf{B}_2}{\sqrt{k_2^4 k_3^2 + k_1^2 k_2^2 k_3^2 + (k_1' k_2 - k_2' k_1)^2}}.$$

Thus, using Theorem 2, the second and the third curvatures of $\overline{\gamma}$ are obtained from

$$\overline{k}_2(s) = \frac{\langle \overline{\mathsf{B}}_1(s), \overline{\gamma}'''(s) \rangle}{\overline{k}_1(s)} \quad \text{and} \quad \overline{k}_3(s) = \frac{\langle \overline{\mathsf{B}}_2(s), \overline{\gamma}^{(4)}(s) \rangle}{\overline{k}_1(s) \overline{k}_2(s)}.$$

□

Theorem 11 *Let γ be a Frenet curve whose curvatures are k_1, k_2, k_3 and let $\widehat{\gamma}$ be the B_1-direction curve of γ. The curvatures of $\widehat{\gamma}$ are given by*

$$\widehat{k}_1 = \sqrt{k_2^2 + k_3^2},$$

$$\widehat{k}_2 = \frac{\sqrt{k_2^4 k_1^2 + k_1^2 k_2^2 k_3^2 + (k_3' k_2 - k_2' k_3)^2}}{k_2^2 + k_3^2},$$

$$\widehat{k}_3 = \frac{\sqrt{k_2^2 + k_3^2} \left(2 (k_2')^2 k_1 k_3 + k_2 k_2' (-2 k_3' k_1 + k_1' k_3) + k_2 \left(-k_1' k_2 k_3' + k_3'' k_1 k_2 - k_2'' k_1 k_3 + k_1^3 k_2 k_3 \right) \right)}{k_2^4 k_1^2 + k_1^2 k_2^2 k_3^2 + (k_3' k_2 - k_2' k_3)^2}.$$

Theorem 12 *Let γ be a Frenet curve whose curvatures are k_1, k_2, k_3 and let $\widetilde{\gamma}$ be its B_2-direction curve. The curvatures of $\widetilde{\gamma}$ are given by*

$$\widetilde{k}_1 = |k_3(s)|, \quad \widetilde{k}_2 = k_2(s), \quad \widetilde{k}_3 = sgn(k_3) k_1(s).$$

Proof Let $\{\widetilde{\mathsf{T}}, \widetilde{\mathsf{N}}, \widetilde{\mathsf{B}}_1, \widetilde{\mathsf{B}}_2, \widetilde{k}_1, \widetilde{k}_2, \widetilde{k}_3\}$ be the Frenet apparatus of $\widetilde{\gamma}$. By the definition of the B_2-direction curve, we can write $\mathsf{B}_2(s) = \widetilde{\gamma}'(s) = \widetilde{\mathsf{T}}(s)$, which yields $\widetilde{\mathsf{T}}'(s) = \mathsf{B}_2'(s) = -k_3 \mathsf{B}_1$. The first curvature of $\widetilde{\gamma}$ is then given by $\widetilde{k}_1 = |k_3(s)|$.

Additionally, the principal normal vector field, the first-binormal vector field, and the second-binormal vector field of $\widetilde{\gamma}$ are obtained as

$$\widetilde{\mathsf{N}} = -\mathrm{sgn}(k_3) \mathsf{B}_1(s), \quad \widetilde{\mathsf{B}}_1 = \mathrm{sgn}(k_3) \mathsf{N}(s), \quad \widetilde{\mathsf{B}}_2 = -\mathsf{T}(s).$$

Thus,

$$\widetilde{k}_2(s) = \frac{\left\langle \widetilde{\mathsf{B}}_1(s), \widetilde{\gamma}'''(s) \right\rangle}{\widetilde{k}_1(s)} = k_2(s), \quad \widetilde{k}_3(s) = \frac{\left\langle \widetilde{\mathsf{B}}_2(s), \widetilde{\gamma}^{(4)}(s) \right\rangle}{\widetilde{k}_1(s)\widetilde{k}_2(s)} = \operatorname{sgn}(k_3)k_1(s).$$

□

Theorem 13 **1)** *Let γ be a Frenet curve in $\mathbb{E}^4$ and $\overline{\gamma}$ be the principal-direction curve of γ. Then γ is a slant helix $\Leftrightarrow$ $\overline{\gamma}$ is a general helix.*

2) *Let γ be a Frenet curve in $\mathbb{E}^4$ and $\widetilde{\gamma}$ be the B_2-direction curve of γ. Then γ is a B_2-slant helix $\Leftrightarrow$ $\widetilde{\gamma}$ is a general helix.*

Proof Let $\{\mathsf{T}, \mathsf{N}, \mathsf{B}_1, \mathsf{B}_2\}$ denote the Frenet frame of γ.

1) By the definition of the principal-direction curve, we have $\mathsf{N}(s) = \overline{\gamma}'(s) = \overline{\mathsf{T}}(s)$. Hence,

γ is a slant helix $\Leftrightarrow$ $\langle \mathsf{N}, \mathbf{u} \rangle = \cos\theta$, ($\theta$=constant, $\mathbf{u}$=constant unit vector)

$\Leftrightarrow$ $\langle \overline{\mathsf{T}}, \mathbf{u} \rangle = \cos\theta$

$\Leftrightarrow$ $\overline{\gamma}$ is a general helix.

2) By the definition of the B_2-direction curve, we have $\mathsf{B}_2(s) = \widetilde{\gamma}'(s) = \widetilde{\mathsf{T}}(s)$. Hence,

γ is a B_2-slant helix $\Leftrightarrow$ $\langle \mathsf{B}_2, \mathbf{v} \rangle = \cos\theta$, ($\theta$=constant, $\mathbf{v}$=constant unit vector)

$\Leftrightarrow$ $\langle \widetilde{\mathsf{T}}, \mathbf{v} \rangle = \cos\theta$

$\Leftrightarrow$ $\widetilde{\gamma}$ is a general helix. □

Definition 11 *Let γ be a Frenet curve whose Frenet apparatus is $\{\mathsf{T}, \mathsf{N}, \mathsf{B}_1, \mathsf{B}_2, k_1, k_2, k_3\}$ in $\mathbb{E}^4$ and $\overline{\gamma}$ be the B_2-direction curve of γ. The curve $\overline{\gamma}$ in $\mathbb{E}^4$ with nonzero curvatures is called a B_2-rectifying curve if its position vector always lies in the orthogonal complement of the principal normal vector of γ.*

For a B_2-rectifying curve $\overline{\gamma}$, we may write

$$\overline{\gamma} = \lambda_1 \mathsf{T} + \lambda_2 \mathsf{B}_1 + \lambda_3 \mathsf{B}_2, \tag{5.1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are differentiable functions.

Theorem 14 *Let $\overline{\gamma}$ be a Frenet curve in $\mathbb{E}^4$ whose curvatures are $\overline{k}_1, \overline{k}_2, \overline{k}_3$. $\overline{\gamma}$ is congruent to a B_2-rectifying curve if and only if there exists a constant c such that*

$$-\left(\frac{c\overline{k}_3}{\overline{k}_2}\right)' + \left(s - c\int \frac{\overline{k}_1\overline{k}_3}{\overline{k}_2} ds\right)\overline{k}_1 = 0. \tag{5.2}$$

Proof ($\Rightarrow$) Let $\overline{\gamma}$ be a B_2-rectifying curve. By the definition, $\overline{\gamma}$ is the B_2-direction curve of the curve γ. Thus, from Theorem 12 we have $\mathsf{T} = -\overline{\mathsf{B}}_2$, $\mathsf{B}_1 = -\operatorname{sgn}(k_3)\overline{\mathsf{N}}$, $\mathsf{B}_2 = \overline{\mathsf{T}}$. Substituting these equations into (5.1) and differentiating yields

$$\begin{cases} \lambda_3' + \operatorname{sgn}(k_3)\lambda_2\overline{k}_1 = 1, \\ -\operatorname{sgn}(k_3)\lambda_2' + \lambda_3\overline{k}_1 = 0, \\ \lambda_1\overline{k}_3 - \operatorname{sgn}(k_3)\lambda_2\overline{k}_2 = 0, \\ \lambda_1' = 0. \end{cases}$$

The last equation gives us $\lambda_1 = c = constant$. Substituting this into the third equation yields $\lambda_2 = \frac{c\overline{k}_3}{\mathrm{sgn}(k_3)\overline{k}_2}$ and, thus, from the first equation we get $\lambda_3 = s - c\int \frac{\overline{k}_1\overline{k}_3}{\overline{k}_2}ds$. Substituting the obtained results into the second equation gives

$$-\left(\frac{c\overline{k}_3}{\overline{k}_2}\right)' + \left(s - c\int \frac{\overline{k}_1\overline{k}_3}{\overline{k}_2}ds\right)\overline{k}_1 = 0.$$

($\Leftarrow$) Let $\overline{\gamma}$ be an arbitrary curve in $\mathbb{E}^4$ whose curvatures satisfy (5.2). Let us consider the vector

$$X = \overline{\gamma} + c\overline{\mathsf{B}}_2 + \frac{c\overline{k}_3}{\overline{k}_2}\overline{\mathsf{N}} - \left(s - c\int \frac{\overline{k}_1\overline{k}_3}{\overline{k}_2}ds\right)\overline{\mathsf{T}}.$$

By differentiating we get $X' = 0$. This means that X is a constant vector. Hence,

$$\overline{\gamma} = -c\overline{\mathsf{B}}_2 - \frac{c\overline{k}_3}{\overline{k}_2}\overline{\mathsf{N}} + \left(s - c\int \frac{\overline{k}_1\overline{k}_3}{\overline{k}_2}ds\right)\overline{\mathsf{T}} + X.$$

If we substitute $\overline{\mathsf{B}}_2 = -\mathsf{T}$, $\overline{\mathsf{N}} = -\mathrm{sgn}(k_3)\mathsf{B}_1$, $\overline{\mathsf{T}} = \mathsf{B}_2$ into the last equation, we obtain

$$\overline{\gamma} = c\mathsf{T} + \mathrm{sgn}(k_3)\frac{c\overline{k}_3}{\overline{k}_2}\mathsf{B}_1 + \left(s - c\int \frac{\overline{k}_1\overline{k}_3}{\overline{k}_2}ds\right)\mathsf{B}_2 + X,$$

i.e. $\overline{\gamma}$ is congruent to a B_2-rectifying curve with a translation by the constant vector X. □

Acknowledgment

The authors would like to thank the reviewer for the valuable comments and suggestions.

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