

## SOME NEW CONTINUITY CONCEPTS FOR METRIC PROJECTIONS

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**1. Introduction.** There has been much recent interest in studying various continuity criteria for the set-valued metric projection onto a set  $V$ . Particular interest has centered around the relationship between these criteria and either the structure of the set  $V$  itself or the geometry of the whole space. See, for example, [3], [4], [7], [8], [10], [16], [17], [18] and [21]. In essentially all of these papers, the concepts of lower semicontinuity (l.s.c.) and/or upper semicontinuity (u.s.c.) for set-valued mappings (as defined, for example, in Hahn [12]) played the key role.

In this note we introduce some simpler and more general "radial" continuity criteria. Roughly speaking, these criteria require that the restriction of the metric projection to certain prescribed line segments be l.s.c. or u.s.c. It turns out that these criteria, which are formally much weaker than l.s.c. or u.s.c., are still strong enough to generalize a number of known results, and weak enough so that many of these theorems now have valid converses (which they did not have under the stronger hypotheses of l.s.c. or u.s.c.).

Full details of proofs along with additional results and related material will appear elsewhere.

Throughout this note  $X$  will denote a (real or complex) normed linear space. For  $x \in X$  and  $r > 0$ , let

$$S(x, r) = \{y \in X : \|x - y\| = r\}.$$

The distance from a point  $x$  to a subset  $V$  of  $X$  is defined by

$$d(x, V) = \inf\{\|x - v\| : v \in V\}.$$

The *metric projection* onto a subset  $V$  of  $X$  is the mapping  $P_V$  which associates with each  $x \in X$  its set of best approximations in  $V$ , i.e.

$$P_V(x) = \{v \in V : \|x - v\| = d(x, V)\}.$$

$V$  is called *proximal* (resp. *Chebyshev*) provided  $P_V(x)$  contains at least (resp. exactly) one point for each  $x \in X$ .  $V$  is called a *sun* if for each  $x \in X$

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and  $v \in P_V(x)$ ,  $v \in P_V(v + \lambda(x - v))$  for every  $\lambda \geq 0$ . The convex hull of a set  $A$  is denoted  $\text{co}(A)$ . The notation  $(x, y)$  is used for the open interval  $\{\lambda x + (1 - \lambda)y : 0 < \lambda < 1\}$ .  $P_V$  is said to be l.s.c. (resp. u.s.c.) at  $x_0$  if, for open set  $W$  with  $W \cap P_V(x_0) \neq \emptyset$  (resp.  $W \supset P_V(x_0)$ ), there exists a neighborhood  $U$  of  $x_0$  such that  $W \cap P_V(x) \neq \emptyset$  (resp.  $W \supset P_V(x)$ ) for every  $x \in U$ .

All other undefined notation or terminology can be found in [11].

**2. ORL continuity.** The first generalization of l.s.c. is

**DEFINITION 2.1.** Let  $V \subset X$  and  $x_0 \in X$ .  $P_V$  is said to be outer radially lower (abbreviated ORL) continuous at  $x_0$  if for each  $v_0 \in P_V(x_0)$  and each open set  $W$  with  $W \cap P_V(x_0) \neq \emptyset$ , there exists a neighborhood  $U$  of  $x_0$  such that  $W \cap P_V(x) \neq \emptyset$  for every  $x$  in  $U \cap \{v_0 + \lambda(x_0 - v_0) : \lambda \geq 1\}$ .  $P_V$  is called ORL continuous if it is ORL continuous at every point.

**THEOREM 2.2.** Let  $V \subset X$  and consider the following statements.

- (1)  $V$  is a sun.
- (2)  $P_V$  is ORL continuous.
- (3) "Local best approximations are global", i.e., for each  $x \in X$ , every local minimum of the function  $\Phi(v) = \|x - v\|$  on  $V$  is a global minimum.
- (4)  $V$  is a "moon" (in the sense of [1]).

Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4).

The implication (3)  $\Rightarrow$  (2) is false in general (e.g. take  $V$  to be the complement of the open unit ball in the Euclidean plane). Also, the implication (4)  $\Rightarrow$  (3) is false in general. However, it is an open question whether (2)  $\Rightarrow$  (1).

We call a space  $X$  an MS-space if every moon in  $X$  is a sun. In such a space all the conditions of Theorem 2.2 are obviously equivalent. In [1], the spaces of type  $C(T)$ ,  $T$  compact Hausdorff (or more generally  $C_0(T)$ ,  $T$  locally compact Hausdorff) and  $l_1(S)$  for any set  $S$ , were shown to be MS-spaces. On the negative side, no strictly convex space is a MS-space. In particular, we can state

**COROLLARY 2.3.** Let  $X$  be an MS-space and  $V \subset X$ . Then  $V$  is a sun if and only if  $P_V$  is ORL continuous.

It is interesting to compare this result with a particular consequence of two theorems of Vlasov ([21, Theorem 7] and [20, Theorem 13]) which establishes the hard part of the following theorem (cf. also Asplund [2] for an alternate proof):

**THEOREM.** A Chebyshev set  $V$  in a Hilbert space is a sun (i.e. is convex) if and only if  $P_V$  is continuous.

It is still unknown whether every Chebyshev set in a Hilbert space is convex. In fact, it is apparently unknown whether there exists a Chebyshev set in *any* space which is not a sun.

**3. IRL continuity.** We now give a second generalization of l.s.c.

**DEFINITION 3.1.** Let  $V \subset X$  and  $x_0 \in X$ .  $P_V$  is said to be inner radially lower (abbreviated IRL) continuous at  $x_0$  if, for every  $v_0 \in P_V(x_0)$  and each open set  $W$  with  $W \cap P_V(x_0) \neq \emptyset$ , there exists a neighborhood  $U$  of  $x_0$  such that  $W \cap P_V(x) \neq \emptyset$  for every  $x$  in  $U \cap \{v_0 + \lambda(x_0 - v_0) : 0 \leq \lambda \leq 1\}$ .  $P_V$  is called IRL continuous if it is IRL continuous at each point.

**THEOREM 3.2.** *If  $P_V(x)$  is convex, then  $P_V$  is IRL continuous at  $x$ . In particular, if  $V$  is convex or Chebyshev,  $P_V$  is IRL continuous.*

Taking  $X$  to be the plane endowed with the maximum norm,  $V$  to be the two point set  $\{(1, 0), (1, \frac{1}{2})\}$  and  $x = (0, 0)$  shows that the converse of Theorem 3.2 is false.

**THEOREM 3.3.** *Let  $V \subset X$  be proximal. Suppose that either  $X$  is smooth or every convex extremal subset of  $S(0, 1)$  is finite-dimensional. Then, for each  $x \in X \setminus V$ , there exists  $v \in P_V(x)$  such that*

$$\text{co}(P_V(y)) \subset S(y, d(y, V))$$

*for every  $y \in (x, v)$ . In particular, the set  $\{x \in X : \text{co}(P_V(x)) \subset S(x, d(x, V))\}$  is dense in  $X$ .*

Some condition like the smoothness of  $X$  or the finite-dimensionality of the faces of  $S(0, 1)$  is essential in Theorem 3.3 as can be seen by considering  $X = L_\infty[0, 1]$  and  $V = S(0, 1)$ .

The following three corollaries are more or less immediate consequences of Theorem 3.3.

**COROLLARY 3.4** (STEČKIN [19]). *Let  $V$  be a proximal subset of a strictly convex space  $X$ . Then the set  $\{x \in X : x \text{ has a unique best approximation in } V\}$  is dense in  $X$ .*

**COROLLARY 3.5.** *Let  $V$  be a proximal subset of a strictly convex space. Then  $P_V$  is IRL continuous if and only if  $V$  is Chebyshev.*

In the special case when  $P_V$  is Hausdorff continuous (resp. l.s.c.), the “only if” part of Corollary 3.5 had been established by Blatter, Morris and Wulbert [4] (resp. Blatter [5]). It is noteworthy, however, that the converse of their results are not valid. This follows from the recent example

of Kripke [14] of a Chebyshev subspace, having a discontinuous metric projection, in a reflexive strictly convex space.

A subset  $V$  is called *boundedly compact* if the intersection of  $V$  with every closed ball is compact.

**COROLLARY 3.6.** *Let  $X$  be a strictly convex and smooth Banach space and let  $V \subset X$  be boundedly compact. The following are equivalent.*

- (1)  $P_V$  is l.s.c.
- (2)  $P_V$  is IRL continuous.
- (3)  $V$  is Chebyshev.
- (4)  $V$  is convex.

This result follows from Corollary 3.5 and the result of Vlasov [20] that in a smooth Banach space every boundedly compact Chebyshev set is convex. The equivalence of (1), (3), and (4) had been observed earlier by Blatter, Morris and Wulbert [4].

**4. ORU continuity.** A generalization of u.s.c. is given by

**DEFINITION 4.1.** Let  $V \subset X$  and  $x_0 \in X$ .  $P_V$  is called outer radially upper (abbreviated ORU) continuous at  $x_0$  if, for each  $v_0 \in P_V(x_0)$  and each open set  $W \supset P_V(x_0)$ , there exists a neighborhood  $U$  of  $x_0$  such that  $W \supset P_V(x)$  for every  $x$  in  $U \cap \{v_0 + \lambda(x_0 - v_0) : \lambda \geq 1\}$ .  $P_V$  is called ORU continuous if it is ORU continuous at each point.

**THEOREM 4.2.** *Let  $V \subset X$  be proximal and suppose  $P_V(x)$  is convex for every  $x \in X$ . If  $P_V$  is ORU continuous, then  $P_V(x)$  is compact for every  $x \in X$ .*

This theorem generalizes one of Singer's [18] who had proved it in the particular case when  $V$  is a subspace and  $P_V$  is u.s.c. Theorem 4.2 is false in general if  $P_V$  is not convex-valued. To see this, we need only take  $X = l_2$  and  $V$  to be the complement of the open unit ball. Then  $P_V$  is ORU continuous but  $P_V(0) = S(0, 1)$  is not compact.

**THEOREM 4.3.** *Let  $V$  be a sun such that  $P_V(x)$  is compact for every  $x \in X$ . Then  $P_V$  is ORU continuous.*

Combining Theorems 4.2 and 4.3 we obtain

**COROLLARY 4.4.** *Let  $V$  be a proximal convex set. Then  $P_V$  is ORU continuous if and only if  $P_V(x)$  is compact for every  $x$ .*

It is worth mentioning that Corollary 4.4 is false with u.s.c. in place of ORU continuity, even if  $V$  is a subspace. This follows since there exist Chebyshev subspaces with discontinuous metric projections. (The first such example was given by Lindenstrauss [15, pp. 87–88].)

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